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Quasi-arithmetic means and ratios of an interval induced from weighted aggregation operations

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Abstract In this paper we deal with weighted quasi-arithmetic means of an interval using utility functions in decision making. The mean values are discussed from the viewpoint of weighted aggregation operators, and they are given as weighted aggregated values of each point in the interval. The properties of the weighted quasi-arithmetic mean and its translation invariance are investigated. For the application in economics, we demonstrate the decision maker's attitude based on his utility by the weighted quasi-arithmetic mean and the aggregated mean ratio. Several examples of the weighted quasi-arithmetic mean and the aggregated mean ratio for various typical utility functions are shown to understand our motivation and for the applications in decision making.

Keywords Quasi-arithmetic mean · Weighted aggregation operator · Utility function · Mean ratio · Translation invariance

1 Introduction

The arithmetic mean of an interval [a, b], where a and b are real numbers, is given by the middle value $(a + b)/2$ of the both endpoints a and b. However, we need to estimate data subjectively in decision making such as management, artificial intelligent and so on, and then we often use utility functions (Fishburn 1970; Fodor and Roubens 1994; Yoshida 2003a, b). This paper deals with quasi-arithmetic means of an interval evaluated under decision maker's subjective utility for the application in economics. In this paper, we discuss weighted quasi-arithmetic means of an interval from the viewpoint of *weighted aggregation operators*, extending the results in Yoshida (2008). Kolmogorov (1930) and Nagumo (1930) studied aggregation operators and they verified under some assumptions that the aggregated value of real numbers $x_1, x_2, \ldots, x_n$ ($x_i \in [0, 1], i = 1, 2, \ldots, n$) is represented as

$$\xi^n(x_1, x_2, \ldots, x_n) = f^{-1}\left(\sum_{i=1}^{n} f(x_i) w_i\right) \tag{1}$$

with a continuous strictly increasing function $f : [0, 1] \mapsto [0, 1]$ and weights $\{w_i | i = 1, 2, \ldots, n\}$ such that $w_i > 0$ ($i = 1, 2, \ldots, n$) and $\sum_{i=1}^{n} w_i = 1$. Starting from this Eq. (1), in Sect. 2 we introduce a mean of an interval by an aggregated value of all points in the interval through a utility function, and we call the value a *weighted quasi-arithmetic mean*. In Sects. 3 and 4, we investigate the fundamental properties of the weighted quasi-arithmetic mean and its translation invariance. Next, in Sect. 5, we introduce an *aggregated mean ratio* of the weighted quasi-arithmetic mean in the interval by an interior ratio, and we demonstrate the relation among the weighted quasi-arithmetic mean, the aggregated mean ratio and the decision maker's preference/attitude based on his utility. In economics, the decision maker's attitudes, for example neutral, risk averse and risk loving, are characterized by Arrow–Pratt index of the utility function (Arrow 1971; Pratt 1964; Gollier 2001; Gollier et al. 2005). This paper characterizes the decision maker's attitudes by the utility functions and the weighted quasi-arithmetic means. Next we discuss the properties of the weighted quasi-arithmetic means and the aggregated mean ratios regarding combinations of utility functions. In

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local/global regions we also investigate the movement of the weighted quasi-arithmetic means and the aggregated mean ratios, which is sensitive with respect to the length of the interval. In Sect. 6, we show several examples of the weighted quasi-arithmetic means and the aggregated mean ratios with various typical utility functions, and we give their relations with the classical quasi-arithmetic means. Finally, in Appendix, we give the proofs of the theorems, the propositions and the lemmas in Sects. 3–5.

2 Weighted aggregation operators

In this section, we introduce weighted quasi-arithmetic means of an interval by weighted aggregation operations from the viewpoint of subjective decision making. Kolmogoroff (1930) and Nagumo (1930) studied the aggregation operators in the form (1), and Aczél (1948) developed the theory regarding weighted aggregation operations. First, we start from the notion of weighted aggregation operations with respect to finite number of variables on [0, 1], and next we construct a weighted quasi-arithmetic mean of an interval step by step. Let n be a fixed positive integer, and let $\xi^n : [0,1]^n \mapsto [0,1]$ be a function. We represent it as $\xi^n(x_1, x_2, \ldots, x_n)$ for $(x_1, x_2, \ldots, x_n) \in [0,1]^n$.

Definition 2.1 [n-ary weighted aggregation operator (Calvo et al. 2002; Calvo and Pradera 2004)] A function $\xi^n : [0,1]^n \mapsto [0,1]$ is called an *n-ary weighted aggregation operator* if it satisfies the following conditions (A.i)–(A.v):

(A.i) $\xi^n(x_1, x_2, \ldots, x_n) \le \xi^n(y_1, y_2, \ldots, y_n)$ whenever $x_i \le y_i$ for all $i = 1, 2, \ldots, n$.

(A.ii) $\xi^n(x_1, x_2, \ldots, x_n) < \xi^n(y_1, y_2, \ldots, y_n)$ if $x_i \le y_i$ for all $i = 1, 2, \ldots, n$ and $x_j < y_j$ for some $j = 1, 2, \ldots, n$.

(A.iii) ξ^n is continuous on $[0,1]^n$.

(A.iv) $\xi^n(x, x, \ldots, x) = x$ for all $x \in [0,1]$.

(A.v) It holds that

$$\begin{aligned}\xi^n(\xi^n(x_{11}, \ldots, x_{1n}), \ldots, \xi^n(x_{n1}, \ldots, x_{nn}))\\ = \quad \xi^n(\xi^n(x_{11}, \ldots, x_{n1}), \ldots, \xi^n(x_{1n}, \ldots, x_{nn})).\end{aligned}$$

We can find another definition of n-ary aggregation in Calvo et al. (2002), Calvo and Pradera (2004), which requires (A.i) only with boundary conditions. However, in this paper we introduce the n-ary aggregation by Definition 2.1 to discuss quasi-arithmetic means (5) with a continuous utility function f. The condition (A.ii) together with (A.i) is said to be *strictly monotone*, and the properties (A.iii), (A.iv) and (A.v) are said to be *continuous*, *idempotent* and *bisymmetrical*, respectively.

The following well-known result regarding the weighted aggregation operations is given by Aczél (1948).

Lemma 2.2 [Aczél (1948)] *A function* $\xi^n : [0,1]^n \mapsto [0,1]$ *satisfies* (A.i)–(A.v) *if and only if there exists a continuous strictly increasing function* $f : [0,1] \mapsto [0,1]$ *and weights* $\{w_i | i = 1, 2, \ldots, n\}$ *such that* $w_i > 0\,(i = 1, 2, \ldots, n)$, $\sum_{i=1}^n w_i = 1$ *and*

$$\xi^n(x_1, x_2, \ldots, x_n) = f^{-1}\left(\sum_{i=1}^n f(x_i) w_i\right) \quad (2)$$

for $(x_1, x_2, \ldots, x_n) \in [0,1]^n$.

Hence, weighted aggregation operators are given as follows.

Definition 2.3 [Weighted aggregation operator (Calvo et al. 2002; Calvo and Pradera 2004)] A function $\xi : \bigcup_{n \ge 1} [0,1]^n \mapsto [0,1]$ is called a *weighted aggregation operator* if it is given by n-ary weighted aggregation operators ξ^n such as $\xi = \xi^n$ on $[0,1]^n$ for each $n = 1, 2, \ldots$.

In this paper, we take a continuous strictly increasing function $f : [0,1] \mapsto [0,1]$ as a decision maker's utility function, and we put a continuous function $w : [0,1] \mapsto (0, \infty)$ as a weighting function. Let n be a positive integer. Define a function $\xi^n : [0,1]^n \mapsto [0,1]$ by

$$\xi^n(x_1, x_2, \ldots, x_n) = f^{-1}\left(\sum_{i=1}^n f(x_i) w(x_i) \bigg/ \sum_{i=1}^n w(x_i)\right)$$

for $(x_1, x_2, \ldots, x_n) \in [0,1]^n$. Then, this term is an n-ary weighted aggregation operator (2) with weights $w_i = w(x_i)/\sum_{j=1}^n w(x_j)$ $(i = 1, 2, \ldots, n)$, and by Lemma 2.2 it has the properties (A.i)–(A.v). Further, by Definition 2.3, a weighted aggregation operator $\xi : \bigcup_{n \ge 1} [0,1]^n \mapsto [0,1]$ is given by

$$\begin{aligned}\xi(x_1, x_2, \ldots, x_n) &= \xi^n(x_1, x_2, \ldots, x_n)\\ &= f^{-1}\left(\sum_{i=1}^n f(x_i) w(x_i) \bigg/ \sum_{i=1}^n w(x_i)\right) \quad (3)\end{aligned}$$

for $(x_1, x_2, \ldots, x_n) \in [0,1]^n$ and $n = 1, 2, \ldots$. Next we construct a weighted quasi-arithmetic mean of an interval under the utility function f from the viewpoint of aggregation of all points in the interval. Let $[a, b]$ be a closed interval satisfying $0 \le a < b \le 1$. Let $\{[c_{i-1}, c_i] | i = 1, 2, \ldots, n\}$ be a partition of the interval $[a, b]$ such that

$$c_i = a + \frac{i(b-a)}{n}$$

for $i = 0, 1, 2, \ldots, n$. Take a point x_i on the interval $[c_{i-1}, c_i]$ such that $x_i \in [c_{i-1}, c_i]$ for each $i = 1, 2, \ldots, n$. From (3), we define a weighted quasi-arithmetic mean of the interval $[a, b]$ as follows:

$$M^f([a,b]) = \lim_{n\to\infty} \xi(x_1,x_2,\ldots,x_n) = \lim_{n\to\infty} f^{-1}\left(\sum_{i=1}^{n} f(x_i)w(x_i) \Big/ \sum_{i=1}^{n} w(x_i)\right). \quad (4)$$

