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Studying interval valued matrix games with fuzzy logic

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Abstract Matrix games have been widely used in decision making systems. In practice, for the same strategies players take, the corresponding payoffs may be within certain ranges rather than exact values. To model such uncertainty in matrix games, we consider interval-valued game matrices in this paper; and extend the results of classical strictly determined matrix games to fuzzily determined interval matrix games. Finally, we give an initial investigation into mixed strategies for such games.

1 Introduction

1.1 Matrix game

Game theory had its beginnings in the 1920s, and significantly advanced at Princeton University through the work of John Nash Dutta (1999), Nash (1950, 1951), and Winston (2004). The simplest game is a zero sum one involving only two players. An $m \times n$ matrix $G = \{g_{ij}\}_{m\times n}$ may be used to model such a two-person zero-sum game. If the row player R uses his ith strategy (row) and the column player C selects her jth choice (column), then R wins (and subsequently C loses) the amount g_{ij}. The objective of R is to maximize his gain while C tries to minimize her loss.

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Example 1 A game is described by the matrix

$$G = \begin{bmatrix} 0 & 6 & -2 & -4 \\ 5 & 2 & 1 & 3 \\ -8 & -1 & 0 & 20 \end{bmatrix} \tag{1}$$

In the game above, the players R and C have three and four possible strategies, respectively. If R chooses his first strategy and C chooses her second, then R wins $g_{12} = 6$ (C loses 6). If R chooses his third strategy and C chooses her first, then R wins $g_{31} = -8$ (R loses 8, C wins 8). In this paper we restrict our attention to such two-person zero-sum games.

1.2 Strictly determined matrix game

If there exists a g_{ij} in a classical $m \times n$ game matrix G such that g_{ij} is simultaneously the minimum value of the ith row and the maximum value of the jth column of G then g_{ij} is called a *saddle value* of the game. If a matrix game has a saddle value it is said to be *strictly determined*. It is well known, Dutta (1999) and Winston (2004), that the optimal strategies for both R and C in a strictly determined game are:

– R should choose any row containing a saddle value, and
– C should choose any column containing a saddle value.

A saddle value is also called the value of the (strictly determined) game. In (1), g_{23} is simultaneously the minimum of the second row and the maximum of the third column. Hence the game is strictly determined and its value is $g_{23} = 1$. The knowledge of an opponent's move provides no advantage since the payoff will always be a saddle value in a strictly determined game.

1.3 Motivation for this work

Matrix games have many useful applications, especially in decision making systems. However, in real world applications, due to certain forms of uncertainty, outcomes of a matrix game may not be a fixed number even though the players do not change their strategies. Hence, fuzzy games have been studied Garagic and Cruz (2003), Russell and Lodwick (2002), Wu and Soo (1998). By noticing the fact that the payoffs may only vary within a designated range for fixed strategies, we propose to use an interval-valued matrix, whose entries are closed intervals, to model such kind of uncertainty.

Throughout the rest of this paper we will use boldface letters to denote (closed and bounded) intervals. For example, $\mathbf{x}$ is an interval. Its greatest lower bound and the least upper bound are denoted by $\underline{x}$ and $\overline{x}$, respectively. We use uppercase letters to denote general matrices. An boldface upper case letter will represent an interval-valued matrix.

In this paper we assume that the intervals in the game matrix $\mathbf{G}$ are closed and bounded intervals of real numbers, and, for this investigation, represent uniformly distributed possible payoffs.

Definition 2 Let $\mathbf{G} = \{\mathbf{g}_{ij}\}$ be an $m \times n$ interval-valued matrix. The matrix $\mathbf{G}$ defines a *zero-sum interval matrix game* provided whenever the row player R uses his ith strategy and the column player C selects her jth strategy, then R wins and C correspondingly loses a common $x \in \mathbf{g}_{ij}$.

Example 3 Consider the following interval game matrix:

$$\mathbf{G} = \begin{bmatrix} [0,1] & [6,7] & [-2,0] & [-4,-2] \\ [5,6] & [2,7] & [1,3] & [3,3] \\ [-8,-5] & [-1,0] & [0,0] & [20,25] \end{bmatrix} \quad (2)$$

In this game, if R chooses row one and C selects column two, then R wins an amount $x \in [6, 7]$ (C loses the same x that R wins).

In this paper, we attempt to extend results of classical matrix games into interval-valued games. In order to accomplish this, we will need to define fuzzy relational operators for intervals in order to compare every pair of possible interval payoffs from a rational game-play perspective. These relational operators for intervals will be developed in Sect. 2. We then study crisply determined and fuzzily determined interval games in Sects. 3 and 4. Since not all intervals games are determined we begin an investigation of mixed strategies for non-determined games. We describe a potential mapping of such an interval game into an interval linear programming problem in Sect. 5, and show how linear interval inequalities can be solved under our definition in Sect. 6. We conclude the paper with Sect. 7.

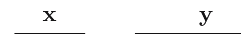

Fig. 1 Non-overlapping intervals, $\mathbf{x} < \mathbf{y}$

2 Comparing intervals

In order to compare strategies and payoffs for an interval game matrix, we need a notion of an interval ordering relation that corresponds to the intuitive notion of a "better possible" outcome/payoff. This will be done by defining the notion of a non-empty interval $\mathbf{x}$ not being a better payoff than a non-empty interval $\mathbf{y}$, i.e. the notion that $\mathbf{x}$ is less than or equal to $\mathbf{y}$. Other approaches to relational orderings of intervals have been developed and extended that define such orderings between some pairs of intervals. Fishburn defined in Fishburn (1985) a concept of interval order corresponding to a special kind of partially ordered set. His context is for the study of the order of vertices in interval graphs. An interval graph refers to a graph (X, ∼) whose points can be mapped into intervals of a linearly ordered set such that, for all distinct x and y, x ∼ y if and only if the intervals assigned to x and y have a nonempty intersection. Allen's paper Allen (1983) in 1983 listed 13 possible cases for the temporal relationships of two time intervals. However, neither compares general intervals nor models such a comparison in our game-theoretic context. Unlike these models, we wish to make every pair of our intervals comparable and to fuzzily quantify the notion of "indifference" in our game-theoretic context except when the two intervals are equal.

