ORIGINAL PAPER

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Regular grammars with truth values in lattice-ordered monoid and their languages

Abstract In this study, we introduce the concept of lattice-valued regular grammars. Such grammars have become a necessary tool for the analysis of fuzzy finite automata. The relationship between lattice-valued finite automata (LA) and lattice-valued regular grammars (LRG) are discussed and we get the following results, for a given LRG, there exists an LA such that they accept the same languages, and vice versa. We also show the equivalence between deterministic lattice-valued regular grammars and deterministic lattice-valued finite automata.

Keywords Fuzzy finite automata · Fuzzy regular grammar · Fuzzy regular language · Lattice-ordered monoid

1 Introduction

The concept of fuzzy automata was introduced by Wee in 1967. Thereafter, there were considerable authors, such as Santos (1975, 1976, 1977), Lee and Zadeh (1969) etc., have contributed in this field. For detail, we refer to Kandel and Lee (1980), and Wechler (1978) and especially the recent ones by Mordeson and Malik (2000, 2002). Also, fuzzy automata have many important applications such as in learning systems, the model of computing with words, pattern recognition and data base theory (Mordeson and Malik 2002; Ying 2002).

Recently, Qiu (2001) and Asveld (2003) proposed to study fuzzy automata based on residuated logic and the

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proposed method provided a tool to study fuzzy automata in the frame of many-valued logic. However, these concepts and conclusions are mostly drawn from fuzzifying preliminarily the classical case, and therefore have poor level structure. To state more precise, automata are the mathematical models to recognize formal languages in the theory of classical computation, and the former proposed fuzzy automata with truth values in unit interval [0, 1] with max–min composition, we call them classical fuzzy automata in this paper. To overcome this kind of problem, we study the automata theory with truth values in the more general structures: lattice-ordered monoids.

It is well known in the classical automata theory (Hopcroft and Ullman 1979) that, the following four approaches to represent a language (regular language, to be precise) $\mathscr{L}$ are equivalent:

(i) $\mathscr{L}$ is recognized by some deterministic finite state automaton.
(ii) $\mathscr{L}$ is recognized by some nondeterministic finite state automaton.
(iii) $\mathscr{L}$ is described by some regular expression.
(iv) $\mathscr{L}$ is generated by some regular grammar.

The same results hold for fuzzy regular languages with truth values in [0, 1] and with max–min composition (Bělohlávek 2002; Kandel and Lee 1980; Malik and Mordeson 2000, 2002; Močkoř 2002; Santos 1975, 1976, 1977; Shen 1996; Thomason and Marinos 1974). For generalized lattice-valued languages, it has been shown in Li(2003) that (i) and (ii) are not equivalent for some truth-value lattice-ordered monoids. The nondeterministic lattice-valued finite automaton (LA) is more powerful than deterministic lattice-valued finite automata (DLA) when it comes to the recognitions of fuzzy languages. An important problem arises as to the description of LA or DLA by regular expressions and regular grammars. Since the problem of regular expressions has been resolved in Li and Pedrycz (2004), this issue is of interest in the study of the relationship between LA and LRG.

2 Some basic concepts

We first give some basic concepts we use in this paper.

Definition 2.1. Given a lattice L, we use $\vee$, $\wedge$ to represent the supremum operation and infimum operation on L, respectively, with 0, 1 being the least and the largest elements. Assume that there is a binary operation $\bullet$ (we call it multiplication) on L such that $(L, \bullet, e)$ is a monoid with identity $e \in L$. We call L a lattice-ordered monoid (some modification of the the notion of lattice-ordered monoid in Birkhoff (1940) if it satisfies the following two conditions:

(i) $\forall\ a \in L,\ a \bullet 0 = 0 \bullet a = 0,$
(ii) $\forall\ a, b, c \in L,\ a \bullet (b \vee c) = (a \bullet b) \vee (a \bullet c),$ and $(b \vee c) \bullet a = (b \bullet a) \vee (c \bullet a).$

For a lattice-ordered monoid, since we only concerned with the multiplication $\bullet$ and finite supremum operation $\vee$, in what follows, a lattice-ordered monoid will be denoted by $(L, \bullet, \vee)$.

We give some examples of lattice-ordered monoids.

Example 2.1. (1) Let $(L, \wedge, \vee)$ be a distributive lattice (Balbes and Dwinger 1974), and *let* $\bullet = \wedge$, then L is a lattice-ordered monoid, and the identity of multiplication is 1.

2. Let $(L, \bullet, \vee)$ be a lattice-ordered monoid, the identity is e. We use $L(n)$ to denote all $n \times n$ matrices with values in L. The multiplication, denoted as $\circ$, is defined as $sup-\bullet$ composition; and $\vee$ is the pointwise-$\vee$. That is, for two $n \times n$ matrices, $A = (a_{ij})$ and $B = (b_{ij})$, with values in L, let $A \circ B = C = (c_{ij})$, then $c_{ij} = \bigvee_{k=1}^{n} (a_{ik} \bullet b_{kj})$, and let $A \vee B = D = (d_{ij})$, then $d_{ij} = a_{ij} \vee b_{ij}$. Then $(L(n), \circ, \vee)$ is also a lattice-ordered monoid, the identity is the diagonal-matrix $E = \text{diag}(e, e, ..., e)$ with e as the diagonal element. In general, the multiplication on $L(n)$ is not commutative, even if the multiplication on L is commutative.

3. Let $\bullet$ be any uninorm (Li and Shi 2000; Yager 1994) on $[0, 1]$, if $0 \bullet 1 = 0$, then $([0, 1], \bullet, \vee)$ is a commutative lattice-ordered monoid. In particular, if $\bullet$ is a t-norm on $[0, 1]$, then $([0,1], \bullet, \vee)$ is a commutative lattice-ordered monoid with identity $e = 1$.

4. Quantales are complete lattice-ordered monoid, where L is a quantale if L is a complete lattice, and it satisfies the following infinite distributive laws (Li et al. 2002; Rosenthal 1990):

$$a \bullet \left(\bigvee_t b_t \right) = \bigvee_t (a \bullet b_t),\ \text{and}\ \left(\bigvee_t b_t \right) \bullet a = \bigvee_t (b_t \bullet a).$$

In the following, we always assume that L is a lattice-ordered monoid.

Definition 2.2. An LA is a five tuple,

$$M = (X, U, \delta, \sigma_0, \sigma_1),$$

where X, U are finite nonempty sets, $\delta: X \times U \to F(X)$, and $\sigma_0, \sigma_1 : X \to L$ are L-fuzzy sets of X. The elements of X are called states, and the elements of U are called (input) symbols. δ is called a fuzzy transition function and σ_0, σ_1 are called fuzzy initial state and fuzzy final state, respectively.