Hence, we can understand the mean (4) as the aggregated value of all points distributed on [*a*, *b*] with the weight *w* under the utility *f* since $\xi(x_1,x_2,\ldots,x_n)$ is an aggregated value of reference points x_i on the small interval $[c_{i-1},c_i]\,(i=1,2,\ldots,n)$ based on the aggregation operator defined by the utility function *f*. By the definition of Riemann integral, we obtain

$$M^f([a,b]) = f^{-1}\left(\int_a^b f(x)w(x)\,\mathrm{d}x \Big/ \int_a^b w(x)\,\mathrm{d}x\right) \quad (5)$$

for $[a,b](\subset[0,1])$ such that $0\le a<b\le 1$ since we have

$$M^f([a,b]) = \lim_{n\to\infty} f^{-1}\left(\sum_{i=1}^{n} f(x_i)w(x_i) \Big/ \sum_{i=1}^{n} w(x_i)\right) = f^{-1}\left(\int_a^b f(x)w(x)\,\mathrm{d}x \Big/ \int_a^b w(x)\,\mathrm{d}x\right).$$

Hence, $M^f([a,b])$ represents a *mean value* given by a real number $c(\in[a,b])$ satisfying

$$f(c)\int_a^b w(x)\,\mathrm{d}x = \int_a^b f(x)w(x)\,\mathrm{d}x$$

in the *mean value theorem*. Now let $D(\subset(-\infty,\infty))$ be an interval. Extending the domain from the closed interval [0, 1] to *D*, in Sects. 3 and 4, we demonstrate the properties of *weighted quasi-arithmetic mean* of closed subintervals of *D* in the form (5) for $[a,b]\subset D(a<b)$, where $f: D\mapsto(-\infty,\infty)$ is a continuous strictly increasing function for utility and $w: D\mapsto(0,\infty)$ is a continuous function for weighting.

3 Weighted quasi-arithmetic means

Let $\mathbb{R}=(-\infty,\infty)$ be the set of all real numbers. For two bounded closed intervals [*a*, *b*] and [*c*, *d*], we give a partial order $\preceq$ as follows:

$$[a,b]\preceq[c,d] \iff a\le c \text{ and } b\le d.$$

Let *D* be a fixed interval which is not a singleton and we call it a domain. Let $\mathcal{I}(D)$ be the set of all nonempty subintervals of *D* and let $\mathcal{C}(D)$ be the set of all nonempty bounded closed subintervals of *D*.

For a continuous strictly increasing function $f: D\mapsto\mathbb{R}$ for utility and a continuous function $w: D\mapsto(0,\infty)$ for weighting, we define the weighted quasi-arithmetic mean $M^f: \mathcal{C}(D)\mapsto D$ by

$$M^f([a,b]) = \begin{cases} f^{-1}\left(\int_a^b f(x)w(x)\mathrm{d}x \Big/ \int_a^b w(x)\mathrm{d}x\right) & \text{if } a<b \\ \lim_{c\downarrow a} f^{-1}\left(\int_a^c f(x)w(x)\mathrm{d}x \Big/ \int_a^c w(x)\mathrm{d}x\right) & \text{if } a=b<\sup D \\ \lim_{c\uparrow b} f^{-1}\left(\int_c^b f(x)w(x)\mathrm{d}x \Big/ \int_c^b w(x)\,\mathrm{d}x\right) & \text{if } a=b=\sup D \end{cases} \quad (6)$$

for $[a,b]\in\mathcal{C}(D)$.

Lemma 3.1 *A weighted quasi-arithmetic mean* $M^f: \mathcal{C}(D)\mapsto D$ *defined by* (6) *has the following properties* (M.i)–(M.iii):

- (M.i) *Let* $[a,b]\in\mathcal{C}(D)$. *Then* $a\le M^f([a,b])\le b$. *Especially,* $M^f([a,a])=a$ *holds for* $a\in D$.
- (M.ii) *Let* $[a,b],[c,d]\in\mathcal{C}(D)$ *such that* $[a,b]\preceq[c,d]$. *Then it holds that* $M^f([a,b])\le M^f([c,d])$.
- (M.iii) *The map* $M^f: \mathcal{C}(D)\mapsto D$ *is continuous, i.e. it holds that*

$$\lim_{n\to\infty} M^f([a_n,b_n]) = M^f([a,b])$$

for $[a,b]\in\mathcal{C}(D)$ *and* $[a_n,b_n]\in\mathcal{C}(D)(n=1,2,\ldots)$ *such that* $\lim_{n\to\infty}a_n=a$ *and* $\lim_{n\to\infty}b_n=b$.

The properties (M.i)–(M.iii) in Lemma 3.1 are corresponding respectively to the *compensative*, *monotone* and *continuous* conditions in Definition 2.1.

Remark 3.2 We note the weighted quasi-arithmetic mean (6) also has the following additional properties as an evaluation of intervals.

- (i) In Lemma 3.1, we may take a continuous *strictly decreasing* function when we deal with a *regret function* instead of the utility function *f*. Then, the corresponding mean is reduced to the one with a strictly decreasing function. Actually we can see the reason as follows. Let $g: D\mapsto\mathbb{R}$ be a continuous *strictly decreasing* function, and let $w: D\mapsto(0,\infty)$ be a continuous function. Define a map $M^g: \mathcal{C}(D)\mapsto D$ in the same way as (6) for *g* instead of *f*. Then, taking a continuous *strictly increasing* function $f=-g: D\mapsto\mathbb{R}$, we obtain $g^{-1}(x)=f^{-1}(-x)\,(x\in[a,b])$and $M^g([a,b])=M^f([a,b])$ for $[a,b]\in\mathcal{C}(D)$ such that $a<b$. The weighted quasi-arithmetic mean M^g is reduced to the same one as (6).
- (ii) Let $f: D\mapsto\mathbb{R}$ be a continuous strictly increasing function, and let a function $g:\mathbb{R}\mapsto\mathbb{R}$ by $g(x)=rx+s$ with $r,s\in\mathbb{R}$ such that $r\ne0$. Put a continuous strictly

monotone function $h : D \mapsto \mathbb{R}$ by $h(x) = g(f(x))$ for $x \in D$. Then, the corresponding mean with the utility function h is reduced to the same one with the utility function f. Actually we can check this fact as follows. Define a map $M^g : \mathcal{C}(D) \mapsto D$ for the function g in the same way as (6) for h instead of f. Then, we obtain $h^{-1}(x) = f^{-1}((x-s)/r)$ and $M^h([a,b]) = M^f([a,b])$ for $[a,b] \in \mathcal{C}(D)$ such that $a < b$.

In the rest of this section we extend the weighted quasi-arithmetic mean (6) of a closed interval to be applicable for a general interval. For intervals $(a,b], [a,b), (a,b) \in \mathcal{I}(D)$, we extend the weighted mean M^f by

$$M^f((a,b]) = \lim_{c \downarrow a} M^f([c,b]) \quad \text{if } (a,b] \in \mathcal{I}(D),$$
$$M^f([a,b)) = \lim_{c \uparrow b} M^f([a,c]) \quad \text{if } [a,b) \in \mathcal{I}(D),$$
$$M^f((a,b)) = \lim_{c \downarrow a} \lim_{d \uparrow b} M^f([c,d]) \quad \text{if } (a,b) \in \mathcal{I}(D).$$

From the definitions we have $M^f((a,b]) = M^f([a,b])$ and $M^f((a,b)) = M^f([a,b))$ if $a \neq \inf D$, and $M^f([a,b)) = M^f([a,b])$ and $M^f((a,b)) = M^f((a,b])$ if $b \neq \sup D$. Next we can also extend the weighted mean M^f regarding the endpoints of D by

$$M^f((a,b)) = \lim_{c \downarrow a} M^f([c,b)) \quad \text{if} \quad (a,b) \in \mathcal{I}(D), \quad a = \inf D, \quad a \notin D,$$
$$M^f((a,b]) = \lim_{c \downarrow a} M^f([c,b]) \quad \text{if} \quad (a,b] \in \mathcal{I}(D), \quad a = \inf D, \quad a \notin D,$$
$$M^f((a,b)) = \lim_{c \uparrow b} M^f((a,c]) \quad \text{if} \quad (a,b) \in \mathcal{I}(D), \quad b = \sup D, \quad b \notin D,$$
$$M^f([a,b)) = \lim_{c \uparrow b} M^f([a,c]) \quad \text{if} \quad [a,b) \in \mathcal{I}(D), \quad b = \sup D, \quad b \notin D.$$

