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Non-commutative fuzzy Galois connections

G. Georgescu, A. Popescu

Abstract Fuzzy Galois connections were introduced by Bělohlávek in [4]. The structure considered there for the set of truth values is a complete residuated lattice, which places the discussion in a "commutative fuzzy world". What we are doing in this paper is dropping down the commutativity, getting the corresponding notion of Galois connection and generalizing some results obtained by Bělohlávek in [4] and [7]. The lack of the commutative law in the structure of truth values makes it appropriate for dealing with a sentences conjunction where the order between the terms of the conjunction counts, gaining thus a temporal dimension for the statements. In this "non-commutative world", we have not one, but two implications ([15]). As a consequence, a Galois connection will not be a pair, but a quadruple of functions, which is in fact two pairs of functions, each function being in a symmetric situation to his pair. Stating that these two pairs are compatible in some sense, we get the notion of strong L-Galois connection, a more operative and prolific notion, repairing the "damage" done by non-commutativity.

Keywords Non-commutative fuzzy logic, Fuzzy Galois connection, Fuzzy relation, Non-commutative conjunction

1
Introduction

As in [4], our discussion concerns only Galois connections between power sets.

A classical (crisp) Galois connection between the power sets of X and Y (shortly, between X and Y) is a pair $(\uparrow, \downarrow)$ of functions $\uparrow: 2^X \longrightarrow 2^Y$, $\downarrow: 2^Y \longrightarrow 2^X$ such that $A_1 \subseteq A_2$ implies $A_2^{\uparrow} \subseteq A_1^{\uparrow}$, $B_1 \subseteq B_2$ implies $B_2^{\downarrow} \subseteq B_1^{\downarrow}$, $A \subseteq A^{\uparrow\downarrow}$, $B \subseteq B^{\downarrow\uparrow}$, for all $A, A_1, A_2 \in 2^X$, $B, B_1, B_2 \in 2^Y$. Galois connections naturally arise when one considers an arbitrary relation between X (the universe of objects) and Y (the universe of attributes). For $A \subseteq X$, $A^{\uparrow}$ is the set of all attributes shared by the objects of A and, for $B \subseteq Y$, $B^{\downarrow}$ is the set of all objects that share the attributes of B. In [14], Ore showed that all the Galois connections between X and Y are induced by binary relations, so Galois connections are in a bijective correspondence with binary relations.

In [4], Bělohlávek generalized the notion of Galois connection to the case where the truth values (degrees) come from a complete residuated lattice, that is a structure $(L, \vee, \wedge, *, \rightarrow, 0, 1)$, where $(L, \vee, \wedge, 0, 1)$ is a complete, bounded lattice, $(L, *, 1)$ is a commutative monoid and $(*, \rightarrow)$ forms a residuated pair (i.e. $x * y \leq z$ iff $x \leq y \rightarrow z$ for all $x, y, z \in L$). Of course, the framework is fuzzy logic and thus, for a fixed set X, called universe, instead of the power set one considers the L-power set, L^X (the set of functions $A : X \longrightarrow L$). For any $x \in X$ and $A \in L^X$, $A(x)$ represents the truth value of "x is in A". An L-relation between the universes X and Y is a function I from $L^{X\times Y}$, $I(x,y)$ showing, for any $(x,y) \in X \times Y$, how much is x in the relation I with y. For $A_1, A_2 \in L^X$, $S(A_1, A_2)$ (in [4], it is denoted $Subs(A_1, A_2)$) is the subsethood degree of A_1 in A_2, namely $\bigvee_{x\in X}(A_1(x) \rightarrow A_2(x))$ (it is, in fact, the truth value of "for all x, x is in A_1 implies x is in A_2"). $A_1 \subseteq A_2$ means that for all $x \in X$, $A_1(x) \leq A_2(x)$, i.e. $S(A_1, A_2) = 1$. An L-Galois connection between X and Y is a pair $(\uparrow, \downarrow)$ of functions $\uparrow: L^X \longrightarrow L^Y$, $\downarrow: L^Y \longrightarrow L^X$ such that $S(A_1, A_2) \leq S(A_2^{\uparrow}, A_1^{\uparrow})$, $S(B_1, B_2) \leq S(B_2^{\downarrow}, B_1^{\downarrow})$, $A \subseteq A^{\uparrow\downarrow}$, $B \subseteq B^{\downarrow\uparrow}$, for all $A, A_1, A_2 \in L^X$, $B, B_1, B_2 \in L^Y$. As in the crisp case, an L-relation I from $L^{X\times Y}$ induces an L-Galois connection between X and Y:

$$A^{\uparrow}(y) = \bigwedge_{x\in X}(A(x) \rightarrow I(x,y))$$

for all $y \in Y$;

$$B^{\downarrow}(x) = \bigwedge_{y\in Y}(B(y) \rightarrow I(x,y))$$

for all $x \in X$. Bělohlávek proved a generalization of Ore's theorem, showing that there are as many L-Galois connections between X and Y as L-relations in $L^{X\times Y}$ (every L-Galois connection being induced by an L-relation). He also provided a representation of L-Galois connections by families of crisp Galois connections with certain properties, namely L-nested systems.

All these can be found in [4]. Now let us suppose that $*$ (the multiplication on L) is not necessarily commutative and that, instead of $\rightarrow$, we have two residua $\rightarrow$ and $\Rightarrow$, satisfying the properties:

$$x * y \leq z \text{ iff } x \leq y \rightarrow z \text{ iff } y \leq x \Rightarrow z$$

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Dedicated to Prof. Ján Jakubík on the occasion of his 80th birthday.

for all $x, y, z \in L$. Then L is a generalized residuated lattice ([15]). The statement "$*$, not being necessarily commutative, gives rise to two residua instead of one" is motivated by the fact that, in both commutative and non-commutative case, the residuum (residua), if there exists (exist), is (are) uniquely determined by $*$. Of course, for each universe X, we have two subsethood degrees between the L-subsets of X, S_1 and S_2, corresponding to the two implications given by the two residua.

An L-Galois connection will consist of a quadruple $(\uparrow, \Uparrow, \downarrow, \Downarrow)$ of functions $\uparrow, \Uparrow: L^X \longrightarrow L^Y$, $\downarrow, \Downarrow: L^Y \longrightarrow L^X$, where $(\uparrow, \Downarrow)$ and $(\Uparrow, \downarrow)$ are both pairs of *complementary functions that satisfy* similar properties as in the commutative case, only with a left-right variation of the subsethood degree considered. In addition, when fulfilled a compatibility condition between these two pairs (demanding that $\uparrow$ and $\Uparrow$, respectively $\downarrow$ and $\Downarrow$, have the same starting point, i.e. coincide on crisp singletons), the L-Galois connection will be called strong.

Section 2 of this paper creates the framework for our discussion (generalized residuated lattices, L-sets, subsethood degrees).

In Sec. 3, we define the L-Galois connections and investigate their properties, following quite closely [4] in order to generalize the results from there. It turns out that most of the generalizations are possible only with strong L-Galois connections; this, together with the fact that strong L-Galois connections (which still generalize the commutative case of L-Galois connections) are precisely the ones induced by L-relations, suggests that this might be the right fuzzy non-commutative notion of Galois connection, although the strongness condition is quite unlike "Galois connection style".

Section 4 treats, in the style of [7], the case where universes come equipped with fuzzy equalities and, consequently, the considered relations and Galois connections must respect those equalities. It is proved that a bijection between L-relations and strong L-Galois connection exists also here.

2 The framework

Definition 2.1 A generalized residuated lattice is a structure $(L, \vee, \wedge, *, \rightarrow, 0, 1)$ such that the following conditions hold:

(GR1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice;
(GR2) $(L, *, 1)$ is a monoid;
(GR3) (residuation)

For all $a, b, c \in L$ we have the equivalences:

$$a \leq b \rightarrow c \quad \text{iff } a * b \leq c \ ,$$

$$a \leq b \Rightarrow c \quad \text{iff } b * a \leq c \ .$$

Obviously, a generalized residuated lattice is a residuated lattice (identifying $\rightarrow$ with $\Rightarrow$) iff $*$ is commutative.