We can regard $\delta(x, u)(y)$ as the grade of membership that the next state of M is y, given that the present state of M is x and the input symbol u is implied. For simplification we denote $\delta(x, u)(y)$ as $\delta(x, u, y)$.

Let U^* denote the set of all words of finite letter over U and let Λ denotes the empty word. Then, U^* is the free monoid generated by U with the concatenation operation. For $\theta \in U^*$, $|\theta|$ denotes the length of θ.

By means of Li (2003), we can extend δ on $X \times U^*$, denoted as $\delta^*: X \times U^* \to F(X)$, in the following forms:

(i) $\forall\ y \in X$, if $y = x$, then $\delta^*(x, \Lambda, y) = e$, otherwise $\delta^*(x, \Lambda, y) = 0$;
(ii) $\forall\ \theta \in U^*,\ u \in U,\ \delta^*(x, \theta u, y) = \bigvee_{z \in X} [\delta^*(x, \theta, z) \bullet \delta(z, u, y)].$

For the extension δ^*, since L is a lattice-ordered monoid, it satisfies the following equation for any $\theta \in U^*$ (Li 2003), if $\theta = \theta_1 \theta_2$, then

$$\delta^*(x, \theta_1\theta_2, y) = \bigcup_{z \in X} [\delta^*(x, \theta_1, z) \bullet \delta^*(z, \theta_2, y)].$$

Definition 2.3. Suppose that $M = (X, U, \delta, \sigma_0, \sigma_1)$ is an LA. Then the L-valued language $f_M \in F(U^*)$ accepted by M or recognized by M is defined as follows,

$$f_M(\theta) = \sigma_0 \circ \delta_\theta \circ \sigma_1 = \bigvee_{x,y \in X} [\sigma_0(x) \bullet \delta^*(x, \theta, y) \bullet \sigma_1(y)].$$

The element $f \in F(U^*)$ is called an L-language on U, an L-language which is accepted by an LA is called an LA-language.

For two LAs M_1 and M_2, we say that they are *equivalent* if they accept the same LA-language, that is, $f_{M_1} = f_{M_2}$.

As to the classical automata theory, we have the notions of deterministic finite automata and nondeterministic finite automata. The notion of an LA is a generalization of the notion of nondeterministic automaton. The LA is nondeterministic in nature , in the following we will present a deterministic counterpart of the notion of an LA.

Definition 2.4. A DLA is a five tuple,

$$M = (X, U, \delta, x_0, \sigma_1),$$

where X, U and σ_1 are the same defined as LA , $x_0 \in X$ is the initial state, $\delta: X \times U \to X$ is called a transition function.

The extension of δ onto U^* is the same as that of classical case. Then, the L-language $f_M \in F(U^*)$ accepted by a DLA is defined as,

$$\forall \theta \in U^*, \quad f_M(\theta) = \sigma_1(\delta^*(x_0, \theta)).$$

However, unlike the classical case, Li (2003) has shown that in general LA and DLA are not equally powerful.

In the classical automata theory we know that regular grammars and finite automata are essentially the same in that either is a specification of a regular language. Since then, it is necessary for us to study the corresponding (distinctive) fuzzy grammars for LA and DLA.

Definition 2.5. (1) A lattice-valued grammar is a quadruple $G = (N, T, P, S)$, where

(i) N is a finite set whose elements are called nonterminal symbols.

(ii) T is a finite set whose elements are called terminal symbols and $T \cap N = \phi$.

(iii) $S \in N$ is called the starting symbol and S does not occur on the right-hand side of any fuzzy productions.

(iv) P is a finite collection of fuzzy productions over $T \cup N$ and

$$P = \left\{ u \xrightarrow{\rho} v \mid u \in (N \cup T)^* N (N \cup T)^*, v \in (N \cup T)^* \right\}, \text{where}$$
$\rho \in L - \{0\}$.

We may interpret $\rho(u, v)$ as the grade of membership that u will be replaced by v.

If $u \xrightarrow{\rho} v$ is a production and $\alpha, \beta \in (N \cup T)^*$, then $\alpha v \beta$ is said to be directly derivable from $\alpha u \beta$. In this case, we write

$$\alpha u \beta \xRightarrow{\rho} \alpha v \beta.$$

If $u_i \in (N \cup T)^*$ for $i = 1, 2, ..., p$ and u_{i+1} is directly derivable from u_i for $i = 1, 2, ..., p-1$, we say that u_p is derivable from u_1 and written as,

$$u_1 \xRightarrow{\rho}{}^* u_p.$$

We call

$$u_1 \xRightarrow{\rho_1} u_2 \xRightarrow{\rho_2} \ldots \xRightarrow{\rho_{p-1}} u_p$$

the derivation chain of u_p (from u_1), where $\rho = \rho_1 \bullet \rho_2 \bullet ... \bullet \rho_{p-1}$.

(2) The L-language $f_G \in F(T^*)$ generated by an L-valued grammar is defined as follows:

$$f_G(\theta) = \bigvee \left\{ \rho : S \xRightarrow{\rho}{}^* \theta \right\}$$

L-valued grammars are classified according to the form of the production rules used (Chomsky hierarchy). These grammars are sometimes described as follows:

Definition 2.6. Let $G = (N, T, P, S)$ be an L-valued grammar.

1. G is unrestricted (of type 0), if there are no restrictions on the form of the production rules. Accordingly, f_G is called L-valued type 0 language.

2. G is context-sensitive if for every $u \xrightarrow{\rho} v \in P$, $u, v \in (T \cup N)^*$, $\rho(u, v) > 0$ implies $|u| \leq |v|$. Accordingly, f_G is called L-valued context-sensitive language.

3. G is context-free if for every $u \xrightarrow{\rho} v \in P$, $u, v \in (T \cup N)^*$, $\rho(u, v) > 0$ implies $|u| \leq |v|$ and $u \in N$. Accordingly, f_G is called L-valued context-free language.

4. G is weak regular if for every $u \xrightarrow{\rho} v \in P$, $u, v \in (T \cup N)^*$, $\rho(u, v) > 0$ implies $u \in N$ and $v \in T^+ B$, $B \in N \cup \{\Lambda\}$ or $u = S$, $v = \Lambda$. Accordingly, f_G is called L-valued weak regular language.

5. G is regular if for every $u \xrightarrow{\rho} v \in P$, $u, v \in (T \cup N)^*$, $\rho(u, v) > 0$ implies $u \in N$ and $v \in TB$, $B \in N \cup \{\Lambda\}$ or $u = S$, $v = \Lambda$. Accordingly, f_G is called L-valued regular language.