4 Translation invariance of the weighted quasi-arithmetic means

In this section, we discuss the translation invariance of the weighted quasi-arithmetic means. Let D be a fixed domain. Let $f : D \mapsto \mathbb{R}$ be a continuous strictly increasing function for utility, and let $w : D \mapsto (0,\infty)$ be a continuous function for weighting. Let $M^f : \mathcal{C}(D) \mapsto D$ be the weighted quasi-arithmetic mean given by (6). For an interval $I (\subset \mathbb{R})$ and a function $h : I \mapsto \mathbb{R}$, we define

$$h(I) = \{h(x) \mid x \in I\}.$$

If $I = [a,b] \in \mathcal{C}(D)$ and h is continuous strictly increasing, then it holds that $h([a,b]) = \{h(x) \mid x \in [a,b]\} = \left[\min_{x\in[a,b]} h(x), \max_{x\in[a,b]} h(x)\right] = [h(a), h(b)]$. Let a strictly increasing function $\varphi : \mathbb{R} \mapsto \mathbb{R}$. Using the above notation, the weighted quasi-arithmetic mean M^f is said to be *translation invariant* for φ if

$$\varphi \circ M^f = M^f \circ \varphi,$$

where $\circ$ is the composition of maps. This follows that $\varphi(M^f([a,b])) = M^f([\varphi(a), \varphi(b)]) = M^f(\varphi([a,b]))$ for $[a,b] \in \mathcal{C}(D)$. The following lemma gives the weighted quasi-arithmetic means with a translation invariance. General translation invariance for aggregation operators is discussed by Lázaro et al. (2004), Mesiar and Rückschlossová (2004).

Proposition 4.1 (Translation-invariance) *The following* (i)–(iii) *hold.*

(i) *Let a domain $D = [0,\infty)$. Assume w satisfy $w(xy) = w(x)w(y)$ for all $x, y \in D$. Put a function $f(x) = x^\gamma$ on D with a positive constant γ. Then, the mean M^f is translation-invariant for a magnification function $\varphi(x) = rx$,i.e. it holds that*

$$r \cdot M^f([a,b]) = M^f([ra, rb])$$

for $[a,b] \subset [0,\infty)$and a positive number r.

(ii) *Let a domain $D = (0,\infty)$. Assume w satisfy $w(xy) = w(x)w(y)$ for all $x, y \in D$. Put a function $f(x) = \gamma \log x$ on D with a positive constant γ. Then, the mean M^f is translation-invariant for a magnification function $\varphi(x) = rx$, i.e. it holds that*

$$r \cdot M^f([a,b]) = M^f([ra, rb])$$

for $[a,b] \subset (0,\infty)$ and a positive number r.

(iii) *Let a domain $D = (-\infty,\infty)$. Assume w satisfy $w(x+y) = w(x)w(y)$ for all $x, y \in D$. Put a function $f(x) = e^{\gamma x}$ on D with a positive constant γ. Then, the mean M^f is translation-invariant for a shift function $\varphi(x) = x + s$, i.e. it holds that*

$$M^f([a,b]) + s = M^f([a+s, b+s])$$

for $[a,b] \subset (-\infty,\infty)$ and a real number s.

5 The weighted quasi-arithmetic means and the aggregated mean ratios

In this section, we introduce a aggregated mean ratio of the weighted quasi-arithmetic mean and we discuss the relation among the weighted quasi-arithmetic mean, the aggregated mean ratio and the decision maker's attitude based on his utility. Let D be a fixed domain. Let $f : D \mapsto \mathbb{R}$ be a continuous strictly increasing function for utility, and let $w : D \mapsto (0,\infty)$ be a continuous function for weighting. Let $M^f : \mathcal{C}(D) \mapsto D$ be the weighted quasi-arithmetic mean given by (6) with a specified f.

Taking the continuous strictly increasing function f as a utility function in decision making, we discuss the decision-maker's judgment by the weighted quasi-arithmetic mean based on his utility. Let $\mathcal{C}(D)_< = \{[a,b] \in \mathcal{C}(D) | a < b\}$. For an interval $[a,b] \in \mathcal{C}(D)_<$, we define an interior ratio $\theta^f(a,b)$ from a position of the weighted quasi-arithmetic mean $M^f([a,b])$ on the interval $[a, b]$ by

$$\theta^f(a,b) = \frac{M^f([a,b]) - a}{b-a}. \tag{7}$$

We call it an *aggregated mean ratio* under the subjective utility f. In this section, we investigate the properties of the ratio θ^f concerning the utility f and we discuss the movement of the ratio $\theta^f(a,b)$ with respect to the end-points a, b of the interval $[a, b]$ in local regions and global regions. Dujmović (1974), Dujmović and Larsen (2007), Dujmović and Nagashima (2006) studied a *conjunction/disjunction degree*, which is a similar type of ratio to (7) in the power case, for computer science. This paper discusses characterizations from the viewpoint of economics. Let $D_\theta = \{(a,b) \in D \times D | a < b\}$. The following lemma is trivial from (M.i) and (M.ii) of Lemma 3.1.

Lemma 5.1 *An aggregated mean ratio θ^f defined by (7) has the following properties* (θ.i) and (θ.ii):

(θ.i). *Let $(a,b) \in D_\theta$. Then it holds that $0 \le \theta^f(a,b) \le 1$.*
(θ.ii) *The map $\theta^f : D_\theta \mapsto [0,1]$ is continuous, i.e. it holds that*

$$\lim_{n\to\infty} \theta^f(a_n,b_n) = \theta^f(a,b)$$

for $(a,b) \in D_\theta$ and $(a_n,b_n) \in D_\theta (n = 1,2,\ldots)$ such that $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$.

For a strictly increasing function $\varphi : \mathbb{R} \mapsto \mathbb{R}$, we define

$$\varphi(a,b) = (\varphi(a), \varphi(b))$$

for $(a,b) \in D_\theta$. Then, the aggregated mean ratio θ^f is said to be *translation invariant* for φ if

$$\theta^f = \theta^f \circ \varphi,$$

where $\circ$ is the composition of maps. This follows that $\theta^f(a,b) = \theta^f(\varphi(a),\varphi(b))$ for $(a,b) \in D_\theta$. The following proposition, which is is trivial from Proposition 4.1, gives the aggregated mean ratios with translation invariance.

Proposition 5.2 (Translation-invariance) *The following* (i)–(iii) *hold.*

(i) *Let a domain $D = [0,\infty)$. Assume w satisfy $w(xy) = w(x)w(y)$ for all $x,y \in D$. Put a function $f(x) = x^\gamma$ on D with a positive constant γ. Then, the ratio θ^f is translation-invariant for a magnification function $\varphi(x) = rx$, i.e. it holds that*

$$\theta^f(a,b) = \theta^f(ra,rb)$$

for $(a,b) \in D_\theta = \{(a,b) | 0 \le a < b < \infty\}$ and a positive number r.

(ii) *Let a domain $D = (0,\infty)$.Assume w satisfy $w(xy) = w(x)w(y)$ for all $x,y \in D$. Put a function $f(x) = \gamma \log x$ on D with a positive constant γ. Then, the ratio θ^f is translation-invariant for a magnification function $\varphi(x) = rx$, i.e. it holds that*

$$\theta^f(a,b) = \theta^f(ra,rb)$$

for $(a,b) \in D_\theta = \{(a,b) | 0 < a < b < \infty\}$ and a positive number r.

(iii) *Let a domain $D = (-\infty,\infty)$. Assume w satisfy $w(x+y) = w(x)w(y)$ for all $x,y \in D$. Put a function $f(x) = e^{\gamma x}$ on D with a positive constant γ. Then, the ratio θ^f is translation-invariant for a shift function $\varphi(x) = x + s$, i.e. it holds that*

$$\theta^f(a,b) = \theta^f(a+s,b+s)$$

for $(a,b) \in D_\theta = \{(a,b) | -\infty < a < b < \infty\}$ and a real number s.

For a closed interval $[a,b] \in \mathcal{C}(D)_<$. Define the *neutral weighted mean* $N(a,b)$ and its aggregated mean ratio $\nu(a,b)$ by

$$N(a,b) = \int_a^b x\,w(x)\mathrm{d}x \Big/ \int_a^b w(x)\mathrm{d}x \tag{8}$$

and

$$\nu(a,b) = \frac{N(a,b)-a}{b-a} = \int_a^b (x-a)w(x)\mathrm{d}x \Big/ \int_a^b (b-a)w(x)\mathrm{d}x. \tag{9}$$

Then we can easily check that $a < N(a,b) < b$ and $0 < \nu(a,b) < 1$. Now we compare weighted quasi-arithmetic means and aggregated mean ratios corresponding to the utilities when $N(a,b)$ and $\nu(a,b)$ are taken as a reference mean and a reference ratio, respectively. Let $g : D \mapsto \mathbb{R}$ be another continuous strictly increasing function for utility. Let $M^g : \mathcal{C}(D) \mapsto D$ be the weighted quasi-arithmetic mean defined by g instead of f in the way of (6) and we put the aggregated mean ratio θ^g for M^g. Then we obtain the following results.

