Soft Comput (2008) 12:739–749
DOI 10.1007/s00500-007-0232-5

ORIGINAL PAPER

Lattice structure on some fuzzy algebraic systems

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Published online: 23 August 2007

Abstract In this paper, we study the lattice structure of some fuzzy algebraic systems such as (G-)fuzzy groups, some fuzzy ordered algebras and fuzzy hyperstructures. We prove that under suitable conditions, these structures form a distributive or modular lattice.

Keywords Distributive (Modular) lattice · (BCC) BCK-algebra · Fuzzy (normal) subgroup · G-fuzzy subgroup · Fuzzy hyperideal

1 Introduction and preliminaries

In the first half of the nineteeth century, George Boole's attempt to formalize propositional logic led to the concept of Boolean algebras. While investigating the axiomatics of Boolean algebras at the end of the nineteeth century, Charles S. Peirce and Ernst Schröder found it useful to introduce the lattice concept that is a partially ordered set $(P, \leq)$ in which for all $x, y \in P$, $\inf\{x, y\}$ and $\sup\{x, y\}$, denoted by $\wedge$ and $\vee$, respectively, exist. Independently, Richard Dedekind's research on ideals of algebraic numbers led to the same discovery. (Boolean) Distributive lattices, i.e., a lattice L satisfies $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$, (or equivalently $(x \vee y) \wedge z \leq x \vee (y \wedge z)$) for all $x, y, z \in L$, have played a many faceted role in the development of lattice theory and it is one of the most extensive and most satisfying chapters of lattice theory. Also, distributive lattices have provided the motivation for many results in general lattice theory. Dedekind also introduced modularity, i.e., a lattice L satisfies $x \geq z \;\Rightarrow x \wedge (y \vee z) = (x \wedge y) \vee z, \forall x, y, z \in L$, for all $x, y \in L$, a weakend form of distributivity.

The study of BCK-algebras was initiated by Imai and Iséki (1966) as a generalization of the concept of set-theoretic difference and propositional calculi. Iséki posed an interesting problem, whether the class of BCK-algebras is a variety. In connection with this problem, Komori introduced in (Komori 1984) a notion of BCC-algebras, as a generalization of BCK-algebras, and proved that the class of all BCC-algebras is not a variety. Dudek (1992) introduced a dual form of BCC-algebras that was introduced by Komori.

Hyperstructure theory was introduced in 1934 by Marty (1934) at the eighth congress of Scandinavian Mathematiciens. He apply the theory to groups and introduced the concept of a hypergroup that is a nonempty set H together with a function $\circ : H \times H \to P(H) \setminus \{\emptyset\}$, so called *binary hyperoperation*, that satisfies:

Associativity $\quad (x \circ y) \circ z = x \circ (y \circ z)$
Reprocuctive rule $x \circ H = H \circ x = H$.

Some mathematiciens apply hyperstructure theory to other subjects of classical pure mathematics and introduced the notions of hyperring, hyperfield, hypermodule and so on. In Jun et al. (2000), applied the hyperstructures to BCK-algebras, and introduced the notion of a hyper BCK-algebra (and

This research partially is supported by the " Fuzzy Systems and its Applications Center of Excelence, Shahid Bahonar University of Kerman, Iran".

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also weak hyper BCK-ideal) which is a generalization of BCK-algebra.

Now, in this paper, we study the lattice structure of the set of all ideals of some ordered algebras, as mentioned in the abstract.

2 Lattice structure on (G-)fuzzy subgroups of a (G-)group

Definition 2.1 (Rosenfeld 1971) Let $(G, \cdot)$ be a group and μ be a fuzzy subset of G. We say that μ is a *fuzzy subgroup* of G if for all $x, y \in G$:

(i) $\mu(x \cdot y) \geq \mu(x) \wedge \mu(y)$,
(ii) $\mu(x^{-1}) = \mu(x)$.

Moreover, if for all $x, y \in G$ we have $\mu(x \cdot y) = \mu(y \cdot x)$, then μ is called a *fuzzy normal subgroup* of G.

Lemma 2.2 (Rosenfeld 1971; Thomas 1997) *Let $(G, \cdot)$ be a group. Then*

(i) *if μ is a fuzzy subgroup of G and $\mu(x) < \mu(y)$, then $\mu(x \cdot y) = \mu(x) = \mu(y \cdot x)$, for all $x, y \in G$.*
(ii) *if $\mathscr{F}_{nt}$ is the set of all fuzzy normal subgroups with the same tip "t" (i.e., $\mu(e) = \nu(e)$, for all $\mu, \nu \in \mathscr{F}_{nt}$) of G and $\mu, \nu \in \mathscr{F}_{nt}$, then*

$$\mu \vee \nu(x) = \bigvee_{x=y\cdot z} \mu(y) \wedge \nu(z)$$

for all $x \in G$.

Theorem 2.3 *$(\mathscr{F}_{nt}, \vee, \cap)$ is a modular lattice.*

Proof By virtue of Lemma 2.2 and that the intersection of any family of fuzzy normal subgroups with the same tip t of G is a fuzzy normal subgroup with the same tip t of G, $(\mathscr{F}_{nt}, \vee, \cap)$ is a lattice. For modularity, let $\mu, \nu, \eta \in \mathscr{F}_{nt}$ be such that $\mu \geq \eta$. We have to prove that $\mu \wedge (\nu \vee \eta) = (\mu \wedge \nu) \vee \eta$. The inequality $\mu \wedge (\nu \vee \eta) \geq (\mu \wedge \nu) \vee \eta$ is obvious. Hence, it is enough to prove that $\mu \wedge (\nu \vee \eta) \leq (\mu \wedge \nu) \vee \eta$. Now, by contrary, suppose that it does not hold. Then there exists $x \in G$ such that

$$(\mu \wedge (\nu \vee \eta))(x) > ((\mu \wedge \nu) \vee \eta)(x)$$

i.e.,

$$\mu(x) \wedge \bigvee_{x=y\cdot z} (\nu(y) \wedge \eta(z)) > \bigvee_{x=y\cdot z} ((\mu \wedge \nu)(y) \wedge \eta(z)).$$

Hence,

$$\mu(x) > \bigvee_{x=y\cdot z} ((\mu \wedge \nu)(y) \wedge \eta(z)) \tag{1}$$

and

$$\bigvee_{x=y\cdot z} (\nu(y) \wedge \eta(z)) > \bigvee_{x=y\cdot z} ((\mu \wedge \nu)(y) \wedge \eta(z)). \tag{2}$$

This implies that, there exist $y_0, z_0 \in G$ such that $x = y_0 \cdot z_0$ and

$$\begin{aligned} \nu(y_0) \wedge \eta(z_0) &> \bigvee_{x=y\cdot z} ((\mu \wedge \nu)(y) \wedge \eta(z)) \\ &\geq \mu(y_0) \wedge \nu(y_0) \wedge \eta(z_0). \end{aligned} \tag{3}$$

Let $a = \nu(y_0) \wedge \eta(z_0)$. Hence, by (3), we have $a > \mu(y_0) \wedge a$. Now, if $\mu(y_0) \wedge a = a$, then $a > a$, which is impossible. Hence, $\mu(y_0) \wedge a = \mu(y_0)$ and so

$$\mu(y_0) \wedge \nu(y_0) \wedge \eta(z_0) = \mu(y_0). \tag{4}$$

By (4), $\eta(z_0) \geq \mu(y_0)$. If $\mu(y_0) = \eta(z_0)$, then by (3),

$$\nu(y_0) \wedge \eta(z_0) > \mu(y_0) \wedge \nu(y_0) \wedge \eta(z_0) = \nu(y_0) \wedge \eta(z_0)$$

which is a contradiction. So $\eta(z_0) > \mu(y_0)$. Moreover, by (1) we have

$$\mu(x) > \mu(y_0) \wedge \nu(y_0) \wedge \eta(z_0) = \mu(y_0).$$

Thus

$$\mu(y_0^{-1}) = \mu(y_0) < \mu(x) = \mu(y_0 \cdot z_0)$$

and so by Lemma 2.2(i),

$$\mu(y_0) = \mu(y_0^{-1}) = \mu(y_0^{-1} \cdot y_0 \cdot z_0) = \mu(z_0).$$

This implies that $\eta(z_0) > \mu(y_0) = \mu(z_0)$ which is a contradiction, because $\eta \leq \mu$. Therefore, $(\mathscr{F}_{nt}, \cap, \vee)$ is a modular lattice. □

Definition 2.4 (Soleimaninasab and Mashinchi 2004) Let G be a nonempty set. Then:

(i) G together with an operation $\cdot : G \times G \to G$ by $(a, b) \mapsto a \cdot b$, called multiplication, is called a *generalized group* or briefly, *G-group* if it satisfies the following conditions:
   (a) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, for all $x, y, z \in G$,
   (b) for all $x \in G$ there exists a unique element $e(x) \in G$ such that $x \cdot e(x) = e(x) \cdot x = x$,
   (c) for all $x \in G$ there exists $x^{-1} \in G$ such that $x \cdot x^{-1} = x^{-1} \cdot x = e(x)$,
(ii) A nonempty subset H of G-group G is called a *G-subgroup* if it itself is a G-group,
(iii) A fuzzy subset A of G-group G is called a *G-fuzzy subgroup* if and only if $A(x \cdot y^{-1}) \geq A(x) \wedge A(y)$, for all $x, y \in G$.

