Soft Comput (2007) 11:593–605
DOI 10.1007/s00500-006-0103-5

ORIGINAL PAPER

Algorithm for solving max-product fuzzy relational equations

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Published online: 9 June 2006
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Abstract Analytical methods are proposed for solving fuzzy linear system of equations when the composition is max-product. These methods provide universal algorithm for computing the greatest solution and the set of all minimal solutions, when the system is consistent. In case of inconsistency, the equations that can not be satisfied are obtained.

Keywords Max-product fuzzy linear equations · Inverse problem

1 Introduction

Inverse problem resolution for fuzzy relational equations and in particular for fuzzy linear systems is subject of great scientific interest. The main results are published in De Baets (2000), Di Nola et al. (1989), Peeva (2006), and Peeva and Kyosev (2004). They demonstrate long and difficult period of investigations for discovering analytical methods and procedures to determine complete solution set, as well as to develop software for computing the greatest and all minimal solutions, see Peeva (2006) and Peeva and Kyosev (2004).

The first and most essential are Sanchez results Sanchez (1976) for the greatest solution of fuzzy relational equations with $\max - \min$ and $\min - \max$ composition. Sanchez gives formulas that permit to determine the potential greatest solution in any of these cases, often used as solvability criteria. After Sanchez results for the greatest solution, attention was paid on the minimal solutions, see Cheng and Wang (2002), Higashi and Klir (1984), Miyakoshi and Shimbo (1986), Pappis and Sugeno (1985), Peeva (1992, 2002, 2006) and Peeva and Kyosev (2004). Universal algorithm and software for solving $\max - \min$ and $\min - \max$ fuzzy relational equations is proposed in Peeva (2002, 2006) and Peeva and Kyosev (2004).

Concerning fuzzy linear system of equations with max-product composition (Di Nola and Lettieri 1989; Di Nola et al. 1989), the results concern greatest solution (Bourke and Fisher 1998), minimal solutions (in some references procedures pretend to yield to minimal solutions, but in fact they yield to some non-minimal solutions as well), estimating time complexity of the problem, applications in optimization problems (Guu and Wu 2002; Loetamonphong and Fang 1999, 2001; Loetamonphong et al. 2002). The relationship with the covering problem is subject of Markovskii (2005), where two methods for solving such fuzzy linear systems (algebraic and with table decomposition) are discussed and an algorithm is proposed, realizing table decomposition method.

Up to now there do not exist satisfactory methods, procedures and software for inverse problem resolution of fuzzy relational equations with max-product composition.

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This paper deals with inverse problem resolution for fuzzy linear system of equations with max-product composition

$$A \odot X = B,$$

where A stands for the matrix of coefficients, X stands for the matrix of unknowns, B is the right-hand side of the system and the max-product composition is written as $\odot$.

Since operations in the fuzzy algebras are different from classical operations, the traditional linear algebra methods – for instance Gaussian elimination method (MacLane and Birkhoff 1979), cannot be used here.

In this paper we extend the methodology developed for max – min fuzzy relational equations (Peeva 2002, 2006; Peeva and Kyosev 2004) for the case of max-product composition – we propose method and software for solving $A \odot X = B$ for unknown X. We obtain as much improvements over the straightforward exhaustive depth search of this NP-hard problem (Markovskii 2005) as possible. Rather than work with the system $A \odot X = B$, we use a matrix, whose elements capture all the properties of the equations. In depth first search, we propose how to drop branches that do not lead to minimal solutions. A sequence of simplification rules is defined, that bring the matrix A into a new form. Once in this form, we apply dominance to remove redundant equations in the system. In this manner we reduce the time complexity of exhaustive search merely by making a more clever choice of the objects over which the search is performed. This provides easily finding the complete solution to the original system.

We propose algorithm and software for computing the greatest and all minimal solutions or for establishing inconsistency of the system $A \odot X = B$.

In Sect. 2 we introduce basic notions. In Sects. 3 and 4 we study fuzzy linear systems of equations and develop a method and procedure for solving them. It provides the algorithm and software that answers to the following questions:

1. What is the complete solution set of a consistent system?
2. If the system is inconsistent, which equations can not be satisfied simultaneously with the other equations in the system?

Section 5 presents inverse problem resolution for max-product fuzzy relational equations. In Sect. 6 we propose software description and some comments on experimental results.

Terminology for computational complexity and algorithms is as in Aho et al. (1976) and Garey and Johnson (1979), for fuzzy sets and fuzzy relations is according to De Baets (2000), Di Nola et al. (1989), Klir et al. (1997), Peeva and Kyosev (2004) and Sanchez (1976), for lattices – as in Grätzer (1978), for algebra – as in MacLane and Birkhoff (1979).

2 Basic notions

Partial order relation on a partially ordered set (poset) P is denoted by the symbol $\leq$. By a *greatest element* of a poset P we mean an element $b \in P$ such that $x \leq b$ for all $x \in P$. The *least element* of P is defined dually.

Set $\mathbb{I}_{\odot} = \langle [0, 1], \vee, \wedge, 0, 1, \odot \rangle$, where $[0, 1]$ is the real unit interval, $\odot$ is the usual product between real numbers and $\vee$, $\wedge$ are respectively defined by

$$a \vee b = \max\{a, b\}, \quad a \wedge b = \min\{a, b\}.$$

Then $\mathbb{I}_{\odot}$ is a complete lattice with universal bounds 0 and 1; it is residuated with respect to $\odot$, being the residuum given by:

$$a \diamond b = \begin{cases} 1, & \text{if } a \leq b \\ \frac{b}{a}, & \text{if } a > b \end{cases}.$$

The algebraic structure $\mathbb{I}_{\odot} = \langle [0, 1], \vee, \wedge, 0, 1, \odot \rangle$ is called *fuzzy algebra*.

2.1 Fuzzy relation compositions – matrix representation

We denote by $F(X)$ the fuzzy sets over the crisp set X. A *fuzzy relation* $R \in F(X \times Y)$ is defined as a fuzzy subset of the Cartesian product $X \times Y$,

$$R = \{ ((x, y), \ \mu_R(x, y)) \}$$

where $(x, y) \in X \times Y$ and $\mu_R : X \times Y \rightarrow [0, 1]$.

The *inverse* (or *transpose*) $R^{-1} = R^t \in F(Y \times X)$ of $R \in F(X \times Y)$ is defined as

$$R^{-1}(y, x) = R(x, y) \quad \text{for all pairs} \quad (y, x) \in Y \times X.$$

For the Cartesian product $X \times Y$ the *first projection* pr_1 and the *second projection* pr_2 are defined (MacLane and Birkhoff 1979) as $pr_1(X \times Y) = X$ and $pr_2(X \times Y) = Y$, respectively.

The fuzzy relations $R \in F(X \times Y)$ and $S \in F(Y \times Z)$, with $pr_2(X \times Y) = pr_1(Y \times Z) = Y$, are called *composable*.

In what follows when we consider compositions, we work with composable fuzzy relations.

Definition 1 *Let the fuzzy relations $R \in F(X \times Y)$ and $S \in F(Y \times Z)$ be given.*

1. *The $\odot$-composition (Di Nola and Lettieri 1989; Di Nola et al. 1989) of R and S is the fuzzy relation $R \odot S \in F(X \times Z)$ with*

$$\mu_{R \odot S}(x, z) = \bigvee_{y \in Y} (\mu_R(x, y) . \mu_S(y, z)), \ (x, z) \in X \times Z.$$

2. *The $\diamond$-composition of R and S is the fuzzy relation $R \diamond S \in F(X \times Z)$ with*

$$\mu_{R \diamond S}(x, z) = \bigwedge_{y \in Y} (\mu_R(x, y) \diamond \mu_S(y, z)), \ (x, z) \in X \times Z.$$

The next Theorem 1 and Theorem 2 are particular cases of Theorem 1 and Theorem 2, respectively, cf. Di Nola and Lettieri (1989), proved in Di Nola and Lettieri (1989) for the general case of a complete lattice, residuated with respect to the product.