Theorem 5.3 *Assume that f and g are C^2 -class functions on D. Let $[a,b] \in \mathcal{C}(D)_<$. Then the following* (i)–(iii) *hold.*

(i) *If f and g satisfy $f''/f' < g''/g'$ on (a,b), it holds that $M^f([a,b]) < M^g([a,b])$ and $\theta^f(a,b) < \theta^g(a,b)$.*
(ii) *If f and g satisfy $f''/f' \le g''/g'$ on (a,b), it holds that $M^f([a,b]) \le M^g([a,b])$ and $\theta^f(a,b) \le \theta^g(a,b)$.*

(iii) *If f is semi-linear, i.e.* $f(x) = rx + s (x \in \mathbb{R})$ *with a positive number r and a real number s, then it holds that* $M^f([a,b]) = N(a,b)$ *and* $\theta^f(a,b) = v(a,b)$.

Corollary 5.4 *Assume that f is a* C^2 *-class function on D. Let* $[a,b] \in \mathcal{C}(D)_<$. *Then the following* (i)–(iv) *hold.*

(i) *If f satisfies* $f'' < 0$ *on* (a,b), *then* $\theta^f(a,b) < v(a,b)$.
(ii) *If f satisfies* $f'' \leq 0$ *on* (a,b), *then* $\theta^f(a,b) \leq v(a,b)$.
(iii) *If f satisfies* $f'' > 0$ *on* (a,b), *then* $\theta^f(a,b) > v(a,b)$.
(iv) *If f satisfies* $f'' \geq 0$ *on* (a,b), *then* $\theta^f(a,b) \geq v(a,b)$.

Remark 5.5

(i) In Corollary 5.4, $f'' = 0$ implies the decision maker's *risk neutral attitude*, $f'' < 0$ denotes the decision maker's *risk averse attitude*, and $f'' > 0$ is the decision maker's *risk loving attitude*. Therefore, when we may choose two functions f and g as decision maker's utilities, Theorem 5.3 implies that the utility f yields more risk averse results than g if $f''/f' \leq g''/g'$ on (a,b). The inequality $\theta^f(a,b) \leq \theta^g(a,b)$ implies that the aggregated mean ratio $\theta^f(a,b)$ is more risk averse than $\theta^g(a,b)$. The term $-f''/f'$ is called the *Arrow–Pratt index* and it implies the degree of absolute risk aversion in economics (Arrow 1971; Pratt 1964).
(ii) Kolesárová (2001) studied relations between $M^f([a,b])$ and f in the power cases.

The assertions in Theorem 5.3 (ii) and Corollary 5.4 (ii) are actually necessary and sufficient. The following Theorem 5.6 and Corollary 5.7 show equivalences regarding the assertion 'if—then' in Theorem 5.3(ii) and Corollary 5.4(ii).

Theorem 5.6 *Assume that f and g are* C^2*-class functions on D. Let* $[a,b] \in \mathcal{C}(D)_<$. *Then the following (a)–(c) are equivalent.*

(a) $f''/f' \leq g''/g'$ *on* (a,b).
(b) $M^f([c,d]) \leq M^g([c,d])$ *for all* $[c,d]$ *satisfying* $[c,d] \subset [a,b]$ *and* $c < d$.
(c) $\theta^f(c,d) \leq \theta^g(c,d)$ *for all* $[c,d]$ *satisfying* $[c,d] \subset [a,b]$ *and* $c < d$.

The following corollary is trivial taking g a semi-linear function in Theorem 5.6.

Corollary 5.7 *Assume that f is a* C^2 *-class function on D. Let* $[a,b] \in \mathcal{C}(D)_<$. *Then the following (a)–(c) are equivalent.*

(a) $f'' \leq 0$ *on* (a,b).
(b) $M^f([c,d]) \leq N(a,b)$ *for all* $[c,d]$ *satisfying* $[c,d] \subset [a,b]$ *and* $c < d$.
(c) $\theta^f(c,d) \leq v(a,b)$ *for all* $[c,d]$ *satisfying* $[c,d] \subset [a,b]$ *and* $c < d$.

In Theorem 5.3, we investigate a combination of two utility functions f and g. Let $h = (f+g)/2$ be a combination of utility functions f and g. Then, the following result implies that the estimation by the utility $h = (f+g)/2$ gives a *middle attitude* by the both utilities f and g.

Proposition 5.8 *Assume that f and g are* C^2 *-class functions on D. Let* $h = (f+g)/2$ *and let* $[a,b] \in \mathcal{C}(D)_<$. *Then the following (i) and (ii) hold.*

(i) *If f and g satisfy* $f''/f' < g''/g'$ *on* (a,b), *then* $M^f([a,b]) < M^h([a,b]) < M^g([a,b])$ *and* $\theta^f(a,b) < \theta^h(a,b) < \theta^g(a,b)$.
(ii) *If f and g satisfy* $f''/f' \leq g''/g'$ *on* (a,b), *then* $M^f([a,b]) \leq M^h([a,b]) \leq M^g([a,b])$ *and* $\theta^f(a,b) \leq \theta^h(a,b) \leq \theta^g(a,b)$.

Next we discuss the movement of the aggregated mean ratio $\theta^f(a,b)$ in the local regions. Let $[a,b] \in \mathcal{C}(D)_<$. The following theorem gives a local property of the ratio $\theta^f(a,b)$ at the neighborhood of the endpoints.

Theorem 5.9 *Assume that f is a* C^2 *-class function on D. Let* $[a,b] \in \mathcal{C}(D)_<$. *Then, it holds that*

$$\lim_{c \downarrow a} \theta^f(a,c) = \lim_{c \downarrow a} v(a,c) = \frac{1}{2}, \tag{10}$$

$$\lim_{c \uparrow b} \theta^f(c,b) = \lim_{c \uparrow b} v(c,b) = \frac{1}{2}. \tag{11}$$

In the spirit of the extension of intervals at the end of Sect. 2, we can investigate the movement of the aggregated mean ratio $\theta^f(a,b)$ in global regions in the following lemma, which is useful for numerical calculation in actual cases (Sect. 5).

Lemma 5.10 *Assume* $(0,\infty)(\subset D)$ *for the domain.*

(i) *For* $b > 0$, *it holds that*

$$\lim_{c \downarrow 0} \theta^f(c,b) = \frac{1}{b} M^f([0,b]). \tag{12}$$

(ii) *If the weighted quasi-arithmetic mean* M^f *is magnification translation invariant, i.e.*

$$r \cdot M^f([a,b]) = M^f([ra,rb]) \tag{13}$$

for $[a,b] \subset D$ *and* $r > 0$, *then for* $a > 0$ *and* $b > 0$ *it holds that*

$$\lim_{c \downarrow 0} \theta^f(c,b) = M^f([0,1]). \tag{14}$$

$$\lim_{c \to \infty} \theta^f(a,c) = M^f([0,1]). \tag{15}$$

6 Examples

In this section, we give examples for weighted quasi-arithmetic means which are presented in the previous sections. When we give a fixed domain D, a continuous strictly increasing function $f : D \mapsto \mathbb{R}$ and a fixed continuous function $w : D \mapsto (0, \infty)$, we can define the weighted quasi-arithmetic mean $M^f([a,b])$ of an interval $[a,b] \in \mathcal{C}(D)$ by (6). We check the movement of the aggregated mean ratio $\theta^f(a,b)$, which is given by (7), with respect to parameters a and b in local regions and global regions in each example. First, we discuss several examples of utility functions f.

Example 6.1 In the following examples, we put the domain $D = (0, \infty)$ and take a weighting function $w(x) = x^\delta$ on $D = (0, \infty)$ with a positive constant δ.

(i) (*Semi-linear case*) Take a function $f(x) = cx + d$ on D with constants c, d such that $c > 0$. From Theorem 5.3(iii), it is trivial that

$$M^f([a,b]) = N(a,b) = \frac{(\delta+1)(b^{\delta+2} - a^{\delta+2})}{(\delta+2)(b^{\delta+1} - a^{\delta+1})}$$

and $\theta^f(a,b) = v(a,b)$ for $[a,b] \subset D$.