Lemma 2.1 In a complete generalized residuated lattice L, the following properties hold for all $a, b, c \in L$, $(a_i)_{i \in I} \subseteq L$, $(b_i)_{i \in I} \subseteq L$, $(a_{ij})_{(i,j) \in I \times J} \subseteq L$:

(1) $a \leq b$ iff $a \rightarrow b = 1$ iff $a \Rightarrow b = 1$;
(2) $a \rightarrow 1 = a \Rightarrow 1 = 1$; $1 \rightarrow a = 1 \Rightarrow a = a$;
(3) $\rightarrow$ and $\Rightarrow$ are antitone in the first and isotone in the second argument;
(4) $*$ is isotone in both arguments;
(5) $a * b \leq a$ and $a * b \leq b$;
(6) $0 * a = a * 0 = 0$;
(7) $\left(\bigvee_{i \in I} a_i\right) * a = \bigvee_{i \in I}(a_i * a)$ and $a * \left(\bigvee_{i \in I} a_i\right) = \bigvee_{i \in I}(a * a_i)$;
(8) $(a \rightarrow b) * a \leq b$ and $a * (a \Rightarrow b) \leq b$;
(9) $a \leq (a \rightarrow b) \Rightarrow b$ and $a \leq (a \Rightarrow b) \rightarrow b$;
(10) $\left(\bigwedge_{i \in I} a_i\right) * a \leq \bigwedge_{i \in I}(a_i * a)$ and $a * \left(\bigwedge_{i \in I} a_i\right) \leq \bigwedge_{i \in I}(a * a_i)$;
(11) $\left(\bigvee_{i \in I} a_i\right) \rightarrow a = \bigwedge_{i \in I}(a_i \rightarrow a)$ and $a \rightarrow \left(\bigwedge_{i \in I} a_i\right) = \bigwedge_{i \in I}(a \rightarrow a_i)$;
(12) $\left(\bigvee_{i \in I} a_i\right) \Rightarrow a = \bigwedge_{i \in I}(a_i \Rightarrow a)$ and $a \Rightarrow \left(\bigwedge_{i \in I} a_i\right) = \bigwedge_{i \in I}(a \Rightarrow a_i)$;
(13) $\bigwedge_{i \in I} \bigwedge_{j \in J} a_{ij} = \bigwedge_{j \in J} \bigwedge_{i \in I} a_{ij} = \bigwedge_{(i,j) \in I \times J} a_{ij}$; $\bigvee_{i \in I} \bigvee_{j \in J} a_{ij} = \bigvee_{j \in J} \bigvee_{i \in I} a_{ij} = \bigvee_{(i,j) \in I \times J} a_{ij}$;
(14) $a \rightarrow (b \Rightarrow c) = b \Rightarrow (a \rightarrow c)$ and $a \Rightarrow (b \rightarrow c) = b \rightarrow (a \Rightarrow c)$;
(15) $*$ is commutative iff $\rightarrow = \Rightarrow$.

Remark 2.1 Of course, many of the properties above (namely those that do not involve family suprema) hold in any generalized residuated lattice, not necessarily complete. However, we are interested only in the case of completeness, since the truth values must be appropriate for universal and existential quantification.

For a set X, call an element from L^X (the set of functions from X to L) an L-fuzzy-subset (L-subset) of X. We shall identify the subsets of X (called crisp subsets, i.e. elements from 2^X) with particular L-subsets of X in the obvious way. For two sets X and Y, an L-relation between X and Y is an element from $L^{X \times Y}$.

In the commutative case ([4]) (i.e. L being a complete residuated lattice), we have, for any $A, B \in L^X$, the subsethood degree $S(A, B)$ of A in B:

$$S(A, B) = \bigwedge_{x \in X} (A(x) \rightarrow B(x)) \ .$$

One can see that, for all $a, b, c \in L^X$,

(1) $S(A, A) = 1$;
(2) $S(A, B) * S(B, C) \leq S(A, C)$,

i.e. $S : L^X \times L^X \Rightarrow L$ is an L-preorder on L^X.

Now we want to define a similar concept of subsethood degree for the case of a generalized residuated lattice $(L, \vee, \wedge, *, \rightarrow, \Rightarrow, 0, 1)$. The existence of two residua $\rightarrow$ and $\Rightarrow$ leads to two indicators of this degree:

$$S_1(A, B) = \bigwedge_{x \in X} (A(x) \rightarrow B(x)) \ ,$$

$$S_2(A, B) = \bigwedge_{x \in X} (A(x) \Rightarrow B(x))$$

for any $A, B \in L^X$. We write $A \subseteq B$ when $A(x) \leq B(x)$ for all $x \in X$.

For the rest of this paper, $(L, \vee, \wedge, *, \rightarrow, \Rightarrow, 0, 1)$ will be a generalized residuated lattice.

Proposition 2.1 For all $A, B, C \in L^X$ the following properties hold:

(1) $A \subseteq B$ implies $S_1(B, C) \leq S_1(A, C)$ and $S_2(B, C) \leq S_2(A, C)$;
(2) $A \subseteq B$ implies $S_1(C, A) \leq S_1(C, B)$ and $S_2(C, A) \leq S_1(C, B)$;
(3) $A \subseteq B$ iff $S_1(A, B) = 1$ iff $S_2(A, B) = 1$;
(4) $S_1(B, C) * S_1(A, B) \leq S_1(A, C)$;
(5) $S_2(A, B) * S_2(B, C) \leq S_2(A, C)$.

Proof. (1): By Lemma 2.1.(3), for any $x \in X$, $A(x) \leq B(x)$ implies $B(x) \rightarrow C(x) \leq A(x) \rightarrow C(x)$ and $B(x) \Rightarrow C(x) \leq A(x) \Rightarrow C(x)$, hence

$$S_1(B, C) = \bigwedge_{x \in X} (B(x) \rightarrow C(x)) \leq \bigwedge_{x \in X} (A(x) \rightarrow C(x)) = S_1(A, C)$$

and, similarly,

$S_1(B, C) \leq S_1(A, C)$.

(2) The proof is similar.

(3) We have the equivalences:

$$\begin{aligned} S_1(A, B) = 1 \quad & \text{iff} \quad \wedge_{x \in X}(A(x) \rightarrow B(x)) = 1 \\ & \text{iff} \quad A(x) \rightarrow B(x) = 1 \text{ for } x \in X \\ & \text{iff} \quad A(x) \leq B(x) \text{ for all } x \in X \\ & \text{iff} \quad A \subseteq B \ . \end{aligned}$$

The equivalence $A \subseteq B$ iff $S_2(A, B) = 1$ can be proved in the same manner.

(4) By Lemma 2.1.(5 and 8), for any $x \in X$, the following inequalities hold:

$$(B(x) \rightarrow C(x)) * (A(x) \rightarrow B(x)) * A(x) \leq (B(x) \rightarrow C(x)) * B(x) \leq C(x) \ .$$

We get, in accordance to (GR3), that

$$(B(x) \rightarrow C(x)) * (A(x) \rightarrow B(x)) \leq A(x) \rightarrow C(x) \ ,$$

for any $x \in X$. Therefore, using Lemma 2.1.(10),

$$\begin{aligned} & S_1(B, C) * S_1(A, B) \\ & = \left[\bigwedge_{x \in X} (B(x) \rightarrow C(x))\right] * \left[\bigwedge_{x \in X} (A(x) \rightarrow B(x))\right] \\ & \leq \bigwedge_{x, y \in X} [(B(x) \rightarrow C(x)) * (A(y) \rightarrow B(y))] \\ & \leq \bigwedge_{x \in X} [(B(x) \rightarrow C(x)) * (A(x) \rightarrow B(x))] \\ & \leq \bigwedge_{x \in X} (A(x) \rightarrow C(x)) \leq S_1(A, C) \ . \end{aligned}$$

(5) has a similar proof.

3 Non-commutative fuzzy Galois connections

In [4], Bělohlávek defines fuzzy Galois connection as a natural generalization of the classical (crisp) one. This section is concerning with a notion of Galois connection corresponding to a fuzzy set theory based on a generalized residuated lattice. The results from [4] are generalized in this section.

Remember that L denotes a generalized residuated lattice.

Definition 3.1 Let X and Y be two arbitrary non-empty sets. An L-Galois connection between X and Y is a quadruple $(\uparrow, \Uparrow, \downarrow, \Downarrow)$ of functions

$\uparrow: L^X \longrightarrow L^Y$, $\Uparrow: L^X \longrightarrow L^Y$, $\downarrow: L^Y \longrightarrow L^X$,

$\Downarrow: L^Y \longrightarrow L^X$

such that the following conditions are fulfilled:

(G1) $S_1(A_1, A_2) \leq S_2(A_2^{\uparrow}, A_1^{\uparrow})$; $S_2(A_1, A_2) \leq S_2(A_2^{\Uparrow}, A_1^{\Uparrow})$;
(G2) $S_1(B_1, B_2) \leq S_2(B_2^{\downarrow}, B_1^{\downarrow})$; $S_2(B_1, B_2) \leq S_1(B_2^{\Uparrow}, B_1^{\uparrow})$;
(G3) $A \subseteq A^{\uparrow\Downarrow}$; $A \subseteq A^{\Uparrow\downarrow}$;
(G4) $B \subseteq B^{\downarrow\Uparrow}$; $B \subseteq B^{\Downarrow\uparrow}$.

