Original paper

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Robust control of systems with fuzzy representation of uncertainties

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Abstract In this paper, an approach is proposed to design robust controllers for uncertain systems with the linguistic uncertainties represented by fuzzy sets. With a provided technique, the fuzzy sets are best approximated by intervals (crisp sets). Then the Kharitonov's theorem is applied to construct a robust *PID* controller for the uncertain plant with time-invariant uncertainties represented by interval models. Also, for the uncertain system with linguistic values of the time-varying uncertainties best approximated by intervals (which are bounded), a robust sliding mode controller is developed to stabilize the uncertain system if the sliding coefficient conditions are satisfied. Moreover, the best approximation intervals are shown to be more related to the possibility distribution of the elements in the universes of discourse of fuzzy sets than the type of membership functions used for fuzzy sets. Examples and simulation results are included to indicate the design approach and the effectiveness of the proposed robust controller.

Keywords Robust control, Fuzzy representation, Uncertain system

1 Introduction

It is known that the exact mathematical model of the controlled plant is not always available in the real world. That is, the mathematical plant model includes uncertainties. The structured (parametric) uncertainties may arise from the major variations in the parameters due to mass series production, to different operating environment, and to aging problem. For example, the aerodynamic coefficients in flight control vary with flight environment [9]. The unstructured uncertainties appear when a parameterized mathematical model fails to specify the system with dynamic models. Many mechanisms (e.g. intervals, lingustic information, etc.) can be used as the representations of uncertainties to describe the difference between the models and real plants.

For the plant parameters which can be measured or estimated, the intervals are usually taken to indicate the uncertainties from the imprecision of the measurement and the improper estimation. The parameter variations are defined by appropriate subsets of parameter space. For a linear system with interval uncertainties, Kharitonov's theorem can be used to analyze the stability of the uncertain system [5]. By using the analysis of stability, a robust controller is able to be designed for the system with uncertain interval parameters. However, the robust controller based on an entire interval may become too conservative to have satisfactory performance. Moreover, the precise invervals for the uncertainties from the variations of parameters are difficult to be specified. Further, the measurements or estimates might not be obtainable all the time in reality. In this case, the linguistic information of the uncertain plant provided by experts becomes important for the control of the uncertain plant.

When the errors between the model structures and the plant dynamic structures occur, the bounds of uncertainty norms represents the unstructured uncertainties. If the norm-bound is known as a priori and the uncertainties satisfy some required conditions, the uncertain plant can be controlled with the sliding mode control techniques [2]. The H^{∞} technique can be applied to the uncertain plant with predefined norm-bounded uncertainties. It is known that the uncertainties might not satisfy the matching conditions, and the required bounds for the norm of uncertainties might not be obtainable, either. Thus, the experts' knowledge as an alternative representation of unstructured uncertainties is great helpful for the design of a robust controller to the variations of the plant dynamic structures.

Nevertheless, it is difficult to combine the linguistic and numerical information [7]. In order to include the linguistic information in the numerical system, a proper interpretation of the linguistic terms is necessary. One of the popular way to interpret the linguistic terms is to represent the linguistic terms as fuzzy sets [8]. For each fuzzy set, a membership function is defined to assign a value (from $[0, 1]$) to every element in the input universe of discourse [3, 4]. The fuzzy representation of uncertainties not only indicates the interval of the variations (by the support of the fuzzy set), but also describe the possibility of each different value in the variation interval (by the membership function). With this concept, Nguyen and Kreinovich design an algorithm to calculate the degree of belief that the chosen control strategy can stabilize the linear system with fuzzy representation of uncertainties [5]. Even the control strategies can be evaluated by the corresponding degree of belief that the control system is stable, it is computational time consuming to develop the satisfactory

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controller with this approach. Recently, to minimize the degree of the difference between a fuzzy set and a approximated crisp set, Nguyen derives a sufficient condition for a crisp set to be the best approximation of a fuzzy set [6]. With the fuzzy sets appropriately approximated by crisp sets, many techniques metioned above can be applied to design a robust controller since the crisp sets are bounded. Since the concept "every element in the defined universe of discourse is possible for the uncertain parameter" is used to be implicitly expressed in the expert's linguistic information, the support of the fuzzy set may come out to be a very large or even an infinite set. However, the experts might not really mean that every possible element for the parameter will occur when the system is in the normal or expected situation. The best crisp approximation of a fuzzy set indicates an interval with high possibility (membership values) elements which may actually represents the region of the regular occurrence of parameter. Because the approximated crisp sets contain only the values of variations with high possibility of occurrence, the controller based on the approximated crisp sets will not lead to a conservative design. Thus, the control system is expected to have satisfactory performance and reasonable robustness to the variations of the system plant. Even so, the sufficient condition in [6] can not specify an unique crisp set. That is, there are more than one crisp set which can satisfies the sufficient condition in [6]. This phenomenon leads to the confusion for the design of a robust controller.

In this paper, robust controllers for uncertain systems are designed. With the uncertainties described by linguistic information and represented as fuzzy sets, the uncertain system is said to be a system with fuzzy Representation of uncertainties (*SFRU*). An approach is provided to implement the idea in [6] for approximating a fuzzy set by an unique crisp set. Thus, a system with fuzzy representation of uncertainties (*SFRU*) becomes a system with interval uncertainties. The best approximation intervals of fuzzy sets are shown to be more related to the possibility distribution of the elements in the universe of discourse of fuzzy sets than the type of membership functions used for fuzzy sets. For a linear system with interval structured (parametric) uncertainties, a robust *PID* controller is designed based on the analyses of the stability of the uncertain control system with Kharitonov's theorem [1]. Moreover, a robust sliding mode controller is designed for the system *SFRU* with time-varying mismatched uncertainties satisfying the sliding coefficient conditions. The simulation results are also provided to show the effectiveness of the proposed robust controller.

