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ORIGINAL ARTICLE

Controllability of networked systems with heterogeneous dynamics

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Abstract

Hao et al. (Int J Robust Nonlinear Control 28(5):1778–1788, 1976) established necessary and sufficient conditions for the controllability of homogeneous networked systems where the individual nodes are linear time-invariant (LTI) systems and the network topology matrix is diagonalizable. In this paper, we consider a class of heterogeneous networked systems having triangularizable network topology. Here, we establish a result which gives necessary and sufficient conditions for controllability of a class of heterogeneous systems, which generalizes the result of Hao et al. (Int J Robust Nonlinear Control 28(5):1778–1788, 1976). Also, we provide some necessary conditions for the controllability of general heterogeneous networked systems having some restrictions on the network topology matrix. Theoretical results are illustrated with numerical examples.

Keywords Controllability · Heterogeneous dynamics · LTI systems · Networked control systems · Network topology

1 Introduction

Controllability is one of the fundamental properties of dynamical control systems introduced by Kalman [20]. Various notions of controllability, like state controllability, structural controllability, etc., are introduced in the literature, and controllability conditions are obtained by many authors both for linear and nonlinear systems [10, 14, 19, 21]. The state controllability deals with the ability of the system to steer the system from an arbitrary initial state to a desired final state using suitable control functions,

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whereas the structural controllability introduced by Lin [22] attempts to set some values to the nonzero parameters in the system matrices such that the resulting system is state controllable in the sense of Kalman. The notion of controllability, whether it is state or structural, has been extensively studied for various types of systems and conditions for controllability have been obtained over the past few decades [17, 24, 26]. Most of these results are for single higher-dimensional control systems. However, in the real-world situation, occurrence of networked control systems is comparatively much larger than that of single stand-alone control systems. In general, modeling complex systems requires a collection of individual systems connected together with an interconnection topology. Controllability of large-scale complex networked systems gives rise to fascinating challenges for various studies. Such studies include different aspects of the systems such as structural complexity, node dynamics, interaction among various nodes, etc. The research on the controllability of networked systems is gaining much attention as it has applications in various fields of science and technology [3, 7, 27].

Many approaches were invoked to study the controllability of a dynamical system over the years. The study of network controllability employs tools like graph theoretic properties of network topology, rank conditions and spectral properties of the system matrices, etc. [1, 12, 18, 23, 29, 30, 39, 41, 42]. The problem of controllability of interconnected systems dates back to the work of Gilbert [9], followed by the works of Callier and Nahum [5] and Fuhrmann [8]. Representation of complex interconnection structures needed the idea of weighted directed graphs to represent the network topology. By dividing the nodes into leaders and followers, some conditions on network topology were derived by Tanner [31], which ensured the controllability of a group of nodes with a single leader. Hara et al. [13] studied networks in which each node is a copy of the same single-input-single-output (SISO) system and obtained necessary and sufficient conditions for the controllability and observability. Later, Wang et al. [32] addressed the controllability problem of networked multi-input-multi-output (MIMO) systems. They established necessary and sufficient conditions for controllability of homogeneous networked systems that involve solution of certain matrix equations. Based on the above work, Wang et al. [33] further derived a necessary and sufficient condition for the state controllability of a homogeneous networked system where communications are performed through one-dimensional connections. They also discussed the controllability of a homogeneous networked system over some particular network topologies such as trees, cycles. Hao et al. [11] derived necessary and sufficient conditions for the controllability of a MIMO homogeneous LTI networked system where the network topology matrix is diagonalizable. Compared to Wang et al. [32], Hao's result is easy to verify as it does not involve solving matrix equations.