Theorem 1 *(Di Nola and Lettieri 1989) Let $R \in F(X \times Y)$ and $T \in F(X \times Z)$ be fuzzy relations, let $\mathbb{S}_\odot$ be the set of all fuzzy relations $S \in F(Y \times Z)$ with $R \odot S = T$. Then,*

1. *$\mathbb{S}_\odot \neq \emptyset$ iff $R^{-1} \diamond T \in \mathbb{S}_\odot$.*
2. *If $\mathbb{S}_\odot \neq \emptyset$ then $R^{-1} \diamond T$ is the greatest element in $\mathbb{S}_\odot$.*

A matrix $A = (a_{ij})_{m \times n}$, with $a_{ij} \in [0, 1]$ for each $i, j, 1 \leq i \leq m, 1 \leq j \leq n$, is called a *membership matrix* (Klir et al. 1997).

In what follows we write 'matrix' instead of 'membership matrix'.

We consider operations with matrices on the fuzzy algebra $\mathbb{I}_\odot$.

Any fuzzy relation $R \in F(X \times Y)$ is representable by a matrix (Di Nola et al. 1989), written for convenience with the same letter $R = (r_{ij})$, where $r_{ij} = \mu_R(x_i, y_j)$ for any $(x_i, y_j) \in X \times Y$.

We stipulate to use the matrix $R = (r_{ij})$ for the fuzzy relation $R \in F(X \times Y)$.

Definition 2 *Let the matrices $A = (a_{ij})_{m \times p}$ and $B = (b_{ij})_{p \times n}$ be given.*

1. *The matrix $C = (c_{ij})_{m \times n} = A \odot B$ is called $\odot$-product of A and B if*

$$c_{ij} = \max_{k=1}^{p} (a_{ik} . b_{kj}), \text{when } 1 \leq i \leq m, 1 \leq j \leq n.$$

2. *The matrix $C = (c_{ij})_{m \times n} = A \diamond B$ is called $\diamond$-product of A and B if*

$$c_{ij} = \min_{k=1}^{p} (a_{ik} \diamond b_{kj}), \text{when } 1 \leq i \leq m, 1 \leq j \leq n.$$

Definition 2 is the matrix representation of the compositions of fuzzy relations as introduced in Definition 1. This permits to manipulate with the matrix products instead of with compositions of fuzzy relations.

Theorem 2 *(Di Nola and Lettieri 1989) Let $A = (a_{ij})_{m \times p}$ and $C = (c_{ij})_{m \times n}$ be given matrices and let $\mathbb{B}_\odot$ be the set of all matrices such that $A \odot B = C$. Then:*

1. *$\mathbb{B}_\odot \neq \emptyset$ iff $A^t \diamond C \in \mathbb{B}_\odot$.*
2. *If $\mathbb{B}_\odot \neq \emptyset$ then $A^t \diamond C$ is the greatest element in $\mathbb{B}_\odot$.*

2.2 Inverse problem

Let $R \in F(X \times Y)$ and $S \in F(Y \times Z)$ be composable fuzzy relations and $R \odot S = T \in F(X \times Z)$ be their composition.

Solving $R \odot S = T$ for the unknown fuzzy relation S, if the fuzzy relations R and T are given, is called *inverse problem resolution* for the fuzzy relational equation $R \odot S = T$.

Inverse problem resolution in case of fuzzy algebras is subject of pure mathematical investigations, where attention is paid on the complete solution set. This is a partially ordered set that is determined (Bourke and Fisher 1998; Di Nola and Lettieri 1989; Markovskii 2005) by the minimal solutions and by the unique maximum solution, bearing in mind the density of ordering also. The property *density of ordering*, see Di Nola and Lettieri (1989), Proposition 4

$$(\forall a, \ b \in [0, \ 1]) \, a < b \quad \Rightarrow \quad (\exists c \in [0, \ 1]) \quad a < c < b$$

in $\mathbb{I}_\odot$ means that the maximum solution and minimal solutions determine complete solution set.

Hence, inverse problem resolution for $\odot$-composite fuzzy equations requires to determine the maximum solution and all minimal solutions.

3 Fuzzy linear systems of equations with $\odot$-composition

We study fuzzy linear systems of equations with $\odot$-composition ($\odot$-FLSE):

$$\left| \begin{array}{lllll} (a_{11}.x_1) & \vee \cdots \vee & (a_{1n}.x_n) & = & b_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (a_{m1}.x_1) & \vee \cdots \vee & (a_{mn}.x_n) & = & b_m \end{array} \right. , \qquad (1)$$

written in the following equivalent matrix form

$A \odot X = B,$

where $A = (a_{ij})_{m \times n}$ stands for the matrix of coefficients, $X = (x_j)_{n \times 1}$ stands for the matrix of unknowns, $B = (b_i)_{m \times 1}$ is the right-hand side of the system. For each $i, 1 \leq i \leq m$ and for each $j, 1 \leq j \leq n$, we have a_{ij}, b_i, $x_j \in [0, 1]$ and the max-product composition is written as $\odot$.

Following the concepts for fuzzy linear systems of equations with max − min composition (Peeva 2006; Peeva and Kyosev 2004), we introduce the corresponding notions for $\odot$-FLSE.

For $X = (x_j)_{n\times1}$ and $Y = (y_j)_{n\times1}$ the inequality $X \leq Y$ means $x_j \leq y_j$ for each j, $1 \leq j \leq n$.

Let us first define solutions for $A \odot X = B$ and give a classification of the $\odot-$FLSE according to the number of its solutions.

Definition 3 *Let the system $A \odot X = B$ in n unknowns be given.*

1. $X^0 = (x_j^0)_{n\times1}$ *with* $x_j^0 \in [0,1]$, *when* $1 \leq j \leq n$, *is called a (point) solution of the system* $A \odot X = B$ *if* $A \odot X^0 = B$ *holds.*
2. *The set of all point solutions* $\mathbb{X}^0$ *of* $A \odot X = B$ *is called complete solution set.*
3. *If* $\mathbb{X}^0 \neq \emptyset$ *then* $A \odot X = B$ *is called consistent, otherwise* $A \odot X = B$ *is called inconsistent.*

In the next exposition we omit the word "point" in "point solution".

Definition 4 *Let the system $A \odot X = B$ in n unknowns be given.*

1. *A solution* $X^0_{\text{low}} \in \mathbb{X}^0$ *is called a lower (minimal) solution of* $A \odot X = B$ *if for any* $X^0 \in \mathbb{X}^0$ *the relation* $X^0 \leq X^0_{\text{low}}$ *implies* $X^0 = X^0_{\text{low}}$, *where* $\leq$ *denotes the partial order, induced in* $\mathbb{X}^0$ *by the order of* [0, 1]. *Dually, a solution* $X^0_{\text{u}} \in \mathbb{X}^0$ *is called an upper (maximal) solution of* $A \odot X = B$ *if for any* $X^0 \in \mathbb{X}$ *the relation* $X^0_{\text{u}} \leq X^0$ *implies* $X^0 = X^0_{\text{u}}$. *When the upper solution is unique, it is called greatest (or maximum) solution.*
2. *The n-tuple* $(X_1, \ldots, X_n)$ *with* $X_j \subseteq [0, 1]$ *for each j,* $1 \leq j \leq n$, *is called an interval solution of the system* $A \odot X = B$ *if any* $X^0 = (x_j^0)_{n\times1}$ *with* $x_j^0 \in X_j$ *for each j,* $1 \leq i \leq n$, *implies* $X^0 = (x_j^0)_{n\times1} \in \mathbb{X}^0$.
3. *Any interval solution of* $A \odot X = B$ *whose components (interval bounds) are determined by a lower solution from the left and by the greatest solution from the right, is called maximal interval solution of* $A \odot X = B$.

In this paper we consider inhomogeneous systems with $b_i \neq 0$ for each $i = 1, ..., m$.

If $A \odot X = B$ is consistent, according to Theorem 2, it has unique maximum solution $X_{\text{gr}} = A^t \diamond B$. The complete solution set is described by the set of all maximal interval solutions. They are determined by all minimal solutions and the maximum one. Since there exists analytical expression for the maximum solution, attention in references is paid on computing minimal solutions.

3.1 Preliminary simplifications

Following the approach for max − min FLSE (Peeva 2002, 2006; Peeva and Kyosev 2004), we propose the first steps for simplifying $\odot-$FLSE so that the complete solution set can be easily found and the size of the instant can be reduced.