The remainder of this paper is organized as follows. The uncertain system (*SFRU*) with linguistic information (which is represented as fuzzy sets) from experts is described in Sect. 2. The sufficient condition (in [6]) for approximating a fuzzy set by a crisp set is introduced in Sect. 3. Also in Sect. 3, Kharitonov's theorem to analyze the stability of the system with interval uncertainties is reviewed. Section 4 presents the proposed approach to implement the best approximation of a fuzzy set by an unique interval. A robust controller is designed for the uncertain system with the application of Kharitonov's theorem in Sect. 5. A robust sliding mode controller is designed for the uncertain system (*SFRU*) with time-varying mismatched uncertainties in Sect. 5. Simulation results are given in Sect. 6. Finally, a conclusion is provided in Sect. 7.

2 An uncertain system *SFRU*

The uncertain system in Fig. 1 is considered to have a linear plant with uncertain parameters in this paper. The only information available for the uncertainties in the plant are the experienced linguistic information from experts. Thus, the uncertain plant can be defined in general with the transfer function as

$$P(s) = \frac{l_m s^m + l_{m-1}s^{m-1} + \cdots + l_1 s^1 + l_0}{L_n s^n + L_{n-1}s^{n-1} + \cdots + L_1 s^1 + L_0}, \quad n > m \ , \tag{1}$$

where $L_i, l_j; i = 0, 1, \dots, n, \ j = 0, 1, \dots, m$, are the uncertain parameters with linguistic information from experts. For example, l_0 may represent

$$l_0 : l_0 \text{ is large} \tag{2}$$

from expert. It is known that the classical techniques can not be applied for the uncertain system with the linguistic information. Also, the fuzzy logic has been shown to be a proper and popular approach to implement the expert knowledge which is in the linguistic forms. Therefore, the linguistic terms in the expert's information is interpreted as fuzzy sets, e.g., the term "large" in Eq. (2) is defined to be a fuzzy set LG with a membership function μ_{LG}. Then from Eq. (2), l_0 can be considered to be equal to the fuzzy set LG, i.e.,

$$l_0 = LG \ .$$

Likewise, the uncertain parameters $L_i, l_j; \ i = 0, 1, \dots, n$, $j = 0, 1, \dots, m$, represent the corresponding fuzzy sets. With the fuzzy sets $L_i, l_j; i = 0, 1, \dots, n, j = 0, 1, \dots, m$, the uncertain plant with the transfer function in Eq. 1 is defined as an uncertain plant with uncertainties represented by fuzzy sets (*SFRU*).

3 Interval approximation of fuzzy sets and Kharitonov's theorem

In this section, the idea in [6] for the best approximation of a fuzzy set by an interval is introduced. Also, the Kharitonov's theorem for the stability analyses of an uncertain system with interval uncertainties is reviewed.

3.1 Interval approximation of fuzzy sets (Nguyen(2000))

It is known that two sets (A and B) are equal if and only if

$$F_A(x) = F_B(x); \forall\, x \in U \iff A \subseteq B \ (F_A(x) \le F_B(x)) \quad \text{and} \quad A \supseteq B \ (F_A(x) \ge F_B(x)) \ , \tag{3}$$

where F_A and F_B are membership functions for A and B, and U is an universe of discourse for A and B. Note that A and B could be fuzzy or crisp sets. Thus, A and B are different ($A \neq B$) if

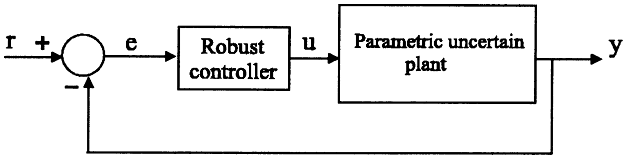

Fig. 1. Block diagram of the closed loop fuzzy uncertain system with a robust controller

$$A \nsubseteq B\ (F_A(x) > F_B\ (x)) \text{ and } A \nsupseteq B\ (F_A(x) < F_B(x)) \ .$$

Let the degree to which A and B are different be

$$\eth(A \neq B) = \max\{\eth(A \nsubseteq B), \eth(A \nsupseteq B)\} \ .$$

With the degree of $A \nsubseteq B$ $(F_A(x) > F_B(x))$ simply calculated by

$$\eth(A \nsubseteq B) = \int_{x \in U} \max\{0, F_A(x) - F_B(x)\} \mathrm{d}x \ ,$$

the degree of $A \neq B$ is

$$\eth(A \neq B) = \max\left\{ \int_{x \in U} \max\{0, F_A(x) - F_B(x)\} \mathrm{d}x, \int_{x \in U} \max\{0, F_B(x) - F_A(x)\} \mathrm{d}x \right\} \ .$$

For a fuzzy set A, to find a crisp set B^* which is the best approximation of A is to find a crisp set B^* such that $\eth(A \neq B^*)$ is minimum for all the crisp set B with the same universe of discourse as A. Since

$$F_B(x) = \begin{cases} 1, & \text{if } x \in B; \\ 0, & \text{if } x \notin B \end{cases} \ ,$$

$$\begin{aligned} \eth(A \nsubseteq B) &= \int_{x \in U} \max\{0, F_A(x) - F_B(x)\} \mathrm{d}x \\ &= \int_{x \in B^c} \max\{0, F_A(x) - F_B(x)\} \mathrm{d}x \\ &= \int_{x \in B^c} F_A(x) \mathrm{d}x \ , \end{aligned} \tag{4}$$

where B^c is the complement set of the crisp set B. Likewise,

$$\eth(A \nsupseteq B) = \int_{x \in B} (1 - F_A(x)) \mathrm{d}x \ .$$

It is easy to find that when the crisp set B becomes larger, the degree of $A \nsubseteq B$ is decreased, however, the degree of $A \nsupseteq B$ is increased. Similarly, the degree of $A \nsubseteq B$ is increased, however, the degree of $A \nsupseteq B$ is decreased when the crisp set B gets smaller. Therefore, the minimum of the degree of $A \neq B$ $(\eth(A \neq B))$ is attained when

$$\eth(A \nsubseteq B) = \eth(A \nsupseteq B) \ .$$

Proposition 1 *For a fuzzy set A, if a crisp set B satisfy*

$$\eth(A \nsubseteq B) = \eth(A \nsupseteq B) \ ,$$

i.e.,

$$\int_{x \in B^c} F_A(x) \mathrm{d}x = \int_{x \in B} (1 - F_A(x)) \mathrm{d}x \ ,$$

then $B = B^*$ is the best crisp approximation set of the fuzzy set A.