For the development of our relational operators in the noted context we assume that a rational player will not prefer an interval $\mathbf{x}$ as in Fig. 1 to interval $\mathbf{y}$, as every possible payoff value $x \in \mathbf{x}$ is less than every payoff value $y \in \mathbf{y}$.

Similarly, we assume that in the case of the intervals in Fig. 2 the player will not prefer interval $\mathbf{x}$ over $\mathbf{y}$ as no value in $\mathbf{x}$ offers a payoff that is greater than what is available in $\mathbf{y}$ and $\mathbf{y}$ offers no payoff that is less than what is allowed in $\mathbf{x}$. So by choosing interval $\mathbf{y}$ over $\mathbf{x}$ maximizes both the least possible and greatest possible payoff.

Finally, in case $\mathbf{x} = \mathbf{y}$ we assume that a rational player will prefer neither over the other. So in these cases , using $\le$ to represent the relation "is not preferred to" we have $\mathbf{x} \le \mathbf{y}$ in the cases represented by Figs. 1 and 2, and each of $\mathbf{x} \le \mathbf{y}$ and $\mathbf{y} \le \mathbf{x}$ for the case of equality. In these cases the preference order exhibits the properties of a total order. Hence these comparisons can be crisply defined as true and are consistent with traditional interval comparison operators.

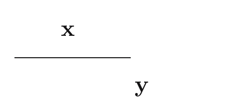

Fig. 2 Overlapping intervals

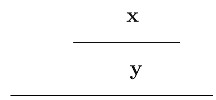

Fig. 3 Nested intervals

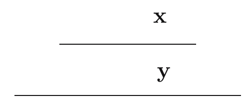

Fig. 4 Nested intervals, $\mathbf{x} \leq \mathbf{y}$ with membership value $\frac{1}{2}$

When $\mathbf{x}$ is completely contained in $\mathbf{y}$ as displayed in Fig. 3, the notion of payoff preference become uncertain, as there exists payoff values in $\mathbf{y}$ which are less than every possible payoff in $\mathbf{x}$ as well as values in $\mathbf{y}$ which are greater than every possible payoff in $\mathbf{x}$. In this case a risk-adverse player may (not necessarily will) prefer $\mathbf{x}$ to $\mathbf{y}$ as $\mathbf{x}$ contains the largest worst possible actual payoff value, whereas a (rational) risk-taking player may prefer $\mathbf{y}$ to $\mathbf{x}$ since $\mathbf{y}$ contains the largest best possible actual payoff. However, for any single game either player may also rationally decide that they are indifferent to the two choices or will choose the other. In other words, the interval payoff preference cannot be determined with classical binary logic. This uncertainty, however, can be well addressed with the theory of fuzzy logic developed by Zadeh Zadeh (1965). Therefore we extend the previous crisp preference comparisons with fuzzy membership.

Such a fuzzy membership extension might be expected to be a continuous one in terms of holding one interval fixed and moving the other in terms of its midpoint and width, but in the presented context no such continuous extension is possible. For if the widths of $\mathbf{x}$ and $\mathbf{y}$ are equal and the two intervals are initially positioned as in Fig. 1, as $\mathbf{x}$ moves to the right, the inequality $\mathbf{x} \leq \mathbf{y}$ is crisply true (having membership value 1 in a fuzzy context) until $\mathbf{x} = \mathbf{y}$, and is crisply false (having membership value 0 in a fuzzy context) afterwards. Hence no membership value of "$\mathbf{x}$ is not preferred to $\mathbf{y}$" will allow for a continuous extension.

To fuzzily quantify the uncertainty discussed above in Fig. 3 we consider the case that the interval $\mathbf{x}$ is positioned with it's left endpoint the same as the left endpoint of $\mathbf{y}$ and $\mathbf{x} \subset \mathbf{y}$. In this case a rational player will crisply prefer $\mathbf{y}$ over $\mathbf{x}$ for the same reasons expressed in the analysis of Fig. 1 and 2. Hence $\mathbf{x} \leq \mathbf{y}$ crisply, and in terms of a fuzzy relational operator the membership value of this relation is 1. On the other hand, when $\mathbf{x}$ is positioned to share it's right endpoint with $\mathbf{y}$, a rational player will crisply prefer $\mathbf{x}$ to $\mathbf{y}$ for the same reason. Hence in this case the membership value of $\mathbf{x} \leq \mathbf{y}$ is 0. We then define the fuzzy membership to be a linear mapping from 1 to 0 as the interval $\mathbf{x}$ "moves" from right to left. The corresponding fuzzy membership values of this relation then can be associated with the notion of the degree of risk-taking that a player may exhibit. However, this relationship is not a probabilistic one, but rather a possible one. For example, a risk-adverse player facing a choice between two such intervals with a $\mathbf{x} \leq \mathbf{y}$ membership value close to 1 may consider the risk of choosing $\mathbf{y}$ over $\mathbf{x}$ in spite of the possibility of receiving an actual payoff less than every value in $\mathbf{x}$. On the other hand, a risk-taking player may choose $\mathbf{y}$ over $\mathbf{x}$ with a small positive membership value of $\mathbf{x} \leq \mathbf{y}$. The linear map

$$f(\mathbf{x}, \mathbf{y}) = \frac{\overline{y} - \overline{x}}{w(y) - w(x)} \tag{3}$$

meets the requirement, where $w(x) = \overline{x} - \underline{x}$ is the width of the interval $\mathbf{x}$.