Step 1 Obtaining the associated matrix in which all zero coefficients corresponds to variables that do not contribute to solve the equation.
Step 2 Computing the index vector to indicate consistency or inconsistency of the system.
Step 3 Rearrangement of the equations in the system non-decreasingly with respect to the components of the index vector.

The first step marks all coefficients that do not contribute to solvability. The last step provides (after some supplementary considerations) automatically fulfilling of some equations. These equations and coefficients are dropped, we made a more clever choice of the objects over which the search is performed.

3.1.1 Step 1. Associated matrix

For the system (1) any coefficient $a_{ij} \geq b_i$ provides a way to satisfy the i-th equation with $a_{ij} . x_j = b_i$, when $x_j = \frac{b_i}{a_{ij}}$. This leads to the idea to distinguish coefficients that contribute for solving the system from these that do not contribute, see (2).

We assign to $A \odot X = B$ a symbolic matrix $A^* = (a^*_{ij})$ with elements a^*_{ij} determined according to the next expression:

$$a^*_{ij} = \begin{cases} S, & \text{if } a_{ij} < b_i \\ E, & \text{if } a_{ij} = b_i \\ G & \text{if } a_{ij} > b_i \end{cases} . \tag{2}$$

The matrix A^* with elements a^*_{ij}, determined by (2), is called *associated matrix* of the system (1). Its elements depend both on A and on B.

The time complexity function for obtaining A^* is O(mn).

Interpretation of A^*

- Any $a^*_{ij} = S$ in A^* corresponds to $a_{ij} < b_i$ in the i-th equation of (1). But $a_{ij} < b_i$ means $a_{ij}.x_j < b_i$ for each $x_j \in [0, 1]$. Hence each $a^*_{ij} = S$ in the i-th row of A^* indicates, that the coefficient a_{ij} do not contribute to satisfy i-th equation of (1).
- Any $a^*_{ij} \neq S$ in (2) corresponds to $a_{ij} \geq b_i \neq 0$ in the i-th equation of (1) that determines a way to satisfy this equation by $x_j = \frac{b_i}{a_{ij}}$. In this case $a_{ij}.x_j = a_{ij}.\frac{b_i}{a_{ij}} = b_i$.

Hence, associated matrix A^* provides first simplification. Rather than work with the system $A \odot X = B$, we use A^*, whose elements capture all the properties of the equations. This reduces the size of the instant and makes easier to solve the original system.

3.1.2 Step 2. IND vector

We introduce a vector $\mathrm{IND} = \mathrm{IND}_{m\times 1}$ to establish consistency of the system. We describe how the components of *IND* depend on A^*. Let we denote by $|G_i|$ the number of elements $a^*_{ij} = G$ and by $|E_i|$ the number of elements $a^*_{ij} = E$ in the i-th row of A^*, $j = 1, ..., n$. Then

$$\mathrm{IND}(i) = |G_i| + |E_i| \tag{3}$$

equals the number of elements $a^*_{ij} \neq S$ in the i-th row of A^*. It means that:

1. If $a^*_{ij} = S$ for each $j = 1, ..., n$ then $\mathrm{IND}(i) = 0$. In this case the i-th equation can not be satisfied and the system is inconsistent.
2. If $a^*_{ij} \neq S$ for some $j = 1, ..., n$ then $\mathrm{IND}(i) = |G_i| + |E_i| \neq 0$. In this case the i-th equation can be satisfied by $|G_i| + |E_i|$ different paths. If $\mathrm{IND}(i) \neq 0$ for each $i = 1, ..., m$ then the system can be either consistent or inconsistent.

Lemma 1 *Let the system $A \odot X = B$ be given. Then we have:*

1. *If $\mathrm{IND}(i) = 0$ for at least one $i = 1, ..., m$ then the system is inconsistent.*
2. *If the system is consistent then the number of its potential minimal solutions does not exceed*

$$\mathrm{PN} = \prod_{i=1}^{m} \mathrm{IND}(i). \tag{4}$$

Here $\mathrm{IND}(i)$ is computed according to (3).

Proof

1. If $IND(i) = 0$ for some $i \in \{1, ..., m\}$ then $a^*_{ij} = S$ for each $j = 1, ..., n$ and $a_{ij} \cdot x_j < b_i$ for each $x_j \in [0, 1]$, the i-th equation can not be satisfied and the system is inconsistent.
2. If the system is consistent then $IND(i) = |G_i| + |E_i| \neq 0$ for each $i \in \{1, ..., m\}$, see (3). Any $a^*_{ij} = G$ or $a^*_{ij} = E$ provides a way to satisfy i-th equation and gives a way to a potential minimal solution as well as lower bound for non-zero components of a minimal solution. Hence all possible ways to satisfy simultaneously all equations does not exceed

$$\mathrm{PN} = \prod_{i=1}^{m} \mathrm{IND}(i).$$

3.1.3 Step 3. Rearrangement of the equations

Two systems are called *equivalent* (MacLane and Birkhoff 1979) if any solution of the first one is a solution of the second one and vice versa. Any interchange of equations in the system $A \odot X = B$ results an equivalent system.

A system $A \odot X = B$, in which the equations are rearranged in such a way that the components of the index vector *IND* are ranked non-decreasingly, i.e.

$$\mathrm{IND}(1) \leq \mathrm{IND}(2) \leq \cdots \leq \mathrm{IND}(m),$$

is said to be in a *normal form*.

4 Solving ⊙-fuzzy linear systems

In this section we propose a unified and exact method and algorithm for solving inhomogeneous ⊙-FLSE of the form $A \odot X = B$, resulting in:

1. A necessary and sufficient condition (Corollary 4) for consistency of the ⊙-FLSE, similar to the general test for consistency of a system (MacLane and Birkhoff 1979) in Linear Algebra.
2. Analytical expressions for the maximum solution and minimal solutions.
3. Algorithm for inverse problem resolution for ⊙-FLSE.

Let the following stipulations be satisfied for inhomogeneous $A \odot X = B$:

1. The system $A \odot X = B$ has coefficient matrix $A = (a_{ij})_{m\times n}$, matrix of unknowns $X = (x_j)_{n\times 1}$, and right-hand side $B = (b_i)_{m\times 1}$ with $b_i \neq 0$ for each $i = 1, ..., m$. Hence it has n unknowns and m equations.
2. The associated matrix A^* for the system $A \odot X = B$ is obtained.
3. Any coefficient $a^*_{ij} = S$ is called S-type coefficient, any $a^*_{ij} = E$ is called E-type coefficient and any $a^*_{ij} = G$ is called G-type coefficient.
4. For each j, $j = 1, \dots, n$, $A^*(j) = (a^*_{ij})_{m\times 1}$ denotes the j-th column of A^* and a^*_{ij} denotes the i-th element ($1 \leq i \leq m$) in $A^*(j)$.

Theorem 3 *Let the system $A \odot X = B$ be given.*

i) If $A^*(j)$ contains G-type coefficient(s) $a^*_{ij} = G$ and

$$\hat{x}_j = \min_{i=1}^{m} \left\{ \frac{b_i}{a_{ij}} \right\}, \quad \text{when } a_{ij} > b_i,$$

then $x_j = \hat{x}_j$ implies in (1):
$a_{ij} . x_j = b_i$ for each i, $1 \leq i \leq m$ when $\frac{b_i}{a_{ij}} = \hat{x}_j$,
$a_{ij} . x_j < b_i$ for each i, $1 \leq i \leq m$ with $\frac{b_i}{a_{ij}} \neq \hat{x}_j$.

ii) If $A^*(j)$ does not contain any G-type coefficient, but it contains E-type coefficient(s) $a^*_{kj} = E$, then $\hat{x}_j = 1$ and $x_j = \hat{x}_j = 1$ implies:
$a_{ij} . x_j = b_i$ for each i, $1 \leq i \leq m$ with $a^*_{ij} = E$,
$a_{ij} . x_j < b_i$ for each i, $1 \leq i \leq m$ with $a^*_{ij} = S$.

iii) If $A^*(j)$ contains neither G- nor E-type coefficient then $\hat{x}_j = 1$ and $x_j = \hat{x}_j = 1$ implies $a_{ij} . x_j < b_i$ for each i, $1 \leq i \leq m$ ($a^*_{ij} = S$ in $A^*(j)$).