Theorem 2.5 *Let $\mathscr{G}_f$ be the set of all G-fuzzy subgroups of G-group G, $A \wedge B = A \cap B$ and*

$$A \vee B = \bigcap_{\alpha\in\Lambda} \{A_\alpha : A_\alpha \in \mathscr{G}_f \text{ and } A, B \leq A_\alpha\},$$

for all $A, B \in \mathscr{G}_f$, where Λ is an indexed set. Then $(\mathscr{G}_f, \vee, \wedge)$ is a modular lattice.

Proof It is easy to see that $A \wedge B = A \cap B \in \mathscr{G}_f$, for all $A, B \in \mathscr{G}_f$. Also, it is obvious that

$$A \vee B = \bigcap_{\alpha \in \Lambda} \{A_\alpha : A_\alpha \in \mathscr{G}_f \text{ and } A, B \leq A_\alpha\} \in \mathscr{G}_f.$$

Moreover, by some modifications we can check that $\vee$ and $\wedge$ satisfy the axioms of a lattice. So, $\mathscr{G}_f$ is a lattice. Now, we show that $\mathscr{G}_f$ is modular. For this, let $A, B, C \in \mathscr{G}_f$ be such that $A \geq C$. It is clear that $(A \wedge B) \vee C \leq A \wedge (B \vee C)$. So, it is enough to prove that $A \wedge (B \vee C) \leq (A \wedge B) \vee C$. Let

$$X = \{A_\beta \in \mathscr{G}_f : A \leq A_\beta\}, \ Y = \{A_\alpha \in \mathscr{G}_f : B, C \leq A_\alpha\},$$
$$Z = \{A_\alpha \in \mathscr{G}_f : A \cap B \leq A_\alpha, C \leq A_\alpha\}$$

and $\alpha, \beta \in \Lambda$. Hence, $X \subseteq \{A_\beta \in \mathscr{G}_f : A \cap B \leq A_\beta\}$ and since $A \geq C$, then $X \subseteq \{A_\alpha \in \mathscr{G}_f : C \leq A_\alpha\}$. This implies that $X \subseteq Z$.

Similarly, $Y \subseteq \{A_\alpha \in \mathscr{G}_f : C \leq A_\alpha\}$ and since $A \cap B \leq B$, then $Y \subseteq \{A_\alpha \in \mathscr{G}_f : A \cap B \leq A_\alpha\}$ and so $Y \subseteq Z$. Now, since $A \in \mathscr{G}_f$, then

$$A = \bigcap_{\beta \in \Lambda} \{A_\beta \in \mathscr{G}_f : A \leq A_\beta\}.$$

Hence,

$$\begin{aligned} A \cap (B \vee C) &= A \cap \bigcap_{\alpha \in \Lambda} \{A_\alpha \in \mathscr{G}_f : B, C \leq A_\alpha\} \\ &= \bigcap_{\beta \in \Lambda} \{A_\beta \in \mathscr{G}_f : A \leq A_\beta\} \cap \bigcap_{\alpha \in \Lambda} \\ &\quad \times \{A_\alpha \in \mathscr{G}_f : B, C \leq A_\alpha\} \\ &= \bigcap_{(\alpha,\beta) \in \Lambda \times \Lambda} \{A_\beta \in \mathscr{G}_f : A \leq A_\beta\} \cap \\ &\quad \times \{A_\alpha \in \mathscr{G}_f : B, C \leq A_\alpha\} \\ &\leq \bigcap_{(\alpha,\beta) \in \Lambda \times \Lambda} \{A_\alpha \in \mathscr{G}_f : A \cap B \leq A_\alpha, C \leq A_\alpha\} \\ &= (A \cap B) \vee C. \end{aligned}$$

Hence, $A \wedge (B \vee C) = A \cap (B \vee C) \leq (A \cap B) \vee C = (A \wedge B) \vee C$, says that $\mathscr{G}_f$ is a modular lattice. □

3 Lattice structure on weak normal F-subpolygroups of an F-polygroup

Definition 3.1 (Zahedi and Hasankhani 1996) Let P be a nonempty set, $I = [0, 1]$, I^P the set of all functions from P into I, $I_*^P = I^P \setminus \{0\}$ and 0 be zero function on $I = [0, 1]$. Then,

- by an *F-hyperoperation* "$*$" on P we mean a function from $P \times P$ to I_*^P,
- let "$*$" be an F-hyperoperation on P. Then $(P, *)$ is called an *F-polygroup* iff

(i) $(x * y) * z = x * (y * z)$,

(ii) there exists an element $e \in P$ such that $x \in supp(x * e \cap e * x), \quad \forall x \in P$

(iii) for each $x \in P$ there exists a unique element $x' \in P$ such that $e \in supp(x * x' \cap x' * x)$

(iv) $z \in supp(x * y) \Rightarrow x \in supp(z * y^{-1}) \Rightarrow y \in supp(x^{-1} * z), \quad \forall x, y, z \in P$

where for fuzzy subset μ of nonempty set X, $supp\mu = \{x \in X : \mu(x) > 0\}$.

Definition 3.2 (Zahedi and Hasankhani 1996) Let P be an F-polygroup and H be a nonempty subset of P. Then, H is called an *F-subpolygroup* of P if $x \in H$ implies that $x^{-1} \in H$ and $supp(x * y) \subseteq H$, for all $x, y \in H$. Moreover, H is called a *weak normal F-subpolygroup* of P if H is an F-subpolygroup of P and $x * H * x^{-1} \leq \chi_H$, for all $x \in P$.

Theorem 3.3 (Zahedi and Hasankhani 1996) *Let H and K be F-subpolygroups of an F-polygroup P and $H \otimes K = \bigcup_{x \in H, y \in K} supp(x * y)$. Then:*

(i) *$H \otimes K$ is an F-subpolygroup of P if and only if $H \otimes K = K \otimes H$,*

(ii) *if K is a weak normal F-subpolygroup of P, then $H \otimes K$ is an F-subpolygroup of P,*

(iii) *if H and K are weak normal F-subpolygroups of P, then $H \otimes K$ is a weak normal F-subpolygroup of P.*

Theorem 3.4 *Let $\mathscr{F}_{wn}$ be the set of all weak normal F-subpolygroups of an F-polygroup P. Then $(\mathscr{F}_{wn}, \otimes, \cap)$ is a modular lattice.*

Proof We first show that $\mathscr{F}_{wn}$ is a lattice. For this let $H, K \in \mathscr{F}_{wn}$. It is easy to see that $H \wedge K = H \cap K \in \mathscr{F}_{wn}$. Moreover, by Theorem 3.3(iii), $H \otimes K \in \mathscr{F}_{wn}$. Now, we prove that $H \vee K = H \otimes K$. Let $x \in H$. Since $e \in K$, then $supp(x * e) \subseteq H \otimes K$. Moreover, by Theorem 3.3(iii), $supp(e * x) \subseteq K \otimes H = H \otimes K$ and so $x \in supp(x * e \cap e * x) = supp(x * e) \cap supp(e * x) \subseteq H \otimes K$. Hence, $H \subseteq H \otimes K$. Similarly, we can show that $K \subseteq H \otimes K$. Now, let $L \in \mathscr{F}_{wn}$ be such that $H, K \subseteq L$ and $x \in H \otimes K$. Thus there exist $h \in H$ and $k \in K$ such that $x \in supp(h * k)$, and since L is an F-subpolygroup of P containing H and K, then $supp(h*k) \subseteq L$. This implies that $x \in L$ and so $H \otimes K \subseteq L$. Hence, $H \vee K = H \otimes K$. Theorefore $\mathscr{F}_{wn}$ is a lattice. For modularity, let $H, K, L \in \mathscr{F}_{wn}$ be such that $H \supseteq L$. We have to prove that $H \cap (K \otimes L) = (H \cap K) \otimes L$. But the relation $H \cap (K \otimes L) \supseteq (H \cap K) \otimes L$, is clear. So, it remains to show that $H \cap (K \otimes L) \subseteq (H \cap K) \otimes L$. Now, let $x \in H \cap (K \otimes L)$. Hence, $x \in H$ and $x \in supp(k * l)$, for $k \in K$ and $l \in L$ and so $k \in supp(x * l^{-1}) \subseteq H$, because

$x \in H$ and $l^{-1} \in L \subseteq H$. Thus $k \in H \cap K$. This implies that $x \in supp(k*l) \subseteq (H \cap K) \otimes L$. Hence, $H \cap (K \otimes L) \subseteq (H \cap K) \otimes L$. Thus, $H \cap (K \otimes L) = (H \cap K) \otimes L$ and so $\mathscr{F}_{wn}$ is modular. □

4 Lattice structure on fuzzy hyperideals of a hyperring

Definition 4.1 (Corsini and Leoreanu 2003) (i) Let $(H, \circ)$ be a hypergroup. Then:

(i) An element $e \in H$ is called an *identity* if for all $a \in H$, $a \in (a \circ e) \cap (e \circ a)$.
(ii) If H has an identity, then for $a \in H$ the element $a' \in H$ is called an *inverse* of a if there exists an identity $e \in H$ such that $e \in (a \circ a') \cap (a' \circ a)$.
(iii) H is called *canonical* if it is commutative (i.e., $a \circ b = b \circ a$, for all $a, b \in H$), has an identity and every element has an inverse.