If L is a residuated lattice then the residua $\rightarrow$ and $\Rightarrow$ coincide hence one obtains the Bělohlávek's notion of fuzzy Galois connection ([4]).

The following proposition provides an alternative definition for L-Galois connections.

Proposition 3.1 A quadruple $(\uparrow, \Uparrow, \downarrow, \Downarrow)$ of functions $\uparrow, \Uparrow: L^X \longrightarrow L^Y$, $\downarrow, \Downarrow: L^Y \longrightarrow L^X$ forms an L-Galois connection iff
$S_1(A, B^{\Downarrow}) = S_2(B, A^{\uparrow})$, $S_2(A, B^{\downarrow}) = S_1(B, A^{\Uparrow})$ for all $A \in L^X$, $B \in L^Y$. (Δ)

Proof. Let $(\uparrow, \Uparrow, \downarrow, \Downarrow)$ be an L-Galois Connection.

First, we prove $S_1(A, B^{\Downarrow}) \leq S_2(B, A^{\uparrow})$. From (G1), we get $S_1(A, B^{\Downarrow}) \leq S_2(B^{\Downarrow\uparrow}, A^{\uparrow})$. Moreover, by (G4), $B \subseteq B^{\Downarrow\uparrow}$; now, applying Proposition 2.1.(1), $S_2(B^{\Downarrow\uparrow}, A^{\uparrow}) \leq S_2(B, A^{\uparrow})$. Thus $S_1(A, B^{\Downarrow}) \leq S_2(B, A^{\uparrow})$. For the converse inequality, we apply (G2) and get $S_2(B, A^{\uparrow}) \leq S_1(A^{\uparrow\Downarrow}, B^{\Downarrow})$; next, by (G3), $A \subseteq A^{\uparrow\Downarrow}$ and thus, applying again Proposition 2.1.(1), $S_2(B, A^{\uparrow}) \leq S_1(A^{\uparrow\Downarrow}, B^{\Downarrow}) \leq S_1(A, B^{\Downarrow})$.

We showed that $S_1(A, B^{\Downarrow}) \leq S_2(B, A^{\uparrow})$. The proof of the fact that $S_1(A, B^{\downarrow}) \leq S_2(B, A^{\Uparrow})$ goes on in a similar manner.

Suppose now, conversely, that Δ hold. We shall prove, for each property from (G1)–(G4), only half of it, the other half having an analogous proof.

(G3): Because $S_2(A^{\uparrow}, A^{\uparrow}) = 1$, we have that $S_1(A, A^{\uparrow\Downarrow}) = S_2(A^{\uparrow}, A^{\uparrow}) = 1$, hence $A \subseteq A^{\uparrow\Downarrow}$.
(G1): $S_2(A_2^{\uparrow}, A_1^{\uparrow}) = S_1(A_1, A_2^{\uparrow\Downarrow})$. But since $A_2 \subseteq A_2^{\uparrow\Downarrow}$ (as showed above) and S_1 is isotone in the second argument, $S_1(A_1, A_2) \leq S_1(A_1, A_2^{\uparrow\Downarrow}) = S_2(A_2^{\uparrow}, A_1^{\uparrow})$.
(G4): Since $S_1(B^{\Downarrow}, B^{\Downarrow}) = 1$, we have $S_2(B, B^{\Downarrow\uparrow}) = S_1(B^{\Downarrow}, B^{\Downarrow}) = 1$, hence $B \subseteq B^{\Downarrow\uparrow}$.
(G2): $S_1(B_2^{\Downarrow}, B_1^{\Downarrow}) = S_2(B_1, B_2^{\Downarrow\uparrow})$. But since $B_2 \subseteq B_2^{\Downarrow\uparrow}$ (as showed above) and S_2 is isotone in the second argument, $S_2(B_1, B_2) \leq S_2(B_1, B_2^{\Downarrow\uparrow}) = S_1(B_2^{\Downarrow}, B_1^{\Downarrow})$.

Now we shall see how a binary L-relation $I \in L^{X \times Y}$ naturally induces an L-Galois connection.

With any L-relation $I \in L^{X \times Y}$, one can associate four functions

$\uparrow_I: L^X \longrightarrow L^Y$, $\Uparrow_I: L^X \longrightarrow L^Y$, $\downarrow_I: L^Y \longrightarrow L^X$, $\Downarrow_I: L^Y \longrightarrow L^X$

defined by

$$A^{\uparrow_I}(y) = \bigwedge_{x\in X}(A(x) \to I(x,y))$$

$$A^{\Uparrow_I}(y) = \bigwedge_{x\in X}(A(x) \Rightarrow I(x,y))$$

$$B^{\downarrow_I}(x) = \bigwedge_{y\in Y}(B(y) \to I(x,y))$$

$$B^{\Downarrow_I}(x) = \bigwedge_{y\in Y}(B(y) \Rightarrow I(x,y))$$

for any $A \in L^X$, $B \in L^Y$ and $x \in X$, $y \in Y$.

Proposition 3.2 For any binary L-relation $I \in L^{X\times Y}$, the quadruple $(\uparrow_I, \Uparrow_I, \downarrow_I, \Downarrow_I)$ is an L-Galois connection.

Proof: We shall prove that the conditions Δ from Proposition 3.1 hold. We have

$$S_1(A, B^{\Downarrow}) = \bigwedge_{x\in X}(A(x) \to B^{\Downarrow}(x))$$
$$= \bigwedge_{x\in X}[A(x) \to \bigwedge_{y\in Y}(B(y) \Rightarrow I(x,y))] \ .$$

Applying successively Lemma 2.1.(11, 14 and 13), we get:

$$S_1(A, B^{\Downarrow}) = \bigwedge_{x\in X}[A(x) \to \bigwedge_{y\in Y}(B(y) \Rightarrow I(x,y))]$$
$$= \bigwedge_{x\in X}\bigwedge_{y\in Y}[A(x) \to (B(y) \Rightarrow I(x,y))]$$
$$= \bigwedge_{y\in Y}\bigwedge_{x\in X}[A(x) \to (B(y) \Rightarrow I(x,y))]$$
$$= \bigwedge_{y\in Y}\bigwedge_{x\in X}[B(y) \Rightarrow (A(x) \to I(x,y))] \ .$$

On the other hand,

$$S_2(B, A^{\uparrow}) = \bigwedge_{y\in Y}(B(y) \Rightarrow A^{\uparrow}(y))$$
$$= \bigwedge_{y\in Y}[B(y) \Rightarrow \bigwedge_{x\in X}(A(x) \to I(x,y))] \ .$$

Applying Lemma 2.1.(12), we get

$$S_2(B, A^{\uparrow}) = \bigwedge_{y\in Y}[B(y) \Rightarrow \bigwedge_{x\in X}(A(x) \to I(x,y))]$$
$$= \bigwedge_{y\in Y}\bigwedge_{x\in X}[B(y) \Rightarrow (A(x) \to I(x,y))] \ .$$

So $S_1(A, B^{\Downarrow}) = S_2(B, A^{\uparrow})$.

The fact that $S_2(A, B^{\downarrow}) = S_1(B, A^{\Uparrow})$ follows similarly.

Remark 3.1 The crisp form of the above reasoning is: $A \subseteq B^{\Downarrow}$ iff $A \subseteq \{x \in X \ / \ x \in B^{\Downarrow}\}$ iff $A \subseteq \{x \in X \ / \ \forall y \in B, (x,y) \in I\}$ iff $\forall x \in A, \forall y \in B, (x,y) \in I$; on the other hand, $B \subseteq A^{\uparrow}$ iff $B \subseteq \{y \in Y \ / \ y \in A^{\uparrow}\}$ iff $B \subseteq \{y \in Y \ / \ \forall x \in A, (x,y) \in I\}$ iff $\forall y \in B, \forall x \in A, (x,y) \in I$; hence, since universal quantifications commute, $A \subseteq B^{\Downarrow}$ iff $B \subseteq A^{\uparrow}$.

Thus, like in the crisp case and in the commutative fuzzy case, relations induce Galois connections ($(\uparrow_I, \Uparrow_I, \downarrow_I, \Downarrow_I)$ is called the L-Galois connection induced by I). Ore ([14]) showed that every crisp Galois connection is induced by some crisp relation. Bělohlávek generalized this result, showing that, when L is a complete residuated lattice, every L-Galois connection is induced by an L-relation. In our non-commutative case, the splitting of implication into two, which leads to the splitting of subsethood degree into two, causes the existence of two "liberty degrees" instead of one in our definition of an L-Galois Connection; that is, the pairs $(\uparrow, \Downarrow)$ and $(\Uparrow, \downarrow)$ are quite independent from each other, making impossible the natural bijective correspondence with L-relations that exists in the commutative case. Thus, one could argue that our definition of L-Galois Connections is two permisive and that for the two pairs $(\uparrow, \Downarrow)$ and $(\Uparrow, \downarrow)$ it should be postulated a compatibility condition. We shall in fact do that, when defining what we shall call a strong L-Galois Connection. Though most of the interesting results hold only for strong L-Galois Connection, we have chosen this variant thinking that the definition of Galois connection should be given only with "traditional tools", like in [8] and [14]. However, though important, this is only a matter of sintax.