3.2 Kharitonov's theorem

Let the characteristic polynomial of an uncertain system with interval parameters be

$$p(s, \tilde{D}) = \sum_{i=0}^{n} D_i s^i = \sum_{i=0}^{n} [d_i^- d_i^+] s^i \ . \tag{5}$$

Associated with the characteristic polynomial $p(s, \tilde{D})$, four Kharitonov polynomials [5] are

$$\begin{aligned} K_1(s) &= d_0^- + d_1^- s + d_2^+ s^2 + d_3^+ s^3 + d_4^- s^4 + \cdots \\ K_2(s) &= d_0^+ + d_1^+ s + d_2^- s^2 + d_3^- s^3 + d_4^+ s^4 + \cdots \\ K_3(s) &= d_0^+ + d_1^- s + d_2^- s^2 + d_3^+ s^3 + d_4^+ s^4 + \cdots \\ K_4(s) &= d_0^- + d_1^+ s + d_2^+ s^2 + d_3^- s^3 + d_4^- s^4 + \cdots \end{aligned} \tag{6}$$

If the element zero is not included in the interval $[d_n^- d_n^+]$, the polynomial $p(s, \tilde{D})$ is a degree invariant polynomial. For a degree invariant characteristic polynomial $p(s, \tilde{D})$, the Kharitonov's theorem is as follows.

Theorem 1 *(Kharitonov's theorem (1978))*
If a characteristic polynomial $p(s, \tilde{D})$ with intervals is degree invariant, the system with the characteristic polynomial $p(s, \tilde{D})$ is stable if and only if the systems with four Kharitonov polynomials as characteristic polynomials are stable.

In the Kharitonov's theorem, the system with the characteristic polynomial $p(s, \tilde{D})$ being stable means that a family of systems which have the characteristic polynomial $p(s, d)$ such that

$$p(s, d) = \sum_{i=0}^{n} d_i s^i; \quad d_i \in [d_i^- d_i^+] \ \forall i \ ,$$

are stable. Note that for the characteristic polynomial $p(s, \tilde{D})$ with possible degree dropping (i.e., $0 \in [d_n^- d_n^+]$), the Theorem 1 can not be applied directly [1]. In this case, not only the characteristic polynomial $p(s, \tilde{D})$ with nonzero coefficient for D_n needs to be discussed, the characteristic polynomial $p(s, \tilde{D})$ with $D_n = 0$ (one degree lower) also needs to be considered. Also, the interval $D_i, i = 1, 2, \ldots, n$, is allowed to be a lumping interval. For a lumping interval D_j,

$$D_j = f(D_0, D_1, \ldots, D_i, \ldots, D_n), \quad j \neq i, \ f(\cdot) \text{ is a function} \ ,$$

the overbounding technique is utilized for simplicity to find the bounds of D_j,

$$d_j^- = \min_{d_0 \in D_0, d_1 \in D_1, \ldots, d_i \in D_i, \ldots, d_n \in D_n} (D_j) \ ,$$

and

$$d_j^+ = \max_{d_0 \in D_0, d_1 \in D_1, \ldots, d_i \in D_i, \ldots, d_n \in D_n} (D_j) \quad ,$$

in this paper.

Even only the sufficient condition of Kharitonov's condition is guaranteed when the overbounding technique is used [1], a robust controller can be designed with the sufficient condition of Kharitonov's condition.

4 Implementation of the interval approximation of fuzzy sets

Since the intervals are crisp sets, the main idea to approximate a fuzzy set A by a crisp set D is to find an interval D which has the minimum degree of difference between D and A [6]. As in Sect. 3.1, the best crisp set (interval) D satisfies

$$\int_{D^c} F_A(x)\mathrm{d}x = \int_D (1 - F_A(x))\mathrm{d}x \tag{7}$$

where D^c is the complement set of the crisp set D and $F_A(x)$ is the membership function of the fuzzy set A. However, it is easy to find that the condition in Eq. 7 is not enough to exactly define a interval D. That is, if the interval D in Eq. (7) is substituted by $D = [d_1 \ d_2]$, then we could only obtain one equation with two variable d_1, d_2. There will be more than one pair of d_1, d_2 which satisfies the condition in Eq. (7). In order to determine an unique interval D, the following approach is proposed for the usually used convex fuzzy set A. Let the interval $[a_1 \ a_2]$ be the region such that

$$F_A(x) = 1; \quad \forall x \in [a_1 \ a_2] \quad ,$$

where $F_A(x)$ is the membership function of A. Also, the center point a_c of the interval $[a_1 \ a_2]$ is specified as

$$a_c = \frac{a_1 + a_2}{2} \quad .$$

Based on the common sense that the best crisp approximation interval D covers the region which includes the points with high membership values, it is reasonable to have $a_c \in [d_1 \ d_2]$. That is,

$$d_1 < a_c < d_2 \quad .$$

Then d_1 and d_2 can be determined by the equations

$$\int_{-\infty}^{d_1} F_A(x)\mathrm{d}x = \int_{d_1}^{a_c} (1 - F_A(x))\mathrm{d}x \tag{8}$$

and

$$\int_{d_2}^{\infty} F_A(x)\mathrm{d}x = \int_{a_c}^{d_2} (1 - F_A(x))\mathrm{d}x \quad . \tag{9}$$

From Eqs. (8) and (9), it is straightforward to see that

$$\int_{-\infty}^{d_1} F_A(x)\mathrm{d}x + \int_{d_2}^{\infty} F_A(x)\mathrm{d}x$$
$$= \int_{d_1}^{a_c} (1 - F_A(x))\mathrm{d}x + \int_{a_c}^{d_2} (1 - F_A(x))\mathrm{d}x \quad ,$$

and it leads to

$$\int_{D^c} F_A(x)\mathrm{d}x = \int_D (1 - F_A(x))\mathrm{d}x \quad .$$

Therefore, the d_1, d_2 obtained from Eqs. (8) and (9) satisfy condition in Eq. (7), and D is a best approximation of A. To detail the approach of approximation, an example is presented next.