(ii) A *hyperring* is a hypergstructure $(R, +, \cdot, 0)$ where, $(R, +)$ is a canonical hypregroup, $(R, \cdot)$ is a semigroup endowed with a two-sided absorbing element 0 and the product is distributive on addition. A nonempty subset S of a hyperring $(R, +, \cdot)$ is said to be a *hyperideal* if $(S, +)$ is a canonical subhypergroup and for all $x \in S$ and $r \in R$ we have $rx, xr \in S$.

Definition 4.2 (Corsini and Leoreanu 2003) Let μ be a fuzzy subset of a hyperring $(R, +, \cdot)$. Then, μ is said to be a *fuzzy hyperideal* of R if for all $x, y \in G$,

$$\bigwedge_{z \in x+y} \mu(z) \geq \mu(x) \wedge \mu(y),\ \ \mu(-x) \geq \mu(x) \text{ and } \mu(x \cdot y) \geq \mu(x) \vee \mu(y).$$

Lemma 4.3 *Let $(H, \circ)$ be a hypergroupoid and μ and ν be fuzzy subsets of H and*

$$(\mu \circ \nu)(z) = \bigvee_{z \in x \circ y} (\mu(x) \wedge \nu(y)).$$

Then for all $t \in [0, 1)$, $(\mu \circ \nu)_{t^>} = \mu_{t^>} \circ \nu_{t^>}$, where for fuzzy subset μ of H, $\mu_{t^>}$ is defined by $\mu_{t^>} = \{x \in H : \mu(x) > t\}$.

Proof Let $z \in H$. Then:

$$\begin{aligned} z \in (\mu \circ \nu)_{t^>} &\Leftrightarrow \bigvee_{z \in x \circ y} (\mu(x) \wedge \nu(y)) > t \\ &\Leftrightarrow \exists x_0, y_0 \in H; z \in x_0 \circ y_0 \text{ and } \mu(x_0) > t, \nu(y_0) > t \\ &\Leftrightarrow \exists x_0, y_0 \in H; z \in x_0 \circ y_0 \text{ and } x_0 \circ y_0 \subseteq \mu_{t^>} \circ \nu_{t^>} \\ &\Leftrightarrow z \in \mu_{t^>} \circ \nu_{t^>}. \end{aligned}$$

□

Lemma 4.4 *Let ρ be a fuzzy subset of a hyperring R. Then ρ is a fuzzy hyperideal of R if and only if for all $t \in [0, 1)$, $\rho_{t^>} \neq \emptyset$ is a hyperideal of R.*

Proof Let ρ be a fuzzy hyperideal of R and $x, y \in \rho_{t^>}$, for $t \in [0, 1)$. Then $\rho(x) > t$ and $\rho(y) > t$ and so for all $u \in x + y$, we have

$$\rho(u) \geq \bigwedge_{z \in x+y} \rho(z) \geq \rho(x) \wedge \rho(y) > t.$$

This implies that $u \in \rho_{t^>}$ and so $x + y \subseteq \rho_{t^>}$. By a similar argument, we can show that $-x \in \rho_{t^>}$. Now, let $u \in \rho_{t^>}$ and $r \in R$. Then, $\rho(u) > t$ and so $\rho(ru) \geq \rho(r) \vee \rho(u) > \rho(r) \vee t \geq t$, which implies that $ru \in \rho_{t^>}$. Similarly, $ur \in \rho_{t^>}$. Hence, $\rho_{t^>}$ is a hyperideal of R.

Conversely, let for $t \in [0, 1)$, $\rho_{t^>} \neq \emptyset$ be a hyperideal of R and $\rho(x) \wedge \rho(y) = t$. Thus $x, y \in \rho_t$ and so for all $s \in [0, t)$ we have $x, y \in \rho_{s^>}$. Hence, $x + y \subseteq \rho_{s^>}$ and so for all $z \in x + y$, we have $\rho(z) > s$, for all $s \in [0, t)$. Thus $\rho(z) \geq t$, for all $z \in x + y$ and so

$$\bigwedge_{z \in x+y} \rho(z) \geq t = \rho(x) \wedge \rho(y).$$

Similarly, we can show that $\rho(-x) \geq \rho(x)$, for all $x \in R$. Now, let $\rho(x) \vee \rho(y) = t$, for $x, y \in R$. Then, $\rho(x) \geq t$ or $\rho(y) \geq t$. Let $\rho(x) \geq t$. Then for all $s \in [0, t)$, $\rho(x) > s$, i.e., $x \in \rho_{s^>}$ and since $\rho_{s^>}$ is a hyperideal of R, then $x \cdot y \in \rho_{s^>}$. Hence, $\rho(x \cdot y) > s$, for all $s \in [0, t)$ and so $\rho(x \cdot y) \geq t = \rho(x) \vee \rho(y)$. Therefore, ρ is a fuzzy hyperideal of R. □

Theorem 4.5 *Let $\mathscr{F}_i$ be the set of all fuzzy hyperideals of R with the same tip "t". Then $(\mathscr{F}_i, +, \cap)$ is a distributive lattice, where for all $\mu, \nu \in \mathscr{F}_i$,*

$$(\mu + \nu)(z) = \bigvee_{z \in x+y} (\mu(x) \wedge \nu(y)).$$

Proof Let $\mu, \nu \in \mathscr{F}_i$. We first show that $\mu + \nu \in \mathscr{F}_i$. For this, by Lemma 4.4, it is enough to prove that for all $t \in [0, 1)$, $(\mu + \nu)_{t^>} \neq \emptyset$ is a hyperideal of R. Since μ and ν are fuzzy hyperideals of R, then by Lemma 4.4, $\mu_{t^>} \neq \emptyset$ and $\nu_{t^>} \neq \emptyset$ are hyperideals of R. Now, we prove that $\mu_{t^>} + \nu_{t^>}$ is a hyperideal of R. For this, let $x, y \in \mu_{t^>} + \nu_{t^>}$, for $x, y \in R$. Then, there exist $u_1, u_2 \in \mu_{t^>}$ and $v_1, v_2 \in \nu_{t^>}$ such that $x \in u_1 + v_1$ and $y \in u_2 + v_2$ and so $x + y \subseteq (u_1 + v_1) + (u_2 + v_2) = (u_1 + u_2) + (v_1 + v_2)$. By definition, $\bigwedge_{a \in u_1+u_2} \mu(a) \geq \mu(u_1) \wedge \mu(u_2) > t$, which shows that $u_1 + u_2 \subseteq \mu_{t^>}$. Similarly, $v_1 + v_2 \subseteq \nu_{t^>}$ and hence, $x + y \subseteq \mu_{t^>} + \nu_{t^>}$. By a similar way, we can show that $-x \in \mu_{t^>} + \nu_{t^>}$, for all $x \in \mu_{t^>} + \nu_{t^>}$. Now, let $x \in \mu_{t^>} + \nu_{t^>}$ and $r \in R$. Then, there exist $u \in \mu_{t^>}$ and $v \in \nu_{t^>}$ such that $x \in u + v$. Now, $rx \in r(u + v) = ru + rv$ and $\mu(ru) \geq \mu(r) \vee \mu(u) > \mu(r) \vee t \geq t$, which implies that $ru \in \mu_{t^>}$. Similarly, $rv \in \nu_{t^>}$ and so $rx \in \mu_{t^>} + \nu_{t^>}$. Similarly, we can show that $xr \in \mu_{t^>} + \nu_{t^>}$. Thus, $\mu_{t^>} + \nu_{t^>}$ is a hyperideal of R and so by Lemma 4.3, $(\mu+\nu)_{t^>}$ is a hyperideal of R. Thus, by Lemma 4.4, $\mu+\nu$ is a fuzzy hyperideal of R.