As a special instance one can note that the membership is 0.5 when the midpoints of $\mathbf{x}$ and $\mathbf{y}$ overlap. If one keeps the interval $\mathbf{y}$ fixed, the midpoints of $\mathbf{x}$ and $\mathbf{y}$ equal and allow the width of $\mathbf{x}$ to vary continuously, there is a pronounced discontinuity in the membership values of $\mathbf{x} \leq \mathbf{y}$ when the widths become equal. However, this discontinuity is not in conflict with the measure of uncertainty of the comparison, as by our definition there is uncertainty in the comparison at all widths of $\mathbf{x}$ except when the intervals are equal (Fig. 4).

Summarizing the above discussion, we extend the crisp comparison operator by defining the fuzzy comparison operator $\preceq$ for two closed and bounded intervals for the "not preferred to" relationship as follow:

Definition 4 Let $\mathbf{x}$ and $\mathbf{y}$ two be nontrivial intervals. The binary fuzzy operator $\preceq$ of $\mathbf{x}$ and $\mathbf{y}$ returns the membership for '$\mathbf{x}$ is not preferred to $\mathbf{y}$' between 0 and 1 as:

$$\mathbf{x} \preceq \mathbf{y} = \begin{cases} 1 & \overline{x} < \underline{y}; \\ 1 & \underline{x} \leq \underline{y} \leq \overline{x} < \overline{y}; \\ \dfrac{\overline{y} - \overline{x}}{w(\mathbf{y}) - w(\mathbf{x})} & \underline{y} < \underline{x} < \overline{x} \leq \overline{y},\, w(\mathbf{x}) \neq w(\mathbf{y}); \\ 1 & \underline{x} = \underline{y},\, w(\mathbf{x}) = w(\mathbf{y}); \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

One can define the dual fuzzy relation "is preferred to" in the analogous way. We will use the symbol $\succeq$ to denote this dual relationship as a reminder of the antisymmetry in the crisp case. Therefore $\succeq$ can be defined in terms of $\preceq$ as follows:

Definition 5 The binary fuzzy operator $\succeq$ of two intervals $\mathbf{x}$ and $\mathbf{y}$ is defined as: $\mathbf{x} \succeq \mathbf{y} = 1$ if $\mathbf{x} = \mathbf{y}$; and $\mathbf{x} \succeq \mathbf{y} = 1 - (\mathbf{x} \preceq \mathbf{y})$ otherwise.

Definition 6 If the value of $\mathbf{x} \preceq \mathbf{y}$ is exactly one or zero, then we say that $\mathbf{x}$ and $\mathbf{y}$ are *crisply comparable*. Otherwise, we say that they are *fuzzily comparable*.

3 Crisply determined interval matrix game

In this section, we extend the concept of classical strictly determined games to interval matrix games whose row and column entries are crisply comparable. In this case we will use $\leq$ and $\geq$ in place of $\succeq$ and $\preceq$ to emphasize the crispness of the appropriate interval comparisons.

Definition 7 Let $\mathbf{G}$ be a $m \times n$ interval game matrix. If there exists a $\mathbf{g}_{ij} \in \mathbf{G}$ such that $\mathbf{g}_{ij}$ is simultaneously crisply less than or equal to $\mathbf{g}_{ik}$ for all $k \in \{1, 2, \ldots, n\}$ and crisply greater than or equal to $\mathbf{g}_{lj}$ for all $l \in \{1, 2, \ldots, m\}$ then the interval $\mathbf{g}_{ij}$ is called a *saddle interval* of the game. An interval matrix game is *crisply determined* if it has a saddle interval.

By the definition above, to determine whether an interval game matrix is crisply determined, one needs only to do the following:

1. For each row ($1 \leq i \leq m$), find the entry $\mathbf{g}_{ij^*}$ that is crisply less than or equal to all other entries in the ith row.
2. For each column ($1 \leq j \leq n$), find the entry $\mathbf{g}_{i^*j}$ that is crisply greater than or equal to all other entries in the jth column.
3. Determine if there is an entry $\mathbf{g}_{i^*j^*}$ that is simultaneously the minimum of the ith row and the maximum of the jth column.
4. If any of the above values cannot be found the game is not crisply determined. Otherwise, it is a crisply determined interval matrix game.

Example 8 Examining the interval game matrix (2), we found that $\mathbf{g}_{14}$, $\mathbf{g}_{23}$, and $\mathbf{g}_{31}$ are the minimum of rows 1, 2, and 3, respectively. And, $\mathbf{g}_{21}$, $\mathbf{g}_{12}$, $\mathbf{g}_{23}$ and $\mathbf{g}_{34}$ are the maximum of columns 1, 2, 3, and 4, respectively. Furthermore, $\mathbf{g}_{23}$ is simultaneously the minimum of the 2^{nd} row and the maximum of the 3^{rd} column. Hence, $\mathbf{g}_{23} = [1, 3]$ is a saddle interval of the game matrix. This is a crisply determined interval matrix game.

Mimicking the optimal strategy for a classical strictly determined game, we have the optimum strategies for both R and C in a crisply determined interval matrix game defined as:

– R should choose any row containing a saddle interval, and
– C should choose any column containing a saddle interval.

In this case the uniqueness of the saddle intervals can be established.

Theorem 9 *If an interval matrix game is crisply determined, its saddle intervals are identical.*

Proof Let $\mathbf{G}$ be a crisply determined interval game matrix, and $\mathbf{g}_{ij}$ and $\mathbf{g}_{lk}$ are saddle intervals. Then, $\mathbf{g}_{ij} \leq \mathbf{g}_{ik} \leq \mathbf{g}_{lk}$, and $\mathbf{g}_{ij} \geq \mathbf{g}_{lj} \geq \mathbf{g}_{lk}$. Hence, $\mathbf{g}_{ij} = \mathbf{g}_{lk}$. □

As in the classical case, the knowledge of an opponent's move provides no advantage since the payoff is assumed to be uniformly distributed within a saddle interval in a strictly determined interval game.