Two L-valued grammars G_1 and G_2 are said to be *equivalent* if they generate the same language, that is, $f_{G_1} = f_{G_2}$.

Theorem 2.1. *L-valued weak regular grammars (LWRG, for short) are equivalent to L-valued regular grammars (LRG, for short). Where the word "equivalent" means that they generate the same classes of L-languages.*

Proof. From the definition, we can easily know that every L-language generated by an LRG can be generated by an LWRG which is the same as the former LRG. In the following we only need to show that every L-language generated by an LWRG can also generated by an LRG.

Let $G_1 = (N_1, T, P_1, S)$ be an LWRG, the language generated by G_1 is denoted as f_{G_1}. We can define an LRG $G = (N, T, P, S)$, such that $f_G = f_{G_1}$.

(i) For each production

$$x \rightarrow a_1 a_2 \ldots a_m y \in P_1, \tag{1}$$

where $a_i \in T_1$ for $i = 1, 2, ..., m$ and $x, y \in N_1$.

When $m = 1$, we have $a_1 \in T$, $x, y \in N_1 \subseteq N$ and $x \rightarrow a_1 y \in P$ as desired.

When $m \geq 2$, $x, y \in N_1 \subseteq N$, we can define new nonterminal symbols $\xi_1, \xi_2, ..., \xi_{m-1}$ in N and denote the set of these symbols as N_0, then $N = N_1 \cup N_0$.

Let ρ_G and ρ_{G_1} denote the grade of membership of productions in G and G_1, respectively. Now we can mimic the production (Eq. 1) by means of productions $x \rightarrow a_1 \xi_1$, $\xi_1 \rightarrow a_2 \xi_2$, ..., $\xi_{m-1} \rightarrow a_m y \in P$ with

$$\rho_G(x \rightarrow a_1 \xi_1) = \rho_G(\xi_1 \rightarrow a_2 \xi_2) = \ldots$$
$$= \rho_G(\xi_{m-2} \rightarrow a_{m-1} \xi_{m-1}) = e;$$
$$\rho_G(\xi_{m-1} \rightarrow a_m y) = \rho_{G_1}(x \rightarrow a_1 a_2 \ldots a_m y).$$

Then $\rho_G(x \rightarrow a_1 a_2 \ldots a_m y) = \rho_G(x \rightarrow a_1 \xi_1) \bullet \rho_G(\xi_1 \rightarrow a_2 \xi_2) \bullet \ldots \bullet \rho_G(\xi_{m-2} \rightarrow a_{m-1} \xi_{m-1}) \bullet \rho_G(\xi_{m-1} \rightarrow a_m y) = \rho_{G_1}(x \rightarrow a_1 a_2 \ldots a_m y)$.

(ii) For each production

$$x \rightarrow b_1 b_2 \ldots b_n \in P_1 \tag{2}$$

where $b_i \in T_1$ for $i = 1, 2, ..., n$ and $x \in N_1$. When $n = 1$, we have $b_1 \in T$ and $X \rightarrow b_1 \in P$ as desired. When $n \geq 2$,

$x \in N_1 \subseteq N$. We can define new nonterminal symbols η_1, η_2, ..., $\eta_{n-1} \in N$ and denote the set of these symbols as N_0, then $N = N_1 \cup N_0$. Now we can mimic the production (Eq. 2) by means of productions

$$x \to b_1\eta_1, \eta_1 \to b_2\eta_2, \ldots \eta_{n-1} \to b_n \in P$$

with $\rho_G(x \to b_1\eta_1) = \rho_G(\eta_1 \to b_2\eta_2) = \ldots = \rho_G(\eta_{n-2} \to b_{n-1}\eta_{n-1}) = e$; $\rho_G(\eta_{n-1} \to b_n) = \rho_{G_1}(x \to b_1b_2\ldots b_n)$. Then $\rho_G(x \to b_1b_2\ldots b_n) = \rho_G(x \to b_1\eta_1) \bullet \rho_G(\eta_1 \to b_2\eta_2) \bullet \ldots \bullet \rho_G(\eta_{n-2} \to b_{n-1}\eta_{n-1}) \bullet \rho_G (\eta_{n-1} \to b_n) = \rho_{G_1}(x \to b_1b_2\ldots b_n)$.

Clearly G thus specified is an L-valued regular grammar. Thus, from Eqs. 1 and 2 we conclude that $f_{G_1} \subseteq f_G$.

To prove the reverse inclusion, suppose that for some $\theta \in T^*$, there is a derivation $S \overset{\rho}{\Rightarrow} \theta$ in G. Let $G_2 = (N, T, P \cup P_1, S)$, where G_2 has both all the productions of G_1 and G , then certainly there is a derivation

$$S \overset{\rho}{\Rightarrow}{}^* \theta \tag{3}$$

in G_2 with $\rho_{G_2}(S \longrightarrow \theta) = \rho_G(S \longrightarrow \theta)$.

We shall show that there is a derivation of θ in G_1 by induction on the number of symbols from $N \setminus N_1$ appearing in the derivation (3). If no such symbols appear then Eq. 3 is already a derivation in G_1. Otherwise the first appearance of a symbol of $N \setminus N_1$ is based either on a production

$$x \to a_1\xi_1 \quad \text{with } \rho_{G_2}(x \to a_1\xi_1) = \rho_G(x \to a_1\xi_1),$$

where $x \to a_1a_2\ldots a_my$ is a production in G_1, or on a production

$$x \to b_1\eta_1 \quad \text{with } \rho_{G_2}(x \to b_1\eta_1) = \rho_G(x \to b_1\eta_1),$$

where $x \to b_1b_2\ldots b_n$ is a production in G_1. Consider the first case, for any of the symbols of $N \setminus N_1$, the only way in which ξ_i can subsequently appear must involve changes from ξ_1 to $a_2\xi_2$, ..., ξ_{m-1} to a_my. Then the sequence of transitions

$$x \to a_1\xi_1, \xi_1 \to a_2\xi_2, \ldots, \xi_{m-1} \to a_my$$

with $\rho_{G_2}(x \to a_1\xi_1)$
$= \rho_G(x \to a_1\xi_1), \rho_{G_2}(\xi_1 \to a_2\xi_2)$
$= \rho_G(\xi_1 \to a_2\xi_2), \ldots, \rho_{G_2}(\xi_{m-1} \to a_my)$
$= \rho_G(\xi_{m-1} \to a_my)$

can be replaced by a single transition $x \to a_1a_2\ldots a_my$ in G_2 with

$$\rho_{G_2}(x \to a_1a_2\ldots a_my)$$
$$= \rho_{G_2}(x \to a_1\xi_1) \bullet \rho_{G_2}(\xi_1 \to a_2\xi_2) \bullet \ldots$$
$$\bullet \rho_{G_2}(\xi_{m-1} \to a_my) = \rho_G(x \to a_1\xi_1)$$
$$\bullet \rho_G(\xi_1 \to a_2\xi_2) \bullet \ldots$$
$$\bullet \rho_G(\xi_{m-1} \to a_my) = \rho_G(x \to a_1a_2\ldots a_my)$$
$$= \rho_{G_1}(x \to a_1a_2\ldots a_my).$$