Now, we prove that $\mu+\nu=\mu\vee\nu$. Let $z\in R$. Since $z\in z+0=0+z$ and $\nu(0)=\mu(0)\geq\mu(x)$, for all $x\in R$, then we have

$$(\mu+\nu)(z)=\bigvee_{z\in x+y}(\mu(x)\wedge\nu(y))\geq\mu(z)\wedge\nu(0)=\mu(z).$$

Similarly, $(\mu+\nu)(z)\geq\nu(z)$ and so $\mu+\nu$ is an upper bound for μ and ν.

Now, let η be a fuzzy hyperideal of R containing μ and ν and $z\in R$. Then there exist $x,y\in R$ such that $z\in x+y$, e.g., we can choose $x=z$ and $y=0$. Hence, $\eta(z)\geq\bigwedge_{u\in x+y}\eta(u)\geq\eta(x)\wedge\eta(y)$ and so for all $x,y\in R$ such that $z\in x+y$ we have

$$\eta(z)\geq\bigvee_{z\in x+y}(\eta(x)\wedge\eta(y))\geq\bigvee_{z\in x+y}(\mu(x)\wedge\nu(y))$$
$$=(\mu+\nu)(z).$$

So, $\mu\vee\nu=\mu+\nu$. Also, this implies that $\mu+\mu=\mu$, for all $\mu\in\mathscr{F}_i$. Now, for distributivity, we prove that $\tau\cap(\rho+\sigma)=(\tau\cap\rho)+(\tau\cap\sigma)$, for all $\rho,\sigma,\tau\in\mathscr{F}_i$. Let $z\in R$. Then:

$$\begin{aligned}((\tau\cap\rho)+(\tau\cap\sigma))(z)&=\bigvee_{z\in x+y}((\tau\cap\rho)(x)\wedge(\tau\cap\sigma)(y))\\
&=\bigvee_{z\in x+y}\Big((\tau(x)\wedge\rho(x))\wedge(\tau(y)\wedge\sigma(y))\Big)\\
&=\bigvee_{z\in x+y}\Big((\tau(x)\wedge\tau(y))\wedge(\rho(x)\wedge\sigma(y))\Big)\\
&=\Big(\bigvee_{z\in x+y}(\tau(x)\wedge\tau(y))\Big)\wedge\Big(\bigvee_{z\in x+y}(\rho(x)\wedge\sigma(y))\Big)\\
&=(\tau+\tau)(z)\wedge(\rho+\sigma)(z)\\
&=\tau(z)\wedge(\rho+\sigma)(z)\\
&=(\tau\cap(\rho+\sigma))(z).\end{aligned}$$

Hence, $\tau\cap(\rho+\sigma)=(\tau\cap\rho)+(\tau\cap\sigma)$. Therefore, $(\mathscr{F}_i,+,\cap)$ is a distributive lattice. □

5 Lattice structure on fuzzy ideals of a BCK-algebra

Definition 5.1 (Meng and Jun 1994) (i) By a *BCK-algebra* we mean a nonempty set X endowed with a binary operation " $*$ " and a constant " 0 " satisfies the following conditions:

(BCK1) $(x*z)*(y*z)\leq x*y$,
(BCK2) $(x*y)*z=(x*z)*y$,
(BCK3) $x*y\leq x$,
(BCK4) $x\leq y$ and $y\leq x$ imply $x=y$,

for all $x,y,z\in X$, where " $\leq$ " is defined by $x\leq y$ if and only if $x*y=0$, for all $x,y\in X$.

(ii) Nonempty subset I of BCK-algebra X is said to be an *ideal* of X if $0\in I$ and $x*y\in I$ and $y\in I$ imply $x\in I$, for all $x,y\in X$.

(iii) Fuzzy subset μ of BCK-algebra X is said to be a *fuzzy ideal* of X if $\mu(0)\geq\mu(x)$ and $\mu(x)\geq\mu(x*y)\wedge\mu(y)$, for all $x,y\in X$.

Note 5.2 (Meng and Jun 1994) Let X be a BCK-algebra. Then,

(i) the following statements hold:
  (a) $x\leq y$ implies that $x*z\leq y*z$, for all $x,y,z\in X$.
  (b) $0*x=0$,
  (c) $x*x=0$,
  (d) it is easy to check that fuzzy subset μ of X is a fuzzy ideal if and only if for all $t\in[0,1]$, the level subset $\mu_t=\{x\in X:\mu(x)\geq t\}\neq\emptyset$ is an ideal of X,
  (e) let μ be a fuzzy subset of X. Then, $\bigcap_{\mu\subseteq\nu}\nu$, where ν is a fuzzy ideal of X, is a fuzzy ideal of X, too. We denote it by $[\mu]$, that is $[\mu]=\{\nu:\mu\subseteq\nu,$ and ν is a fuzzy ideal of $X\}$.

Lemma 5.3 *Let μ be a fuzzy subset of BCK-algebra X. Then μ is a fuzzy ideal of X if and only if*

$$(x*y)*z=0\ \textit{implies that}\ \mu(x)\geq\mu(y)\wedge\mu(z)\tag{5}$$

for all $x,y,z\in X$.

Proof Let μ be a fuzzy ideal of X, $(x*y)*z=0$, for $x,y,z\in X$ and $\mu(y)\wedge\mu(z)=t$. Thus $\mu(y)\geq t$ and $\mu(z)\geq t$ and so $y,z\in\mu_t$. Now, since $(x*y)*z=0\in\mu_t$ and by Note 5.2(ii), μ_t is an ideal of X, then $x*y\in\mu_t$. Similarly, since $y\in\mu_t$, then $x\in\mu_t$ and so

$$\mu(x)\geq t=\mu(y)\wedge\mu(z).$$

Conversely, let the condition (5) holds. Since $(0*x)*x=0$, for all $x\in X$, then $\mu(0)\geq\mu(x)\wedge\mu(x)=\mu(x)$, for all $x\in X$. Moreover, since by (BCK2), for all $x,y\in X$ we have

$$(x*(x*y))*y=(x*y)*(x*y)=0,$$

then by hypothesis, $\mu(x)\geq\mu(x*y)\wedge\mu(y)$, which implies that μ is a fuzzy ideal of X. □

Corollary 5.4 *Let μ be a fuzzy subset of BCK-algebra X. Then μ is a fuzzy ideal of X if and only if*

$$(\cdots((x*a_1)*a_2)*\cdots)*a_n=0\ \textit{implies that}\ \mu(x)$$
$$\geq\mu(a_1)\wedge\mu(a_2)\wedge\cdots\wedge\mu(a_n)$$

for all $x,a_1,a_2,\ldots,a_n\in X$.

Proof The proof follows from Lemma 5.3, by induction on n. □

Lemma 5.5 *Let μ and ν be fuzzy ideals of BCK-algebra X and*

$$\eta(x) = \vee\{\mu \cup \nu(a_1) \wedge \mu \cup \nu(a_2) \wedge \cdots \wedge \mu \cup \nu(a_n) : (\cdots((x * a_1) * a_2) * \cdots) * a_n = 0, \text{ for some } a_1, a_2, \ldots, a_n \in X\}$$

Then $\eta = [\mu \cup \nu]$ and so $[\mu \cup \nu] = \mu \vee \nu$.