(ii) (*Power case*) Take functions $f(x) = x^\gamma$ and $w(x) = x^\delta$ on $D = (0, \infty)$ with constants γ, δ such that $\gamma \neq 0$. Then, for $[a,b] \subset D$ such that $a < b$, we can check

$$M^f([a,b]) = \left(\frac{(\delta+1)(b^{\gamma+\delta+1} - a^{\gamma+\delta+1})}{(\gamma+\delta+1)(b^{\delta+1} - a^{\delta+1})}\right)^{1/\gamma}$$

if $\delta \neq -1, \gamma + \delta \neq -1$. Hence, we obtain

$$\lim_{\delta \to -\gamma-1} M^f([a,b]) = ab\left(\frac{\gamma(\log b - \log a)}{b^\gamma - a^\gamma}\right)^{1/\gamma},$$

$$\lim_{\delta \to -1} M^f([a,b]) = \left(\frac{\gamma(\log b - \log a)}{b^\gamma - a^\gamma}\right)^{-1/\gamma}.$$

From Corollary 5.4 we also have

$$\theta^f(a,b) \lesseqqgtr v(a,b) \quad \text{if } \gamma \lesseqqgtr 1.$$

From Theorem 5.9, it holds that $\lim_{b \downarrow a} \theta^f(a,b) = \lim_{a \uparrow b} \theta^f(a,b) = 1/2$, and from Lemmas 5.10 we obtain that

$$\lim_{a \downarrow 0} \theta^f(a,b) = \lim_{b \to \infty} \theta^f(a,b) = \left(\frac{\delta+1}{\gamma+\delta+1}\right)^{1/\gamma}.$$

They are drawn in Figs. 1 and 2.

(iii) (*Logarithmic case*) Take a concave function $f(x) = \gamma \log x$ on D with a positive constant γ. Then, for $[a,b] \subset D$ such that $a < b$, we can check

$$M^f([a,b]) = \exp\left(\frac{b^{\delta+1}\log b - a^{\delta+1}\log a}{b^{\delta+1} - a^{\delta+1}} - \frac{1}{\delta+1}\right),$$

and Corollary 5.4 implies $\theta^f(a,b) < v(a,b)$. Theorem 5.9 implies $\lim_{b \downarrow a} \theta^f(a,b) = \lim_{a \uparrow b} \theta^f(a,b) = 1/2$, and from Lemmas 5.10 we obtain that

$$\lim_{a \downarrow 0} \theta^f(a,b) = \lim_{b \to \infty} \theta^f(a,b) = \exp\left(-\frac{1}{\delta+1}\right).$$

Figure 3 illustrates these movements.

(iv) (*Exponential case*) Take a convex function $f(x) = e^{-\gamma x}$ on D with a positive constant γ. Then, for $[a,b] \subset D$ such that $a < b$, we can check

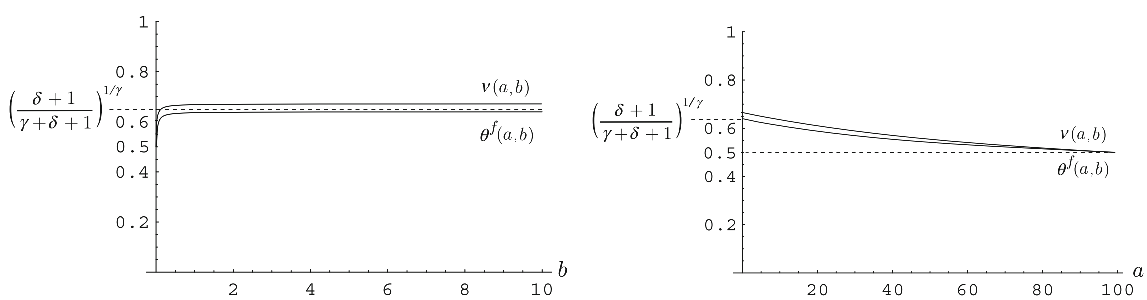

Fig. 1 The map $b \mapsto \theta^f(a,b)$ at $a = 0.01$ and the map $a \mapsto \theta^f(a,b)$ at $b = 100$ when $f(x) = \sqrt{x}$ and $w(x) = x$ ($\gamma = 1/2, \delta = 1$)

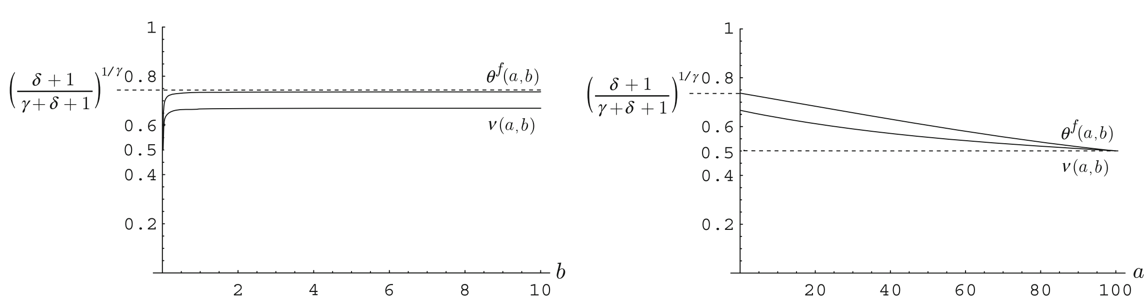

Fig. 2 The map $b \mapsto \theta^f(a,b)$ at $a = 0.01$ and the map $a \mapsto \theta^f(a,b)$ at $b = 100$ when $f(x) = x^3$ and $w(x) = x$ ($\gamma = 3, \delta = 1$)

$$M^f([a,b]) = \frac{1}{\gamma}\log\left(\frac{-\gamma^{\delta+1}(b^{\delta+1}-a^{\delta+1})}{(\delta+1)(\Gamma(\delta+1,\gamma b)-\Gamma(\delta+1,\gamma a))}\right)$$

and Corollary 5.4 implies $v(a,b)<\theta^f(a,b)$, where $\Gamma(\delta+1,c)=\int_c^\infty t^\delta e^{-t}dt$ for $c\geq 0$. From Theorem 5.9, we obtain $\lim_{b\downarrow a}\theta^f(a,b)=\lim_{a\uparrow b}\theta^f(a,b)=1/2$, and from Lemma 5.10 we obtain that

$$\lim_{a\downarrow 0}\theta^f(a,b) = \frac{1}{\gamma b}\log\left(\frac{-\gamma^{\delta+1}b^{\delta+1}}{(\delta+1)(\Gamma(\delta+1,\gamma b)-\Gamma(\delta+1))}\right)$$

for $b>0$, where $\Gamma(\delta+1)=\int_0^\infty t^\delta e^{-t}dt$. We can also check

$$\lim_{b\to\infty}\theta^f(a,b)=\lim_{b\to\infty}\frac{1}{b}M^f([0,b])=0.$$

These results are shown in Fig. 4.

(v) (*Square root case*) Take a convex function $f(x)=s\sqrt{x/r}$ on D with positive constants r,s. Then, for $[a,b]\subset D$ such that $a<b$, we can check

$$M^f([a,b]) = \frac{4r^3(\delta+1)^2\left(b^\delta(b/r)^{3/2}-a^\delta(a/r)^{3/2}\right)^2}{(2\delta+3)^2(b^{\delta+1}-a^{\delta+1})^2}$$

and Corollary 5.4 implies $\theta^f(a,b)<v(a,b)$. Theorem 5.9 implies $\lim_{b\downarrow a}\theta^f(a,b)=\lim_{a\uparrow b}\theta^f(a,b)=1/2$, and Fig. 5 shows the movements with

$$\lim_{a\downarrow 0}\theta^f(a,b)=\lim_{b\to\infty}\theta^f(a,b)=\frac{16}{25}.$$

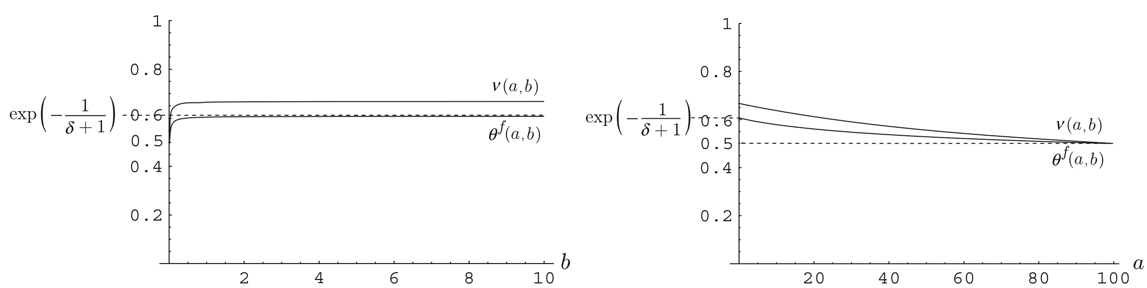

Fig. 3 The map $b\mapsto\theta^f(a,b)$ at $a=0.01$ and the map $a\mapsto\theta^f(a,b)$ at $b=100$ when $f(x)=\log x$ and $w(x)=x$ $(\gamma=1,\delta=1)$