The proof follows from the definition of the associated matrix, its relationship with the system (1) and expression (2).

Corollary 1 *For any consistent system $A \odot X = B$,*

$$X_{\text{gr}} = A^t \diamond B = \hat{X} = (\hat{x}_j)_{n\times 1}$$

where $\hat{x}_j$, $1 \leq j \leq n$, are computed according to Theorem 3.

Corollary 2 *If $a^*_{ij} = S$ for each $i = 1, ..., m$, then $\check{x}_j = 0$ in any minimal solution $\check{X} = (\check{x}_j)_{n\times 1}$ of the consistent system $A \odot X = B$.*

Proof If $A^*(j)$ contains only S-type coefficients then the component $x_j \in [0, 1]$ has no influence on satisfiability of equations, see Theorem 3, iii). Then in all minimal solutions $\check{x}_j = 0$.

Corollary 3 *If $\check{X} = (\check{x}_j)_{n\times 1}$ is a minimal solution of the consistent system $A \odot X = B$, then for each $j = 1, ..., n$ either $\check{x}_j = 0$ or $\check{x}_j = \hat{x}_j$.*

Proof Since $b_i > 0$ for each $i = 1, ..., m$, the system can not have zero solutions, it has some set of non-zero minimal solutions. The component $\check{x}_j$ of a minimal solution has zero value, if $A^*(j)$ contains only S-type coefficients, see Corollary 2. The other components of a minimal solution cannot take their minimal values independently from each other – the equations in the consistent system $A \odot X = B$ must be satisfied simultaneously. If $A^*(j)$ contains E- or G-type coefficients then in a minimal solution the component $\check{x}_j$ may take either zero value or the value $\check{x}_j = \hat{x}_j = \min_{i=1}^{m} \left\{ \frac{b_i}{a_{ij}} \right\}$, when $a_{ij} \geq b_i$, see Theorem 3 i), ii).

4.1 Selected elements

Theorem 3 and its Corollaries 2, 3 prove that all S-type coefficients do not contribute for solving the system and there may exist redundant coefficients of type G and E in the system. We propose a selection of all coefficients that contribute to solve the system. All other coefficients are called non-essential for solvability procedure and we drop them.

Definition 4 *Let the system $A \odot X = B$ with associated matrix A^* be given.*

1. *If $A^*(j) = (a^*_{ij})_{m\times 1}$ contains G-type coefficient $a^*_{kj} = G$, such that*

$$\frac{b_k}{a_{kj}} = \min_{i=1,}^{m} \left\{ \frac{b_i}{a_{ij}} \right\} \quad \text{when } \ a_{ij} > b_i,$$

then each G-type coefficient a^*_{ij} in $A^*(j)$ with $\frac{b_i}{a_{ij}} = \frac{b_k}{a_{kj}}$ is called selected.

2. *If $A^*(j) = (a^*_{ij})_{m\times 1}$ does not contain G-type coefficient, but it contains E-type coefficient(s), then all E-type coefficients in $A^*(j)$, namely $a^*_{ij} = E$ when $1 \leq i \leq m$, are called selected.*
3. *If $A^*(j)$ does not contain neither G-, nor E-type coefficient, then there does not exist selected coefficient in $A^*(j)$.*

From Theorem 3 we obtain

Corollary 4 *Let the system $A \odot X = B$ be given.*

1. *It is consistent if and only if for each i,* $1 \le i \le m$*, there exists at least one selected coefficient* a^*_{ij}*, otherwise it is inconsistent.*
2. *If the system is consistent then*

$$X_{\text{gr}} = A^t \diamond B \tag{5}$$

is its unique maximal (i.e. greatest, or maximum) solution.

3. *The time complexity function for establishing the consistency of the system and for computing* X_{gr} *is* O(mn).

4.2 Help matrix and dominance matrix

Now we propose the next simplification steps.

Step 4 Obtain a help matrix H from selected coefficients.

Step 5 Using a dominance relation over the rows in H, form a dominance matrix D that provides easier way to compute all minimal solutions.

4.2.1 Step 4. help matrix

We introduce a *help matrix* $H = (h_{ij})_{m\times n}$ with elements

$$h_{ij} = \begin{cases} 1 & \text{if } a^*_{ij} \text{ is selected} \\ 0 & \text{otherwise} \end{cases}. \tag{6}$$

We upgrade the components of the vector IND $=$ $\text{IND}_{m\times 1}$ to establish the consistency of the system and to diminish the potential number PN (see (4)) of minimal solutions. Now the i-th component IND(i) of IND equals the number of selected coefficients in the i-th equation of the system, i.e.

$$\text{IND}(i) = \sum_{j=1}^{n} h_{ij}. \tag{7}$$

If there are no selected coefficients in the i-th equation, then IND$(i) = 0$ and the system is inconsistent, see Corollary 4.1).

Obviously, now the potential number PN1 of minimal solutions will be diminished in comparison with PN, i.e.

$$\text{PN1} = \prod_{i=1}^{m} \text{IND}(i) \le PN. \tag{8}$$

4.2.2 Step 5. Dominance matrix

In order to determine the minimal solutions of a $\odot$-FLSE, a suitable dominance relation for the rows of the help matrix H is introduced.

Definition 5 *Let* $h_l = (h_{lj})$ *and* $h_k = (h_{kj})$ *be the l-th and the k-th rows, respectively, in the help matrix H. If for each j,* $1 \le j \le n$*,* $h_{lj} \le h_{kj}$*, then*

h_l is said to be a dominant row to h_k in H;
h_k is redundant row with respect to h_l for solving the system (1).

If h_k is redundant row with respect to h_l for solving (1) it means that:

- k-th equation is automatically satisfied whenever l-th equation is satisfied.
- It is meaningless to investigate the k-th equation, because it will not lead to smaller solution than the l-th equation.
- When we eliminate k-th equation from next consideration we cut redundant branches from the search (they not lead to minimal solutions), making a more clever choice of the objects over which the search is performed.

Using Definition 5, we introduce a *dominance matrix* $D = (d_{ij})$ obtained from H as described below. If the row h_l dominates the row h_k in H, then in D:

We preserve all elements of the row h_l , i.e. $d_{ij} = h_{lj}$ for $j = 1, ..., n$. This preserves non-redundant (or essential for solution procedure) equation.
We replace all elements of the row h_k by 0, i.e. $d_{kj} = 0$ for $j = 1, ..., n$. This eliminates redundant equations and also removes redundant branches of the search.

We again upgrade the components of the vector IND,

$$\text{IND}(i) = \sum_{j=1}^{n} d_{ij}.$$

Now the i-th component IND(i) equals the number of non-redundant selected coefficients in the i-th equation of the system.

Next, the potential number PN2 of minimal solutions will be diminished in comparison with PN1 and PN, i.e.

$$\text{PN2} = \prod_{i=1,\text{IND}(i)\neq 0}^{m} \text{IND}(i) \le \text{PN1} \le \text{PN}. \tag{9}$$

4.3 Finding minimal solutions

From dominance matrix $D = (d_{ij})$ we go to the next simplification. We form a matrix $M = (m^*_{ij})$ indicating non-redundant elements for solving (1). First we remove all zero rows (redundant equations) and all zero columns (non-essential coefficients) from D. From the rest, we obtain:

$$m^*_{ij} = \begin{cases} \frac{b_i}{a_{ij}} & \text{if } h^*_{ij} = 1 \\ 0 & \text{if } h^*_{ij} = 0 \end{cases} \tag{10}$$

In what follows we work with the matrix M.

4.3.1 Algebraic properties

We expand the possible irredundant paths (called coverings in Markovskii (2005), i.e. different ways to satisfy simultaneously equations of the system) using the matrix M and the algebraic properties of the logical sums, as described below.

If the element $m^*_{ij} \neq 0$, we symbolize this with $\left\langle \frac{m^*_{ij}}{j} \right\rangle$. In this case $a_{ij} . m^*_{ij} = b_i$ and hence $\hat{x}_j = m^*_{ij}$ gives a lower bound to fulfill the i-th equation of the system; $\check{x}_j = \hat{x}_j = m^*_{ij}$ is the minimum value for the j-th component.