Proof We have to prove that η is a fuzzy ideal of X. For this, by Lemma 5.3, it is enough to show that if $(x * y) * z = 0$, then $\eta(x) \geq \eta(y) \wedge \eta(z)$. Let $x, y, z \in X$ be such that $(x * y) * z = 0$ and $\epsilon > 0$. Since, $y * y = 0$, then there exist $a_1, a_2, \ldots, a_n \in X$ such that

$$(\ldots((y * a_1) * a_2) * \cdots) * a_n = 0$$

and

$$\eta(y) - \epsilon < \mu \cup \nu(a_1) \wedge \mu \cup \nu(a_2) \wedge \cdots \wedge \mu \cup \nu(a_n).$$

Similarly, there exist $b_1, b_2, \ldots, b_m \in X$ such that $(\cdots((z * b_1) * b_2) * \cdots) * b_m = 0$ and

$$\eta(z) - \epsilon < \mu \cup \nu(b_1) \wedge \mu \cup \nu(b_2) \wedge \cdots \wedge \mu \cup \nu(b_m).$$

Now, since $(x * y) * z = 0$ and so $x * y \leq z$, then by Note 5.2(i)(a), we have

$$\begin{aligned}((\cdots((x * b_1) * b_2) * \cdots) * b_m) * y &= (\cdots(((x * y) * b_1) * b_2) * \cdots) * b_m \\ &\leq (\cdots((z * b_1) * b_2) * \cdots) * b_m = 0.\end{aligned}$$

This implies that $((\cdots((x * b_1) * b_2) * \cdots) * b_m) * y = 0$ and so $(\cdots((x * b_1) * b_2) * \cdots) * b_m \leq y$. Similarly, by Note 5.2(i)(a),

$$(\cdots((((\cdots((x * b_1) * b_2) * \cdots) * b_m) * a_1) * a_2) * \cdots) * a_n \leq (\cdots((y * a_1) * a_2) * \cdots) * a_n = 0.$$

Hence,

$$(\cdots((((\cdots((x * b_1) * b_2) * \cdots) * b_m) * a_1) * a_2) * \cdots) * a_n = 0$$

and so by (BCK2),

$$(\cdots((((\cdots((x * a_1) * a_2) * \cdots) * a_n) * b_1) * b_2) * \cdots) * b_m = 0.$$

Thus,

$$\begin{aligned}\eta(x) &\geq \mu \cup \nu(a_1) \wedge \cdots \wedge \mu \cup \nu(a_n) \\ &\quad \wedge \mu \cup \nu(b_1) \wedge \cdots \wedge \mu \cup \nu(b_m) \\ &> (\eta(y) - \epsilon) \wedge (\eta(z) - \epsilon) \\ &= (\eta(y) \wedge \eta(z)) - \epsilon.\end{aligned}$$

Therefore, $\eta(x) \geq \eta(y) \wedge \eta(z)$. Now, since $x * x = 0$, then $\mu(x) \leq \mu \cup \nu(x) \leq \eta(x)$. Hence, $\mu \leq \eta$. Similarly, $\nu \leq \eta$, which shows that η is an upper bound for μ and ν. Now, let τ be a fuzzy ideal of X containing μ and ν and $x \in X$. Thus,

$$\begin{aligned}\eta(x) &= \vee\{\mu \cup \nu(a_1) \wedge \mu \cup \nu(a_2) \wedge \cdots \wedge \mu \cup \nu(a_n) : \\ &\qquad (\cdots((x * a_1) * a_2) * \ldots) * a_n = 0, \\ &\qquad \text{for some } a_1, a_2, \ldots, a_n \in X\} \\ &\leq \vee\{\tau(a_1) \wedge \tau(a_2) \wedge \cdots \wedge \tau(a_n) : \\ &\qquad (\cdots((x * a_1) * a_2) * \ldots) * a_n = 0, \\ &\qquad \text{for some } a_1, a_2, \ldots, a_n \in X\} \\ &\leq \vee\tau(x) \quad \text{by Corollary 5.4} \\ &= \tau(x).\end{aligned}$$

Therefore, $\eta = [\mu \cup \nu]$ and so $\mu \vee \nu = [\mu \vee \nu]$. □

Theorem 5.6 *Let $\mathscr{F}\mathscr{I}(X)$ be the set of all fuzzy ideals of BCK-algebra X. Then $(\mathscr{F}\mathscr{I}(X), \vee, \cap)$ is a distributive lattice.*

Proof By Lemma 5.5 and that the intersection of any two fuzzy ideal of X is again a fuzzy ideal of X, $(\mathscr{F}\mathscr{I}(X), \vee, \cap)$ is a lattice. For distributivity, it is enough to show that for all $\mu, \rho, \sigma \in \mathscr{F}\mathscr{I}(X)$,

$$\mu \wedge (\rho \vee \sigma) \leq (\mu \wedge \rho) \vee (\mu \wedge \sigma).$$

Because, the converse inequality is obvious. Let $\mu, \rho, \sigma \in \mathscr{F}\mathscr{I}(X), x \in X$ and $\epsilon > 0$. Then there exist $a_1, a_2, \ldots, a_n \in X$, such that

$$(\cdots((x * a_1) * a_2) * \cdots) * a_n = 0$$

and

$$\rho \vee \sigma(x) < \epsilon + \wedge(\rho \cup \sigma(a_1), \rho \cup \sigma(a_2), \ldots, \rho \cup \sigma(a_n)).$$

We observe that $a_1, a_2, \ldots, a_n$ always exist, because $x * x = 0$ holds in any BCK-algebra and we can choose $a_1 = x$, $a_2 = \cdots = a_n = 0$. Now, by the definition of $\rho \cup \sigma$ we have $\rho \cup \sigma(a_i) < \rho(a_i) + \epsilon$ or $\sigma(a_i) + \epsilon$. Without loss of generality, we can suppose that

$$\begin{aligned}&\rho \cup \sigma(a_1) < \rho(a_1) + \epsilon \\ &\vdots \\ &\rho \cup \sigma(a_i) < \rho(a_i) + \epsilon \\ &\rho \cup \sigma(a_{i+1}) < \sigma(a_{i+1}) + \epsilon \\ &\vdots \\ &\rho \cup \sigma(a_n) < \sigma(a_n) + \epsilon.\end{aligned}$$

Because, if $\rho \cup \sigma(a_j) < \sigma(a_j) + \epsilon$, for $j \in \{1, 2, \ldots, i\}$, then we can restrict the set $\{1, 2, \ldots, i\}$ to a set $I = \{1, 2, \ldots, l\} \subseteq \{1, 2, \ldots, i\}$ and rearrange $a_i's$ such that for all $j \in I$, $\rho \cup \sigma(a_j) < \rho(a_j) + \epsilon$. Hence,

$$\rho \vee \sigma(x) < 2\epsilon + \wedge(\rho(a_1), \ldots, \rho(a_i), \sigma(a_{i+1}), \ldots, \sigma(a_n))$$

and so

$$\mu \wedge (\rho \vee \sigma)(x) < 2\epsilon + \wedge(\mu(x) \wedge \rho(a_1), \dots, \mu(x) \wedge \rho(a_i), \mu(x) \wedge \sigma(a_{i+1}), \dots, \mu(x) \wedge \sigma(a_n)).$$

Now, let

$$b_n = (\cdots((x * a_1) * a_2) * \cdots) * a_{n-1}$$
$$b_{n-1} = ((\cdots((x * a_1) * a_2) * \cdots) * a_{n-2}) * b_n$$
$$\vdots$$
$$b_1 = (\cdots((x * b_n) * b_{n-1}) * \cdots) * b_2.$$

Hence,

$$(\cdots((x * b_1) * b_2) * \cdots) * b_n = ((\cdots((x * b_n) * b_{n-1}) * \cdots) * b_2) * b_1 = b_1 * b_1 = 0.$$

Moreover, from the above equalities we deduce that $b_i \leq x$, for all $i \in \{1, 2, \dots, n\}$ and since μ is a fuzzy ideal of X, then $\mu(x) \leq \mu(b_i)$, for all $i \in \{1, 2 \dots, n\}$. Also,

$$b_n * a_n = ((\cdots((x * a_1) * a_2) * \cdots) * a_{n-1}) * a_n = 0$$

which implies that $b_n \leq a_n$. Similarly, for all $i \in \{1, 2, \dots, n\}$ we have $b_i \leq a_i$ and so $\rho(a_i) \leq \rho(b_i)$, for all $i \in \{1, 2, \dots, n\}$. Thus,

$$\mu(x) \wedge \rho(a_1) \leq \mu(b_1) \wedge \rho(b_1)$$
$$\vdots$$
$$\mu(x) \wedge \rho(a_i) \leq \mu(b_i) \wedge \rho(b_i)$$
$$\mu(x) \wedge \sigma(a_{i+1}) \leq \mu(b_{i+1}) \wedge \sigma(b_{i+1})$$
$$\vdots$$
$$\mu(x) \wedge \sigma(a_n) \leq \mu(b_n) \wedge \sigma(b_n)$$

and so

$$\mu \wedge (\rho \vee \sigma)(x) < 2\epsilon + \wedge(\mu \wedge \rho(b_1), \dots, \mu \wedge \rho(b_i), \mu \wedge \sigma(b_{i+1}), \dots, \mu \wedge \sigma(b_n)).$$

Obviously,

$$\mu \wedge \rho(b_i) \leq ((\mu \wedge \rho) \cup (\mu \wedge \sigma))(b_i) \leq [(\mu \wedge \rho) \cup (\mu \wedge \sigma))(b_i)$$

and similarly,

$$\mu \wedge \sigma(b_i) \leq [(\mu \wedge \rho) \cup (\mu \wedge \sigma))(b_i).$$

So,

$$\mu \wedge (\rho \vee \sigma)(x) < 2\epsilon + \bigwedge_{i=1}^{n}[(\mu \wedge \rho) \cup (\mu \wedge \sigma))(b_i) \leq 2\epsilon + [(\mu \wedge \rho) \cup (\mu \wedge \sigma))(x).$$

Since ϵ is arbitrary, then

$$\mu \wedge (\rho \vee \sigma) \leq (\mu \wedge \rho) \vee (\mu \wedge \sigma).$$

Thus,

$$\mu \wedge (\rho \vee \sigma) = (\mu \wedge \rho) \vee (\mu \wedge \sigma)$$

which shows that $(\mathscr{F}\mathscr{I}(X), \vee, \cap)$ is distributive. □

6 Lattice structure on fuzzy ideals of a BCC-algebra

We first give some preliminaries about BCC-algebras.