Example 1 Let the fuzzy set A with the membership function $F_A(x)$ (the solid line in Fig. 2),

$$F_A(x) = \exp(-x^2)$$

Also, the best crisp approximation of A, D, is defined as $D = [d_1 \ d_2]$. From Eq. (7), we have

$$\int_{-\infty}^{d_1} F_A(x)\mathrm{d}x + \int_{d_2}^{\infty} F_A(x)\mathrm{d}x = \int_{d_1}^{d_2} (1 - F_A(x))\mathrm{d}x \tag{10}$$

Since the membership function is symmetrical to the axis $x = 0$, it is reasonable to have

$$d_1 = -d_2 \quad ,$$

and Eq. (10) becomes

$$\int_{-\infty}^{-d_2} F_A(x)\mathrm{d}x + \int_{d_2}^{\infty} F_A(x)\mathrm{d}x = \int_{-d_2}^{d_2} (1 - F_A(x))\mathrm{d}x \quad .$$

By knowing that

$$\int_{-\infty}^{\infty} F_A(x)\mathrm{d}x = \sqrt{\pi} \quad ,$$

we can find that $d_2 = .5\sqrt{\pi}$ and as in Fig. 3, the best crisp approximation of the fuzzy set A is

$$D = \left[-.5\sqrt{\pi} \quad .5\sqrt{\pi}\right] \quad .$$

Since the concept "every element in the defined universe of discourse is possible for the uncertain parameter" is used to be implicitly expressed in the expert's linguistic information, the support of the fuzzy set may come out to be a very large or even an infinite set. However, the experts might not really mean that every possible element for the parameter will occur when the system is in the normal or expected situation. The best crisp approximation of a fuzzy set indicates an interval with high possibility (membership values) elements which may actually represents the region of the regular occurrence of parameter. It can be seen from Fig. 2 that the interval D is smaller than the support of the fuzzy set A, and the elements in interval D. As it is known that there are many types of membership functions can be assigned to the same fuzzy set. Thus, to see the difference between the types of membership functions for the same fuzzy set is important. The next example is provided to show the difference between the best crisp aprroximation of three different types of membership functions for the same fuzzy set.

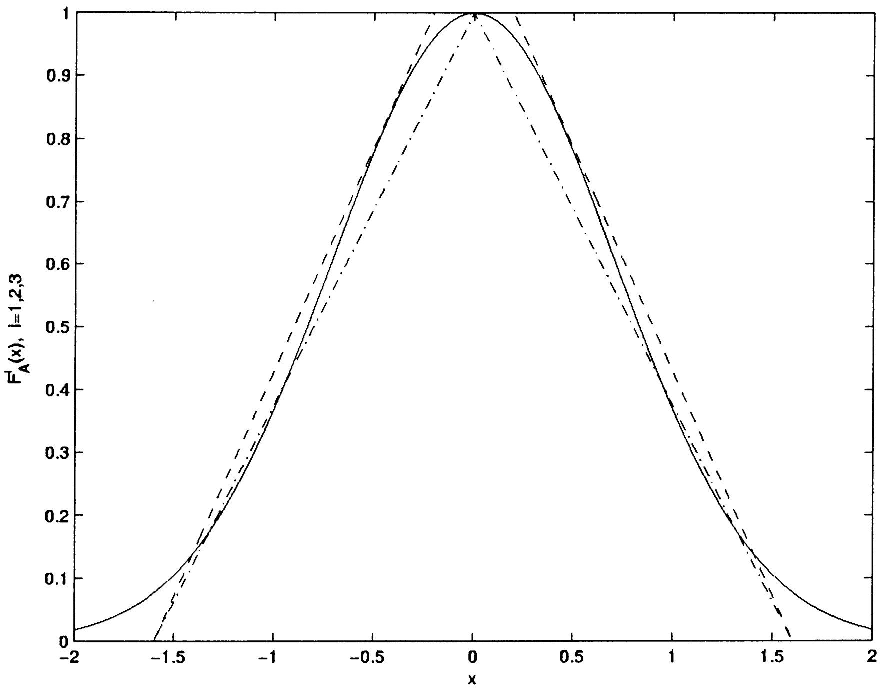

Fig. 2. Membership functions $F_A(x)$

Example 2 Let the linguistic information "around zero" is represented by the fuzzy set A with three different membership functions $F^1_A(x), F^2_A(x), F^3_A(x)$. As the solid line in Fig. 2, the bell-shaped membership function is defined as

$$F^1_A(x) = \exp(-x^2) \ ,$$

the trapezoidal membership function

$$F^2_A(x) = \begin{cases} 1, & \text{if } -0.2 \le x \le 0.2; \\ 1 + (x + .2)/1.4, & \text{if} -1.6 \le x \le -0.2; \\ 1 + (x - .2)/1.4, & \text{if } 0.2 \le x \le 1.6; \\ 0, & \text{otherwise} \end{cases} ,$$

is dash-lined, and the dash-dotted line is the triangular membership function with

$$F^3_A(x) = \begin{cases} 1 + x/1.6, & \text{if } -1.6 \le x \le 0; \\ 1 - x/1.4, & \text{if} 0 < x \le 1.6; \\ 0, & \text{otherwise} \end{cases} .$$

Note that the $F^2_A(x)$ and $F^3_A(x)$ are defined to have the possibility distributions (membership values for elements) be closed to the possibility distribution of $F^1_A(x)$. From Example 1, the best crisp approximation of the fuzzy set A with $F^1_A(x)$ is the interval $D_1 = \left[-.5\sqrt{\pi} \quad .5\sqrt{\pi}\right] \approx [-0.88\ 0.88]$. It is straightforward to get the best crisp approximation intervals

$$D_2 = [-0.9\ 0.9], \quad D_3 = [-0.8\ 0.8] \ ,$$

for the fuzzy set A with $F^2_A(x)$ and $F^3_A(x)$, respectively. In this example, we find that the shape of the membership function does not make much difference in the size of the best crisp approximation intervals. Also, for the representation of the linguistic information by a fuzzy set, the possibility distribution is implicitly shown to be the most important factor which contribute to the size of the best crisp approximation intervals.