All the works discussed above considered networked systems having the same dynamics in each node. However, in real-life applications, all nodes need not possess the same dynamics. Zhou [42] studied a networked system where every subsystem is permitted to have different dynamics. A necessary and sufficient condition for the controllability of a heterogeneous networked system was derived from Popov–Belevitch–Hautus (PBH) rank condition by Wang et al. [34]. They also attempted to extend the results obtained by Wang et al. [32] for homogeneous systems to heterogeneous systems. Later, Xiang et al. [36] extended this work and derived a necessary and sufficient condition for the controllability of a particular type of heterogeneous system in terms of some rank conditions. The obtained results are for a system in which the state matrices of the individual nodes are of a special form. Along with the necessary and sufficient conditions for controllability, several other necessary conditions for controllability were also derived based on the properties of the individual nodes. Inspired by this work, some necessary conditions for controllability of heterogeneous systems were derived in Ajayakumar and George [2]. The notion of structural controllability of large-scale networked systems is also studied by many authors [4, 6, 25, 37, 39]. A brief survey of recent advances in the study of controllability of networked linear dynamical systems can be seen in Xiang et al. [35].

Most of the available controllability results for the networked systems are for homogeneous LTI systems. This paper provides necessary and sufficient conditions for the controllability of a heterogeneous system model and discusses the connection between networked topology and the controllability of the whole networked system. Our result generalizes the work of Hao et al. [11], which was for the controllability of homogeneous LTI networked systems, enabling us to extend the scope of study into a larger class of systems. Compared to the result of Xiang et al. [36], the condition in this paper is easier to verify as it does not require solving matrix equations. The paper is organized as follows. Some preliminaries are given in Sect. 2. The controllability problem is formulated in Sect. 3. In Sect. 4, we prove necessary and sufficient condition for the controllability of the heterogeneous networked system formulated in Sect. 3, and some controllability results of the networked system over some specific topologies are also established. The derived results are substantiated with examples. Conclusion and future scope of the work are given in Sect. 5.

2 Preliminaries

In this paper, we make use of the following notations. Let $\mathbb{R}^{m\times n}$ denotes the set of $m \times n$ real matrices and by I we denote the identity matrix. Let $\{e_1, e_2, \dots, e_n\}$ be the canonical basis for $\mathbb{R}^n$. Let $diag\{a_1, a_2, \dots, a_n\}$ denotes diagonal matrix of order n with diagonal entries $a_1, a_2, \dots, a_n$ and $uppertriang\{a_1, a_2, \dots, a_n\}$ denotes an upper triangular matrix of the form

$$\begin{bmatrix} a_1 & * & * & \dots & * \\ 0 & a_2 & * & \dots & * \\ 0 & 0 & a_3 & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_n \end{bmatrix}$$

By *blockuppertriang*$\{A_1, A_2, \dots A_n\}$, we denote a block upper triangular matrix of the form

$$\begin{bmatrix} A_1 & * & * & \dots & * \\ 0 & A_2 & * & \dots & * \\ 0 & 0 & A_3 & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_n \end{bmatrix}$$

where $A_1, A_2, \dots, A_n$ are matrices. Let $\sigma(A)$ denotes the eigen spectrum of a matrix A.

The following lemmas will be used in the subsequent sections of this paper.

Lemma 1 [15] *Let A and B be similar matrices, that is, there exists a nonsingular matrix P such that $PBP^{-1} = A$. If ν is a left eigenvector of A with respect to the eigenvalue λ, then νP is an eigenvector of B with respect to the eigenvalue λ.*

Lemma 2 [16] *Let $A \otimes B$ denotes the Kronecker product of two matrices A and B. We use the following properties of Kronecker product in this paper.*

(i) $(A \otimes B)(C \otimes D) = (AC \otimes BD)$
(ii) $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ *if A and B are invertible.*
(iii) $(A + B) \otimes C = A \otimes C + B \otimes C$
(iv) $A \otimes (B + C) = A \otimes B + A \otimes C$
(v) $A \otimes B = 0$ *if and only if* $A = 0$ *or* $B = 0$

Lemma 3 [30] *A linear time-invariant control system characterized by the pair of matrices (A, B) is controllable if and only if left eigenvectors of A are not orthogonal to columns of B, i.e., $\nu A = \lambda \nu$ implies that $\nu B \neq 0$.*