Definition 6.1 (Dudek 2000; Dudek and Jun 1999; Dudek and Zhang 1998) Let G be a nonempty set. Then,

(i) algebra $(G, *, 0)$ of type (2,0) is said to be a *BCC-algebra* if it satisfies the following axioms:
(1) $((x * y) * (z * y)) * (x * z) = 0$,
(2) $0 * x = 0$,
(3) $x * 0 = x$,
(4) $x * y = 0$ and $y * x = 0$ imply $x = y$,
for all $x, y, z \in G$.
Any BCC-algebra may be viewed as a partially ordered set with the order " $\leq$ " defined by

$$x \leq y \text{ iff } x * y = 0$$

which has the following properties:
(a) $x * y \leq x$.
(b) $x \leq y$ implies that $x * z \leq y * z$ and $z * y \leq z * x$.
for all $x, y, z \in G$.

(ii) Nonempty subset I of BCC-algebra G is said to be a *BCC-ideal* if $0 \in I$ and $(x * y) * z \in I$ and $y \in I$ imply that $x * z \in I$.
It is well-known that any BCC-ideal of a BCC-algebra is a BCK-ideal.

(iii) BCC-algebra G is said to be *positive implicative* if for all $x, y, z \in G$

$$(x * y) * z = (x * z) * (y * z).$$

(iv) Fuzzy subset μ of BCC-algebra G is said to be a *fuzzy BCC-ideal* if $\mu(0) \geq \mu(x)$, for all $x \in G$ and $\mu(x*z) \geq \mu((x * y) * z) \wedge \mu(y)$, for all $x, y, z \in G$.
It is easy to see that if μ is a fuzzy BCC-ideal of BCC-algebra G and $x \leq y$, then $\mu(x) \geq \mu(y)$, for all $x, y \in G$.

Lemma 6.2 (Dudek and Jun 1999; Dudek et al. 2001) *Let μ be a fuzzy subset of BCC-algebra G. Then μ is a fuzzy BCC-ideal of G if and only if for all $t \in [0, 1)$, $\mu_{t^>} = \{x \in G : \mu(x) > t\} \neq \emptyset$ is a BCC-ideal of G.*

Lemma 6.3 (Dudek and Jun 1999) (i) *In a BCC-algebra every fuzzy BCC-ideal is a fuzzy BCK-ideal.*
(ii) *In a BCK-algebra every fuzzy BCK-ideal is a fuzzy BCC-ideal.*

Lemma 6.4 *Let μ be a fuzzy BCC-ideal of BCC-algebra G. Then*

$$(\cdots((x * a_1) * a_2) * \cdots) * a_n = 0 \textit{ implies that } \mu(x) \geq \mu(a_1) \wedge \mu(a_2) \wedge \cdots \wedge \mu(a_n)$$

for all $x, a_1, a_2, \ldots, a_n \in G$.

Proof The proof is similar to the proof of Corollary 5.4. □

Lemma 6.5 *Every positive implicative* BCC*-algebra is a* BCK*-algebra.*

Proof By Corollary 1 of Dudek (1992), it is enough to prove that $x * (x * y) \leq y$, for all $x, y \in G$. For this, let $x, y \in G$. Then,

$$
\begin{aligned}
(x * (x * y)) * y &= (x * y) * ((x * y) * y) \\
&= (x * y) * ((x * y) * (y * y)) \\
&= (x * y) * ((x * y) * 0) \\
&= (x * y) * (x * y) = 0
\end{aligned}
$$

and so $x * (x * y) \leq y$, completes the proof. □

Lemma 6.6 *Let* μ *and* ν *be fuzzy* BCC*-ideals of positive implicative* BCC*-algebra* G*. Then fuzzy subset*

$$
\begin{aligned}
\eta(x) = \vee\{&\mu \cup \nu(a_1) \wedge \mu \cup \nu(a_2) \wedge \cdots \wedge \mu \cup \nu(a_n) : \\
&(\cdots((x * a_1) * a_2) * \cdots) * a_n = 0, \\
&\text{for some } A_1, a_2, \ldots, a_n \in G\}.
\end{aligned}
$$

is the fuzzy BCC*-ideal of* G *generated by* $\mu \cup \nu$ *and so* $\mu \vee \nu = [\mu \cup \nu]$.

Proof Since by Lemma 6.3(i), μ and ν are fuzzy BCK-ideals of G and by Lemma 6.5, G is a BCK-algebra, then by Lemma 5.5, η is a fuzzy BCK-ideal of G. Hence, by Lemma 6.3(ii), η is a fuzzy BCC-ideal of G. Moreover, similar to the proof of Lemma 5.5, $[\mu \cup \nu] = \eta = \mu \vee \nu$. □

Theorem 6.7 *Let* G *be positive implicative* BCC*-algebra and* $\mathscr{F}\mathscr{I}(G)$ *be the set of all fuzzy* BCC*-ideals of* G*. Then* $(\mathscr{F}\mathscr{I}(G), \vee, \cap)$ *is a distributive lattice.*

Proof Since, by Lemma 6.5, G is a BCK-algebra, then the proof follows from Theorem 5.6. □

7 Lattice structure on fuzzy weak hyper BCK-ideals of a hyper BCK-algaebra

Definition 7.1 (Jun and Xin 2001; Jun et al. 2000) (i) By a *hyper* BCK*-algebra* we mean a nonempty set H endowed with a hyperoperation "$\circ$" and a constant 0 that satisfies the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x \circ H \ll \{x\}$,
(HK4) $x \ll y$ and $y \ll x$ imply $x = y$,

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$. In such case, we call "$\ll$" the *hyperorder* on H.

(ii) The set $S(H) = \{x \in H : x \circ x = \{0\}\}$ is called the *BCK-part* of H.
(iii) Hyper BCK-algebra H is said to be *quasi alternatively hyper* BCK*-algebra of type 1* if $(x \circ y) \circ y = x \circ (y \circ y)$, for all $x, y \in H$.
(iv) Nonempty subset I of hyper BCK-algebra H is said to be a *weak hyper BCK-ideal* if $0 \in I$ and $x \circ y \subseteq I$ and $y \in I$ imply $x \in I$, for all $x, y \in H$.
(v) Fuzzy subset μ of hyper BCK-algebra H is called *fuzzy weak hyper BCK-ideal* if $\mu(0) \geq \mu(x)$ and $\mu(x) \geq (\bigwedge_{a \in x \circ y} \mu(a)) \wedge \mu(y)$, for all $x, y \in H$.

Lemma 7.2 *Let* μ *be a fuzzy subset of hyper* BCK*-algebra* H *and for all* $t \in [0, 1]$,

$$\mu_t = \{x \in H : \mu(x) \geq t\} \text{ and } \mu_{t^>} = \{x \in H : \mu(x) > t\}.$$

Then the following statements are equivalent:

(i) μ *is a fuzzy weak hyper* BCK*-ideal of* H,
(ii) $\mu_t \neq \emptyset$ *is a weak hyper* BCK*-ideal of* H*, for all* $t \in [0, 1]$,
(iii) $\mu_{t^>} \neq \emptyset$ *is a weak hyper* BCK*-ideal of* H*, for all* $t \in [0, 1)$.