Definition 10 The *value interval* of a strictly determined interval game is its saddle interval. A strictly determined interval game is *fair* if its saddle interval is symmetric respect to zero, *i. e.* in the form of $[-a, a]$ for $a \geq 0$.

From Example 8 we know that $\mathbf{g}_{23}$ is a saddle interval of the matrix game (2). However, the midpoint of $\mathbf{g}_{23}$ is 2. Hence, the game is unfair since the row player has an average advantage of 2.

4 Fuzzily determined interval matrix games

For a general interval game matrix, the crisp comparability may not be satisfied for all intervals in the same row (or column). Hence we now must extend interval comparability to define the fuzzy memberships of an interval $\mathbf{v}_i$ being a minimum and a maximum of an interval vector $\mathbf{V}$; and then we define the notion of a least and greatest interval in $\mathbf{V}$.

Definition 11 Let $\mathbf{V} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an interval vector. The fuzzy membership of $\mathbf{v}_i$ being a least interval in $\mathbf{V}$ is defined as $\mu(\mathbf{v}_i) = \min_{1 \leq j \leq n}\{\mathbf{v}_i \prec \mathbf{v}_j\}$. And, a least interval of the vector $\mathbf{V}$ is defined as an interval whose μ value is largest, i.e. $\mathbf{v}_{i^*} = \max_{1 \leq i \leq n} \mu(\mathbf{v}_i)$.

Likewise, the fuzzy membership of $\mathbf{v}_i$ being a maximum interval in $\mathbf{V}$ is $\nu(\mathbf{v}_i) = \min_{1 \leq j \leq n}\{\mathbf{v}_i \succeq \mathbf{v}_j\}$. Similarly, a greatest interval of the vector $\mathbf{V}$ is $\mathbf{v}_{i^*} = \max_{1 \leq i \leq n} \nu(\mathbf{v}_i)$.

Example 12 Find the least and the greatest intervals for the interval vector $\mathbf{V} = \{[2, 5], [3, 7], [4, 5]\}$.

Solution We notice that $\mathbf{v}_2$ and $\mathbf{v}_3$ are not crisply comparable. By Definition 11, we have $\mu([2, 5]) = 1$, $\nu([2, 5]) = 0$; $\mu([3, 7]) = 0$, $\nu([3, 7]) = \frac{2}{3}$; and $\mu([4, 5]) = 0$, $\nu([4, 5]) = \frac{1}{3}$. Hence, the least interval of the vector $\mathbf{V}$ is $\mathbf{v}_1 = [2, 5]$ with membership 1; and the greatest interval of $\mathbf{V}$ is $\mathbf{v}_2 = [3, 7]$ with membership $\frac{2}{3}$.

Notice, however, that unlike real valued games, the least and/or greatest interval of a vector is not necessarily unique. This can happen only when unequal intervals share the same midpoint, as the next example shows.

Example 13 Given the interval vector $\mathbf{V} = \{[2, 5], [3, 6], [4, 5]\}$ we find that the least interval of the vector $\mathbf{V}$ is $\mathbf{v}_1 = [2, 5]$ with membership 1. However, as $\nu([2, 5]) = 0$, $\nu([3, 6]) = \frac{1}{2}$, and $\nu([4, 5]) = \frac{1}{2}$ each of [3, 6] and [4, 5] is a greatest interval with membership value $\frac{1}{2}$.

Definition 11 provides us a way to fuzzily determine least and greatest intervals for any interval vectors. We are now able to define fuzzily determined interval matrix game as follows:

Definition 14 Let $\mathbf{G}$ be an $m \times n$ interval game matrix. If there is a $\mathbf{g}_{ij} \in \mathbf{G}$ such that $\mathbf{g}_{ij}$ is simultaneously a least and a greatest interval for the ith row and the jth column of $\mathbf{G}$, respectively, then $\mathbf{G}$ is a *fuzzily determined interval game*. We also call such $\mathbf{g}_{ij}$ a *fuzzy saddle interval* of the game with its membership as $\min\{\mu(\mathbf{g}_{ij}), \nu(\mathbf{g}_{ij})\}$.

It is obvious that the crisply determined interval game defined in Definition 7 is just a special case of fuzzily determined interval game with 1 as its membership. The game value of a fuzzily determined interval game can be reasonably defined as its fuzzy saddle interval with respect to its membership.

For the convenience of computer implementations, we summarize our discussion as the algorithm below.

Algorithm 15

1. Initialization:
   (a) Input interval game matrix $\mathbf{G} = \{\mathbf{g}_{ij}\}_{m\times n}$
   (b) Initialize `FuzzilyDetermined` as `false`
2. Calculation:
   (a) Evaluate $\mu(\mathbf{g}_{ij})$ and $\nu(\mathbf{g}_{ij})$ for all $i = 1$ to m and $j = 1$ to n
   (b) For each of $i = 1$ to m, find j^* such that $\mu(\mathbf{g}_{ij^*}) = \max_{1\le j\le n}\{\mu(\mathbf{g}_{ij})\}$. Note: j^* depends on i
   (c) For each of $j = 1$ to n, find i^* such that $\nu(\mathbf{g}_{i^*j}) = \max_{1\le i\le m}\{\nu(\mathbf{g}_{ij})\}$. Note: i^* depends on j
3. Checking: For each of $i = 1$ to m and corresponding j^*, check if $\mathbf{g}_{ij^*}$ is also a greatest interval for the j^* column. If so,
   (a) Update `FuzzilyDetermined` to `true`
   (b) Record $\mathbf{g}_{ij^*}$ as a fuzzy saddle interval with its membership $\min\{\mu(\mathbf{g}_{ij^*}), \nu(\mathbf{g}_{ij^*})\}$.
4. Finding results:
   (a) If `FuzzilyDetermined` is `false`, the interval game is not fuzzily determined
   (b) Otherwise, the interval game is fuzzily determined. And, return the fuzzy saddle interval that has the largest membership among all recorded fuzzy saddle intervals. Note: the game is crisply determined if the resulting membership is 1.