Similarly, in the second case the derivation must involve subsequently changes from η_1 to $b_2\eta_2$, ..., η_{n-1} to b_n, and these n transitions can be replaced by a single transition in G_2 from x to $b_1b_2\ldots b_n$ with

$$\rho_{G_2}(x \to b_1b_2\ldots b_n)$$
$$= \rho_{G_2}(x \to b_1\eta_1) \bullet \rho_{G_2}(\eta_1 \to b_2\eta_2) \bullet \ldots$$
$$\bullet \rho_{G_2}(\eta_{n-1} \to b_n)$$
$$= \rho_{G_2}(x \to b_1\eta_1) \bullet \rho_G(\eta_1 \to b_2\eta_2) \bullet \ldots$$
$$\bullet \rho_G(\eta_{n-1} \to b_n) = \rho_G(x \to b_1b_2\ldots b_n)$$
$$= \rho_{G_1}(x \to b_1b_2\ldots b_n).$$

In both cases, the derivation (3) is replaced by one with fewer occurrences of symbols from $N \setminus N_1$ with $\rho_{G_2}(\theta) = \rho_G(\theta)$, then it follows that $f_{G_2} \subset f_{G_1}$. Hence certainly $f_G \subset f_{G_1}$.

Thereby, $f_G = f_{G_1}$, say, LWRG and LRG are *equivalent* as far as the languages generated by them are concerned. $\square$

Example 2.2. Suppose $G_1 = (N_1, T, P_1, S)$ is an LWRG, where $N_1 = \{S, A, B\}$, $T = \{a, b, c\}$,

$$P_1 = \left\{ S \overset{\rho_1}{\to} abA, A \overset{\rho_2}{\to} bbB, B \overset{\rho_3}{\to} cA, A \overset{\rho_4}{\to} ac, B \overset{\rho_5}{\to} bc \right\}.$$

Construct an LRG $G = (N, T, P, S)$, where

$$P = \left\{ S \overset{e}{\to} a\xi_1, \xi_1 \overset{\rho_1}{\to} bA, A \overset{e}{\to} b\xi_2, \xi_2 \overset{\rho_2}{\to} bB, B \overset{\rho_3}{\to} cA, A \overset{e}{\to} a\eta_1, \right.$$
$$\left. \eta_1 \overset{\rho_4}{\to} c, B \overset{e}{\to} b\eta_2, \eta_2 \overset{\rho_5}{\to} c \right\} \text{ and } N = N_1 \cup \{\xi_1, \xi_2, \eta_1, \eta_2\}.$$

Clearly, for $\theta = ab^3cac$ the derivation of θ in G_1 is as follows:

$$S \overset{\rho_1}{\to} abA \overset{\rho_2}{\to} abbbB \overset{\rho_3}{\to} ab^3cA \overset{\rho_4}{\to} ab^3cac,$$

so $f_{G_1}(\theta) = \rho_1 \bullet \rho_2 \bullet \rho_3 \bullet \rho_4$.

The derivation of θ in G is as follows:

$$S \overset{e}{\to} a\xi_1 \overset{\rho_1}{\to} abA \overset{e}{\to} ab^2\xi_2 \overset{\rho_2}{\to} ab^3B \overset{\rho_3}{\to}$$
$$ab^3cA \overset{e}{\to} ab^3ca\eta_1 \overset{\rho_4}{\to} ab^3cac,$$

then $f_G(\theta) = e \bullet \rho_1 \bullet e \bullet \rho_2 \bullet \rho_3 \bullet e \bullet \rho_4 = \rho_1 \bullet \rho_2 \bullet \rho_3 \bullet \rho_4 = f_{G_1}(\theta)$.

3 Lattice-valued grammars and LA

In Theorems 3.1 and 3.2, we shall show that L-valued regular languages coincide with LA-languages.

Theorem 3.1 *Let $G = (N, T, P, S)$ be an L-valued regular grammar, then there exists an LA M such that $f_M = f_G$.*

Proof. Constructing an LA $M = (X, U, \delta, \sigma_0, \sigma_1)$, where $U = T$, $X = N \cup \{Z\}$, $\sigma_0 = e/S$, $\sigma_1 = (e/Z) + (f_G(\Lambda)/S)$ and

$$\delta(x,u,y) = \begin{cases} \rho(x \to uy) & x,y \in N \\ \rho(x \to u) & x \in N, y = Z \\ 0 & \text{otherwise} \end{cases}$$

For any $\sigma, \tau \in (T \cup N)^*$, $\rho(\sigma \to \tau) = \vee\{\rho : \sigma \overset{\rho}{\underset{*}{\Rightarrow}} \tau\}$. We show $f_G = f_M$ below.

Let $\theta \in T^*$, if $\theta = \Lambda$, then $f_M(\theta) = f_G(\Lambda)$ as desired.

When $\theta \neq \Lambda$, suppose $\theta = u_1 u_2 ... u_n$, where $u_i \in T$ for $i = 1, 2, ..., n$. If $S \overset{\rho}{\underset{*}{\Rightarrow}} \theta$ there must exist some derivation of θ with the form

$$S \overset{\rho_1}{\Rightarrow} u_1 A_1 \overset{\rho_2}{\Rightarrow} u_1 u_2 A_2 \Rightarrow \ldots \overset{\rho_{n-1}}{\Rightarrow} u_1 u_2 \ldots u_{n-1} A_{n-1} \overset{\rho_n}{\Rightarrow}$$

$$u_1 u_2 \ldots u_{n-1} u_n,$$

where $A_{i-1} \overset{\rho_i}{\to} u_i A_i \in P$ for $i = 1, 2, ..., n$ and $A_0 = S$, $A_n = u_n$ with $\rho = \rho_1 \bullet \rho_2 \bullet ... \bullet \rho_n$.

Corresponding to the given LA, by the definition of δ, we get that

$$\delta^*(S, \theta, Z) \geq \delta(S, u_1, A_1) \bullet \delta(A_1, u_2, A_2) \bullet \ldots$$
$$\bullet \delta(A_{n-1}, u_n, Z)$$
$$= \rho_1 \bullet \rho_2 \bullet \ldots \bullet \rho_n = \rho,$$

then $f_M(\theta) = \sigma_0 \circ \delta_\theta \circ \sigma_1 = \sigma_0(S) \bullet \delta^*(S, \theta, Z) \bullet \sigma_1(Z) = \delta^*(S, \theta, Z) \geq \rho$.