3 Model formulation

Consider a heterogeneous networked linear time-invariant system with N nodes, where the ith node is described by the following differential equation:

$$\dot{x_i}(t) = A_i x_i(t) + \sum_{j=1}^{N} c_{ij} H x_j(t) + d_i B u_i(t), \quad i = 1, 2, \dots, N \tag{1}$$

where $x_i(t) \in \mathbb{R}^n$ is the state vector; $u_i(t) \in \mathbb{R}^m$ is the external control vector; $A_i \in \mathbb{R}^{n \times n}$ is the state matrix of node v_i; $B \in \mathbb{R}^{n \times m}$ is the control matrix, with $d_i = 1$ if node v_i is under control, otherwise $d_i = 0$. $c_{ij} \in \mathbb{R}$ represents the coupling strength between the nodes v_i and v_j with $c_{ij} \neq 0$ if there is a communication from node v_j to node v_i, otherwise $c_{ij} = 0, i, j = 1, 2, \dots, N$ and $H \in \mathbb{R}^{n \times n}$ is the inner coupling matrix describing the interconnections among the states $x_j, j = 1, 2, \dots, N$ of the nodes.

Let

$$C = \left[c_{ij}\right] \in \mathbb{R}^{N\times N} \quad \text{and} \quad D = diag\{d_1, d_2, \dots, d_N\} \tag{2}$$

denote the network topology and external input channels of networked system (1), respectively. Denote the whole state of the networked system by $X = \left[x_1^T, \dots, x_N^T\right]^T$ and the total external control input vector by $U = \left[u_1^T, \dots, u_N^T\right]^T$.

Now, using the Kronecker product notation, networked system (1) can be reduced into the following compact form:

$$\dot{X}(t) = FX(t) + GU(t) \tag{3}$$

where

$$F = A + C \otimes H, \ \ G = D \otimes B \tag{4}$$

and $A = blockdiag\{A_1, A_2, \dots, A_N\}$.

If the state node matrices $A_1, A_2, \dots, A_N$ are identical, that is, $A_i = \tilde{A}$, then system (1) becomes a homogeneous networked system. Hao et al. [11] have proved the following necessary and sufficient condition for controllability of such homogeneous networked systems.

Theorem 1 [11] *Consider a homogeneous networked system with a diagonalizable network topology matrix C. Let $\sigma(C) = \{\lambda_1, \lambda_2, \dots, \lambda_N\}$. Then networked system (1) is controllable if and only if the following conditions are satisfied.*

(i) *(C, D) is controllable;*
(ii) *$(\tilde{A} + \lambda_i H, B)$ is controllable, for $i = 1, 2, \dots, N$; and*
(iii) *If matrices $\tilde{A} + \lambda_{i_1} H, \dots, \tilde{A} + \lambda_{i_p} H$ $\left(\lambda_{i_k} \in \sigma(C),\ for\ k = 1, \dots, p,\ p > 1\right)$ have a common eigenvalue ρ, then $(t_{i_1} D) \otimes (\xi_{i_1}^1 B), \dots, (t_{i_1} D) \otimes (\xi_{i_1}^{\gamma_{i_1}} B), \dots, (t_{i_p} D) \otimes (\xi_{i_p}^1 B), \dots, (t_{i_p} D) \otimes (\xi_{i_p}^{\gamma_{i_p}} B)$ are linearly independent, where t_{i_k} is the left eigenvector of C corresponding to the eigenvalue λ_{i_k}; $\gamma_{i_k} \geq 1$ is the geometric multiplicity of ρ for $A + \lambda_{i_k} H$; $\xi_{i_k}^l$ $(l = 1, \dots, \gamma_{i_k})$ are the left eigenvectors of $A + \lambda_{i_k} H$ corresponding to ρ, $k = 1, \dots, p$.*

In this paper, we will relax the diagonalizability condition of the network topology matrix C for the homogeneous system and also we derive necessary and sufficient condition for heterogeneous system under more relaxed condition on the network topology.

4 Main results

4.1 Controllability in a general network topology

In this section, we investigate the controllability of (3) under certain network topologies. Suppose that the network topology matrix C is triangularizable. That is, there exists a nonsingular matrix T such that $TCT^{-1} = J$, where $J = uppertriang\{\lambda_1, \lambda_2, \dots, \lambda_N\}$ is the Jordan Canonical Form of C. Let $\sigma(A_i + \lambda_i H) =$

$\{\mu_i^1, \ldots, \mu_i^{q_i}\}$ denotes the set of eigenvalues of $A_i + \lambda_i H, i = 1, 2, \ldots, N$ and $\xi_{ij}^k, k = 1, \ldots, \gamma_{ij}$ be the left eigenvectors of $A_i + \lambda_i H$ corresponding to $\mu_i^j, j = 1, \ldots, q_i, i = 1, \ldots, N$, where $\gamma_{ij} \geq 1$ is the geometric multiplicity of the eigenvalue μ_i^j.