The concept of a fuzzily determined interval game in Definition 14 can be further generalized. For each $\mathbf{g}_{ij} \in \mathbf{G}$, the membership of $\mathbf{g}_{ij}$ being simultaneously a least and a greatest interval for the ith row and the jth column of G can be defined as $\phi(\mathbf{g}_{ij}) = \min\{\mu(\mathbf{g}_{ij}), \nu(\mathbf{g}_{ij})\}$. The entries of $\mathbf{G}$ with the largest value of ϕ can be considered as fuzzy saddle intervals. Therefore, for any interval game matrix, one can find its fuzzy saddle intervals with the membership of the largest value of ϕ. However, it may not make any practical sense if the membership value is too small.

There are many applications of classical game theory to problems in decision theory and finance. In particular, the following is an example how interval Nash games may apply to determine optimal investment strategies.

Example 16 Consider the case of an investor making a decision as to how to invest a non-divisible sum of money when the economic environment may be categorized into a finite number of states. There is no guarantee that any single value (return on the investment) can adequately model the payoff for any one of the economic states. Hence it is more realistic to assume that each payoff lies in some interval.

For this example it is assumed that the decision of such an investor can be modeled under the assumption that the economic environment (or nature) is, in fact, a rational "player" that will choose an optimal strategy. Suppose that the options for this player are: strong economic growth, moderate economic growth, no growth nor shrinkage, and moderate shrinkage(negative growth). For the investor player the options are: invest in bonds, invest in stocks, and invest in a guaranteed fixed return account. In this case clearly a single value for the payoff of either investment in bonds or stock cannot be realistically modeled by a single value representing the percent of return. Hence a game matrix with interval payoff values better represents the view of the game from both players' perspectives.

Consider then the following interval game matrix for this scenario, where the percentage of return is represented in decimal form.

	Bonds	Stocks	Fixed
Strong	[0.11, 0.136]	[0.125, 0.158]	[0.045, 0.045]
Moderate	[0.083, 0.122]	[0.08, 0.11]	[0.045, 0.045]
None	[0.049, 0.062]	[0.02, 0.042]	[0.045, 0.045]
Negative	[0.022, 0.03]	[–0.04, 0.015]	[0.045, 0.045]

The intervals in each row and column are strictly comparable to each other, and using the techniques described earlier one finds that the game is strictly determined, with the value of the game the trivial interval [0.045, 0.045]. This corresponds to the actions of those investors who do not have any insight as to what the economy may do in a given time period and who cannot take high risks.

5 Toward optimal mixed strategies through linear programming

As in the case of classical matrix games, there is no guarantee that an interval valued matrix game is crisply or fuzzily determined. For non-determined interval matrix game, one needs to find an optimal mixed strategy for each player. For non-determined interval valued matrix games we will assume that these mixed strategies are represented by crisp probability values, whose sum for each player is exactly equal to one. Hence the goal is to describe a context in which each player can choose an optimal mixed strategy from the set of all possible mixed strategies.

In the classical zero-sum matrix game context the problem of finding an optimal mixed strategy solution can be mapped to an equivalent linear programming problem. We will now investigate such a transformation for interval valued games and present the resulting linear programming problems to be solved.

Suppose $\mathbf{G} = (\mathbf{g}_{ij})$ is an $m \times n$ interval game matrix and the column player C chooses column j as her strategy. If $P = [p_1, p_2, \ldots, p_m]$ is the row player's mixed strategy then the *expected value* for the row player, given C's given strategy, is the interval $\mathbf{v}$ defined

$$\mathbf{v} = p_1 \cdot \mathbf{g}_{1j} + p_2 \cdot \mathbf{g}_{2j} + \cdots + p_m \cdot \mathbf{g}_{mj} = \sum_{i=1}^{m} p_i \cdot \mathbf{g}_{ij}.$$

To find the row player's optimal strategy we use the "max-min" principle of traditional zero sum matrix games, namely to find the largest minimum expected value/payoff. Hence we need to find a "maximum" value $\mathbf{v}$ and the corresponding mixed strategy P so that $p_1 \cdot \mathbf{g}_{1j} + p_2 \cdot \mathbf{g}_{2j} + \cdots + p_m \cdot \mathbf{g}_{mj} \succeq \mathbf{v}$ for each $1 \le j \le n$. The corresponding system to solve is:

System 17 Maximize $\mathbf{v}$ subject to

$$\begin{aligned}
x_1 \cdot \mathbf{g}_{11} + x_2 \cdot \mathbf{g}_{21} + \cdots + x_m \cdot \mathbf{g}_{m1} &\succeq \mathbf{v}\\
x_1 \cdot \mathbf{g}_{12} + x_2 \cdot \mathbf{g}_{22} + \cdots + x_m \cdot \mathbf{g}_{m2} &\succeq \mathbf{v}\\
&\vdots\\
x_1 \cdot \mathbf{g}_{1n} + x_2 \cdot \mathbf{g}_{2n} + \cdots + x_m \cdot \mathbf{g}_{mn} &\succeq \mathbf{v}\\
\sum_{i=1}^{m} x_i &= 1\\
x_1, x_2, \ldots, x_m &\ge 0
\end{aligned}$$

As the entries of the game matrix $\mathbf{G}$ represents the gains to the row player, the column player attempts to minimize her losses. Therefore she attempts to find the smallest maximum expected value, and the corresponding (dual) system for her is:

System 18 Minimize $\mathbf{v}$ subject to

$$\begin{aligned}
x_1 \cdot \mathbf{g}_{11} + x_2 \cdot \mathbf{g}_{12} + \cdots + x_n \cdot \mathbf{g}_{1n} &\preceq \mathbf{v}\\
x_1 \cdot \mathbf{g}_{21} + x_2 \cdot \mathbf{g}_{22} + \cdots + x_n \cdot \mathbf{g}_{2n} &\preceq \mathbf{v}\\
&\vdots\\
x_1 \cdot \mathbf{g}_{m1} + x_2 \cdot \mathbf{g}_{m2} + \cdots + x_n \cdot \mathbf{g}_{mn} &\preceq \mathbf{v}\\
\sum_{i=1}^{n} x_i &= 1\\
x_1, x_2, \ldots, x_m &\ge 0
\end{aligned}$$

In the classical game theory context one can assume that each of the payoffs are positive, as an appropriate linear shift of the payoff values does not affect the characteristics of the game. In the case of interval valued games a similar shift to make each of the interval payoffs positive (i.e. the left end-point of each interval entry in the game matrix is positive) can be employed. This shift, as will be shown, does not affect the characteristics of the game.