Thus, we have shown that $f_G(\theta) \leq f_M(\theta)$ for any $\theta \in U^*$, that is, $f_G \leq f_M$.

Conversely, if $f_M(\theta) = \delta^*(S, \theta, Z) \geq \delta(S, u_1, A_1) \bullet \delta(A_1, u_2, A_2) \bullet ... \bullet \delta(A_{n-1}, u_n, Z)$. For any $A_i \in X$, let $\delta(S, u_1, A_1) = \rho_1, ..., \delta(A_{n-1}, u_n, A_n) = \rho_n$, and let $\rho = \rho_1 \bullet \rho_2 \bullet ... \bullet \rho_n$, then there exists corresponding derivation $S \overset{\rho}{\underset{*}{\Rightarrow}} \theta$. Then it shows that if $\rho \leq f_M(\theta)$, then $\rho \leq f_G(\theta)$, that is, $f_M \leq f_G$.

Therefore $f_M = f_G$, that is, L-valued languages can be recognized by LA. $\square$

We give an example to illustrate the proof of the above theorem.

Example 3.1. Let $L = [0, 1]$, $\bullet$ is the multiplication of real numbers, $G = (N, T, P, S)$ is an LRG, where $N = \{S, A, B\}$, $T = \{a, b\}$. The productions in P are as follows:

$$P: S \overset{0.5}{\to} aA, S \overset{0.6}{\to} aB, A \overset{0.7}{\to} aA, B \overset{0.4}{\to} aB, A \overset{0.3}{\to} b, B \overset{0.8}{\to} b.$$

Construct an LA, $M = (X, U, \delta, \sigma_0, \sigma_1)$, where

$$U = \{a, b\}, X = \{S, A, B, Z\}, \sigma_0 = \frac{e}{S}, \sigma_1 = \frac{e}{Z} + \frac{f_G(\Lambda)}{S},$$

the transition function with the form

$$\delta(S, a, A) = 0.5 \quad \delta(S, a, B) = 0.6 \quad \delta(A, a, A) = 0.7,$$
$$\delta(B, a, B) = 0.4 \quad \delta(A, b, Z) = 0.3 \quad \delta(B, b, Z) = 0.8.$$

For $\theta = a^3 b$, the derivation of θ are as follows

(i) $S \overset{0.5}{\to} aA \overset{0.7}{\to} aaA \overset{0.7}{\to} aaaA \overset{0.3}{\to} aaab.$
(ii) $S \overset{0.6}{\to} aB \overset{0.4}{\to} aaB \overset{0.4}{\to} aaaB \overset{0.8}{\to} aaab.$

So $f_G(a^3 b) = (0.5 \times 0.7 \times 0.7 \times 0.3) \vee (0.6 \times 0.4 \times 0.4 \times 0.8) = 0.0768$, $f_M(a^3 b) = \delta^*(S, a^3 b, Z) = [\delta(S, a, A) \bullet \delta(A, a, A) \bullet \delta(A, a, A) \bullet \delta(A, b, Z)] \vee [\delta(S, a, B) \bullet \delta(B, a, B) \bullet \delta(B, a, B) \bullet \delta(B, b, Z)] = 0.0768$. Thereby $f_M(\theta) = f_G(\theta)$.

Theorem 3.2 Let $M = (U, X, \delta, \sigma_0, \sigma_1)$ be an LA, there exists an LRG such that $f_G = f_M$.

Proof. We define an LRG $G = (N, T, P, S)$ as follows $T = U$, $N = X \cup \{S\}$. The productions in P are defined as follows:

(i) $x \overset{\rho_1}{\to} uy \in P$ for each $\delta(x, u, y) \neq 0$, where $\rho_1 = \delta(x, u, y)$.
(ii) If $\delta(x, u, y) \neq 0$ and $\sigma_1(y) \neq 0$, then $x \overset{\rho_2}{\to} u \in P$, where $\rho_2 = \bigvee_{y \in X} [\delta(x, u, y) \bullet \sigma_1(y)]$.
(iii) If $\delta(x, u, y) \neq 0$ and $\sigma_0(x) \neq 0$, then $S \overset{\rho_3}{\to} uy \in P$, where $\rho_3 = \bigvee_{x \in X} [\sigma_0(x) \bullet \delta(x, u, y)]$.
(iv) If $\delta(x, u, y) \neq 0$, $\sigma_0(x) \neq 0$ and $\sigma_1(y) \neq 0$, then $S \overset{\rho_4}{\to} u \in P(u \in U \cup \{\Lambda\})$, where $\rho_4 = \bigvee_{x,y \in X} [\sigma_0(x) \bullet \delta(x, u, y) \bullet \sigma_1(y)]$.

Obviously, such a grammar G is an LRG. We show that $f_G = f_M$ below.

Let $\theta \in T^*$, if $\theta = \Lambda$, then $f_G(\Lambda) = \rho(S \to \Lambda) = f_M(\Lambda)$.

When $\theta \neq \Lambda$, suppose $\theta = u_1 u_2 ... u_n$, where $u_i \in T$ for $i = 1, 2, ..., n$.

If $S \overset{\rho}{\underset{*}{\Rightarrow}} \theta$ there must exist some derivation of θ with the form

$$S \overset{\rho_1}{\Rightarrow} u_1 A_1 \overset{\rho_2}{\Rightarrow} u_1 u_2 A_2 \Rightarrow \ldots \overset{\rho_{n-1}}{\Rightarrow} u_1 u_2 \ldots u_{n-1} A_{n-1}$$
$$\overset{\rho_n}{\Rightarrow} u_1 u_2 \ldots u_{n-1} u_n = \theta,$$

where $A_{i-1} \overset{\rho_i}{\to} u_i A_i \in P, i = 1, 2, \ldots n$ and $A_0 = S, A_n = u_n$ with $\rho = \rho_1 \bullet \rho_2 \bullet \ldots \bullet \rho_n$. From the definition of P, we know that

$$f_M(\theta) = \sigma_0 \circ \delta_\theta \circ \sigma_1 = \bigvee_{A_0, A_n \in X} [\sigma_0(A_0) \bullet \delta^*(S, \theta, A_n)]$$
$$\geq \bigvee_{A_0, A_n \in X} [\sigma_0(A_0) \bullet \delta(S, u_1, A_1) \bullet \delta(A_1, u_2, A_2)$$
$$\bullet \ldots \bullet \delta(A_{n-1}, u_n, A_n) \bullet \sigma_1(A_n)]$$
$$= \bigvee_{A_0 \in X} [\sigma_0(A_0) \bullet \delta(S, u_1, A_1)] \bullet \delta(A_1, u_2, A_2)$$
$$\bullet \ldots \bullet \bigvee_{A_n \in X} [\delta(A_{n-1}, u_n, A_n) \bullet \sigma_1(A_n)]$$
$$= \rho_1 \bullet \rho_2 \bullet \ldots \bullet \rho_n = \rho$$

Thus, we have shown that $f_G(\theta) \leq f_M(\theta)$ for any $\theta \in U^*$, that is, $f_G \leq f_M$.