For each i, $1 \leq i \leq m$, the elements $m^*_{ij} \neq 0$ in M mark the potential lower bounds of different ways, to satisfy the i-th equation of the system, written M_i and symbolized by the sign $\sum$:

$$M_i = \sum_{1 \leq j \leq n} \left\langle \frac{m^*_{ij}}{j} \right\rangle. \tag{11}$$

For instance, if there exist two ways to satisfy the second equation of a system – by $x_4 = 0.5$ and $x_5 = 0.25$, it is symbolized as

$$M_2 = \left\langle \frac{m^*_{24}}{4} \right\rangle + \left\langle \frac{m^*_{25}}{5} \right\rangle = \left\langle \frac{0.5}{4} \right\rangle + \left\langle \frac{0.25}{5} \right\rangle.$$

We have to consider equations simultaneously, i.e., to compute the *concatenation* W of all ways, symbolized by the sign $\prod$:

$$W = \prod_{1 \leq i \leq m} \left(\sum_{1 \leq j \leq n} \left\langle \frac{m^*_{ij}}{j} \right\rangle \right). \tag{12}$$

For instance,

$$\begin{aligned} M_2.M_3 &= \left(\left\langle \frac{m^*_{24}}{4} \right\rangle + \left\langle \frac{m^*_{25}}{5} \right\rangle \right) . \left\langle \frac{m^*_{31}}{1} \right\rangle \\ &= \left(\left\langle \frac{0.5}{4} \right\rangle + \left\langle \frac{0.25}{5} \right\rangle \right) . \left\langle \frac{0.4}{1} \right\rangle \end{aligned}$$

means that there exist two ways satisfy the second equation of a system – by $x_4 = 0.5$ and $x_5 = 0.25$, and one way to satisfy the third equation by $x_1 = 0.4$ and also we consider these equations simultaneously.

In order to compute complete solution set, it is important to determine different ways to satisfy simultaneously equations of the system. To achieve this aim we list the properties of concatenation (12).

Concatenation is *distributive* with respect to addition, i.e.

$$\begin{aligned} &\left\langle \frac{m^*_{i_1j_1}}{j_1} \right\rangle \left(\left\langle \frac{m^*_{i_2j_2}}{j_2} \right\rangle + \left\langle \frac{m^*_{i_2j_3}}{j_3} \right\rangle \right) \\ &= \left\langle \frac{m^*_{i_1j_1}}{j_1} \right\rangle \left\langle \frac{m^*_{i_2j_2}}{j_2} \right\rangle + \left\langle \frac{m^*_{i_1j_1}}{j_1} \right\rangle \left\langle \frac{m^*_{i_2j_3}}{j_3} \right\rangle . \end{aligned} \tag{13}$$

This analytical expression demonstrates simultaneous satisfaction of both equations (i_1, i_2) by selected elements in two different ways – the first way, that corresponds to the first summand, is by the selected elements $m^*_{i_1j_1}$ and $m^*_{i_2j_2}$ in rows i_1, i_2 and columns j_1, j_2, respectively; the second way corresponds to the second summand and it is formed by the selected elements $m^*_{i_1j_1}, m^*_{i_2j_3}$.

Concatenation is *commutative*:

$$\left\langle \frac{m^*_{i_1j_1}}{j_1} \right\rangle \left\langle \frac{m^*_{i_2j_2}}{j_2} \right\rangle = \left\langle \frac{m^*_{i_2j_2}}{j_2} \right\rangle \left\langle \frac{m^*_{i_1j_1}}{j_1} \right\rangle. \tag{14}$$

This provides the validity of Step 3 – rearrangement of equations in the $\odot$-FLSE.

The next property is called *absorption for multiplication*:

$$\left\langle \frac{m^*_{i_1j_1}}{j_1} \right\rangle \left\langle \frac{m^*_{i_2j_1}}{j_1} \right\rangle = \left\langle \frac{m^*_{i_1j_1}}{j_1} \right\rangle \tag{15}$$

Expression (15) gives the lower solution for simultaneous satisfying of two different equations i_1 and i_2, when selected coefficients belong to the same column j_1. Hence, expanding along the non-zero elements in the the i-th row, we automatically satisfy all equations in the system, having the same m^*_{ij}. It is clear that this property reduces the number of the ways that have to be investigated.

We apply (13), (14), (15) to expand the parentheses in (12). We obtain the set of ways, from which we extract the minimal solutions:

$$W = \sum_{(j_1, \cdots, j_m)} \left\langle \frac{m^*_{i_1j_1}}{j_1} \right\rangle \left\langle \frac{m^*_{i_2j_2}}{j_2} \right\rangle \cdots \left\langle \frac{m^*_{i_mj_m}}{j_m} \right\rangle. \tag{16}$$

We simplify (16) according to the next described *absorption for addition* (missing $\left\langle \frac{m^*_{ij}}{j} \right\rangle$ are supposed to be $\left\langle \frac{0}{j} \right\rangle$):

$$\left\langle \frac{m^*_{i_1j_1}}{j_1} \right\rangle \cdots \left\langle \frac{m^*_{i_mj_m}}{j_m} \right\rangle + \left\langle \frac{m^*_{s_1j_1}}{j_1} \right\rangle \cdots \left\langle \frac{m^*_{s_mj_m}}{j_m} \right\rangle = \begin{cases} \left\langle \frac{m^*_{i_1j_1}}{j_1} \right\rangle \cdots \left\langle \frac{m^*_{i_mj_m}}{j_m} \right\rangle, \text{if } m^*_{i_tj_t} \le m^*_{s_tj_t} \text{ for } t = 1, \dots, m \\ \text{unchanged, otherwise} \end{cases} . \tag{17}$$

From two compatible point solutions with respect to the relation $\le$, expression (17) selects the smaller, because complete solution set $\mathbb{X}^0$ is a poset, see Di Nola et al. (1989).

Property (17) provides reduction of the number of terms in (16) that we investigate to obtain lower solutions. In particular,

$$\left\langle \frac{m^*_{i_1j_1}}{j_1} \right\rangle \cdot \left\langle \frac{m^*_{i_2j_2}}{j_2} \right\rangle + \left\langle \frac{m^*_{i_1j_1}}{j_1} \right\rangle \cdot \left\langle \frac{m^*_{i_2j_2}}{j_2} \right\rangle = \left\langle \frac{m^*_{i_1j_1}}{j_1} \right\rangle \cdot \left\langle \frac{m^*_{i_2j_2}}{j_2} \right\rangle . \tag{18}$$

A property called *combined absorption* follows from (15), (17) and (18):

$$\left\langle \frac{m^*_{i_1j_1}}{j_1} \right\rangle \left[\left\langle \frac{m^*_{i_2j_1}}{j_1} \right\rangle + \left\langle \frac{m^*_{i_2j_2}}{j_2} \right\rangle \right] = \left\langle \frac{m^*_{i_1j_1}}{j_1} \right\rangle \left\langle \frac{m^*_{i_2j_1}}{j_1} \right\rangle + \left\langle \frac{m^*_{i_1j_1}}{j_1} \right\rangle \left\langle \frac{m^*_{i_2j_2}}{j_2} \right\rangle = \left\langle \frac{m^*_{i_1j_1}}{j_1} \right\rangle . \tag{19}$$

After simplifying (16) according to (17)–(19) any term

$$\left\langle \frac{m^*_{i_1j_1}}{j_1} \right\rangle \left\langle \frac{m^*_{i_2j_2}}{j_2} \right\rangle \cdots \left\langle \frac{m^*_{i_mj_m}}{j_m} \right\rangle$$

determines a minimal solution $\check{X} = (\check{x}_j)$, with components (obtained after expanding brackets in (12) by rules (13)–(20)), see also Corollary 3:

$$\check{x}_{j_t} = \begin{cases} m^*_{i_tj_t} = \hat{x}_{j_t} & \text{if } m^*_{i_tj_t} \ne 0 \text{ in (16)} \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

Corollary 5 *For any consistent $\odot$-FLSE the minimal solutions are computable and the set of all its minimal solutions is finite.*

4.3.2 Method based on expansion along the non-zero elements of M

The proposed formalism in Sect. 4.3.1 provides the next quite simple method based on expansion along the non-zero elements of the row in M.