Theorem 19 *Suppose $\mathbf{G} = (\mathbf{g}_{ij})$ is an $m \times n$ interval game matrix and $c > 0$. The interval $\mathbf{v}$ is a row player's optimal mixed strategy expected value with strategy distribution $P = [p_1, p_2, \ldots, p_m]$ if and only if $\mathbf{v} + [c, c]$ is a corresponding optimal value with strategy distribution P for the row player in the game $\mathbf{G}' = (\mathbf{g}_{ij} + [c, c])$.*

Proof If $P = [p_1, p_2, \ldots, p_m]$ is a strategy distribution and $1 \le j \le n$ then

$$\sum_{i=1}^{m} x_i(\mathbf{g}_{ij} + [c, c]) = \sum_{i=1}^{m} (x_i \cdot \mathbf{g}_{ij} + x_i \cdot [c, c]) = \sum_{i=1}^{m} x_i \mathbf{g}_{ij} + [c, c] \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} x_i \mathbf{g}_{ij} + [c, c].$$

Hence maximizing $\sum_{i=1}^{m} x_i(\mathbf{g}_{ij} + [c, c]) \ge \mathbf{v}$ is equivalent to maximizing $\sum_{i=1}^{m} x_i \mathbf{g}_{ij} + [c, c] \ge \mathbf{v}$. A similar result follows immediately for the column player. □

Continuing, as the entries in $\mathbf{G}$ can be assumed to be positive, we have $\mathbf{v} > 0$. However, the width of $\mathbf{v}$ in general can vary. In order to "normalize" the width of $\mathbf{v}$ in order to investigate a method for solving these interval systems we will now assume that $\mathbf{v}$ is a trivial interval, i.e. the width of $\mathbf{v}$ is zero. Hence $\mathbf{v}$ can be simultaneously viewed as an interval and real number. Hence, in this case, dividing each of the inequalities in System 19 by $\mathbf{v}$ and treating the resulting quotients $\frac{x_k}{\mathbf{v}}$ as a new real valued variable z_k, we noticed that maximizing $\mathbf{v}$ is equivalent to minimizing $\frac{1}{\mathbf{v}} = \frac{\sum_{i=1}^{m} x_i}{\mathbf{v}} = \sum_{i=1}^{m} z_i$ since $\sum_{i=1}^{m} x_i = 1$. Therefore System 17 can be converted into an "interval" linear programming problem:

System 20 Minimize $z_1 + z_2 + \cdots + z_m$ subject to

$$z_1 \cdot \mathbf{g}_{11} + z_2 \cdot \mathbf{g}_{21} + \cdots + z_m \cdot \mathbf{g}_{m1} \succeq 1$$
$$z_1 \cdot \mathbf{g}_{12} + z_2 \cdot \mathbf{g}_{22} + \cdots + z_m \cdot \mathbf{g}_{m2} \succeq 1$$
$$\vdots$$
$$z_1 \cdot \mathbf{g}_{1n} + z_2 \cdot \mathbf{g}_{2n} + \cdots + z_m \cdot \mathbf{g}_{mn} \succeq 1$$
$$z_1, z_2, \ldots, z_m \geq 0$$

where the "1" is the interval [1, 1]. After this linear programming problem is solved for the values $z_1, z_2, \ldots z_m$ the final values of $x_1, x_2, \ldots, x_m$ and $\mathbf{v}$ can be quickly found.

To optimize her strategy the row player will attempt to find a strategy distribution $P^* = [p_1^*, p_2^*, \ldots, p_m^*]$ and a largest value for $\mathbf{v}$ so that for any strategy distribution Q for the column player we will have $P^*\mathbf{G}Q^T \succeq \mathbf{v}$ for a fixed relational membership value α, treating $\mathbf{v}$ as a trivial interval. In other words, the row player must solve this system (for a fixed relational membership value $0 < \alpha \leq 1$).

In a similar fashion, the column player will attempt to find a strategy distribution $Q^* = [q_1^*, q_2^*, \ldots, q_n^*]$ and a smallest value for $\mathbf{w} \geq 0$ so that for any strategy distribution P for the row player we will have $P\mathbf{G}(Q^*)^T \preceq \mathbf{w}$ for the same membership value α. Therefore the corresponding system will be:

System 21 Maximize $z_1 + z_2 + \cdots + z_m$ subject to

$$z_1 \cdot \mathbf{g}_{11} + z_2 \cdot \mathbf{g}_{12} + \cdots + z_n \cdot \mathbf{g}_{1n} \preceq 1$$
$$z_1 \cdot \mathbf{g}_{21} + z_2 \cdot \mathbf{g}_{22} + \cdots + z_n \cdot \mathbf{g}_{2n} \preceq 1$$
$$\vdots$$
$$z_1 \cdot \mathbf{g}_{m1} + z_2 \cdot \mathbf{g}_{m2} + \cdots + z_n \cdot \mathbf{g}_{mn} \preceq 1$$
$$z_1, z_2, \ldots, z_n \geq 0$$

When these systems are solved, the values of P^*, Q^*, v, and w are determined.