Conversely, if

$$f_M(\theta) = \sigma_0 \circ \delta_\theta \circ \sigma_1 = \bigvee_{A_1, ..., A_n \in X} (\sigma_0(A_0) \bullet \delta(S, u_1, A_1)$$
$$\bullet \delta(A_1, u_2, A_2) \bullet \ldots \bullet \delta(A_{n-1}, u_n, A_n) \bullet \sigma_1(A_n))$$
$$\geq \bigvee_{A_0 \in X} [\sigma_0(A_0) \bullet \delta(S, u_1, A_1)] \bullet \delta(A_1, u_2, A_2)$$
$$\bullet \ldots \bullet \bigvee_{A_n \in X} [\delta(A_{n-1}, u_n, A_n) \bullet \sigma_1(A_n)].$$

For any $A_1, ..., A_n \in X$, let $\bigvee_{A_0 \in X} [\sigma_0(A_0) \bullet \delta(S, u_1, A_1)] \bullet \delta(A_1, u_2, A_2) \bullet \ldots \bullet \bigvee_{A_n \in X} [\delta(A_{n-1}, u_n, A_n) \bullet \sigma_1(A_n)] = \rho_1 \bullet \rho_2 \bullet \ldots \bullet \rho_n = \rho$, then there exists corresponding derivation $S \overset{\rho}{\underset{*}{\Rightarrow}} \theta$. Thus it shows that if $\rho \leq f_M(\theta)$, then $\rho \leq f_G(\theta)$, that is, $f_M(\theta) \leq f_G(\theta)$ for any $\theta \in U^*$, that is, $f_M \leq f_G$.

Therefore $f_M = f_G$, that is, the languages recognized by LA are L-valued regular languages. $\square$

We give an example to illustrate the proof of the above theorem.

Example 3.2 Let $L = [0, 1]$, $\bullet$ is the multiplication of real numbers, $M = (U, X, \delta, \sigma_0, \sigma_1)$ is an LA where $U = \{u_1, u_2\}$, $X = \{x_1, x_2, x_3\}$, $\sigma_0 = (0.9/x_1) + (0.5/x_2)$, $\sigma_1 = (0.5/x_2) + (0.8/x_3)$,

$$\delta(x_1, u_1, x_2) = 0.8, \delta(x_2, u_1, x_3) = 0.7, \delta(x_3, u_1, x_3)$$
$$= 0.4, \delta(x_3, u_2, x_2) = 0.5.$$

Construct an LRG $G = (N, T, P, S)$, where $N = \{x_1, x_2, x_3\}$, $T = \{u_1, u_2\}$. The productions in P are as follows:

$$S \xrightarrow{0.36} u_1, S \xrightarrow{0.72} u_1 x_2, S \xrightarrow{0.35} u_1 x_3, x_1 \xrightarrow{0.8} u_1 x_2, x_2 \xrightarrow{0.7} u_1 x_3, x_3$$
$$\xrightarrow{0.4} u_1 x_3, x_3 \xrightarrow{0.5} u_2 x_2, x_1 \xrightarrow{0.4} u_1, x_2 \xrightarrow{0.56} u_1, x_3 \xrightarrow{0.32} u_1, x_3 \xrightarrow{0.25} u_2.$$

For $\theta = u_1^2 u_2 u_1$,

$$f_M(\theta) = \sigma_0 \circ \delta_\theta \circ \sigma_1$$
$$= [\sigma_0(x_1) \bullet \delta^*(x_1, \theta, x_3)$$
$$\bullet \sigma_1(x_3)] \bigvee [\sigma_0(x_2) \bullet \delta^*(x_2, \theta, x_3)$$
$$\bullet \sigma_1(x_3)] = 0.14112.$$

Correspondingly , the derivation of θ in G are as follows:

(i) $S \xrightarrow{\rho_1} u_1 x_2 \xrightarrow{\rho_2} u_1 u_1 x_3 \xrightarrow{\rho_3} u_1 u_1 u_2 x_2 \xrightarrow{\rho_4} u_1 u_1 u_2 u_1$.

(ii) $S \xrightarrow{\rho_1'} u_1 x_3 \xrightarrow{\rho_2'} u_1 u_1 x_3 \xrightarrow{\rho_3'} u_1 u_1 u_2 x_2 \xrightarrow{\rho_4'} u_1 u_1 u_2 u_1$.

Furthermore in (i), we have,

$$\rho_1 = \sigma_0(x_1) \bullet \delta(x_1, u_1, x_2) = 0.9 \times 0.8 = 0.72,$$
$$\rho_2 = \delta(x_2, u_1, x_3) = 0.7,$$
$$\rho_3 = \delta(x_3, u_2, x_2) = 0.5,$$
$$\rho_4 = \delta(x_2, u_1, x_3) \bullet \sigma_1(x_3) = 0.56,$$
$$\rho = \rho_1 \bullet \rho_2 \bullet \rho_3 \bullet \rho_4 = 0.72 \times 0.7 \times 0.5$$
$$\times 0.56 = 0.14112,$$

in (ii)

$$\rho_1' = \sigma_0(x_2) \bullet \delta(x_2, u_1, x_3) = 0.5 \times 0.7 = 0.35,$$
$$\rho_2' = \delta(x_3, u_1, x_3) = 0.4,$$
$$\rho_3' = \delta(x_3, u_2, x_2) = 0.5,$$
$$\rho_4' = \delta(x_2, u_1, x_3) \bullet \sigma_1(x_3) = 0.56,$$
$$\rho' = \rho_1' \bullet \rho_2' \bullet \rho_3' \bullet \rho_4' = 0.35 \times 0.4 \times 0.5 \times 0.56 = 0.0392.$$

Then $f_G(\theta) = \rho \vee \rho' = 0.14112$, thus $f_M(\theta) = f_G(\theta)$.