1. Take the non-zero elements of i-th row (for $i = 1, \dots, m$) of M and form the sum M_i, see (11).
2. Expand M: from each element $m^*_{ij} \ne 0$ in M_i we form a summand, consisting of $\left\langle \frac{m^*_{ij}}{j} \right\rangle$, multiplied by a submatrix M_{ij} of M; M_{ij} is obtained as follows: we delete in M the i-th row and the j-th column, see (15), as well as all rows with the same $m^*_{ij} \ne 0$ – they are automatically satisfied, see (18). From the resulting submatrix we remove redundant rows, zero rows and zero columns.
3. If $i > m$-stop, otherwise take the next i.

Example Following the above theoretical background, we solve the system given by Markovskii in 2005 $A \odot X = B$ with:

$$\left| \begin{array}{l} (0.2 \,.\, x_2) \vee (0.05 \,.\, x_3) \vee (0.4 \,.\, x_5) = 0.1 \\ (0.1 \,.\, x_1) \vee (.6 \,.\, x_2) \vee (.3 \,.\, x_3) \vee (.2 \,.\, x_5) \vee (.2 \,.\, x_6) = 0.3 \\ (0.8 \,.\, x_1) \vee (0.48 \,.\, x_2) \vee (0.24 \,.\, x_3) \vee (0.48 \,.\, x_4) = 0.24 \\ (0.3 \,.\, x_1) \vee (0.4 \,.\, x_4) \vee (0.8 \,.\, x_5) \vee (0.15 \,.\, x_6) = 0.2 \\ (0.12 \,.\, x_3) \vee (0.2 \,.\, x_4) \vee (0.48 \,.\, x_5) \vee (0.1 \,.\, x_6) = 0.12 \\ (0.5 \,.\, x_1) \vee (0.3 \,.\, x_2) \vee (0.1 \,.\, x_4) \vee (0.6 \,.\, x_5) = 0.15 \end{array} \right. .$$

In order to make the exposition clear, equations are marked as e_1–e_6, unknowns are x_1–x_6:

$$A = \left(\begin{array}{lllllll} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ e_1 & 0 & 0.2 & 0.05 & 0 & 0.4 & 0 \\ e_2 & 0.1 & 0.6 & 0.3 & 0 & 0.2 & 0.2 \\ e_3 & 0.8 & 0.48 & 0.24 & 0.48 & 0 & 0 \\ e_4 & 0.3 & 0 & 0 & 0.4 & 0.8 & 0.15 \\ e_5 & 0 & 0 & 0.12 & 0.2 & 0.48 & 0.1 \\ e_6 & 0.5 & 0.3 & 0 & 0.1 & 0.6 & 0 \end{array} \right), \quad B = \left(\begin{array}{ll} e_1 & 0.1 \\ e_2 & 0.3 \\ e_3 & 0.24 \\ e_4 & 0.2 \\ e_5 & 0.12 \\ e_6 & 0.15 \end{array} \right).$$

For associated matrix according to (2) we have:

$$A^* = \left(\begin{array}{lllllll} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ e_1 & S & G & S & S & G & S \\ e_2 & S & G & E & S & S & S \\ e_3 & G & G & E & G & S & S \\ e_4 & G & S & S & G & G & S \\ e_5 & S & S & E & G & G & S \\ e_6 & G & G & S & S & G & S \end{array} \right),$$

Computing IND as given in Sect. 3.1.2.

IND(1) = 2, because in the first row of A^* there exist two G-type coefficients; IND(2) = 2, because in the second row of A^* there exist one G-type and one E-type coefficients; IND(3) = 4, because in the third row of A^* there exist three G-type coefficients and one E-type coefficient, etc. for IND(4) = 3, IND(5) = 3, IND (6) = 3. The system can be either consistent or inconsistent and it may have no more than $\text{PN} = \prod_{i=1}^{6} \text{IND}(i) = 2.2.4.3.3.3 = 432$ potential minimal solutions, see (4).

For computing H and X_{gr} we apply Theorem 3:

For $j=1$, $A^*(1)$ contains three G-type coefficients, namely a^*_{31}, a^*_{41} and a^*_{61}. We apply Theorem 3 i) to compute $X_{gr}(1)=\hat{x}_1$, to select essential and to drop non-essential coefficients.

$$\hat{x}_1 = \min_{i=1}^{6}\left\{\frac{b_3}{a_{31}}, \frac{b_4}{a_{41}}, \frac{b_6}{a_{61}}\right\} = \min\left\{\frac{0.24}{0.8}, \frac{0.2}{0.3}, \frac{0.15}{0.5}\right\} = 0.3.$$

Since $\hat{x}_1 = 0.3$ we put $X_{gr}(1)=\hat{x}_1=0.3$. This selection is based on

$$\frac{b_3}{a_{31}} = \frac{b_6}{a_{61}} = 0.3 < \frac{b_4}{a_{41}},$$

hence selected coefficients from the first column are $a^*_{31}=a^*_{61}$, see Definition 4 and we put $h_{31}=h_{61}=1$ according to (6). It means that we have found a way to satisfy e_3 and e_6 by the terms $a_{31}\cdot\hat{x}_1$ and $a_{61}\cdot\hat{x}_1$, respectively.

For $j=2$ we also apply Theorem 3 i) and by analogy with previous inference, selected coefficients from the second column are $a^*_{12}=a^*_{22}=a^*_{32}=a^*_{62}$, hence $h_{12}=h_{22}=h_{32}=h_{62}=1$ and $X_{gr}(2)=0.5$.

For $j=3$ there do not exist G-type coefficients, we apply Theorem 3 ii) and select all E-type coefficients $a^*_{23}=a^*_{33}=a^*_{53}$, leading to $h_{23}=h_{33}=h_{53}=1$, and $X_{gr}(3)=1$.

For $j=4$ we apply Theorem 3 i), $a^*_{34}=a^*_{44}$ are selected, $h_{34}=h_{44}=1$, $X_{gr}(4)=0.5$.

For $j=5$ we apply Theorem 3 i), select $a^*_{15}=a^*_{45}=a^*_{55}=a^*_{65}$, $h_{15}=h_{45}=h_{55}=h_{65}=1$, $X_{gr}(5)=0.25$.

For $j=6$ apply Theorem 3 iii), because $A^*(6)$ contains only S-type coefficients, x_6 does not contribute for solving the system, $X_{gr}(6)=1$.

The result is summarized as:

$$H = \begin{pmatrix} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ e_1 & 0 & 1 & 0 & 0 & 1 & 0 \\ e_2 & 0 & 1 & 1 & 0 & 0 & 0 \\ e_3 & 1 & 1 & 1 & 1 & 0 & 0 \\ e_4 & 0 & 0 & 0 & 1 & 1 & 0 \\ e_5 & 0 & 0 & 1 & 0 & 1 & 0 \\ e_6 & 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathrm{IND} = \begin{pmatrix} 2 \\ 2 \\ 4 \\ 2 \\ 2 \\ 3 \end{pmatrix},$$

$$X_{gr} = \begin{pmatrix} 0.3 \\ 0.5 \\ 1 \\ 0.5 \\ 0.25 \\ 1 \end{pmatrix}.$$

Notice that the components of the vector IND are upgraded and computed from H. Since $\mathrm{IND}(i) \neq 0$ for $i=1,\ldots,6$, the system is consistent, see Corollary 4. It has no more than PN1 = 2.2.4.2.2.3 = 192 potential minimal solutions, see current IND-components.