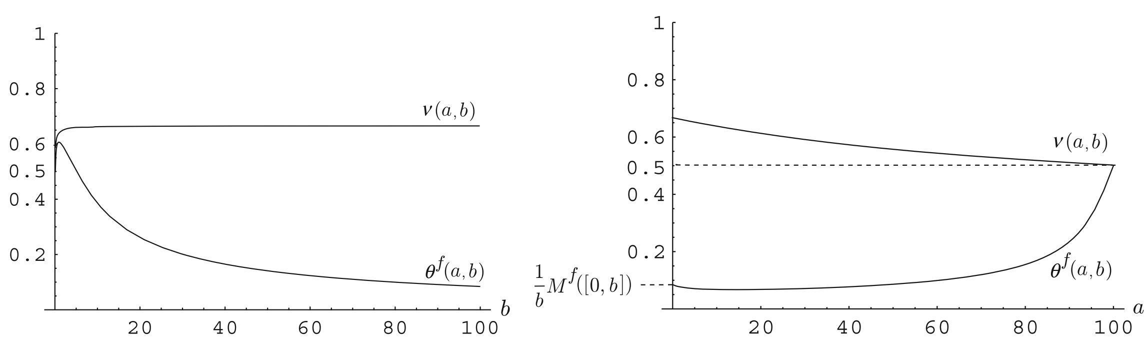

Fig. 4 The map $b\mapsto\theta^f(a,b)$ at $a=0.1$ and the map $a\mapsto\theta^f(a,b)$ at $b=100$ when $f(x)=e^{-x}$ and $w(x)=x(\gamma=1,\delta=1)$

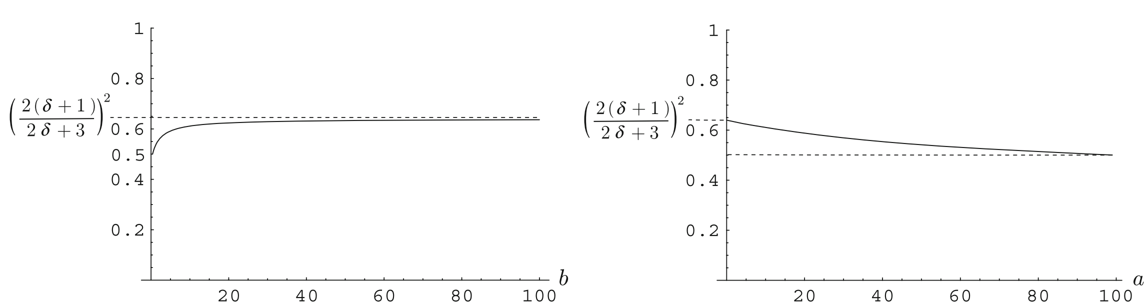

Fig. 5 The map $b\mapsto\theta^f(a,b)$ at $a=1$ and the map $a\mapsto\theta^f(a,b)$ at $b=100$ when $f(x)=3\sqrt{x/2}$ and $w(x)=x$ $(r=2,s=3)$

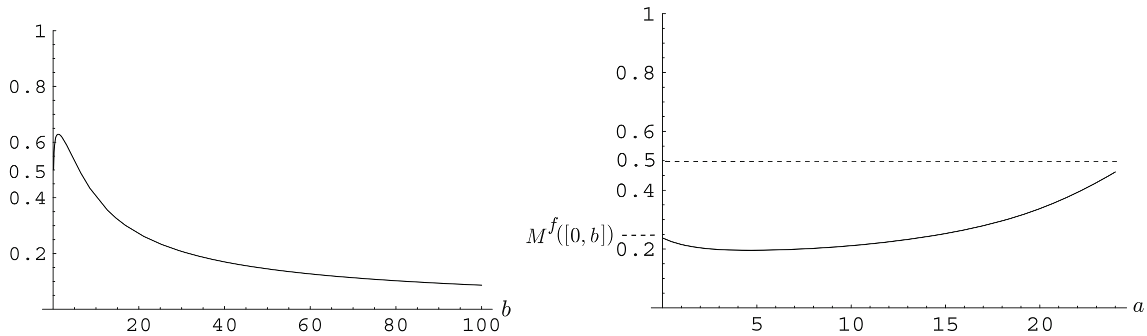

Fig. 6 The map $b \mapsto \theta^f(a,b)$ at $a = 0.1$ and the map $a \mapsto \theta^f(a,b)$ at $b = 25$ when $f(x) = 1/(1+\mathrm{e}^{-x})$ and $w(x) = x\,(\gamma = 1, \delta = 1)$

(vi) (*Sigmoid case*) Take a sigmoid function $f(x) = 1/(1+\mathrm{e}^{-\gamma x})$ on D with a positive constant γ. Then, for $[a,b] \subset D$ such that $a < b$, we can check

$$M^f([a,b]) = -\frac{1}{\gamma}\log\left(\frac{b^{\delta+1}-a^{\delta+1}}{(\delta+1)\int_a^b x^\delta/(1+\mathrm{e}^{-\gamma x})\mathrm{d}x} - 1\right)$$

and Corollary 5.4 implies $\theta^f(a,b) < v(a,b)$. Theorem 5.9 implies $\lim_{b\downarrow a}\theta^f(a,b) = \lim_{b\downarrow a} v(a,b) = 1/2$ and we can also check that

$$\lim_{a\downarrow 0}\theta^f(a,b) = -\frac{1}{\gamma}\log\left(\frac{b^{\delta+1}}{(\delta+1)\int_0^b x^\delta/(1+\mathrm{e}^{-\gamma x})\mathrm{d}x} - 1\right)$$

and $\lim_{b\to\infty}\theta^f(a,b) = 0$ (Fig. 6).

Table 1 shows the weighted quasi-arithmetic means $M^f([a,b])$ and the corresponding index f''/f' for Examples 6.1(i)–(vi). Next we give two examples as an application of Theorem 5.3. The following examples show the cases that decision maker's two attitudes f and g are changing with regions in their comparison.

The weighted aggregation operators for weighted quasi-arithmetic means are also related to WOWA operators in Torra and Godo (2002), where they have studied weights not for quasi-arithmetic means but for weighted means.

Table 1 Quasi-arithmetic means in Example 6.1

f	f''/f'	$M^f([a,b])$
$cx + d(c > 0)$	0	$\frac{(\delta+1)(b^{\delta+2}-a^{\delta+2})}{(\delta+2)(b^{\delta+1}-a^{\delta+1})}$
$x^\gamma(\gamma > 0)$	$\frac{\gamma-1}{x}$	$\frac{(\delta+1)(b^{\gamma+\delta+1}-a^{\gamma+\delta+1})}{(\gamma+\delta+1)(b^{\delta+1}-a^{\delta+1})}$
$\gamma\log x(\gamma > 0)$	$-\frac{1}{x}$	$\exp\left(\frac{b^{\delta+1}\log b - a^{\delta+1}\log a}{b^{\delta+1}-a^{\delta+1}} - \frac{1}{\delta+1}\right)$
$\mathrm{e}^{-\gamma x}(\gamma > 0)$	$-\gamma$	$\frac{1}{\gamma}\log\left(\frac{-\gamma^{\delta+1}(b^{\delta+1}-a^{\delta+1})}{(\delta+1)(\Gamma(\delta+1,\gamma b)-\Gamma(\delta+1,\gamma a))}\right)$
$s\sqrt{\frac{x}{r}}(r,s > 0)$	$-\frac{r}{2x}$	$\frac{4r^3(\delta+1)^2\left(b^\delta(b/r)^{3/2}-a^\delta(a/r)^{3/2}\right)^2}{(2\delta+3)^2(b^{\delta+1}-a^{\delta+1})^2}$
$\frac{1}{1+\mathrm{e}^{-\gamma x}}(\gamma > 0)$	$\frac{\gamma(1-\mathrm{e}^{-\gamma x})}{1+\mathrm{e}^{-\gamma x}}$	$-\frac{1}{\gamma}\log\left(\frac{b^{\delta+1}-a^{\delta+1}}{(\delta+1)\int_a^b x^\delta/(1+\mathrm{e}^{-\gamma x})\mathrm{d}x} - 1\right)$

Example 6.2

(i) (*Logarithmic case and square root case*) Take concave functions $f(x) = \log x$ and $g(x) = \sqrt{x}$ on $D = (0,\infty)$ (Example 6.1(ii), (iii)). Then we have

$$\frac{f''(x)}{f'(x)} = -\frac{1}{x} < -\frac{1}{2x} = \frac{g''(x)}{g'(x)}$$

for $x \in D$. From Theorem 5.3, we obtain $\theta^f(a,b) < \theta^g(a,b)$ for $[a,b] \in \mathcal{C}(D)$ such that $a < b$, where $\theta^f(a,b)$ is the aggregated mean ratio given by $f(x) = \log x$ and $\theta^g(a,b)$ is the aggregated mean ratio given by $g(x) = \sqrt{x}$. This shows that $f(x) = \log x$ is more risk averse than $g(x) = \sqrt{x}$ as utilities.