Corollary 3.1 *L-valued regular grammars are equivalent to LAs.*

Proof. Straightforward by Theorem 3.1 and Theorem 3.2. $\square$

By Theorem 2.1 and Corollary 3.1 we can easily get that

Corollary 3.2 *L-valued automata are equivalent to L-valued weak regular grammars.*

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4 Characterization of DLA by *L*-valued regular grammars

Definition 4.1 (1) $G = (N, T, P, S)$ is called an L-valued deterministic regular grammar (DLRG, for short) if all the productions in P have only three forms,

$$A \xrightarrow{e} uB \text{ or } A \xrightarrow{\rho} u \text{ or } S \xrightarrow{\rho} \Lambda, \quad \text{where } A, B \in N, u \in T,$$
$$\text{and } \rho \in L - \{0\}.$$

Accordingly, f_G is called an L-valued deterministic regular language. Obviously, DLRG belongs to a special kind of LRG.

Similarly, for $u_1, ..., u_p \in (N \cup T)^*$, if $u_i \in (N \cup T)^*$ for $i = 1, 2, ..., p$ and u_{i+1} is directly derivable from u_i for $i = 1, 2, ..., p-1$, we say that u_p is derivable from u_1 and written as,

$$u_1 \xrightarrow{\rho} {}^* u_p$$

We call

$$u_1 \xrightarrow{\rho_1} u_2 \xrightarrow{\rho_2} \ldots \xrightarrow{\rho_{p-1}} u_p$$

the derivation chain of u_p (from u_1). However, here $\rho_1 = \rho_2 = ... = \rho_{p-2} = e$ and $\rho = \rho_1 \bullet \rho_2 \bullet ... \bullet \rho_{p-1} = \rho_{p-1}$.

(2) The L-language $f_G \in F(T^*)$ generated by an L-valued grammar is defined as follows:

$$f_G(\theta) = \bigvee \left\{ \rho : S \xrightarrow{\rho} {}^* \theta \right\}$$

Remark 4.1 Suppose that G is a DLRG, then the range of images set of f is finite, that is, the set $R_f = \{f_G(\theta) | \theta \in T^*\}$ is a finite set of L.

Since $f_G(\theta) = \vee \{\rho : S \xrightarrow{\rho} {}^* \theta\}$, it is clear that R_f is completely determined by the productions with form like $A \xrightarrow{\rho} u$. Since P is finite, such ρs are finite.

Lemma 4.1 (Li 2003) *Let $M = (X, U, \delta, x_0, \sigma_1)$ be an LA, where $\delta: X \times U \to 2^X$, $\sigma: X \to L$, $x_0 \in X$, $f_M(\theta) = \bigvee [\sigma_1(x) : x \in \delta^*(x_0, \theta)]$, then there exists a DLA, $N = (Y, U, \eta, y_0, \tau)$, such that $f_M = f_N$.*

Proof. Take $Y = 2^X$, and $\eta: Y \times U \to Y$ is defined as, $\eta(Z, u) = \cup_{x \in Z} \delta(x, u)$, $y_0 = \{x_0\}$, $\tau: Y \to L$ is taken as, $\tau(Z) = \bigvee_{x \in Z} \sigma(x)$. Then $N = (Y, U, \eta, y_0, \tau)$ is a DLA, and $\eta^*(y_0, \theta) = \delta^*(x_0, \theta)$. Thus $f_N(\theta) = \tau(\eta^*(y_0, \theta)) = \bigvee_{x \in \eta^*(y_0, \theta)} \sigma(x) = \bigvee_{x \in \delta^*(x_0, \theta)} \sigma(x) = f_M(\theta)$, that is, $f_N = f_M$. $\square$

Theorem 4.1 *L-valued deterministic regular languages can be accepted by DLA.*

Proof. Suppose $G = (N, T, P, S)$ be a DLRG, and construct an LA $M = (X, U, \delta, x_0, \sigma_1)$ where $U = T$, $x_0 = S$. Check all the productions in G, and let P_1 denote the set

whose elements are with form of $A \xrightarrow{\rho} u$, that is, $P_1 = \left\{ x \xrightarrow{\rho_i} u | i = 1, 2, \ldots, m \right\}$, where m is just the number of such productions.

For any $(x, u) \in NT$, define δ as follows

$$\delta(x, u) = \begin{cases} \left\{ y | x \xrightarrow{e} uy \in P \right\} \cup \{Z_i\} & x \xrightarrow{\rho} u \in P_1 \\ \left\{ y | x \xrightarrow{e} uy \in P \right\} & x \xrightarrow{\rho} u \notin P_1 \end{cases}$$

and define σ_1 as: $\sigma_1 = (\rho_i \ (x \rightarrow u)/Z_i) + (f_G(\Lambda)/S)$. Furthermore, $X = N \cup \{ Z_i\}$ for $i = 1, 2, ..., m$.

Let $\theta \in T^*$, if $\theta = \Lambda$, then, $f_M(\theta) = \sigma_1 (\delta(x_0, \Lambda)) = f_G(\Lambda)$.

When $\theta \neq \Lambda$, suppose that $\theta = u_1 u_2 \ldots u_n$ where $u_i \in T$ for $i = 1, 2, ..., n$. If $S \overset{*}{\underset{\rho}{\Rightarrow}} \theta$, there must exist some derivation of the form

$$S \xrightarrow{e} u_1 A_1 \xrightarrow{e} u_1 u_2 A_2 \Rightarrow \ldots \xrightarrow{e} u_1 u_2 \ldots u_{n-1} A_{n-1}$$
$$\xrightarrow{\rho} u_1 u_2 \ldots u_{n-1} u_n,$$

where $A_{i-1} \xrightarrow{e} u_i A_i \in P$, for $i = 1, 2, ..., n-1$, $A_0 = S$ and $A_{n-1} \xrightarrow{\rho} u_n \in P$. Furthermore $\rho \ (S \rightarrow \theta) = \rho \ (A_{n-1} \rightarrow u_n) = \rho$.

Corresponding to the LA, by the definition of δ and σ_1, we get that $A_1 \in \delta(x_0, u_1)$; $A_2 \in \delta(A_1, u_2)$; ...; $A_{n-1} \in \delta(A_{n-2}, u_{n-1})$; $Z_t \in \delta(A_{n-1}, u_n)$ for $1 \leq t \leq m$.

Then $f_M(\theta) = \vee \{\sigma_1(Z) : Z \in \delta^* (x_0, \theta)\} \geq \sigma_1(Z_t) = \rho(A_{n-1} \rightarrow u_n) = \rho$. Thus, we have shown that $f_G(\theta) \leq f_M(\theta)$ for any $\theta \in U^*$, that is, $f_G \leq f_M$.