With the interval approximation of fuzzy sets, the uncertain system with the uncertainties described by linguistic information and represented as fuzzy sets is described by the system containing interval parameters. The stability of the uncertain system with interval parameters is analyzed and the robust controller is designed in the following section.

5 Design of a robust controller

In this section, the Kharitonov's theorem is first applied to enable the analyses of the stability of the uncertain system with interval parameters. Then a robust controller is defined. Also, the design of the robust sliding mode controller is detailed for the time-varying uncertainties in this section.

5.1 Definition of a robust controller

As in the Fig. 1, the control system is a closed loop system with a controller G included. For a system *SFRU* with its fuzzy information approximated to be intervals, the transfer function of the uncertain plant, $P(s)$, in Eq. (1) becomes a transfer function with intervals,

$$P(s, \tilde{D}_l, \tilde{D}_L) = \frac{\sum_{j=0}^{m} D_{lj} s^j}{\sum_{i=0}^{n} D_{Li} s^i}, \quad n > m \ , \tag{11}$$

where $D_{lj}, D_{Lj}, j = 1, 2, \ldots, m$ are intervals, and $\tilde{D}_l, \tilde{D}_L$ represent the collections of intervals. To clarify the

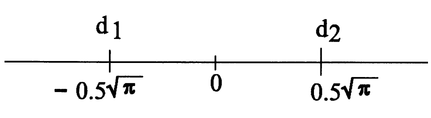

Fig. 3. Approximated interval of the fuzzy set A with bell shape membership function

meaning of a robust controller, the following definitions are provided.

Definition 1 *(A family Q_p of plants) Let a family of plants Q_p be defined as the collection of the plants which have the transfer functions $p_c(s)$ such that*

$$p_c(s) = \frac{\sum_{j=0}^{m} d_{lj} s^j}{\sum_{i=0}^{n} d_{Li} s^i}, \quad n > m \ ,$$

and

$$d_{Li} \in D_{Li} \quad and \quad d_{lj} \in D_{lj}; \ \forall i,j \ .$$

Definition 2 *A controller G is said to be a robust controller if every plant in the family Q_p can be stabilized by the controller G.*

5.2 Design of a robust *PID* controller

Let the interval transfer function of the uncertain plant $P(s)$ with the fuzzy set approximated by intervals be $P(s, \tilde{D}_l, \tilde{D}_L)$ in Eq. (11). In this subsection, a robust *PID* controller for a uncertain plant with the interval transfer function $P(s, \tilde{D}_l, \tilde{D}_L)$ is designed. From the discussion in two previous subsection, the idea to design a robust *PID* controller $G(s)$ is to have the interval characteristic polynomial satisfy the Kharitonov's theorem. For an example, consider an uncertain plant with two uncertain parameters, q_1 and q_2. The transfer function of the uncertain plant is

$$P(s) = \frac{1}{s^3 + 2s^2 + \alpha s + \beta} \tag{12}$$

where

$$\alpha = -1 - q_1 - q_2 - q_2^2 \ ,$$

and

$$\beta = -q_2^2 - 3q_2 - q_1 - q_1 q_2 - 2 \ .$$

Assume that the only information available for the values of the uncertain parameters q_1, q_2 is linguistic information. From expert's experience, the uncertain parameters q_1, q_2 of the three order plant are both around zero. Note that although the expert uses the same linguistic term for the uncertain parameters q_1, q_2, the "around zero" for q_1 might not have exactly the same meaning as the "around zero" for q_2. With the linguistic term "around zero" represented as fuzzy sets A_{q_1}, A_{q_2} for q_1, q_2, the triangular type membership functions $F_{A_{q_1}}, F_{A_{q_2}}$ are adopted (for simplicity) and shown in Fig. 4. To approximate the fuzzy sets in the uncertain plant with intervals D_1, D_2, it is easy to find that (see Sect. 4)

$$D_1 = [-0.3 \ 0.3], \quad D_2 = [-0.2 \ 0.2] \ .$$

With the overbounding technique in Sect. 3.2, α and β can be represented by intervals

$$D_\alpha = [-1.54 \ -0.5], \quad D_\beta = [-3 \ -1.04] \ ,$$

respectively. Then, the interval transfer function of the uncertain plant is

$$P(s, D) = \frac{1}{s^3 + 2s^2 + D_\alpha s + D_\beta} \ . \tag{13}$$

Since $P(s, D)$ is a type 0 plant, the *PID* controller with transfer function $G(s)$

$$G(s) = k_p + k_i/s + k_d s \ .$$

is designed to eliminate the steady state error. The closed loop interval transfer function $H(s, D)$ is