Consider the L-Galois connection $(\uparrow_I, \Uparrow_I, \downarrow_I, \Downarrow_I)$ induced by some L-relation. The two pairs $(\uparrow_I, \Downarrow_I)$ and $(\Uparrow_I, \downarrow_I)$ are, in this case, strongly connected, in fact each one of them uniquely determining the other. This will lead us to the notion of strong L-Galois Connection. But let us first investigate some properties of L-Galois connections between two fixed sets X and Y.

We make the following denotation:
(for any $a \in A$ and $x \in X$)
$\{a|x\}$ is the function from L^X defined by:

$$\{a|x\}(x') = \begin{cases} 0, & \text{if } x' \neq x \\ a, & \text{if } x' = x \end{cases}$$

for all $x' \in X$.

Let $(\uparrow, \Uparrow, \downarrow, \Downarrow)$ be an L-Galois Connection.

Lemma 3.1 For any $a \in L$, $x \in X$ and $y \in Y$, we have:

(1) $\{1|x\}^{\uparrow}(y) = \{1|y\}^{\Downarrow}(x)$;
(2) $\{1|x\}^{\Uparrow}(y) = \{1|y\}^{\downarrow}(x)$.

Proof.
(1) We know that $S_1(\{1|x\}, \{1|y\}^{\Downarrow}) = S_2(\{1|x\}^{\uparrow}, \{1|y\})$.
This means that

$$\bigwedge_{x'\in X}(\{1|x\}(x') \to \{1|y\}^{\Downarrow}(x'))$$
$$= \bigwedge_{y'\in Y}(\{1|x\}^{\uparrow}(y') \to \{1|y\}(y')) \ ,$$

that is

$1 \to \{1|y\}^{\Downarrow}(x) = 1 \Rightarrow \{1|x\}^{\uparrow}(y)$,
that is $\{1|y\}^{\Downarrow}(x) = \{1|x\}^{\uparrow}(y)$.
(2) has a similar proof.

Lemma 3.2 For any $a \in L$, $x \in X$ and $y \in Y$, we have:

(1) $a \Rightarrow \{1|y\}^{\downarrow}(x) \leq \{a|x\}^{\Uparrow}(y)$;
(2) $a \to \{1|y\}^{\Downarrow}(x) \leq \{a|x\}^{\uparrow}(y)$;
(3) $a \Rightarrow \{1|x\}^{\uparrow}(y) \leq \{a|y\}^{\Downarrow}(x)$;
(4) $a \to \{1|x\}^{\Uparrow}(y) \leq \{a|y\}^{\downarrow}(x)$.

Proof. We shall only prove (1), the rest being provable analogously.

By the definition of S_2,

$$S_2(\{a|x\}, \{\{1|y\}^{\downarrow}(x)|x\}) = \bigwedge_{x' \in X} [\{a|x\}(x') \Rightarrow \{\{1|y\}^{\downarrow}(x)|x\}(x')] = a \Rightarrow \{1|y\}^{\downarrow}(x) \ ,$$

hence, by (G1),

$$a \Rightarrow \{1|y\}^{\downarrow}(x) \leq S_1(\{\{1|y\}^{\downarrow}(x)|x\}^{\Uparrow}, \{a|x\}^{\Uparrow}) \ .$$

On the other hand, since $\{\{1|y\}^{\downarrow}(x)|x\} \subseteq \{1|y\}^{\downarrow}$, we have, by (G3),

$$1 = \{1|y\}(y) \leq \{1|y\}^{\downarrow\Uparrow}(y) \leq \{\{1|y\}^{\downarrow}(x)|x\}^{\Uparrow}(y) \ ,$$

therefore $\{\{1|y\}^{\downarrow}(x)|x\}^{\Uparrow}(y) = 1$.

Thus,

$$S_1(\{\{1|y\}^{\downarrow}(x)|x\}^{\Uparrow}, \{a|x\}^{\Uparrow}) = \bigwedge_{y' \in Y} [\{1|y\}^{\downarrow}(x)|x^{\Uparrow}(y') \to \{a|x\}^{\Uparrow}(y')] \leq \{1|y\}^{\downarrow}(x)|x^{\Uparrow}(y) \to \{a|x\}^{\Uparrow}(y) = 1 \to \{a|x\}^{\Uparrow}(y) = \{a|x\}^{\Uparrow}(y) \ .$$

Therefore $a \Rightarrow \{1|y\}^{\downarrow}(x) \leq \{a|x\}^{\Uparrow}(y)$.

Lemma 3.3 For any $a \in L$, $x \in X$ and $y \in Y$, we have:

(1) $a \Rightarrow \{1|x\}^{\Uparrow}(y) = \{a|x\}^{\Uparrow}(y)$;
(2) $a \to \{1|x\}^{\uparrow}(y) = \{a|x\}^{\uparrow}(y)$;
(3) $a \Rightarrow \{1|y\}^{\Downarrow}(x) = \{a|y\}^{\Downarrow}(x)$;
(4) $a \to \{1|y\}^{\downarrow}(x) = \{a|y\}^{\downarrow}(x)$.

Proof. We only prove (1). Let $a \in A$, $x \in X$ and $y \in Y$. From Lemma 3.1.(2) and Lemma 3.2.(1), we immediately get $a \Rightarrow \{1|x\}^{\Uparrow}(y) \leq \{a|x\}^{\Uparrow}(y)$. For the converse inequality, notice that $a = S_2(\{1|x\}, \{a|x\})$. Now, from (G1),

$$a \leq S_1(\{a|x\}^{\Uparrow}, \{1|x\}^{\Uparrow}) \leq \{a|x\}^{\Uparrow}(y) \to \{1|x\}^{\Uparrow}(y) \ .$$

From (GR3), we get, successively, that

$$a * \{a|x\}^{\Uparrow}(y) \leq \{1|x\}^{\Uparrow}(y)$$

and

$$\{a|x\}^{\Uparrow}(y) \leq a \Rightarrow \{1|x\}^{\Uparrow}(y) \ .$$

Lemma 3.4 L^X and L^Y are complete lattices with respect to the component-wise inclusion in which infima and suprema are the component-wise infima and suprema from L. Moreover, for any $(A_j)_{j\in J} \subseteq L^X$ and $(B_j)_{j\in J} \subseteq L^Y$, we have:

(1) $\left(\bigvee_{j\in J} A_j\right)^{\uparrow} = \bigwedge_{j\in J} A_j^{\uparrow}$;

(2) $\left(\bigvee_{j\in J} A_j\right)^{\Uparrow} = \bigwedge_{j\in J} A_j^{\Uparrow}$;

(3) $\left(\bigvee_{j\in J} B_j\right)^{\downarrow} = \bigwedge_{j\in J} B_j^{\downarrow}$;

(4) $\left(\bigvee_{j\in J} B_j\right)^{\Downarrow} = \bigwedge_{j\in J} B_j^{\Downarrow}$.

Proof. That L^X and L^Y are complete lattices with component-wise suprema and infima follows immediately from the fact that L is a complete lattice.

Further, we only prove (1). Let $B \in L^Y$ and $(A_j)_{j\in J} \subseteq L^X$. Applying, successively, Lemma 2.1.(11, 13 and 12), we get

$$\begin{aligned} S_2\left(B, \left(\bigvee_{j\in J} A_j\right)^{\uparrow}\right) &= S_1\left(\bigvee_{j\in J} A_j, B^{\Downarrow}\right) = \bigwedge_{x\in X}\left[\left(\bigvee_{j\in J} A_j(x)\right) \to B^{\Downarrow}(x)\right] \\ &= \bigwedge_{x\in X}\bigwedge_{j\in J}(A_j(x) \to B^{\Downarrow}(x)) = \bigwedge_{j\in J}\bigwedge_{x\in X}(A_j(x) \to B^{\Downarrow}(x)) \\ &= \bigwedge_{j\in J} S_1(A_j, B^{\Downarrow}) = \bigwedge_{j\in J} S_2(B, A_j^{\uparrow}) \\ &= \bigwedge_{j\in J}\bigwedge_{x\in X}(B(x) \Rightarrow A_j^{\uparrow}(x)) = \bigwedge_{x\in X}\bigwedge_{j\in J}(B(x) \Rightarrow A_j^{\uparrow}(x)) \\ &= \bigwedge_{x\in X}\left[B(x) \Rightarrow \left(\bigwedge_{j\in J} A_j^{\uparrow}(x)\right)\right] = S_2\left(B, \bigwedge_{j\in J}(A_j^{\uparrow})\right) \ . \end{aligned}$$

In particular, $B \subseteq \left(\bigvee_{j\in J} A_j\right)^{\uparrow}$ iff $B \subseteq \bigwedge_{j\in J}(A_j^{\uparrow})$, so $\left(\bigvee_{j\in J} A_j\right)^{\uparrow} = \bigwedge_{j\in J} A_j^{\uparrow}$.