If each interval $\mathbf{g}_{ij}$ is interpreted as a trapezoidal fuzzy number each of the two previous systems becomes a fuzzy linear programming problem with a crisp objective function and fuzzy constraints. Several techniques for solving such fuzzy systems have been developed, including Fuller and Zimmermann (1993). These techniques define the notion of an (approximate) optimal solution in a fuzzy context. However, it is still worthwhile to develop direct techniques to solve interval linear programming problems, computing exact interval solutions whenever possible. Hence we continue to address the development of such a general theory.

6 Solving interval inequalities

In order to solve the systems described in the previous section we must determine techniques for solving interval inequalities in general.

6.1 Simple inequalities

We will first consider the simplest case, namely to maximize the real value z where $z \cdot \mathbf{x} \preceq \mathbf{y}$ where each of $\mathbf{x}$ and $\mathbf{y}$ is a positive interval. Clearly if both $\mathbf{x}$ and $\mathbf{y}$ are trivial intervals then the maximum value of z is $\frac{\mathbf{y}}{\mathbf{x}}$. Now consider the case when at least one is not trivial. As interval comparisons require a fuzzy comparison operator, we will consider the following restatement of this linear inequality problem:

System 22 Given $0 < \alpha \leq 1$ and intervals $\mathbf{x}$ and $\mathbf{y}$ find the maximum value of z where $z \cdot \mathbf{x} \preceq \mathbf{y}$ with membership value not less than α.

We will represent the relationship between $z \cdot \mathbf{x}$ and $\mathbf{y}$ in a planar context where an interval $\mathbf{v}$ is represented by the ordered pair $(m(\mathbf{v}), w(\mathbf{v}))$ where $m(\mathbf{v})$ is the midpoint of the interval and $w(\mathbf{v})$ is its width. Since this analysis considers only positive intervals, i.e. $m(\mathbf{v}) < w(\mathbf{v})$, the corresponding point in this coordinate system must lie below the diagonal in Fig. 5 below.

As the mapping $f(z) = z \cdot \mathbf{x}$ is linear, it is easy to see that as z varies the interval $z \cdot \mathbf{x}$ moves on the line from (0, 0) through $(m(\mathbf{x}), w(\mathbf{x}))$. The dynamics of how the interval $z \cdot \mathbf{x}$ "moves through" the interval $\mathbf{y}$ has 3 general cases that must be considered. To distinguish between these cases consider the value of z where the midpoint of $z \cdot \mathbf{x}$ equals to the midpoint of $\mathbf{y}$. This value can easily be computed to be $\frac{\underline{y}+\overline{y}}{\underline{x}+\overline{x}}$, which will be denoted by c. One of three situations can occur for the relationship of $c \cdot \mathbf{x}$ to $\mathbf{y}$: $c \cdot \mathbf{x} \subset \mathbf{y}$ and $c \cdot \mathbf{x} \neq \mathbf{y}$ (corresponds to the line from (0, 0) through $(m(\mathbf{x}), w(\mathbf{x}))$ in the above figure intersecting the vertical line containing $(m(\mathbf{y}), w(\mathbf{y}))$ below that point), $c \cdot \mathbf{x} = \mathbf{y}$ (corresponds to the points (0, 0), $\mathbf{x}$ and $\mathbf{y}$ being collinear in the above figure), and $\mathbf{y} \subset c \cdot \mathbf{x}$ and $c \cdot \mathbf{x} \neq \mathbf{y}$ (corresponds to the line from (0, 0) through $(m(\mathbf{x}), w(\mathbf{x}))$ in the above figure intersecting the vertical line containing $(m(\mathbf{y}), w(\mathbf{y}))$ above that point).

Consider the case that $c \cdot \mathbf{x} = \mathbf{y}$. Clearly $z = c$ is the maximum value as $c \cdot \mathbf{x} \leq \mathbf{y}$ crisply, and if $\epsilon > 0$ then $(c+\epsilon)\mathbf{x} \geq \mathbf{y}$ crisply so that $(c+\epsilon)\mathbf{x} \preceq \mathbf{y}$ has membership value 0.

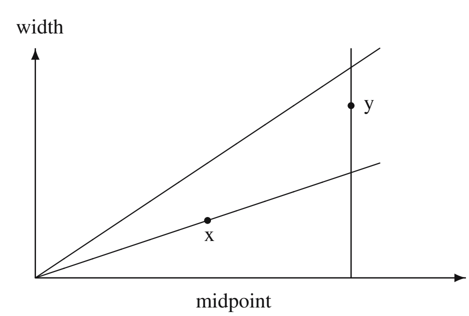

Fig. 5 Graphical Representation of $z \cdot \mathbf{x} \prec \mathbf{y}$

Next consider the case that $\mathbf{y} \subset c \cdot \mathbf{x}$ and $c \cdot \mathbf{x} \neq \mathbf{y}$. Hence we see that $c\underline{x} < \underline{y}$ and $\overline{y} < c\overline{x}$. As the membership values of $z \cdot \mathbf{x} \preceq \mathbf{y}$ is non-increasing as z increases we need only find the value of z such that the membership value of $z \cdot \mathbf{x} \preceq \mathbf{y}$ is equal to α. Hence we to solve the equation $\frac{\underline{y}-z\underline{x}}{(\underline{y}-z\underline{x})+(z\overline{x}-\overline{y})} = \alpha$ for z. Doing so one finds that $z = \frac{\underline{y}+\alpha(\overline{y}-\underline{y})}{\underline{x}+\alpha(\overline{x}-\underline{x})}$. Therefore this is the largest value of z that satisfies the initial inequality with membership not less than α. Notice that in the special case of $\alpha = 1$ we get the optimal value $z = \overline{y}/\overline{x}$ which corresponds to the value where the left endpoints of $z \cdot \mathbf{x}$ and $\mathbf{y}$ are equal, which is where the crisp comparisons become fuzzy.