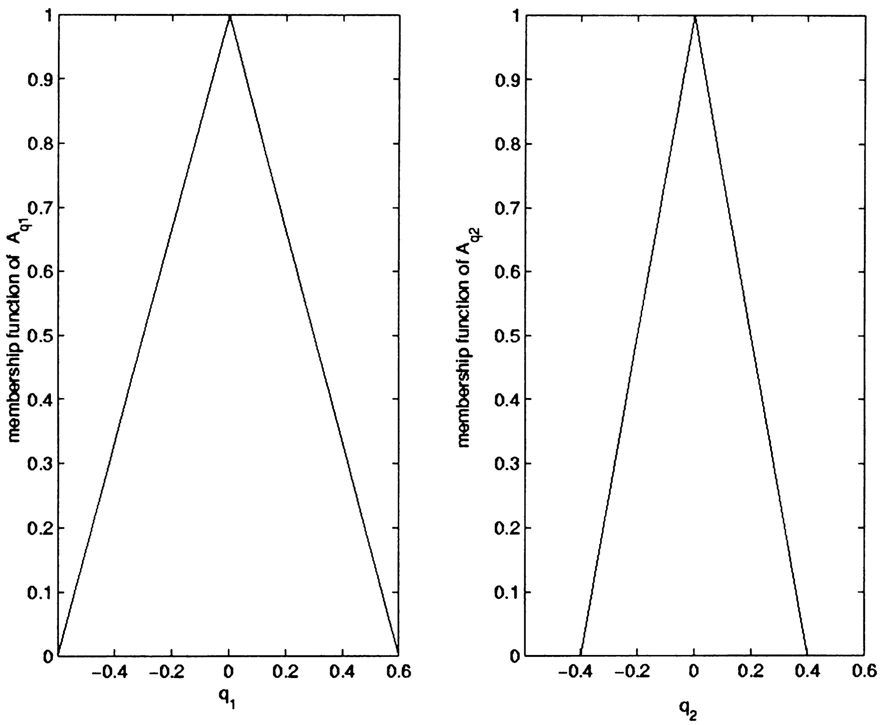

Fig. 4. Membership functions of "around zero" for q_1 and q_2

$$H(s,D) = \frac{G(s)P(s,D)}{1+G(s)P(s,D)}$$
$$= \frac{(k_p + k_i/s + k_d s)1/(s^3+2s^2+D_\alpha s+D_\beta)}{1+(k_p+k_i/s+k_d s)/(s^3+2s^2+D_\alpha s+D_\beta)}$$
$$= \frac{(k_i + k_p s + k_d s^2)}{s^4+2s^3+(k_d+D_\alpha)s^2+(k_p+D_\beta)s+k_i} \quad (14)$$

The interval characteristic polynomial, $P_c(s,D)$, of the closed loop control system is equal to the denominator of $H(s,D)$, i.e.,

$$P_c(s,D) = s^4+2s^3+(k_d+D_\alpha)s^2+(k_p+D_\beta)s+k_i$$
$$= s^4+2s^3+(k_d+[-1.54\ \ -0.5])s^2 + (k_p+[-3\ \ -1.04])s+k_i \ . \quad (15)$$

Associated with the interval polynomial in Eq. (15), four Kharitonov's polynomials are

$$K_1(s) = k_i+(k_p-3)s+(k_d-0.5)s^2+2s^3+s^4$$
$$K_2(s) = k_i+(k_p-1.04)s+(k_d-1.54)s^2+2s^3+s^4$$
$$K_3(s) = k_i+(k_p-3)s+(k_d-1.54)s^2+2s^3+s^4$$
$$K_4(s) = k_i+(k_p-1.04)s+(k_d-0.5)s^2+2s^3+s^4 \ .$$

With the Routh Hurwitz criterion, the conditions for the systems with the Kharitonov's polynomials as characteristic polynomials to be stable are

$$\begin{cases} k_i > 0 \\ k_p > 3 \\ k_d > \max\{(\frac{4k_i}{k_p-3}+k_p-2)/2, \\ \qquad (\frac{4k_i}{k_p-1.04}+k_p+2.04)/2, \\ \qquad (\frac{4k_i}{k_p-3}+k_p+0.08)/2, \\ \qquad (\frac{4k_i}{k_p-1.04}+k_p-0.04)/2\} \ . \end{cases} \quad (16)$$

Then, it is easy to determine the coefficients for the *PID* controller, for example, if k_i and k_p are selected to be $k_i = 1$ and $k_p = 4$, then

$$k_d > 4.04 \ .$$

That is, if the *PID* controller is designed to have transfer function

$$G(s) = 4 + 1/s + k_d s$$

with $k_d > 4.04$, then the uncertain control system with the interval characteristic polynomial $P_c(s,D)$ is stable. Therefore, the proposed approach is shown to be able to design a robust *PID* controller to stabilize the uncertain system with only linguistic information available for the uncertain parameters q_1 and q_2.

5.3
Design of a robust sliding mode controller

In this section, the same uncertain plant is assumed to be the same as in Eq. (12) with two uncertain parameters considered as time-varing uncertainties. In this case, the parameters $q_1(t)$ and $q_2(t)$ are functions of time. The linguistic information from experts is the only information we can obtain for the parameters $q_1(t)$ and $q_2(t)$. Although the parameters are time-varying, the values of $q_1(t)$ and $q_2(t)$ are indicated (by experts) to be around zero. Let the linguistic information "around zero" for $q_1(t)$ and $q_2(t)$ be represented by the fuzzy sets A_{q_1} and A_{q_2} with membership functions $F_{A_{q_1}}$ and $F_{A_{q_2}}$ shown in Fig. 4. As calculated in Sect. 5.2, the best crisp approximation of the fuzzy sets A_{q_1} and A_{q_2} results in the intervals

$$D_1 = [-0.3\ 0.3], \quad D_2 = [-0.2\ 0.2] \ , \quad (17)$$

respectively. Let the plant in Eq. (12) be represented as a state equation

$$\dot{x}(t) = (A + \Delta A(t))x(t) + Bu(t) \quad (18)$$

with

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} ,$$

and

$$\Delta A(t) = \begin{bmatrix} 0 & q_1 & q_1 \\ 0 & q_2 & q_2 \\ 0 & 0 & -q_2 \end{bmatrix} . \quad (19)$$