Definition 3.2 An L-Galois connection $(\uparrow, \Uparrow, \downarrow, \Downarrow)$ is called **strong** if it satisfies one of the following equivalent conditions:

(i) $\{1|x\}^{\uparrow} = \{1|x\}^{\Uparrow}$ for all $x \in X$.
(ii) $\{1|y\}^{\downarrow} = \{1|y\}^{\Downarrow}$ for all $y \in Y$.

That (i) and (ii) are equivalent one can see from Lemma 3.1.

Proposition 3.3 An L-Galois connection is strong iff it is induced by some L-relation. There is a bijective correspondence between the set of L-relations and the set of strong L-Galois connections (between X and Y).

Proof. Let $I \in L^{X\times Y}$ be an L-relation. We know that it naturally induces an L-Galois Connection, $C_I = (\uparrow_I, \Uparrow_I, \downarrow_I, \Downarrow_I)$. Let us show that this L-Galois connection is strong. Take $x \in X$ and $y \in Y$. Since

$$\{1|x\}^{\uparrow}(y) = \bigwedge_{x'\in X}[\{1|x\}(x') \to I(x',y)]$$
$$= 1 \to I(x,y) = I(x,y)$$

and

$$\{1|x\}^{\Uparrow}(y) = \bigwedge_{x'\in X}[\{1|x\}(x') \Rightarrow I(x',y)]$$
$$= 1 \Rightarrow I(x,y) = I(x,y) \;,$$

it follows that $\{1|x\}^{\uparrow}(y) = \{1|x\}^{\Uparrow}(y)$. Thus, for any $x \in X$, $\{1|x\}^{\uparrow} = \{1|x\}^{\Uparrow}$.

Consider now $(\uparrow, \Uparrow, \downarrow, \Downarrow)$ a strong L-Galois connection and denote it with C. We define $I_C \in L^{X\times Y}$ as follows: for any $x \in X$ and $y \in Y$,

$$I_C(x,y) = \{1|x\}^{\uparrow}(y) = \{1|x\}^{\Uparrow}(y) = \{1|y\}^{\downarrow}(x)$$
$$= \{1|y\}^{\Downarrow}(x) \;.$$

This definition is correct according to the definition of strong L-Galois connection and Lemma 3.1.

Let us prove now that the two correspondences from above ($I \mapsto C_I$ and $C \mapsto I_C$) between L-relations and L-Galois connections are invertible and inverse to each other.

Consider $I \in L^{X\times Y}$ and let $x \in X$ and $y \in Y$. Then

$$I_{C_I}(x,y) = \{1|x\}^{\uparrow_I}(y) = \bigwedge_{x'\in X}(\{1|x\}(x') \to I(x',y))$$
$$= 1 \to I(x,y) = I(x,y) \;.$$

So $I_{C_I} = I$.

Now, take $C = (\uparrow, \Uparrow, \downarrow, \Downarrow)$ to be an L-Galois Connection. We show that $C = C_{I_C}(= (\uparrow_{I_C}, \Uparrow_{I_C}, \Downarrow_{I_C}, \downarrow_{I_C}))$. Remember that L^X and L^Y are complete lattices with respect to the component-wise inclusion in which infima and suprema are the component-wise infima and suprema from L. Let $A \in L^X$ and $B \in L^Y$. We have that: $A = \bigvee_{x\in X}\{A(x)|x\}$, so, according to Lemma 3.4.(1), $A^{\uparrow} = \bigwedge_{x\in X}(\{A(x)|x\}^{\uparrow})$.

Let $y \in Y$. From above and Lemma 3.3.(2), we get

$$A^{\uparrow}(y) = \bigwedge_{x\in X}(\{A(x)|x\}^{\uparrow})(y)$$
$$= \bigwedge_{x\in X}(A(x) \to \{1|x\}^{\uparrow}(y)) \;.$$

On the other hand,

$$A^{\uparrow_{I_C}}(y) = \bigwedge_{x\in X}(A(x) \to I_C(x,y))$$
$$= \bigwedge_{x\in X}(A(x) \to \{1|x\}^{\uparrow}(y)) \;.$$

Thus $A^{\uparrow_{I_C}} = A^{\uparrow}$. That $A^{\Uparrow_{I_C}} = A^{\Uparrow}$, for any $A \in L^X$ and $B^{\downarrow_{I_C}} = B^{\downarrow}$ and $B^{\Downarrow_{I_C}} = B^{\Downarrow}$, for any $B \in L^Y$, follow analogously.

Notice that, for a strong L-Galois Connection, the corresponding L-relation I_C can be defined in terms of any of the four functions from C (in fact, it was sufficient the restriction of one function to crisp singletons). This immediately gives:

Corollary 3.1 If $(\uparrow, \Uparrow, \downarrow, \Downarrow)$ and $(\uparrow', \Uparrow', \downarrow', \Downarrow')$ are two strong L-Galois connections, then the following statements are equivalent:

(1) $(\uparrow, \Uparrow, \downarrow, \Downarrow) = (\uparrow', \Uparrow', \downarrow', \Downarrow')$.
(2) $\uparrow = \uparrow'$.
(3) $\uparrow$ and $\uparrow'$ coincide on crisp singletons.
(4) $\Uparrow = \Uparrow'$.
(5) $\Uparrow$ and $\Uparrow'$ coincide on crisp singletons.
(6) $\downarrow = \downarrow'$.
(7) $\downarrow$ and $\downarrow'$ coincide on crisp singletons.
(8) $\Downarrow = \Downarrow'$.
(9) $\Downarrow$ and $\Downarrow'$ coincide on crisp singletons.

Actually, we can tell more about the above bijective correspondence, namely that, considering the partial order $\subseteq$ on $L^{X\times Y}$ and the component-wise $\subseteq$ on the set of strong L-Galois Connections between X and Y, this correspondence is an order isomorphism.

Proposition 3.4 Let $(\uparrow_1, \Uparrow_1, \downarrow_1, \Downarrow_1)$ $(\uparrow_2, \Uparrow_2, \downarrow_2, \Downarrow_2)$ be two strong L-Galois Connections between X and Y and I_1, I_2 the corresponding L-relations. Then the following are equivalent:

(1) $I_1 \subseteq I_2$;
(2) For each $A \in L^X$, $B \in L^Y$, we have $A^{\uparrow_1} \subseteq A^{\uparrow_2}$ and $B^{\Downarrow_1} \subseteq B^{\Downarrow_2}$;
(3) For each $A \in L^X$, $B \in L^Y$, we have $A^{\Uparrow_1} \subseteq A^{\Uparrow_2}$ and $B^{\downarrow_1} \subseteq B^{\downarrow_2}$;

Proof. "(1) implies (2)" and "(1) implies (3)" are easy consequences of $\to$ and $\Rightarrow$ being isotone in the second argument. Let us prove "(2) implies (1)". Particularizing (2), we get that, for any $x \in X$ and $y \in Y$, $\{1|x\}^{\uparrow_1}(y) \leq \{1|x\}^{\uparrow_2}(y)$, that is $I_1(x,y) \leq I_2(x,y)$. "(3) implies (1)" follows similarly.

The equivalence between (2) and (3) together with Corollary 3.1 immediately give:

Corollary 3.2 Fix, as usual, X and Y. Consider these five partially ordered sets:

- The set of strong L-Galois connections with the component-wise inclusion;
- The set of functions $\uparrow$ such that there exists the triple $(\Uparrow, \downarrow, \Downarrow)$ making $(\uparrow, \Uparrow, \downarrow, \Downarrow)$ a strong L-Galois connection, together with the component-wise inclusion;
- The set of functions $\Uparrow$ such that there exists the triple $(\uparrow, \downarrow, \Downarrow)$ making $(\uparrow, \Uparrow, \downarrow, \Downarrow)$ a strong L-Galois connection, together with the component-wise inclusion;
- The set of functions $\downarrow$ such that there exists the triple $(\uparrow, \Uparrow, \Downarrow)$ making $(\uparrow, \Uparrow, \downarrow, \Downarrow)$ a strong L-Galois connection, together with the component-wise inclusion;
- The set of functions $\Downarrow$ such that there exists the triple $(\uparrow, \Uparrow, \downarrow)$ making $(\uparrow, \Uparrow, \downarrow, \Downarrow)$ a strong L-Galois connection, together with the component-wise inclusion.