Considering the last case of $c \cdot \mathbf{x} \subset \mathbf{y}$ and $c \cdot \mathbf{x} \neq \mathbf{y}$, we once again must find the value of z so that the membership value of $z \cdot \mathbf{x} \preceq \mathbf{y}$. But as $\mathbf{y}$ properly contains $z \cdot \mathbf{x}$ once the left endpoint of the two intervals agree, the portion of the interval $\mathbf{y}$ to the right of $z \cdot \mathbf{x}$ must be considered. Hence in a symmetrical fashion from the previous case the equation $\frac{\overline{y}-z\overline{x}}{(z\underline{x}-\underline{y})+(\overline{y}-z\overline{x})} = \alpha$ must be solved for z. Doing so generates the maximum value for z to be the expression $\frac{\overline{y}-\alpha(\overline{y}-\underline{y})}{\overline{x}-\alpha(\overline{x}-\underline{x})}$. Therefore we arrive at the following theorem.

Theorem 23 *If each of* $\mathbf{x}$ *and* $\mathbf{y}$ *is a positive interval and* $0 < \alpha \leq 1$ *then there is a maximum value of the real valued variable* z *such that* $z \cdot \mathbf{x} \preceq \mathbf{y}$ *with fuzzy membership value not less than* α.

Example 24 Solve the fuzzy linear programming problem for $\alpha = 0.9$:

maximize z subject to

$$z[1, 2] \preceq [3, 5]$$
$$z \geq 0$$

The value of z so that the midpoints are equal is $c = \frac{3+5}{1+2} = \frac{8}{3}$. In this case $[3, 5]$ is a proper subset of $c[1, 2] = [8/3, 16/3]$. So the maximum value of z that satisfies the inequality with the stated membership cut value is $z = \frac{3+0.9(2)}{1+0.9(1)} = \frac{4.8}{1.9} = 2.526315\cdots$.

6.2 Extending to more general cases

Let each of z_1 and z_2 be a real-valued variable, each of $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{y}$ be a positive interval, and $0 < \alpha \leq 1$. Consider the interval inequality $z_1 \cdot \mathbf{x}_1 + z_2 \cdot \mathbf{x}_2 \prec \mathbf{y}$ and the objective function $z_1 + z_2$. Let the interval binary operator $\ominus$ be defined as $\mathbf{x} - \mathbf{y} = [\underline{x} - \underline{y}, \overline{x} - \overline{y}]$. If z_1 is held constant between 0 and the corresponding maximum value of c that satisfies $c \cdot \mathbf{x}_1 \preceq \mathbf{y}$ (setting $z_2 = 0$ and solving the resulting simpler case using the fuzzy membership value α), then the maximum value of z_2 that satisfies the inequality $z_2 \cdot \mathbf{x}_2 \preceq (\mathbf{y} \ominus z_1 \cdot \mathbf{x}_1)$ using the membership value α can be determined by the above algorithm. The resulting value for z_2, in each of the three cases, is clearly a function of z_1, call it $z2_{\max(z_1)}$. Hence the original objective function can be rewritten as $z_1 + z2_{\max(z_1)}$, which can be seen to be a continuous function of z_1. Therefore the objective function must obtain a maximum value on the interval $[0, c]$ which then can be used to determine the solution to the initial interval linear programming problem.

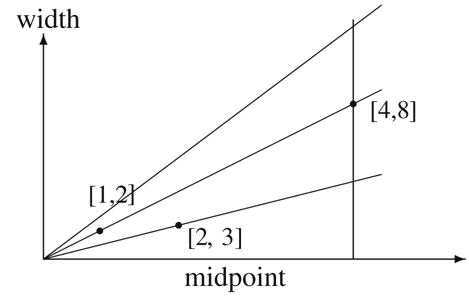

Fig. 6 $x[1, 2] + y[2, 3] \prec [4, 8]$

The following is a simple example that illustrates this approach.

Example 25 Solve the fuzzy linear programming problem for $\alpha = 0.9$:

maximize $x + y$ subject to

$$x[1, 2] + y[2, 3] \prec [4, 8]$$
$$z \geq 0$$

Solution We first consider the inequality $x[1, 2] \prec [4, 8]$. Note that the two intervals are collinear in the interval midpoint-radius plane, so setting the two midpoints equal we get $c = 4$. Therefore we must consider the resulting inequality $y[2, 3] \prec ([4, 8] \ominus x[1, 2])$ or $y[2, 3] \prec [4 - x, 8 - 2x]$ for each x in $[0, 4]$. In the interval midpoint-radius plane the interval $[2, 3]$ lies below the line containing $[1, 2]$ and $[4, 8]$ and hence the line containing $(0, 0)$ and the interval $[2, 3]$ intersects the vertical line containing $[4 - x, 8 - 2x]$ below that point. See Fig. 6.

Therefore, for each value of x in $[0, 4]$ the corresponding value of y is

$$y = \frac{(8 - 2x) - 0.9(8 - 2x - (4 - x))}{3 - 0.9(3 - 2)} v$$
$$= \frac{(2 - 0.9)(4 - x)}{3 - 0.9} = \frac{1.1(4 - x)}{2.1}.$$

We must optimize the objective function $x + y = x + \frac{1.1(4-x)}{2.1}$ on the interval $[0, 4]$. The derivative of this function is $1 - \frac{1.1}{2.1}$ which is positive. Therefore the maximum value of the objective function occurs when $x = 4$ and $y = 0$.

7 Conclusion

A model for crisply and fuzzily determined interval valued Nash games has been developed using an appropriate fuzzy interval comparison operator. This model parallels the classical game context in a closely analogous way. Also, the theory of optimal mixed strategies for interval valued games has been introduced, once again mimicking the classical model of converting the game into a linear programming problem.

In order to use interval linear programming techniques to find optimal mixed strategies in interval games, some assumptions must be made relative to the expected value interval **v**. Assuming that this interval is trivial generates corresponding linear programming problems that can be quickly solved. However, as the expected value of the game corresponds to a linear combination of the entries in the game matrix, this assumption appears to be unrealistic. We are continuing the work of extending the context to expected value intervals of positive diameter.

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