(ii) (*Logarithmic case and exponential case*) Take convex utility functions $f(x) = \log x$ and $g(x) = 1 - \mathrm{e}^{-x}$ on $D = (0,\infty)$ (Example 6.1(ii), (iv)). Then we can easily check that

$$\frac{f''(x)}{f'(x)} = -\frac{1}{x} \lesseqgtr -1 = \frac{g''(x)}{g'(x)} \qquad \text{if } x \lesseqgtr 1$$

for $x \in D$. From Theorem 5.3, we obtain $\theta^f(a,b) < \theta^g(a,b)$ for $[a,b] \subset (0,1]$ such that $a < b$ and we also obtain $\theta^f(a,b) > \theta^g(a,b)$ for $[a,b] \subset [1,\infty)$ such that $a < b$, where $\theta^f(a,b)$ is the aggregated mean ratio given by $f(x) = \log x$ and $\theta^g(a,b)$ is the aggregated mean ratio given by $g(x) = 1 - \mathrm{e}^{-x}$. This shows that $f(x) = \log x$ is more risk averse than $g(x) = 1 - \mathrm{e}^{-x}$ in the region $(0,1)$ and that $f(x) = \log x$ is more risk loving than $g(x) = 1 - \mathrm{e}^{-x}$ in the region $(1,\infty)$. This example shows that decision makers' attitudes are comparable in each local area using the index f''/f'.

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Appendix

In this section, we give the proofs of the theorems, the propositions and the lemmas in Sects. 3–5.

Proof of Lemma 3.1 Let $[a,b] \in \mathcal{C}(D)$ satisfying $a<b$. (M.i) Since f and f^{-1} are strictly increasing, we have

$$M^f([a,b]) = f^{-1}\left(\int_a^b f(x)w(x)\mathrm{d}x \bigg/ \int_a^b w(x)\mathrm{d}x\right) \le f^{-1}\left(\int_a^b f(b)w(x)\mathrm{d}x \bigg/ \int_a^b w(x)\mathrm{d}x\right) = f^{-1}(f(b)) = b.$$

Thus we get $M^f([a,b]) \le b$. In the same way, we also have $M^f([a,b]) \ge a$. Therefore, we obtain $a \le M^f([a,b]) \le b$. This inequality also implies that $M^f([a,a]) = a$ for $a \in D$ together with the definition (6). (M.ii) Put a function

$$F(t) = \int_a^t f(x)w(x)\mathrm{d}x \bigg/ \int_a^t w(x)\mathrm{d}x$$

for $t > a$. Then we have

$$F'(t) = \frac{f(t)w(t)\int_a^t w(x)\mathrm{d}x - w(t)\int_a^t f(x)w(x)\mathrm{d}x}{\left(\int_a^t w(x)\mathrm{d}x\right)^2} = \frac{w(t)\int_a^t (f(t)-f(x))w(x)\mathrm{d}x}{\left(\int_a^t w(x)\mathrm{d}x\right)^2} > 0$$

since f is strictly increasing on D and $w > 0$ on D. Thus the map $t \in (a,\infty) \cap D \mapsto F(t)$ is strictly increasing. We also put a function

$$G(t) = \int_t^b f(x)w(x)\mathrm{d}x \bigg/ \int_t^b w(x)\mathrm{d}x$$

for $t<b$. Then we have

$$G'(t) = \frac{-f(t)w(t)\int_t^b w(x)\mathrm{d}x + w(t)\int_t^b f(x)w(x)\mathrm{d}x}{\left(\int_t^b w(x)\mathrm{d}x\right)^2} = \frac{w(t)\int_t^b (f(x)-f(t))w(x)\mathrm{d}x}{\left(\int_t^b w(x)\mathrm{d}x\right)^2} > 0$$

since f is strictly increasing on D and $w > 0$ on D. Thus the map $t \in (-\infty,b) \cap D \mapsto G(t)$ is also strictly increasing. Let $[a,b],[c,d] \in \mathcal{C}(D)$ satisfying $[a,b] \preceq [c,d]$. From the above results and the definition (6), we have $f(M^f([a,b])) \le f(M^f([a,d])) \le f(M^f([c,d]))$. Therefore, we get $M^f([a,b]) \le M^f([c,d])$ since f^{-1} is strictly increasing. (M.iii) The continuity is trivial from the definition (6). Therefore, the proof is completed. □

Proof of Proposition 4.1

(i) Let r be a positive number and let $[a,b] \subset [0,\infty)$. Then we have

$$\begin{aligned} r \cdot M^f([a,b]) &= r \cdot \left(\int_a^b x^\gamma w(x)\mathrm{d}x \bigg/ \int_a^b w(x)\mathrm{d}x\right)^{1/\gamma} \\ &= \left(\int_a^b (rx)^\gamma w(x)\mathrm{d}x \bigg/ \int_a^b w(x)\mathrm{d}x\right)^{1/\gamma} \\ &= \left(\int_{ra}^{rb} x^\gamma w(x/r)\mathrm{d}x \bigg/ \int_{ra}^{rb} w(x/r)\mathrm{d}x\right)^{1/\gamma} \\ &= \left(\int_{ra}^{rb} x^\gamma w(x)\mathrm{d}x \bigg/ \int_{ra}^{rb} w(x)\mathrm{d}x\right)^{1/\gamma} \\ &= M^f([ra,rb]). \end{aligned}$$

(ii) Let r be a positive number and let $[a,b] \subset (0,\infty)$. Then we have

$$\begin{aligned} &r \cdot M^f([a,b]) \\ &= \exp\left(\frac{1}{\gamma}\cdot\int_a^b \gamma\log x\cdot w(x)\mathrm{d}x \bigg/ \int_a^b w(x)\mathrm{d}x + \log r\right) \\ &= \exp\left(\frac{1}{\gamma}\cdot\int_a^b \gamma\log(rx)\cdot w(x)\mathrm{d}x \bigg/ \int_a^b w(x)\mathrm{d}x\right) \\ &= \exp\left(\frac{1}{\gamma}\cdot\int_{ra}^{rb} \gamma\log x\cdot w(x/r)\mathrm{d}x \bigg/ \int_{ra}^{rb} w(x/r)\mathrm{d}x\right) \\ &= \exp\left(\frac{1}{\gamma}\cdot\int_{ra}^{rb} \gamma\log x\cdot w(x)\mathrm{d}x \bigg/ \int_{ra}^{rb} w(x)\mathrm{d}x\right) \\ &= M^f([ra,rb]). \end{aligned}$$

(iii) Let s be a real number and let $[a,b] \subset (-\infty,\infty)$. Then we have

$$\begin{aligned} &M^f([a,b]) + s \\ &= \frac{1}{\gamma}\log\left(\int_a^b e^{\gamma x} w(x)\mathrm{d}x \bigg/ \int_a^b w(x)\mathrm{d}x\right) + s \\ &= \frac{1}{\gamma}\log\left(\int_a^b e^{\gamma(x+s)} w(x)\mathrm{d}x \bigg/ \int_a^b w(x)\mathrm{d}x\right) \\ &= \frac{1}{\gamma}\log\left(\int_{a+s}^{b+s} e^{\gamma x} w(x-s)\mathrm{d}x \bigg/ \int_{a+s}^{b+s} w(x-s)\mathrm{d}x\right) \\ &= \frac{1}{\gamma}\log\left(\int_{a+s}^{b+s} e^{\gamma x} w(x)\mathrm{d}x \bigg/ \int_{a+s}^{b+s} w(x)\mathrm{d}x\right) \\ &= M^f([a+s,b+s]). \end{aligned}$$

Thus we obtain this lemma. □

Proof of Theorem 5.3

(i) Let f and g satisfy $f''/f' < g''/g'$ on (a,b). Define

$$\begin{aligned}H(y) &= \int_a^y f(x)w(x)\mathrm{d}x - f(M^g([a,y]))\int_a^y w(x)\mathrm{d}x\\ &= \int_a^y f(x)w(x)\mathrm{d}x\\ &\quad - f\left(g^{-1}\left(\int_a^y g(x)w(x)\mathrm{d}x \Big/ \int_a^y w(x)\mathrm{d}x\right)\right)\\ &\quad \times \int_a^y w(x)\mathrm{d}x\end{aligned}$$

for $y \in (a,b]$. Then we have

$$\begin{aligned}H'(y) &= \frac{w(y)}{g'(M^g([a,y]))}\\ &\quad \times (g'(M^g([a,y]))(f(y) - f(M^g([a,y])))\\ &\quad - f'(M^g([a,y]))(g(y) - g(M^g([a,y])))).\end{aligned}$$

On the other hand, the map $x \mapsto g'(x)/f'(x)$ is increasing on (a,b) since

$$\left(\frac{g'(x)}{f'(x)}\right)' = \frac{g''(x)f'(x) - g'(x)f''(x)}{(f'(x))^2} > 0$$

for $x \in (a,b)$ from $f' > 0$, $g' > 0$ and $f''/f' < g''/g'$ on (a,b). Therefore, we get

$$\frac{g'(M^g([a,y]))}{f'(M^g([a,y]))} < \frac{g'(\xi)}{f'(\xi)},$$

where a constant $\xi \in (M^g([a,y],y)$ is given by

$$\frac{g'(\xi)}{f'(\xi)} = \frac{g(y) - g(M^g([a,y]))}{f(y) - f(M^g([a,y]))}$$

from Cauchy's mean value theorem. Thus we get

$$\frac{g'(M^g([a,y]))}{f'(M^g([a,y]))} < \frac{g(y) - g(M^g([a,y]))}{f(y) - f(M^g([a,y]))}.$$