The five structures are isomorphic (and, of course, isomorphic to $(L^{X\times Y}, \subseteq)$).

So, for an L-Galois connection to have its components correlated, we had to postulate that $\uparrow$ and $\Uparrow$ (or, same thing, $\downarrow$ and $\Downarrow$) coincide on crisp singletons. This condition could be seen as a remedy for the "non-commutativity syndrome", which splits everything in two. Thus, the "healthy" version of L-Galois connection is the strong one. Notice also that strong L-Galois connections still generalize L-Galois connections from the commutative case. Our interest in the rest of the paper will be focused on strong L-Galois connections.

The next proposition says that, in non-trivial cases, our notion of strong L-Galois connection coincides with the one from [4] if and only if L is commutative.

Proposition 3.5 Suppose X, $Y \neq \emptyset$. Then the following are equivalent:

(i) L has the operation $*$ commutative;
(ii) For any strong L-Galois connection $(\uparrow, \Uparrow, \downarrow, \Downarrow)$ between X and Y, $\uparrow=\Uparrow$ and $\downarrow=\Downarrow$.

Proof. (i) implies (ii): Suppose $*$ is commutative, and hence, from Lemma 2.1.(15), $\rightarrow=\Rightarrow$. Let I be the corresponding L-relation. We have that, for any $A \in L^X$, $B \in L^Y$ and $x \in X$, $y \in Y$,

$$A^{\uparrow}(y) = \bigwedge_{x\in X}(A(x) \rightarrow I(x,y)) = \bigwedge_{x\in X}(A(x) \Rightarrow I(x,y)) = A^{\Uparrow}(y)$$

and

$$B^{\downarrow}(x) = \bigwedge_{y\in Y}(B(y) \rightarrow I(x,y)) = \bigwedge_{y\in Y}(B(y) \Rightarrow I(x,y)) = B^{\Downarrow}(x) \ .$$

So $\uparrow=\Uparrow$ and $\downarrow=\Downarrow$.

(ii) implies (i): Let $a, b \in L$. Take $A \in L^X$ as the constant function a and $I \in L^{X\times Y}$ as the constant function b. According to the hypothesis, $\uparrow_I=\Uparrow_I$ and thus, for an $y \in Y$,

$$a \rightarrow b = \bigwedge_{x\in X}(A(x) \rightarrow I(x,y) = \bigwedge_{x\in X}(A(x) \Rightarrow I(x,y) = a \Rightarrow b \ .$$

Applying again Lemma 2.1.(15), we get $*$ commutative.

In the rest of the section, generalizing another result from [4], we shall get a representation of strong L-Galois Connections by (crisp) Galois connections.

Like in [4], for $A \in L^X$ (and same goes for $A \in L^Y$, or $A \in L^{X\times Y}$) and $a \in L$, the a-cut of A is the crisp set ${}^aA = \{x \in X \ / \ A(x) \geq a\}$. As already mentioned, we view crisp subsets of X (i.e. elements from 2^X) as particular cases of L-subsets of X from L^X.

Lemma 3.5 Let $I \in L^{X\times Y}$, $(\uparrow, \Uparrow, \downarrow, \Downarrow)$ the (strong) L-Galois connection induced by I and, for any $a \in A$, $(^{\wedge_a}, ^{\vee_a})$ the crisp Galois connection between X and Y induced by the crisp relation ${}^aI$. Then:

(1) For any $a \in L$, $A \in 2^X$ and $B \in 2^Y$, we have:
$${}^a(A^{\uparrow}) = {}^a(A^{\Uparrow}) = A^{\wedge_a}; {}^a(B^{\downarrow}) = {}^a(B^{\Downarrow}) = B^{\vee_a} \ .$$
(2) For any $a \in L$, $A \in L^X$ and $B \in L^Y$, we have:
$${}^a(A^{\uparrow}) = \bigcap_{b\in L}({}^bA)^{\wedge_{a*b}}; \ {}^a(B^{\Downarrow}) = \bigcap_{b\in L}({}^bB)^{\vee_{b*a}} \ ;$$
$${}^a(A^{\Uparrow}) = \bigcap_{b\in L}({}^bA)^{\wedge_{b*a}}; \ {}^a(B^{\downarrow}) = \bigcap_{b\in L}({}^bB)^{\vee_{a*b}} \ .$$

Proof.
(1): For any $y \in Y$, we have that

$$\begin{aligned} y \in {}^a(A^{\uparrow}) &\text{ iff } A^{\uparrow}(x) \geq a \text{ iff } \bigwedge_{x\in X}(A(x) \rightarrow I(x,y)) \geq a \\ &\text{iff (}A\text{ being crisp) } \bigwedge_{x\in A}(1 \rightarrow I(x,y)) \geq a \\ &\text{iff } \bigwedge_{x\in A} I(x,y) \geq a \\ &\text{iff } (x,y) \in {}^aI \text{ for all } x \in A \text{ iff } x \in A^{\wedge_a} \ . \end{aligned}$$

Thus ${}^a(A^{\uparrow}) = A^{\wedge_a}$. Analogously, ${}^a(A^{\Uparrow}) = A^{\wedge_a}$ and ${}^a(B^{\downarrow}) = {}^a(B^{\Downarrow}) = B^{\vee_a}$.
(2): Let $y \in Y$. We have that

$$\begin{aligned} y \in {}^a(A^{\uparrow}) &\text{ iff } A^{\uparrow}(y) \geq a \\ &\text{iff } \bigwedge_{x\in X}(A(x) \rightarrow I(x,y)) \geq a \\ &\text{iff for all } x \in X, a \leq A(x) \rightarrow I(x,y) \\ &\text{iff for all } x \in X, a * A(x) \leq I(x,y). \quad \# \end{aligned}$$

On the other hand,

$$\begin{aligned} &y \in \bigcap_{b\in L}({}^bA)^{\wedge_{a*b}} \text{ iff} \\ &\quad \text{for all } b \in L, \text{ for all } x \in {}^bA, I(x,y) \geq a * b \text{ iff} \\ &\text{for all } x \in X, \text{ for all } b \in L, b \leq A(x) \text{implies } a * b \\ &\qquad \leq I(x,y). \quad \#\# \end{aligned}$$

It is immediate that ## implies # (taking b as $A(x)$). Conversely, suppose # holds and let $x \in X$ and $b \in B$ such that $b \leq A(x)$. Then $a * b \leq a * A(x) \leq I(x,y)$. So # implies ##. This means that $y \in {}^a(A^{\uparrow})$ iff $y \in \bigcap_{b\in L}({}^bA)^{\wedge_{a*b}}$, for any arbitrary $y \in Y$. Thus ${}^a(A^{\uparrow}) = \bigcap_{b\in L}({}^bA)^{\wedge_{a*b}}$ The other three equalities from (2) follow in the same way.

The definition of an L-nested system is taken from [4]:

Definition 3.3 A system $\{(^{\wedge_a}, ^{\vee_a}) \ / \ a \in L\}$ of crisp Galois connections is called L-**nested** if:

(a) for each $a, b \in L$, $a \leq b$, $A \in 2^X$, $B \in 2^Y$, we have $A^{\wedge_b} \subseteq A^{\wedge_a}$, $B^{\vee_b} \subseteq B^{\vee_a}$;
(b) for every $x \in X, y \in Y$, the set $\{a \in L \ / \ y \in \{x\}^{\wedge_a}\}$ has the greatest element.

Proposition 3.6 Let X and Y be two sets. There is a bijective correspondence from the set of strong L-Galois Connections between X and Y to the set of L-nested systems of crisp Galois connections between X and Y.

Proof. Let $C = (\uparrow, \Uparrow, \downarrow, \Downarrow)$ be an L-Galois connection. Define $S_C = \{(\wedge_a, \vee_a) \ / \ a \in L\}$, where, if $a \in L$, $A^{\wedge_a} = {}^a(A^{\uparrow}) = {}^a(A^{\Uparrow})$ and $B^{\vee_a} = {}^a(B^{\Downarrow}) = {}^a(B^{\Downarrow})$ for any $A \in 2^X$, $B \in 2^Y$. The definition is correct according to Lemma 3.5.(1). We show that S_C is an L-nested system. Let I be the L-relation that corresponds to C. If $a \leq b$, then ${}^bI \subseteq {}^aI$. Applying Lemma 3.5.(1), $(\wedge_a, \vee_a)$ and $(\wedge_b, \vee_b)$ are the crisp Galois connections induced by ${}^aI$ and ${}^bI$; thus, from Proposition 3.4. (that treats, in particular, the crisp case), we get $A^{\wedge_b} \subseteq A^{\wedge_a}$ and $B^{\vee_b} \subseteq B^{\vee_a}$ for all $A \in 2^X$, $B \in 2^Y$. In addition, for any $x \in X$ and $y \in Y$, the set $\{a \in L \ / \ y \in \{x\}^{\wedge_a}\}$ is precisely $\{a \in L \ / \ I(x,y) \geq a\}$, which has the greatest element, namely $I(x,y)$.