Proof (i) ⇒ (ii) See Jun and Xin (2001).
(ii) ⇒ (iii) By the definition of $\mu_{t^>}$ we have

$$
\begin{aligned}
\mu_{t^>} &= \{x \in H : \mu(x) > t\} = \bigcup_{s \in (t,1]} \{x \in H : \mu(x) \geq s\} \\
&= \bigcup_{s \in (t,1]} \mu_s.
\end{aligned}
$$

Now, let $\mu_{t^>} \neq \emptyset$, for $t \in [0, 1)$. Then for some $s \in (t, 1]$, $\mu_s \neq \emptyset$ and since by (ii), $0 \in \mu_s$, then $0 \in \mu_t$. Let $x \circ y \subseteq \mu_{t^>}$ and $y \in \mu_{t^>}$, for $x, y \in H$. Then, for all $a \in x \circ y$, there exists $s \in (t, 1]$ such that $a \in \mu_s$. Since the set $\{s : s \in (t, 1]\}$ is a chain, then the set $\{\mu_s : s \in (t, 1]\}$ is a chain and so there exists $s_1 \in (t, 1]$ such that $a \in \mu_{s_1}$, for all $a \in x \circ y$, which implies that $x \circ y \subseteq \mu_{s_1}$. Also, since $y \in \mu_{t^>}$, then there exists $s_2 \in (t, 1]$ such that $y \in \mu_{s_2}$. Now, $\mu_{s_1} \subseteq \mu_{s_2}$ or $\mu_{s_2} \subseteq \mu_{s_1}$. W.L.O.G, let $\mu_{s_2} \subseteq \mu_{s_1}$. Then $x \circ y \subseteq \mu_{s_1}$ and $y \in \mu_{s_1}$ and since, μ_{s_1} is a weak hyper BCK-ideal of H, then $x \in \mu_{s_1} \subseteq \mu_{t^>}$. Thus, $\mu_{t^>}$ is a weak hyper BCK-ideal of H.

(iii) ⇒ (i) It is easy to see that $\mu(0) \geq \mu(x)$, for all $x \in H$. Now, let

$$\left(\bigwedge_{a \in x \circ y} \mu(a)\right) \wedge \mu(y) = t,$$

for $x, y \in H$. Then for all $a \in x \circ y$, $\mu(a) \geq t$ and $\mu(y) \geq t$. This implies that for all $s \in [0, t)$, $\mu(a) > s$ and $\mu(y) > s$. Hence, $a \in \mu_{s^>}$ and so $x \circ y \subseteq \mu_{s^>}$. Also $y \in \mu_{s^>}$, for all

$s \in [0, t)$. Thus by (iii), $x \in \mu_{s^>}$ and so $\mu(x) > s$, for all $s \in [0, t)$, which implies that $\mu(x) \geq t$. Hence,

$$\mu(x) \geq \left(\bigwedge_{a \in x \circ y} \mu(a) \right) \wedge \mu(y).$$

Therefore, μ is a fuzzy weak hyper BCK-ideal of H. □

Lemma 7.3 *Let μ be a fuzzy weak hyper BCK-ideal of hyper BCK-algebra H. Then*

$$(\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_n = \{0\} \text{ implies that } \mu(x) \geq \mu(a_1) \wedge \mu(a_2) \cdots \wedge \mu(a_n),$$

for all $x, a_1, \ldots, a_n \in H$.

Proof We prove the lemma by induction on n. Let $n = 2$, $(x \circ a_1) \circ a_2 = \{0\}$, for $x, a_1, a_2 \in H$ and $\mu(a_1) \wedge \mu(a_2) = t$. Then $a_1 \in \mu_t$ and $a_2 \in \mu_t$. Since $(x \circ a_1) \circ a_2 = \{0\} \subseteq \mu_t$, $a_2 \in \mu_t$ and by Lemma 7.2, μ_t is a weak hyper BCK-ideal of H, then $x \circ a_1 \subseteq \mu_t$. Similarly, we get $x \in \mu_t$ and so $\mu(x) \geq t = \mu(a_1) \wedge \mu(a_2)$. Now, let the lemma holds for $n = k - 1$ and

$$(\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_k = \{0\}$$

for $a_1, a_2, \ldots, a_k \in H$. Then by (HK2) we have

$$(\cdots((x \circ a_k) \circ a_1) \circ \cdots) \circ a_{k-1} = \{0\}.$$

Let $u \in x \circ a_k$. Then $(\cdots((u \circ a_1) \circ a_2) \circ \cdots) \circ a_{k-1} = \{0\}$ and so

$$\mu(u) \geq \mu(a_1) \wedge \mu(a_2) \wedge \cdots \wedge \mu(a_{k-1}).$$

Hence,

$$\bigwedge_{a \in x \circ a_k} \mu(a) \geq \mu(a_1) \wedge \mu(a_2) \wedge \cdots \wedge \mu(a_{k-1}).$$

On the other hand, since μ is a fuzzy weak hyper BCK-ideal of H, then

$$\mu(x) \geq \left(\bigwedge_{a \in x \circ a_k} \mu(a) \right) \wedge \mu(a_k) \geq \mu(a_1) \wedge \mu(a_2) \wedge \cdots \wedge \mu(a_{k-1}) \wedge \mu(a_k).$$

Therefore, the induction is complete. □

Theorem 7.4 *Let H be a hyper BCK-algebra and f be a fuzzy subset of H that satisfies the BCK-part condition, i.e.,*

$$\forall x \in H \setminus S(H), \quad f(x) = 0.$$

Then fuzzy subset μ which is defined by

$$\mu(x) = \vee\{f(a_1) \wedge \cdots \wedge f(a_n) : (\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_n = \{0\}, \text{ for some } a_1, a_2, \ldots, a_n \in H\}$$

is the fuzzy weak hyper BCK-ideal of H generated by f and we denote it by $[f]_w$.

Proof We have to prove that μ is a fuzzy weak hyper BCK-ideal of H. For this, by Lemma 7.2, it is enough to prove that $\mu_{t^>} \neq \emptyset$ is a weak hyper BCK-ideal of H. Since, $0 \circ x = \{0\}$ for all $x \in H$, then it is easy to see that $\mu(0) \geq \mu(x)$, for all $x \in H$. Now, let $x \circ y \subseteq \mu_{t^>}$ and $y \in \mu_{t^>}$, for $x, y \in H$ and $t \in [0, 1)$. Then $\mu(y) > t$ and for all $a \in x \circ y$, $\mu(a) > t$ and so by the definition of μ,

$$\vee\{f(a_1) \wedge \cdots \wedge f(a_n) : (\cdots((a \circ a_1) \circ a_2) \circ \cdots) \circ a_n = \{0\}, \text{ for some } a_1, a_2, \ldots, a_n \in H\} > t$$

and

$$\vee\{f(b_1) \wedge \cdots \wedge f(b_m) : (\cdots((y \circ b_1) \circ b_2) \circ \cdots) \circ b_m = \{0\}, \text{ for some } b_1, b_2, \ldots, b_m \in H\} > t.$$

Hence, there exist $a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_l \in H$ such that

$$(\cdots((a \circ a_1) \circ a_2) \circ \cdots) \circ a_k = \{0\}, \qquad \forall a \in x \circ y \tag{6}$$

$$(\cdots((y \circ b_1) \circ b_2) \circ \cdots) \circ b_l = \{0\} \tag{7}$$

and

$$f(a_1) \wedge \cdots \wedge f(a_k) > t \text{ and } f(b_1) \wedge \cdots \wedge f(b_l) > t \tag{8}$$

Now, let $u \in (\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_k$. Then

$$\begin{aligned} u \circ y &\subseteq ((\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_k) \circ y \\ &= ((\cdots(((x \circ y) \circ a_1) \circ a_2) \circ \cdots) \circ a_k \qquad \text{by (HK2)} \\ &= \bigcup_{a \in x \circ y} (\cdots((a \circ a_1) \circ a_2) \circ \cdots) \circ a_k \\ &= \{0\} \qquad \text{by (6)} \end{aligned}$$

which implies that $u \circ y = \{0\}$. Hence,

$$\begin{aligned} &(\cdots((u \circ b_1) \circ b_2) \circ \cdots) \circ b_l \\ &= ((\cdots((u \circ b_1) \circ b_2) \circ \cdots) \circ b_l) \circ \{0\} \\ &= ((\cdots((u \circ b_1) \circ b_2) \circ \cdots) \circ b_l) \circ (\cdots((y \circ b_1) \circ b_2) \\ &\quad \circ \cdots) \circ b_l \qquad \text{by (7)} \\ &\ll ((\cdots((u \circ b_1) \circ b_2) \circ \cdots) \circ b_{l-1}) \circ ((\cdots((y \circ b_1) \circ b_2) \\ &\quad \circ \cdots) \circ b_{l-1} \qquad \text{by (HK1)} \\ &\vdots \\ &\ll u \circ y = \{0\} \end{aligned}$$

and so

$$(\cdots((u \circ b_1) \circ b_2) \circ \cdots) \circ b_l = \{0\}.$$

Since, $u \in (\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_k$ is arbitrary, then

$$(\cdots(((\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_k) \circ b_1) \circ \cdots) \circ b_l = \{0\}.$$

Also, by (8) we have

$$f(a_1) \wedge \cdots \wedge f(a_k) \wedge f(b_1) \wedge \cdots \wedge f(b_l) > t.$$