From $f' > 0$, $g' > 0$ and the above equality regarding $H'(y)$, this implies $H'(y) < 0$ for $y \in (a,b)$, Together with $H(a) = 0$, we obtain $H(b) < 0$ for $b > a$. Thus we have

$$\int_a^b f(x)w(x)\mathrm{d}x < f(M^g([a,b]))\int_a^b w(x)\mathrm{d}x.$$

Since, f^{-1} is increasing, we obtain

$$\begin{aligned}M^f([a,b]) &= f^{-1}\left(\int_a^b f(x)w(x)\mathrm{d}x \Big/ \int_a^b w(x)\mathrm{d}x\right)\\ &< M^g([a,b]).\end{aligned}$$

This also implies

$$\begin{aligned}\theta^f(a,b) &= \frac{M^f([a,b]) - a}{b-a}\\ &< \frac{M^g([a,b]) - a}{b-a}\\ &= \theta^g(a,b).\end{aligned}$$

Thus (i) holds. We also obtain (ii) in the same way as (i). The proof of (iii) is straightforward. Therefore, the proof of this theorem is completed. □

Proof of Theorem 5.6 $(a) \Rightarrow (b)$: Let $[c,d]$ satisfy $[c,d] \subset [a,b]$ and $c < d$. Then, we obtain (b) from (a) applying Theorem 5.3(ii) to $f''/f' \le g''/g'$ on (c,d).$(b) \Rightarrow (a)$: Let M^f and M^g satisfy $M^f(I) \le M^g(I)$ for all closed intervals $I(\subset [a,b])$. Suppose that $f''/f' \le g''/g'$ does not hold on (a,b). Then there exits an closed interval $[c,d]$ such that $[c,d] \subset [a,b]$, $c < d$ and $f''/f' > g''/g'$ on (c,d). By Theorem 5.3(i) we have $M^f([c,d]) > M^g([c,d])$, and this contradicts $M^f(I) \le M^g(I)$ with $I = [c,d]$. Therefore, we obtain $f''/f' \le g''/g'$ on (a,b). Thus, the equivalence between (a) and (b) hold. This theorem holds since (b) and (c) are also equivalent clearly. □

Proof of Proposition 5.8

(i) Let f and g satisfy $f''/f' < g''/g'$ on (a,b). For $h = (f+g)/2$, we have

$$\begin{aligned}\frac{h''}{h'} &= \frac{f''+g''}{f'+g'}\\ &= \frac{f''}{f'}\cdot\frac{f'}{f'+g'} + \frac{g''}{g'}\cdot\frac{g'}{f'+g'}\\ &< \frac{g''}{g'}\cdot\frac{f'}{f'+g'} + \frac{g''}{g'}\cdot\frac{g'}{f'+g'}\\ &= \frac{g''}{g'}.\end{aligned}$$

Therefore, we get $h''/h' < g''/g'$. In the same way, we also have $f''/f' < h''/h'$. From Theorem 5.3(i), we obtain $M^f([a,b]) < M^h([a,b]) < M^g([a,b])$ and $\theta^f(a,b) < \theta^h(a,b) < \theta^g(a,b)$. We also obtain (ii) in the same way as (i). Therefore the proof is completed. □

Proof of Theorem 5.9 Fix any $a \in D$. First, we have

$$\begin{aligned}\lim_{b\downarrow a} v(a,b) &= \lim_{b\downarrow a}\frac{1}{(b-a)\int_a^b w(x)\mathrm{d}x}\int_a^b (x-a)w(x)\mathrm{d}x\\ &= \lim_{b\downarrow a}\frac{(b-a)w(b)}{\int_a^b w(x)\mathrm{d}x + (b-a)w(b)}\\ &= \lim_{b\downarrow a}\frac{w(b)}{\int_a^b w(x)\mathrm{d}x\Big/(b-a) + w(b)}\\ &= \frac{w(a)}{w(a)+w(a)} = \frac{1}{2}.\end{aligned}$$

Therefore, it holds that $\lim_{b\downarrow a} v(a,b) = 1/2$. Next, by Taylor expansion, we have

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(c(x))(x-a)^2$$

for $x \in (a,\infty)\cap D$, where $c(x)$ satisfies $a<c(x)<x$. For $b \in D$ such that $a<b$, it implies that

$$\begin{aligned}
&\frac{\int_a^b f(x)w(x)\mathrm{d}x \Big/ \int_a^b w(x)\mathrm{d}x - f(a)}{b-a} \\
&= \frac{1}{(b-a)\int_a^b w(x)\mathrm{d}x}\int_a^b (f(x)-f(a))w(x)\mathrm{d}x \\
&= \frac{1}{(b-a)\int_a^b w(x)\mathrm{d}x} \\
&\quad \times \int_a^b \left(f'(a)(x-a) + \frac{1}{2}f''(c(x))(x-a)^2\right)w(x)\mathrm{d}x.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\frac{1}{(b-a)\int_a^b w(x)\mathrm{d}x}\int_a^b f'(a)(x-a)w(x)\mathrm{d}x \\
&= f'(a)v(a,b)
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{1}{(b-a)\int_a^b w(x)\mathrm{d}x}\int_a^b \frac{1}{2}f''(c(x))(x-a)^2 w(x)\mathrm{d}x\right| \\
&\le \frac{1}{(b-a)\int_a^b w(x)\mathrm{d}x}\int_a^b K(b-a)^2 w(x)\mathrm{d}x \\
&\le K(b-a)
\end{aligned}$$

with a positive constant $K = \max_{y\in[a,b]} |f''(y)|/2$. Thus we get

$$\begin{aligned}
&\lim_{b\downarrow a}\frac{\int_a^b f(x)w(x)\mathrm{d}x \Big/ \int_a^b w(x)\mathrm{d}x - f(a)}{b-a} \\
&\quad = f'(a)\lim_{b\downarrow a} v(a,b). \qquad (16)
\end{aligned}$$

Next, we put a function

$$F(t) = \int_a^t f(x)w(x)\mathrm{d}x \Bigg/ \int_a^t w(x)\mathrm{d}x,$$

for $t \in (a,\infty)\cap D$. Then we have

$$\begin{aligned}
&\frac{f^{-1}\left(\int_a^b f(x)w(x)\mathrm{d}x \Big/ \int_a^b w(x)\mathrm{d}x\right) - a}{\int_a^b f(x)w(x)\mathrm{d}x \Big/ \int_a^b w(x)\mathrm{d}x - f(a)} \\
&\quad = \frac{f^{-1}(F(b)) - a}{F(b) - f(a)}. \qquad (17)
\end{aligned}$$

Since $\lim_{b\to a} F(b) = f(a)$ and $\lim_{t\to f(a)} f^{-1}(t) = a$, we get

$$\begin{aligned}
\lim_{b\to a}\frac{f^{-1}(F(b)) - a}{F(b)-f(a)} &= \lim_{t\to f(a)}\frac{f^{-1}(t)-a}{t-f(a)} \\
&= \left.\frac{1}{f'(f^{-1}(t))}\right|_{t=f(a)} \\
&= \frac{1}{f'(a)}. \qquad (18)
\end{aligned}$$

These equalities (16), (17) and (18) imply

$$\begin{aligned}
\lim_{b\downarrow a}\theta^f(a,b) &= \lim_{b\downarrow a}\frac{f^{-1}\left(\int_a^b f(x)w(x)\mathrm{d}x \Big/ \int_a^b w(x)\mathrm{d}x\right) - a}{b-a} \\
&= f'(a)\lim_{b\downarrow a} v(a,b)\frac{1}{f'(a)} \\
&= \lim_{b\downarrow a} v(a,b) \\
&= \frac{1}{2}.
\end{aligned}$$

Thus we obtain (10). We can easily check (11) in a similar way. Therefore, we obtain this theorem. □

Proof of Lemma 5.10

(i) Fix any $b > 0$. (12) is trivial from (M.iii). (ii) If M^f satisfies (13), then

$$\begin{aligned}
\lim_{a\downarrow 0}\theta^f(a,b) &= \frac{1}{b}M^f([0,b]) \\
&= M^f([0,1]) \\
&= f^{-1}\left(\int_0^1 f(x)w(x)\mathrm{d}x \Bigg/ \int_0^1 w(x)\mathrm{d}x\right)
\end{aligned}$$

for $b > 0$. Thus we have (14). Finally fix any $a \in D$. From (13) and (M.iii),

$$\begin{aligned}
\lim_{b\to\infty}\theta^f(a,b) &= \lim_{b\to\infty}\frac{M^f([a,b]) - a}{b-a} \\
&= \lim_{b\to\infty}\frac{1}{b}M^f([a,b]) \\
&= \lim_{b\to\infty} M^f([a/b,1]) \\
&= M^f([0,1]).
\end{aligned}$$

Thus we also obtain (15). Therefore, we get this lemma. □

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