Next we obtain dominance matrix. The rows e_3 and e_6 are redundant, they are dominated by e_2 and e_1, respectively. We obtain for the dominance matrix D:

$$D = \begin{pmatrix} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ e_1 & 0 & 1 & 0 & 0 & 1 & 0 \\ e_2 & 0 & 1 & 1 & 0 & 0 & 0 \\ e_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ e_4 & 0 & 0 & 0 & 1 & 1 & 0 \\ e_5 & 0 & 0 & 1 & 0 & 1 & 0 \\ e_6 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We remove redundant rows and columns and upgrade IND:

$$M = \begin{pmatrix} & x_2 & x_3 & x_4 & x_5 \\ e_1 & 0.5 & 0 & 0 & 0.25 \\ e_2 & 0.5 & 1 & 0 & 0 \\ e_4 & 0 & 0 & 0.5 & 0.25 \\ e_5 & 0 & 1 & 0 & 0.25 \end{pmatrix}, \quad \mathrm{IND} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}.$$

From current IND it follows that the system is in normal form and it has no more than PN2 = 16 minimal solutions.

Expansion along M follows. In the first row of M there exist two non-zero elements, namely $m^*_{12}=0.5$ and $m^*_{15}=0.25$. When we select $\check{x}_2 = m^*_{12} = 0.5$, we satisfy equations e_1 and e_2, selecting $\check{x}_5 = m^*_{15} = 0.25$ we satisfy e_1, e_4 and e_5. For this reason these equations are excluded from the corresponding summands:

$$\left\langle\frac{0.5}{2}\right\rangle\cdot\begin{pmatrix} & x_3 & x_4 & x_5 \\ e_4 & 0 & 0.5 & 0.25 \\ e_5 & 1 & 0 & 0.25 \end{pmatrix} + \left\langle\frac{0.25}{5}\right\rangle\cdot\begin{pmatrix} & x_2 & x_3 & x_4 \\ e_2 & 0.5 & 1 & 0 \end{pmatrix}.$$

In the second summand we exclude the zero column for x_4:

$$\left\langle\frac{0.5}{2}\right\rangle\cdot\begin{pmatrix} & x_3 & x_4 & x_5 \\ e_4 & 0 & 0.5 & 0.25 \\ e_5 & 1 & 0 & 0.25 \end{pmatrix} + \left\langle\frac{0.25}{5}\right\rangle\cdot\begin{pmatrix} & x_2 & x_3 \\ e_2 & 0.5 & 1 \end{pmatrix}$$

Expanding the last expression and having in mind absorption (18) we obtain:

$$\begin{aligned} &\left[\left\langle\frac{0.5}{2}\right\rangle\cdot\left\langle\frac{1}{3}\right\rangle\cdot\left\langle\frac{0.5}{4}\right\rangle + \left\langle\frac{0.5}{2}\right\rangle\cdot\left\langle\frac{0.25}{5}\right\rangle\right] \\ &\quad + \left[\left\langle\frac{0.5}{2}\right\rangle\cdot\left\langle\frac{0.25}{5}\right\rangle + \left\langle\frac{1}{3}\right\rangle\cdot\left\langle\frac{0.25}{5}\right\rangle\right] \\ &\quad = \left\langle\frac{0.5}{2}\right\rangle\cdot\left\langle\frac{1}{3}\right\rangle\cdot\left\langle\frac{0.5}{4}\right\rangle \\ &\quad + \left\langle\frac{0.5}{2}\right\rangle\cdot\left\langle\frac{0.25}{5}\right\rangle + \left\langle\frac{1}{3}\right\rangle\cdot\left\langle\frac{0.25}{5}\right\rangle. \end{aligned} \tag{21}$$

According to (21) the minimal solutions are:

$$\check{X}_1 = (0\ 0.5\ 1\ 0.5\ 0\ 0)^t, \quad \check{X}_2 = (0\ 0.5\ 0\ 0\ 0.25\ 0)^t,$$
$$\check{X}_3 = (0\ 0\ 1\ 0\ 0.25\ 0)^t.$$

Hence the maximal interval solutions, respectively complete solution set, are:

$$X_{\max 1} = \begin{pmatrix} [0,0.3] \\ 0.5 \\ 1 \\ 0.5 \\ [0,0.25] \\ [0,1] \end{pmatrix}, \quad X_{\max 2} = \begin{pmatrix} [0,0.3] \\ 0.5 \\ [0,1] \\ [0,0.5] \\ 0.25 \\ [0,1] \end{pmatrix},$$

$$X_{\max 3} = \begin{pmatrix} [0,0.3] \\ [0,0.5] \\ 1 \\ [0,0.5] \\ 0.25 \\ [0,1] \end{pmatrix}.$$

The minimal solutions for this example are determined by Markovskii (2005), using covering. We devise here a computational scheme, extending the approach for max − min FLSE (Peeva 2006) for $\odot$−composition. Instead of work with the system $A \odot X = B$, we use the associated matrix A^*, whose elements preserve all the properties of the initial system. Manipulation on A^* is carried out to select some coefficients and to determine the help matrix H. Using dominance relation, we bring H into a new form D. Once in this form, solution to the system is easily found.

4.4 The algorithm

Conventional reasoning to solve $\odot$−FLSE leads to combinatorial problem, see Markovskii (2005). Using the theoretical background from Sects. 3 and 4, we devise algorithm that computes maximal and all minimal solutions (without listing duplications of minimal solutions or non-minimal solutions) and that is smaller time consuming in comparison with the algorithms given in Guu and Wu (2002), Loetamonphong and Fang (1999, 2001), Loetamonphong et al. (2002) and Markovskii (2005).

Algorithm for solving $A \odot X = B$.

1. Enter the matrices $A_{m\times n}$ and $B_{m\times 1}$.
2. Compute $A^* = (a^*_{ij})$ with a^*_{ij} according to (2).
3. Compute H, IND, X_{gr}.
4. Transform the system in normal form.
5. If IND$(i) = 0$ for some $i = 1, \dots, m$, then the system is inconsistent and the equation(s) with IND$(i) = 0$ can not be satisfied simultaneously with the other equation(s) in the system.
   Go to Step 9.
6. If IND$(i) = 1$ for each $i = 1, \dots, m$, the system is consistent with unique: maximum solution, minimum solution and maximal interval solution; X_{gr} contains the maximum solution; X_{low} is determined according expression (20); $X_{\max}$ is determined by X_{low} on the left and by X_{gr} on the right.
   Go to Step 9.
7. Compute the dominance matrix $D = (d_{ij})_{m\times n}$ as described in Sect. 4.2.2.
8. Compute the matrix M with elements computed by (10). Expand M along non-zero elements by rows as given in Sect. 4.3.2. Simplify W according to algebraic properties in Sect 4.3.1.
9. The system is consistent, X_{gr} contains the maximum solution. Determine the minimal solutions according to expressions (12)–(20). Obtain the maximal interval solutions by minimal solutions and by maximum solution.
10. End.

The algorithm for solving $\odot$-FLSE is provided by Theorem 3 and its Corollaries, algebraic-logic properties of the terms as described in Sect. 4.3 and expansion along M. Based on simplifications, help and dominance matrices, as well as the matrix M, the algorithm has smaller computational complexity in comparison with the algorithms proposed in Guu and Wu (2002), Loetamonphong and Fang (1999, 2001), Loetamonphong et al. (2002), Markovskii (2005).

Theorem 4 *For $A \odot X = B$:*

- *(i) It is solvable in polynomial time whether the system is consistent or not.*
- *(ii) If the system is consistent the maximum solution, the minimal solutions and the maximal interval solutions are computable.*
- *(iii) For inconsistent system we can determine the equations that can not be satisfied by $A^t \diamond B$.*

By this theoretical background in MATLAB workspace we develop software for computing the complete solution set or for establishing inconsistency of the system $A \odot X = B$. Software is available free for educational and research purposes only, upon request to the authors.

5 Solving fuzzy relational equations

Let $R \in F(X \times Y)$, $Q \in F(Y \times Z)$ and $T \in F(X \times Z)$ be fuzzy relations on $\mathbb{I}_\odot$. The equation

$$R \odot Q = T, \tag{22}$$

where one of the fuzzy relations on the left is unknown, and the other relation on the left and the relation T are given, is called $\odot$-*fuzzy relational equation* ($\odot$-FRE). If the relations are over finite support, we can assign to any relation a matrix and thus (22) can be interpreted as fuzzy matrix equation on $\mathbb{I}\odot$.

Let the fuzzy relations R, Q and T be over finite support, and the composition $R \odot Q = T$ makes sense.