Consider now $S = \{({}^{\wedge_a}, {}^{\vee_a}) \ / \ a \in L\}$ an L-nested system. Define $C_S = (\uparrow_S, \Uparrow_S, \downarrow_S, \Downarrow_S)$, with $\uparrow_S, \Uparrow_S \colon L^X \longrightarrow L^Y$ and $\downarrow_S, \Downarrow_S \colon L^Y \longrightarrow L^X$ as follows: for any $A \in L^X$, $B \in L^Y$, $x \in X$, $y \in Y$,

$$A^{\uparrow_S}(y) = \bigvee \left\{ a \ / \ y \in \bigcap_{b \in L} ({}^bA)^{\wedge_{a*b}} \right\};$$

$$B^{\downarrow_S}(x) = \bigvee \left\{ a \ / \ x \in \bigcap_{b \in L} ({}^bB)^{\vee_{a*b}} \right\};$$

$$A^{\Uparrow_S}(y) = \bigvee \left\{ a \ / \ y \in \bigcap_{b \in L} ({}^bA)^{\wedge_{b*a}} \right\};$$

$$B^{\Downarrow_S}(x) = \bigvee \left\{ a \ / \ x \in \bigcap_{b \in L} ({}^bB)^{\vee_{b*a}} \right\} .$$

Let us show that C_S is a strong L-Galois connection. Consider the L-relation I defined by

$$I(x,y) = \bigvee \{a \mid y \in \{x\}^{\wedge_a}\} = \bigvee \{a \mid x \in \{y\}^{\wedge_a}\}$$

for all $x \in X$, $y \in Y$ and $(\uparrow_I, \Uparrow_I, \downarrow_I, \Downarrow_I)$ the corresponding L-Galois Connection. We shall prove that $(\uparrow_S, \Uparrow_S, \downarrow_S, \Downarrow_S) = (\uparrow_I, \Uparrow_I, \downarrow_I, \Downarrow_I)$. From the definition of I, $({}^{\wedge_a}, {}^{\vee_a})$ is the crisp Galois connection induced by ${}^aI$. Thus, according to Lemma 3.5(2), for any $A \in L^X$ and $y \in Y$,

$$A^{\uparrow_I}(y) = \bigvee \{a \ / \ y \in {}^a(A^{\uparrow_I})\}$$
$$= \bigvee \left\{ a \ / \ y \in \bigcap_{b \in L} ({}^bA)^{\wedge_{a*b}} \right\} = A^{\uparrow}(y) \ .$$

Similarly, we get $A^{\Uparrow_I} = A^{\Uparrow_S}$, $A^{\downarrow_I} = A^{\downarrow_S}$, $A^{\Downarrow_I} = A^{\downarrow_S}$.

Thus we defined two mappings $(C \mapsto S_C)$ and $S \mapsto C_S$ between strong L-Galois Connections and L-nested systems. Let us show that these mappings are invertible and inverse to each other.

Let $C = (\uparrow, \Uparrow, \downarrow, \Downarrow)$ an L-Galois connection and let $A \in L^X$, $y \in Y$. It is obvious that $A^{\uparrow} = \bigvee \{a \ / \ x \in {}^a(A^{\uparrow})\}$. By Lemma 3.5.(2),

$$A^{\uparrow} = \bigvee \{a \ / \ x \in^a (A^{\uparrow})\} = \bigvee \left\{ a \ / \ y \in \bigcap_{b \in L} ({}^bA)^{\wedge_{a*b}} \right\} \ ,$$

where $\wedge_{a*b}$ is the first component of the crisp Galois connection induced by the relation ${}^aI$, which is, via Lemma 3.5.(1), exactly the first component of the element from S_C corresponding to $a * b$. And similar statements can be proved for $\Uparrow$, $\downarrow$ and $\Downarrow$, obtaining that $C = C_{S_C}$.

Finally, let $S = \{({}^{\wedge_a}, {}^{\vee_a}) \ / \ a \in L\}$ an L-nested system. Let $A \in 2^X$ and $a \in L$. We know, from Lemma 3.5.(1), that ${}^a(A^{\uparrow_C}) = A^{\wedge_a}$ and the similar statements for $\Uparrow_C$, $\downarrow_C$, and $\Downarrow_C$, meaning together that $S = S_{C_S}$.

Remark 3.2 Notice that we did not have to enrich, because of non-commutativity, the notion of L-nested system. Thus, this notion is "commutative-free", like those of fuzzy sets and fuzzy relations.

4 Galois connections modulo equality

In what follows, we shall prove a generalization of the theorem stating the one-to-one correspondence between strong L-Galois Connections and L-relations. Instead of just sets, we shall take sets with L-equalities. This will also generalize Proposition 7 from [7].[1]

Definition 4.1 Let U be a set. A binary L-relation $\approx \in Ł^{U \times U}$ (in infixed notation) is called L-equality on U if, for all $x, y, z \in U$,

(i) $(x \approx x) = 1$ (reflexivity);
(ii) $(x \approx y) = (y \approx x)$ (symmetry);
(iii) $((x \approx y) * (y \approx z)) \wedge ((y \approx z) * (x \approx y)) \leq (x \approx z)$ (transitivity)
(iv) $(x \approx y) = 1$ implies $x = y$.

Definition 4.2 Let $(X, \approx_X)$ and $(Y, \approx_Y)$ two sets with L-equalities. An L-relation R between $(X, \approx_X)$ and $(Y, \approx_Y)$ is an L-relation between X and Y which is compatible with $\approx_X$ and $\approx_Y$, i.e., for all $x_1, x_2 \in X$, $y_1, y_2 \in Y$,

$$(x_1 \approx_X x_2) * R(x_1, y_1) * (y_1 \approx_Y y_2) \leq R(x_2, y_2)$$

and

$$(y_1 \approx_X y_2) * R(x_1, y_1) * (x_1 \approx_Y x_2) \leq R(x_2, y_2) \ .$$

Denote with $L^{(X, \approx_X) \times (Y, \approx_Y)}$ the set of L-relations between $(X, \approx_X)$ and $(Y, \approx_Y)$.

Definition 4.3 Let $(X, \approx_X)$ be a set with L-equality. An L-subset $A \in L^X$ is said to be:

– left extensional w.r.t. $(\approx_X)$ if, for all $x, x' \in X$,

$A(x) * (x \approx_X x') \leq A(x')$;

– right extensional w.r.t. $\approx$ if, for all $x, x' \in X$,

$(x' \approx_X x) * A(x) \leq A(x')$;

– extensional w.r.t. $\approx_X$ if it is both left and right extensional w.r.t. $\approx_X$.

Denote with $L_l^{(X, \approx_X)}$ ($L_r^{(X, \approx_X)}$, $L^{(x, \approx_X)}$) the set of L-sets that are left extensional (respectively right extensional, extensional) w.r.t. $\approx_X$.

Definition 4.4 Let $(X, \approx_X)$ and $(Y, \approx_Y)$ be two sets with L-equalities. An L-Galois connection between $(X, \approx_X)$ and $(Y, \approx_Y)$ is a quadruple $(\uparrow, \Uparrow, \downarrow, \Downarrow)$ of functions

$$\uparrow \colon L_l^{(X, \approx_X)} \longrightarrow L_r^{(Y, \approx_Y)} \ , \ \Uparrow \colon L_r^{(X, \approx_X)} \longrightarrow L_l^{(Y, \approx_Y)},$$
$$\downarrow \colon L_l^{(Y, \approx_Y)} \longrightarrow L_r^{(X, \approx_X)} \ , \ \Downarrow \colon L_r^{(Y, \approx_Y)} \longrightarrow L_l^{(X, \approx_X)}$$

[1] In [7], the author consider the proof of this proposition a simple adaptation of the one that does not consider equalities on the universes and leaves it to the reader. In the non-commutative case, the proof will encounter a few technical difficulties, so we give it here.

such that:

(a) $S_1(A_1,A_2) \leq S_2(A_2^{\uparrow}, A_1^{\uparrow})$; $S_2(A_1,A_2) \leq S_1(A_2^{\Uparrow}, A_1^{\Uparrow})$;
(b) $S_1(B_1,B_2) \leq S_2(B_2^{\downarrow}, B_1^{\downarrow})$; $S_2(B_1,B_2) \leq S_1(B_2^{\Downarrow}, B_1^{\Downarrow})$;
(c) $A \subseteq A^{\uparrow\Downarrow}$; $A \subseteq A^{\Uparrow\downarrow}$;
(d) $B \subseteq B^{\downarrow\Uparrow}$; $B \subseteq B^{\Downarrow\uparrow}$.