Thus, the regular form of the state equation is

$$\dot{z}(t) = \begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} \Delta A_1(t) \\ \Delta A_2(t) \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u(t)$$

where

$$A_{11} = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}; \quad A_{12} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} ,$$
$$A_{21} = [0 \quad 0], \quad A_{22} = -1, \quad B_2 = 1 \ ,$$

and

$$\Delta A_1(t) = \begin{bmatrix} 0 & q_1(t) & q_1(t) \\ 0 & q_2(t) & q_2(t) \end{bmatrix} ;$$
$$\Delta A_2(t) = [0 \quad 0 \quad -q_2(t)] \ .$$

It can be seen that $\Delta A_2(t)$ satisfies the conventional match condition and

$$\Delta A_1(t) = DF(t)E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} [0 \quad 1 \quad 1] \quad (20)$$

where

$$F(t)^T F(t) = q_1^2(t) + q_2^2(t) \le 1 \text{ (see Eq. 17)} \ .$$

Thus, the the sliding coefficient matrix C can be design to be

$$C = [\,C_1 \quad C_2\,] = [0 \quad 1 \quad 1], \quad C_2 = 1;$$

and the matrix C satisfies the sliding coefficient condition [2] that

$$E = E_a C, \quad E_a = 1 \ .$$

Also, C_2B_2 is invertable, since

$$C_2B_2 = 1 \ .$$

As stated in the work of Chan [2], the system with time-varying uncertainties is guaranteed to reach the sliding mode with the output of the sliding controller u,

$$u(t) = u_{eq}(t) + \Delta u(t) \text{with}$$

$$u_{eq}(t) = -(C_2B_2)^{-1}(C_1(A_{11}z_1(t) + A_{12}z_2(t)) + C_2(A_{21}z_1(t) + A_{22}z_2(t))) \text{ and}$$

$$\Delta u(t) = u_m(t) - \frac{1}{2}(C_2B_2)^{-1}(C_1DD^TC_1^TS + E_a^TE_aS) - (C_2B_2)^{-1}(k_1 sgn(S) + k_2S)$$

$$u_m(t) = \begin{cases} \frac{-B_2^TC_2^TS}{\|B_2^TC_2^TS\|+0.0001}\rho, & \text{if } S \neq 0, \\ 0, & \text{if } S = 0 \end{cases} , \tag{21}$$

where $S = Cz(t)$, and $\rho = 1$ is a norm bound of ΔA_2, i.e., $\|\Delta A_2\| \leq \rho$. Since the uncertain system with sliding mode controller is guaranteed to reach the sliding mode and the control system is asymptotically stable on the sliding surface, the uncertain system with time-varying parameters is stable.

6
Simulation results

In this section, the simulation results are provided for the fuzzy uncertain systems in the examples in Sects. 5.2 and 5.3.

6.1
The uncertain system with *PID* controller

Let the linguistic transfer function $P(s)$ of the uncertain plant with linguistic parameter q_1 and q_2 be the same as Eq. (12),

$$P(s) = \frac{1}{s^3 + 2s^2 + \alpha s + \beta}$$

where α and β are specified as in Sect. 5.2. With the proposed approach, the best crisp approximation of fuzzy sets is obtained for the linguistic information. This make the system with linguistic information be able to be stabilized. If the *PID* controller

$$G(s) = k_p + k_ds + k_i/s \ ,$$

is designed to have

$$k_i = 1, \quad k_p = 8, \quad k_d = 20 \ ,$$

then the stability condition in Eq. (16) is satisfied, and the fuzzy uncertain system is expected to be stable. Figure 5 shows the system performance with zero reference input ($r = 0$), output initial value $y(0) = 0.8$ and different values of q_1 and q_2 ($q_1 = 0.3, q_2 = 0.2$ (dash line), $q_1 = -0.3$, $q_2 = 0.2$ (dash-dotted line), $q_1 = -0.3, q_2 = -0.2$ (solid line)). The uncertain system is shown to be robust to the variations of the parameters in Fig. 5.

6.2
The uncertain system with sliding mode controller

To implement the example in Sect. 5.3, the time varying uncertain parameters $q_1(t)$ and $q_2(t)$ are assumed to be

$$q_1(t) = 0.3\sin(0.1t),$$
$$q_2(t) = 0.2\cos(0.1t) \ . \tag{22}$$

The simulation is provided with zero reference input ($r = 0$), and the initial conditions

$$x(0) = [0.5 \ 0.25 \ 0.01]^T \ .$$

Figure 6 shows the open loop uncertain system is unstable, and the Fig. 7 indicates the unstable uncertain

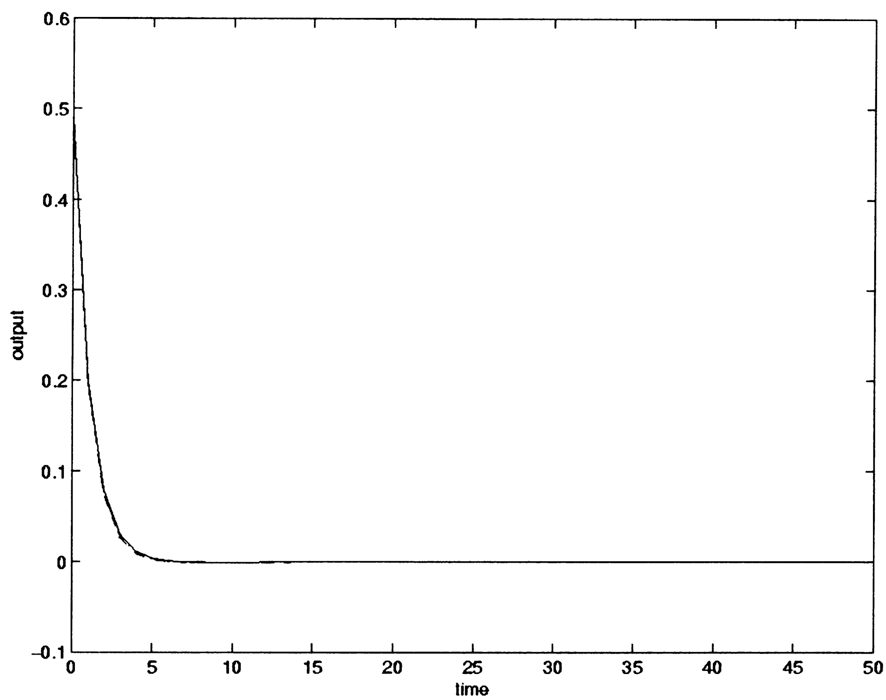

Fig. 5. Performance of the uncertain system with $q_1 = .3$, $q_2 = .2$ (dash line), $q_1 = -.3, q_2 = .2$ (dash-dotted line), $q_1 = -.3, q_2 = -.2$ (solid line)