Theorem 5 *$R \odot Q = T$ is solvable for Q if R and T are given.*

Proof Follows from Theorem 3 and the Algorithm. In order to solve (22), we represent relations by the corresponding matrices:

$$R_{m\times n} \odot Q_{n\times p} = T_{m\times p}. \tag{23}$$

We split $Q_{n\times p}$ and $T_{m\times p}$ in (23) by columns and apply the equivalence:

$$R_{m\times n} \odot Q_{n\times p} = T_{m\times p} \Leftrightarrow \left| \begin{array}{l} R_{m\times n} \odot Q^{(1)} = T^{(1)} \\ \quad \cdots \qquad\qquad \cdots \\ R_{m\times n} \odot Q^{(p)} = T^{(p)} \end{array} \right. . \tag{24}$$

$Q^{(j)}$, $j = 1,...,p$, denotes the j-th column of Q and $T^{(j)}, j = 1,...,p$ – j-th column of T. Now, instead of (23) we solve p fuzzy linear systems, having the same matrix $R_{m\times n}$, with the Algorithm.

Solving fuzzy matrix equation $A \odot B = C$ for A or FRE $R \odot Q = T$ for R, requires first to transpose the equation.

Hence, applying the Algorithm for FLSE, we can solve: fuzzy matrix equation of the form $A \odot B = C$ for B (or for A); FRE of the form $R \odot Q = T$ for Q (or for R).

The algorithm is realized in MATLAB and gives the complete solution set when the Eq. (22) has solution, otherwise it establishes inconsistency.

6 Software description and some experimental results

We develop software, based on this method and algorithm in MATLAB workspace.

The algebraic-logical approach and matrix based approach are programmed as alternative programming techniques. The algebraic-logical approach uses the MATLAB library published in Peeva and Kyosev (2004) and available under General Public License for construction and operation with terms. This approach has advantages – it operates only with essential (non-zero) elements of the matrix, not wasting computational time for checking duplicated or non-minimal solutions (see absorptions in Sect. 4.3.1), so directly whole branches of redundant solutions are cut.

The matrix approach is based on the operation with and within matrices, without building new structures. Applying dominance rules before each new sub-step can speed up the calculation process, and thus the method seems to be preferable for larger systems.

Theoretically both methods are equivalent. Which one is faster depends on the properties of the instant. A comparison between computational times at this moment is not suitable, because the MATLAB Environment has a set of pre-compiled functions for matrix operations, which are very fast. In contrast, our MATLAB Library with implementation of the algebraic-logical approach is currently used as not compiled set of functions, which are working slower.

We include some prints from MATLAB session. For the above Example they confirm the same results as these in Markovskii (2005):

```
Help matrix:
   0.00   0.50   0.00   0.00   0.25   0.00
   0.00   0.50   1.00   0.00   0.00   0.00
   0.30   0.50   1.00   0.50   0.00   0.00
   0.00   0.00   0.00   0.50   0.25   0.00
   0.00   0.00   1.00   0.00   0.25   0.00
   0.30   0.50   0.00   0.00   0.25   0.00

Greatest Solution transposed =
   0.30   0.50   1.00   0.50   0.25   1.00

Dominance matrix initial
   0.00   1.00   0.00   0.00   1.00   0.00
   0.00   1.00   1.00   0.00   0.00   0.00
   1.00   1.00   1.00   1.00   0.00   0.00
   0.00   0.00   0.00   1.00   1.00   0.00
   0.00   0.00   1.00   0.00   1.00   0.00
   1.00   1.00   0.00   0.00   1.00   0.00

Significant rows from the dominance
matrix
   0.00   1.00   0.00   0.00   1.00   0.00
   0.00   1.00   1.00   0.00   0.00   0.00
   0.00   0.00   0.00   1.00   1.00   0.00
   0.00   0.00   1.00   0.00   1.00   0.00
   1.00   1.00   1.00   1.00   0.00   0.00

Minimal Solutions - transposed
        0   0.50   1.00   0.50      0      0
        0   0.50      0      0   0.25      0
        0      0   1.00      0   0.25      0

Short solution summary: s =
        exists: 1
           low: [3x6 double]
      sol_numb: 3
           Xgr: [0.3000 0.5000 1 0.5000
                 0.2500 1]
```

```
 Ind: [5x1 double]
 hlp: [6x6 double]
   A: [6x6 double]
   B: [6x1 double]
   d: [5x6 double]
```

The presented structure consists information about the input matrix and data from different solution steps. More detailed solution summary is also available, where also solution times for the different routines are saved.

7 Conclusions

In this paper we develop exact method and universal algorithm for solving max-product fuzzy linear systems of equations and max-product fuzzy relational equations.

Various applications of inverse problem for max – product composition in finite fuzzy machines, as inference engine, for fuzzy modeling, for some optimization problems, are possible. They will be subject of next publications.

Acknowledgements The authors are grateful to the referees for their valuable suggestions.

References

Aho A, Hopcroft J, Ullman J (1976) The design and analysis of computer algorithms. Addison-Wesley, London

Cheng L, Wang P (2002) Fuzzy relational equations (I): the general and specialized solving algorithms. Soft Comput 6:428–435

Bourke M, Fisher D (1998) Solution algorithms for fuzzy relational equations with max-product composition. Fuzzy Sets Syst 94:61–69

De Baets B (2000) Analytical solution methods for fuzzy relational equations, in the series: In: Dubois D, Prade H (eds) The handbooks of fuzzy sets series, vol. 1, Kluwer, Dordrecht, pp 291–340

Di Nola A, Lettieri A (1989) Relation equations in residuated lattices. Rendiconti del Circolo Matematico di Palermo, s. II, XXXVIII, pp 246–256

Di Nola A, Pedrycz W, Sessa S, Sanchez E (1989) Fuzzy relation equations and their application to knowledge engineering. Kluwer Dordrecht

Garey M, Johnson D (1979) Computers and intractability. A guide to the theory of NP-completeness. Freeman San Francisco

Grätzer G (1978) General lattice theory. Akademie-Verlag, Berlin

Guu SM, Wu Y-K (2002) Minimizing a linear objective function with fuzzy relation equation constraints. Fuzzy Optimiz Decision Making 4(1):347–360

Higashi M, Klir G (1984) Resolution of finite fuzzy relation equations, Fuzzy Sets Syst 13(1):65–82

Klir G, Clair U, Yuan B (1997) Fuzzy set theory foundations and applications. Prentice Hall, Englewood Cliffs

Loetamonphong J, Fang S.-C (1999) An efficient solution procedure for fuzzy relational equations with max-product composition. IEEE Trans Fuzzy Syst 7(4):441–445

Loetamonphong J, Fang S.-C (2001) Optimization of fuzzy relation equations with max-product composition. Fuzzy Sets Syst 118(3):509–517

Loetamonphong J, Fang S.-C, Young R (2002) Multi-objective optimization problems with fuzzy relation equation consrtaints. Fuzzy Sets Syst 127(3):141–164

MacLane S, Birkhoff G (1979) Algebra. Macmillan, New York

Markovskii A (2005) On the relation between equations with max-product composition and the covering problem. Fuzzy Sets Syst 153:261–273

Miyakoshi M, Shimbo M (1986) Lower solutions of systems of fuzzy equations. Fuzzy Sets Syst 19:37–46

Pappis C, Sugeno M (1985) Fuzzy relational equations and the inverse problem. Fuzzy Sets Syst 15:79–90

Peeva K (1992) Fuzzy linear systems. Fuzzy Sets and Systems 49:339-355

Peeva K (2002) Fuzzy linear systems – Theory and applications in Artificial Intelligence areas. DSc Thesis, Sofia, pp 239 (in Bulgarian)

Peeva K (2006) Universal algorithm for solving fuzzy relational equations. Italian J Pure Appl Math 9:9–20

Peeva K, Kyosev Y (2004) Fuzzy relational calculus-theory, applications and software (with CD-ROM). In the series Advances in Fuzzy Systems - Applications and Theory, vol 22, World Scientific Publishing Company, Software downloadable from http://www.mathworks.com/matlabcentral/fileexchange/loadFile.do?objectId=6214

Sanchez E (1976) Resolution of composite fuzzy relation equations. Inform Control 30:38–48