Conversely, if $f_M(\theta) = \vee [\sigma_1(Z) : Z \in \delta^* (x_0, \theta)] \geq \sigma_1(Z_t) = \rho(A_{n-1} \underset{*}{\rightarrow} u_n) = \rho$, then there exist corresponding derivation $S \overset{*}{\underset{\rho}{\Rightarrow}} \theta$, it implies that if $\rho \leq f_M(\theta)$, then $\rho \leq f_G(\theta)$, that is, $f_M \leq f_G$.

Therefore, $f_M = f_G$. From lemma 4.1, f_M is *equivalent* to a DLA language, thus f_G can be accepted by a DLA. $\square$

Theorem 4.2 *The languages accepted by DLA are L-valued deterministic regular languages.*

Proof. Let $M = (X, U, \delta, x_0, \sigma_1)$ be a DLA, define an LRG $G = (N, T, P, S)$, where $T = U$, $N = X$, $S = x_0$.

Check all the transition functions in M, for every $y = \delta(x, u)$ where $x, y \in X$, $u \in U$, we define a corresponding production: $x \xrightarrow{e} uy \in P$. Let $Y = \{x | \sigma_1(x) \neq 0\}$, $Y \subseteq X$.

If the transition functions satisfy that $\delta(x, u) \in Y$, then we define another kind of productions: $x \xrightarrow{\rho_i} u$ and $\rho_i(x \rightarrow u) = \sigma_1(Z_i)$, $Z_i \in Y$, $i = 1, 2, ..., m$, where $m = |Y|$.

Clearly, such a grammar G is an DLRG. We show $f_M = f_G$ below.

Let $\theta \in T^*$, if $\theta = \Lambda$, then $f_G (\theta) = \rho(S \rightarrow \Lambda) = f_M(\Lambda)$.

When $\theta \neq \Lambda$, suppose that $\theta = u_1, u_2 \ldots u_n$, where $u_i \in T$ for $i = 1, 2, ..., n$, then the process that θ is accepted by M can be denoted as

$$\delta(x_0, u_1) = A_1; \delta(A_1, u_2) = A_2; \ldots, \delta(A_{n-2}, u_{n-1})$$
$$= A_{n-1}; \delta(A_{n-1}, u_n) = Z(Z \in Y),$$

so $f_M(\theta) = \sigma_1(\delta^*(x_0, \theta)) = \sigma_1(Z)$.

Correspondingly, from the construction of G we know that, if $S \overset{*}{\underset{\rho}{\Rightarrow}} \theta$ there must exist some derivation of θ with form,

$$S \xrightarrow{e} u_1 A_1 \xrightarrow{e} u_1 u_2 A_2 \Rightarrow \ldots$$
$$\xrightarrow{e} u_1 u_2 \ldots u_{n-1} A_{n-1} \xrightarrow{\rho} u_1 u_2 \ldots u_{n-1} u_n,$$

where $A_{i-1} \xrightarrow{e} u_i A_i \in P, i = 1, 2, \ldots, n-1, A_0 = S = x_0$; $A_{n-1} \xrightarrow{\rho} u_n \in P$. Furthermore, $\rho \ (S \rightarrow \theta) = \rho(A_{n-1} \rightarrow u_n) = \sigma_1(Z)$, thus $f_G = f_M$. $\square$

Example 4.1 Let $L = [0, 1]$, $\bullet$ is the multiplication of real numbers, $M = (X, U, \delta, x_0, \sigma_1)$ is a DLA, where $U = \{u_1, u_2\}$, $X = \{x_1, x_2, x_3, x_4\}$, $x_0 = x_1$, $\sigma_1 = (0.5/x_2) + (0.8/x_3)$, $\delta(x_1, u_1) = x_2$, $\delta(x_2, u_1) = x_3$, $\delta(x_3, u_1) = x_4$, $\delta(x_4, u_2) = x_3$, $\delta(x_3, u_2) = x_2$.

Construct an LRG $G = (N, T, P, S)$, where $N = \{x_1, x_2, x_3, x_4\}$, $T = \{u_1, u_2\}$, $S = x_1$. The fuzzy productions in P are as follows:

(a) $x_1 \xrightarrow{e} u_1 x_2, x_2 \xrightarrow{e} u_1 x_3, x_3 \xrightarrow{e} u_1 x_4, x_4 \xrightarrow{e} u_2 x_3, x_3 \xrightarrow{e} u_2 x_2$ and
(b) $x_1 \xrightarrow{\rho_1} u_1, x_2 \xrightarrow{\rho_2} u_1, x_4 \xrightarrow{\rho_2} u_2, x_3 \xrightarrow{\rho_1} u_2$

and $\rho_1 = \sigma_1(x_2) = 0.5$, $\rho_2 = \sigma_1(x_3) = 0.8$.

For $\theta = u_1^3 u_2$, its derivations in G are as follows, $x_1 \xrightarrow{e} u_1 x_2 \xrightarrow{e} u_1^2 x_3 \xrightarrow{e} u_1^3 x_4 \xrightarrow{\rho_2} u_1^3 u_2$, then $f_G(\theta) = \rho_2 = 0.8$, and $f_M(\theta) = \sigma_1 \ (\delta^*(x_1, \theta)) = \sigma_1(x_3) = 0.8$, thus $f_M(\theta) = f_G(\theta)$.

Thereby, from theorems 4.1 and 4.2 we can immediately get the following conclusion:

Corollary 4.1. *For an L-language f over a finite set U, f can be accepted by a DLA iff f can be generated by a DLRG.*

In Li (2003), a sufficient and necessary condition has been given for the lattice-ordered monoid L, which to ensure that any LA is equivalent to some DLA, then we can obtain another similar conclusion for LRG and DLRG.

Corollary 4.2. *For any LRG G, there exists a DLRG G' such that $f_G = f_{G'}$ iff the lattice-ordered monoid L satisfies the following conditions, for any finite subset L' of L, the subalgebra of $(L, \bullet, \vee)$ generated by L' is finite.*

Distributive lattice always satisfies the conditions stated in Corollary 4.2, that is to say, for any distributive lattice L, if L' is a finite subset of L, then the sublattice of L generated by L' is also finite. Then we deduce the following corollary.

Corollary 4.3 *If $(L, \wedge, \vee)$ is a distributive lattice, then any LRG are equivalent to an DLRG.*

Conclusion

Regular grammars with truth values in lattice-ordered monoid and the languages generated by them are

studied in this paper. We show the *equivalence* between LA and LRG in the sense of recognizing fuzzy languages. Since, LA are not equivalent to DLA in the power of recognizing fuzzy languages, we define a special regular grammar DLRG and show that this kind of grammars are equivalent to DLA.

Of course, much more work should be done in this direction. Some other related work such as minimization of LA, the special (algebraic) properties of LA languages from the point of view of level structure, and the formal model of computing with words (Ying 2002) based on LRG and DLRG, and so on, should be studied in the future work.

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