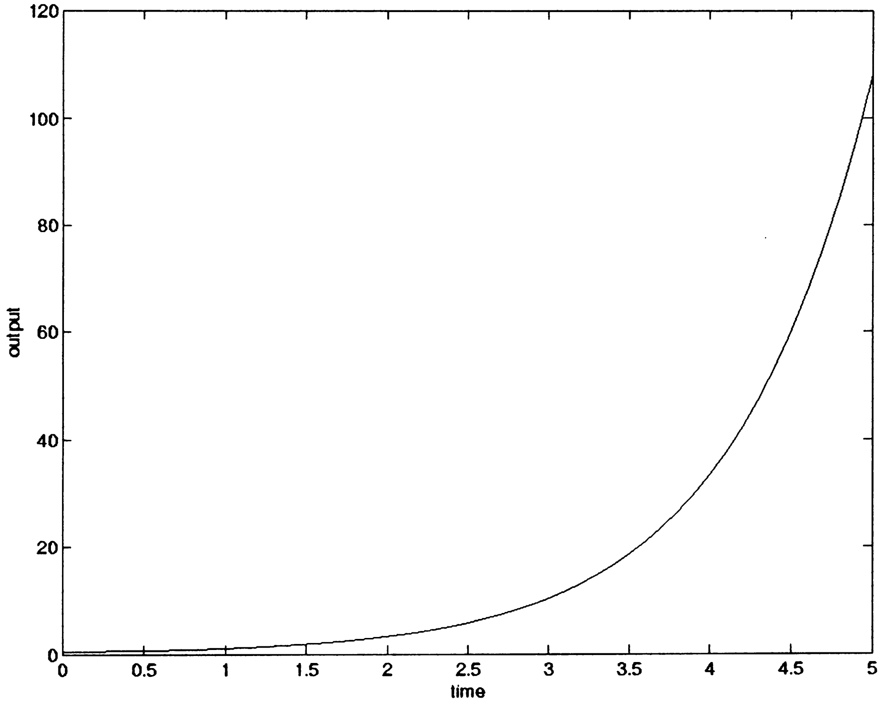

Fig. 6. Output of the open loop system (x_1)

system is stabilized when the sliding mode control is applied.

7 Conclusions

An approach to design a robust controller for an uncertain system with linguistic information from experts is proposed. The linguistic information is represented as fuzzy sets and the uncertain system with fuzzy sets is defined as a system with fuzzy representation of uncertainties. An effective approach is proposed to implement the idea in [6] for best approximating the fuzzy sets with crisp intervals. With the fuzzy sets best approximated by intervals (crisp sets), Kharitonov's theorem is applied to construct a robust *PID* controller for a fuzzy uncertain system. Also, for the linguistic time-varying uncertainties, a robust sliding mode controller is designed for the uncertain system with the best crisp approximation of fuzzy representation of the linguistic uncertainties. Examples and simulation results are included to indicate the design approach and the effectiveness of the proposed robust controller.

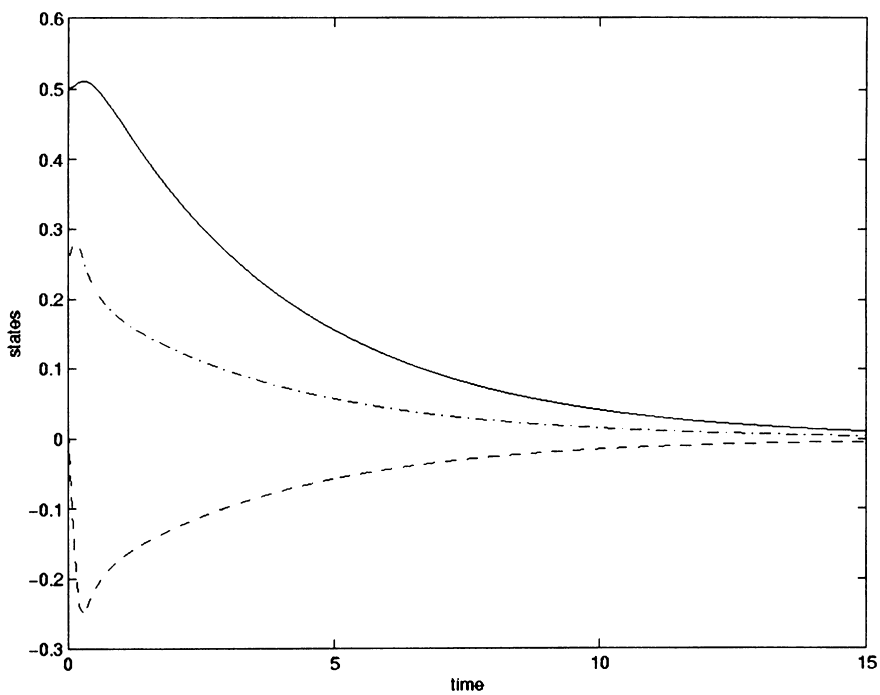

Fig. 7. States of the close loop system: x_1 (solid line), x_2 (dash-dot line), x_3 (dash line)

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