Hence,

$$\mu(x) = \vee\{f(a_1) \wedge \cdots \wedge f(a_m) : (\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_m = \{0\}, \text{ for some } a_1, a_2, \ldots, a_m \in H\} > f(a_1) \wedge \cdots \wedge f(a_k) \wedge f(b_1) \wedge \cdots \wedge f(b_l) > t.$$

This shows that $x \in \mu_{t^>}$. Hence, $\mu_{t^>}$ is a weak hyper BCK-ideal of H and so by Lemma 7.2, μ is a fuzzy weak hyper BCK-ideal of H. Now, let $x \in H$. If $x \in H \setminus S(H)$, then by hypothesis, $f(x) = 0 \leq \mu(x)$. Let $x \in S(H)$. Then $x \circ x = \{0\}$ and so

$$\mu(x) = \vee\{f(a_1) \wedge \cdots \wedge f(a_n) : (\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_n = \{0\}, \text{ for some } a_1, \ldots, a_n \in H\} \geq f(x).$$

Thus $\mu(x) \geq f(x)$ and so $f \subseteq \mu$. Now, let ν be a fuzzy weak hyper BCK-ideal of H such that $f \subseteq \nu$ and $x \in H$. Then

$$\begin{aligned}\mu(x) &= \vee\{f(a_1) \wedge \cdots \wedge f(a_n) : (\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_n = \{0\}, \text{ for some } a_1, \ldots, a_n \in H\}\\ &\leq \vee\{\nu(a_1) \wedge \cdots \wedge \nu(a_n) : (\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_n = \{0\}, \text{ for some } a_1, \ldots, a_n \in H\}\\ &\leq \vee\{\nu(x)\} \quad \text{by Lemma 7.3}\\ &= \nu(x).\end{aligned}$$

Therefore, $\mu = [f]_w$. □

Now, we have the following result.

Corollary 7.5 *Let H be a hyper BCK-algebra and f and g be fuzzy weak hyper BCK-ideals of H with the same tip t (i.e., f(0)=g(0)) and that satisfy the BCK-part condition. Then for all $x \in H$,*

$$f \vee g(x) = [f \cup g]_w(x) = \vee\{f \cup g(a_1) \wedge \cdots \wedge f \cup g(a_n) : (\cdots((x \circ a_1) \circ \cdots) \circ a_n = \{0\}\}.$$

Theorem 7.6 *Let H be a quasi alternatively hyper BCK-algebra of type 1 and $\mathscr{F}^w_{p,t}(H)$ be the set of all fuzzy weak hyper BCK-ideals of H with the same tip t and that satisfy the BCK-part condition. Then $(\mathscr{F}^w_{p,t}(H), \vee, \cap)$ is a distributive lattice.*

Proof It is easy to see that the intersection of any two fuzzy weak hyper BCK-ideals of H is again a fuzzy weak hyper BCK-ideal of H and so $\mu \wedge \nu = \mu \cap \nu$ and by Corollary 7.5, $\mu \vee \nu = [\mu \cup \nu]_w$. So, $(\mathscr{F}^w_{p,t}(H), \vee, \cap)$ is a lattice. For distributivity, we prove that $\mu \wedge (\nu \vee \sigma) \leq (\mu \wedge \nu) \vee (\mu \wedge \sigma)$, for all $\mu, \nu, \sigma \in \mathscr{F}^w_{p,t}(H)$. The converse is obvious. Let $\mu, \nu, \sigma \in \mathscr{F}^w_{p,t}(H)$, $x \in H$ and $\epsilon > 0$. If $x \in H \setminus S(H)$, since μ satisfies the BCK-part condition, then $\mu(x) = 0$ and so $\mu \wedge (\nu \vee \sigma)(x) = 0$. Hence, the distributive inequality holds. Let $x \in S(H)$. Then, $x \circ x = \{0\}$. Hence, we can choose $a_1, a_2, \ldots, a_n \in H$ such that

$$(\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_n = \{0\}$$

and

$$\nu \vee \sigma(x) < \epsilon + \wedge(\nu \cup \sigma(a_1), \ldots, \nu \cup \sigma(a_n)).$$

Similar to the proof of Theorem 5.6, we can show that

$$\mu \wedge (\nu \vee \sigma)(x) < 2\epsilon + \wedge(\mu(x) \wedge \nu(a_1), \ldots, \mu(x) \wedge \nu(a_i), \mu(x) \wedge \sigma(a_{i+1}), \ldots, \mu(x) \wedge \sigma(a_n)).$$

Now, let $\mu(x) > \mu(a_j)$, for some $j \in \{1, 2, \ldots, n\}$ and

$$u \in (\cdots((x \circ a_1) \circ \cdots) \circ a_{j-1}) \circ a_{j+1}) \circ \cdots) \circ a_n.$$

Since H is a quasi alternatively hyper BCK-algebra of type 1, then

$$\begin{aligned}u \in u \circ 0 \subseteq u \circ (a_j \circ a_j) &= (u \circ a_j) \circ a_j\\ &\subseteq (((\cdots((x \circ a_1) \circ \cdots) \circ a_{j-1}) \circ a_{j+1}) \circ \cdots) \circ a_n) \circ a_j) \circ a_j\\ &= ((\cdots((x \circ a_1) \circ \cdots) \circ \cdots) \circ a_n) \circ a_j \text{ by (HK2)}\\ &= 0 \circ a_j = \{0\}\end{aligned}$$

and so $u = 0$. This implies that

$$(\cdots((x \circ a_1) \circ \cdots) \circ a_{j-1}) \circ a_{j+1}) \circ \cdots) \circ a_n = \{0\}.$$

By continuing this process, for each a_j, $j \in \{1, 2, \ldots, n\}$, such that $\mu(x) > \mu(a_j)$, we can omit a_j and by a new arrangement we get that

$$(\cdots((x \circ a'_1) \circ \cdots) \circ a'_k) \circ b'_1) \circ \cdots) \circ b'_l = \{0\}$$

where $\{a'_1, \ldots, a'_k\} \subseteq \{a_1, \ldots, a_i\}$, $\{b'_1, \ldots, b'_l\} \subseteq \{a_{i+1}, \ldots, a_n\}$, $\mu(x) \leq \mu(a'_i)$ and $\mu(x) \leq \mu(b'_i)$, for all $i \in \{1, \ldots, k\} \cup \{1, \ldots, l\}$. Now,

$$\begin{aligned}&\wedge(\mu(x) \wedge \nu(a_1), \ldots, \mu(x) \wedge \nu(a_i), \mu(x) \wedge \sigma(a_{i+1}), \ldots, \mu(x) \wedge \sigma(a_n))\\ &\leq \wedge(\mu(a'_1) \wedge \nu(a'_1), \ldots, \mu(a'_k) \wedge \nu(a'_k), \mu(b'_1) \wedge \sigma(b'_1), \ldots, \mu(b'_l) \wedge \sigma(b'_l))\end{aligned}$$

and so

$$\begin{aligned}\mu \wedge (\nu \vee \sigma)(x) &< 2\epsilon + \wedge(\mu(x) \wedge \nu(a_1), \ldots, \mu(x) \wedge \nu(a_i), \mu(x) \wedge \sigma(a_{i+1}), \ldots, \mu(x) \wedge \sigma(a_n))\\ &< 2\epsilon + \wedge(\mu(a'_1) \wedge \sigma(a'_1), \ldots, \mu(a'_k) \wedge \sigma(a'_k), \mu(b'_1) \wedge \nu(b'_1), \ldots, \mu(b'_l) \wedge \nu(b'_l))\\ &\leq 2\epsilon + [(\mu \wedge \nu) \cup (\mu \wedge \sigma)]_w(x).\end{aligned}$$

Thus $\mu \wedge (\nu \vee \sigma) \leq (\mu \wedge \nu) \vee (\mu \wedge \sigma)$. Hence, $(\mathscr{F}^w_{p,t}(H), \vee, \cap)$ is a distributive lattice. □

8 Conclusion

We prove that the set of all fuzzy (G-)subgroups of a (G-)group and the set of all weak normal F-subpolygroups of an F-polygrup form a modular lattice. Also, we show that the set of all fuzzy hyperideals with the same tip "t" of a hyperring is a distributive lattice and so is a modular lattice. Moreover, the set of all fuzzy ideals of a BCK-algebra is a distributive lattice. Finally, we prove that the set of all fuzzy weak hyper BCK-ideals of a quasi alternatively hyper BCK-algebra with the same tip "t" and that satisfy the BCK-part condition forms a distributive lattice. But, there are an open problem.

Open problem. Whether the set of all fuzzy hypersubalgebras of a hyperring and also the set of all fuzzy (strong) hyper BCK-ideals of a hyper BCK-algebra forms a distributive lattice or even a modular lattice.

Acknowledgments The authors would like to thank the referees for the valuable suggestions and corrections for improvement of this paper.

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