Consider now a crisp singleton from L^X, that is an L-set of the form $\{1|x\}$, with $x \in X$ and $\approx$ an L-equality on X. We have that, for each $x, x' \in X$,

$$\{1|x\}(x') * (x' \approx x'') = \begin{cases} 0, \text{ if } x \neq x', \\ x \approx x'', \text{ otherwise.} \end{cases}$$

Hence $\{1|x\}$ is left extensional w.r.t. $\approx$. Analogously, $\{1|x\}$ is also right extensional w.r.t. $\approx$. Thus the following definition makes sense:

Definition 4.5 An L-Galois connection between $(X, \approx_X)$ and $(Y, \approx_Y)$ is called strong if one of the following equivalent conditions holds:

(1) $\uparrow$ and $\Uparrow$ coincide on the crisp singletons of L^X;
(2) $\downarrow$ and $\Downarrow$ coincide on the crisp singletons of L^Y.

The two above conditions are equivalent because, for each $x \in X$ and $y \in Y$, we have $\{1|x\}^{\uparrow}(y) = \{1|y\}^{\Downarrow}(x)$ and $\{1|x\}^{\Uparrow}(y) = \{1|y\}^{\downarrow}(x)$.
(this has a proof similar to the one of Lemma 3.1)

Proposition 4.1 Let $(X, \approx_X)$ and $(Y, \approx_Y)$ be two sets with L-equalities. Then there are as many strong L-Galois connections between $(X, \approx_X)$ and $(Y, \approx_Y)$ as L-relations from $L^{(X,\approx_X)\times(Y,\approx_Y)}$.

Proof. Let $(\uparrow, \Uparrow, \downarrow, \Downarrow)$ *be an* **L**-*Galois connection between* $(\mathbf{X}, \approx_X)$ *and* $(\mathbf{Y}, \approx_Y)$. *Define* $\mathbf{I} \in \mathbf{L}^{X\times Y}$ *by*

$$I(x,y) = \{1|x\}^{\uparrow}(y) = \{1|x\}^{\Uparrow}(y) = \{1|y\}^{\downarrow}(x) = \{1|y\}^{\Downarrow}(x)$$

for all $x \in X$, $y \in Y$. We show that I is in $L^{(X,\approx_X)\times(Y,\approx_Y)}$. Let $x, x' \in X$, $y, y' \in Y$. We apply, successively, the left extensionality of $\{1|x\}^{\Uparrow}$, the fact that $\{1|x\}^{\Uparrow}(y') = \{1|y'\}^{\downarrow}(x)$, the symmetry of $\approx_X$ and the right extensionality of $\{1|y'\}^{\downarrow}$:

$$\begin{aligned}
&(x \approx_X x') * I(x,y) * (y \approx_Y y') \\
&\quad = (x \approx_X x') * (\{1|x\}^{\Uparrow}(y) * (y \approx_Y y')) \\
&\quad \leq (x \approx_X x') * \{1|x\}^{\Uparrow}(y') = (x \approx_X x') * \{1|y'\}^{\downarrow}(x) \\
&\quad = (x' \approx_X x) * \{1|y'\}^{\downarrow}(x) \leq \{1|y'\}^{\downarrow}(x') = I(x',y') \ .
\end{aligned}$$

Analogously,

$$\begin{aligned}
&(y \approx_Y y') * I(x,y) * (x \approx_X x') \\
&\quad = (y' \approx_Y y) * \{1|x\}^{\uparrow}(y) * (x \approx_X x') \\
&\quad \leq \{1|x\}^{\uparrow}(y') * (x \approx_X x') \\
&\quad = \{1|y'\}^{\Downarrow}(x) * (x \approx_X x') \leq \{1|y'\}^{\Downarrow}(x') = I(x',y') \ .
\end{aligned}$$

Consider now $I \in L^{(X\approx_X)\times(Y\approx_Y)}$. Define $(\uparrow, \Uparrow, \downarrow, \Downarrow)$ as follows:

$$\begin{aligned}
A_1^{\uparrow}(y) &= \bigwedge_{x\in X}(A_1(x) \to I(x,y));\ A_2^{\Uparrow}(y) \\
&= \bigwedge_{x\in X}(A_2(x) \Rightarrow I(x,y)); \\
B_1^{\downarrow}(x) &= \bigwedge_{y\in Y}(B_1(y) \to I(x,y));\ B_2^{\Downarrow}(y) \\
&= \bigwedge_{y\in Y}(B_2(y) \Rightarrow I(x,y))
\end{aligned}$$

for all $A_1 \in L_l^{(X,\approx_X)}$, $A_2 \in L_r^{(X,\approx_X)}$, $B_1 \in L_l^{(Y,\approx_Y)}$, $B_2 \in L_r^{(Y,\approx_Y)}$, $x \in X$, $y \in Y$.

From a similar reason as in the proof of Proposition 3.3, we have that conditions (a)–(d) from Definition 4.3 hold. It remains to show that, if $A \in L_r^{(X,\approx_X)}$, then $A^{\uparrow} \in L_l^{(Y,\approx_Y)}$ and the other three corresponding statements for $\Uparrow, \downarrow, \Downarrow$. We only prove the one mentioned above, the rest following similarly. Take $y, y' \in Y$. We have that, for each $x \in X$,

$$\begin{aligned}
&(y \approx_Y y') * (A(x) \to I(x,y)) * A(x) \leq (y \approx_Y y') * I(x,y) \\
&\quad = (y \approx_Y y') * I(x,y) * (x \approx_X x) \leq I(x,y')
\end{aligned}$$

(we applied Lemma 2.1.(8) and the compatibility of I). Further, by residuation, we have:
for each $x \in X$, $(y \approx_Y y') * A(x) \to I(x,y)) \leq A(x) \to I(x,y')$ #.

We now apply, successively, the symmetry of $\approx_Y$, Lemma 2.1.(10) and #:

$$\begin{aligned}
&(y' \approx_Y y) * A^{\uparrow}(y) \\
&\quad = (y \approx_Y y') * A^{\uparrow}(y) = (y \approx_Y y') * \bigwedge_{x\in X}(A(x) \to I(x,y)) \\
&\quad \leq \bigwedge_{x\in X}[(y \approx_Y y') * (A(x) \to I(x,y))] \\
&\quad \leq \bigwedge_{x\in X}(A(x) \to I(x,y')) = A^{\uparrow}(y') \ .
\end{aligned}$$

Thus we have defined two mappings between strong L-Galois connections and compatible L-relations. By definition, the two mappings are both restrictions and corestrictions of the mappings defined in the proof of Proposition 3.3, hence they are, like those, bijective and inverse to each other.

References

1. **Abrusci VM, Ruet P** (2000) Non-commutative logic: the multiplicative frequent. Annals of Pure and Apll. Logic, **101**: 29–64
2. **Baudot R** (2000) Non-commutative logic programming language NoCLog. In: Symposium LCCS 2000(Santa Barbara), Short presentation, pp. 3
3. **Baudot R, Fouquré C** Theorem prover with focusing in non-commutative logic, www.lipn.univ-paris13.fr/baudot/publi.html
4. **Bělohlávek R** (1999) Fuzzy Galois Connections. Math Logic Quarterly **45**(4): 497–504
5. **Bělohlávek R** (2001) Lattices of fixed points of Fuzzy Galois Connections. Math Logic Quarterly **47**(1): 111–116
6. **Bělohlávek R** (2000) Similarity relations in concept lattices. J Logic Comput, **10**(6): 823–845

7. **Bělohlávek R** Concept lattices and order in fuzzy logic. Annals of Pure and Appl Logic
8. **Birkhoff G** (1967) Lattice Theory. AMS Coll Publ 25, Providence, RI
9. **Flondor P, Georgescu G, Iorgulescu A** (2001) Pseudo t-norms and pseudo BL-algebras. Soft Computing **5**: 335–371
10. **Georgescu G, Popescu A** (2002) Concept lattices and similarity in non-commutative fuzzy logic. Fundamenta informatical **53**(1): 23–54
11. **Hájek P** (1998) Metamathematics of fuzzy logic. Kluwer Academic Publishers, Dordrecht
12. **Hajek P** Fuzzy logic with non-commutative conjunctions
13. **Hajek P** Observations on non-commutative fuzzy logic. To appear in Soft Computing
14. **Ore O** (1944) Galois connections Trans AMS **55**: 493–513
15. **Turunen E** (1999) Mathematics behind fuzzy logic. Physica-Verlag, Heidelberg
16. **Wille R** (1982) Restructuring lattice theory: an approach based on hierarchies of concepts. In: Rival I (ed.), Ordered Sets, Reidel, Dordrecht-Boston, pp. 445–470
17. **Zadeh L** (1965) Fuzzy sets. Information and control **8**: 338–353