We investigate the controllability of original system (1) in terms of the eigenvalues and left eigenvectors of the matrix F in compact form (3). When the network topology matrix C is triangularizable with triangularizing matrix T and if $T \otimes I$ commutes with A, we characterize the eigenvalues and left eigenvectors of F in terms of the eigenvalues and left eigenvectors of $A_i + \lambda_i H, \ i = 1, 2, \ldots, N$ as shown in the following theorem.

Theorem 2 *Let T be the triangularizing matrix for the network topology matrix C and suppose $T \otimes I$ commutes with A. Let (μ_i^j, ξ_{ij}^k) denotes the left eigenpair of $A_i + \lambda_i H$. Then the following statements hold true.*

(i) *The eigenspectrum of F is the union of eigenspectrum of $A_i + \lambda_i H$, where $i = 1, 2, \ldots, N$. That is,*

$$\sigma(F) = \cup_{i=1}^N \sigma(A_i + \lambda_i H) = \left\{\mu_1^1, \ldots, \mu_1^{q_1}, \ldots, \mu_N^1, \ldots, \mu_N^{q_N}\right\}$$

(ii) *If J is a diagonal matrix, then $e_i T \otimes \xi_{ij}^k, k = 1, \ldots, \gamma_{ij}$ are the left eigenvectors of F corresponding to the eigenvalue $\mu_i^j, j = 1, \ldots, q_i, i = 1, \ldots, N$.*

(iii) *If J contains a Jordan block of order $l \geq 2$ for some eigenvalue λ_{i_0} of C with $\xi_{ij}^k H = 0$ for all $i = i_0, i_0+1, \ldots, i_0+l-1, j = 1, 2, \ldots, q_i, k = 1, 2, \ldots, \gamma_{ij}$, then $e_i T \otimes \xi_{ij}^k, k = 1, \ldots, \gamma_{ij}$ are the left eigenvectors of F corresponding to the eigenvalue $\mu_i^j, i = 1, 2, \ldots, N, j = 1, 2, \ldots, q_i$.*

Proof (i) By hypothesis, T is a nonsingular matrix such that $TCT^{-1} = J$, where $J = uppertriang\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ is the Jordan Canonical Form of C. Let

$$\tilde{F} = (T \otimes I)F(T^{-1} \otimes I) = (T \otimes I)(A + C \otimes H)(T^{-1} \otimes I)$$

As $T \otimes I$ commutes with A, we have

$$\begin{aligned}
\tilde{F} &= A(T \otimes I)(T^{-1} \otimes I) + (T \otimes I)(C \otimes H)(T^{-1} \otimes I) \\
&= A + (TCT^{-1} \otimes H) \\
&= A + J \otimes H \\
&= A + uppertriang\{\lambda_1, \lambda_2, \ldots, \lambda_N\} \otimes H \\
&= blockuppertriang\{A_1 + \lambda_1 H, \ldots, A_N + \lambda_N H\}
\end{aligned}$$

Since $\tilde{F}$ and F have same eigenvalues, we get

$$\sigma(F) = \{\mu_1^1, \ldots, \mu_1^{q_1}, \ldots, \mu_N^1, \ldots, \mu_N^{q_N}\}$$

(ii) Let $\xi_{ij}^k, k = 1, \ldots, \gamma_{ij}$ be the left eigenvectors of $A_i + \lambda_i H$ corresponding to $\mu_i^j, j = 1, \ldots, q_i, i = 1, \ldots, N$. If J is a diagonal matrix, $\tilde{F}$ is a block diagonal matrix and hence $e_i \otimes \xi_{ij}^k, k = 1, \ldots, \gamma_{ij}$ are left eigenvectors of $\tilde{F}$ corresponding to $\mu_i^j, j = 1, \ldots, q_i, i = 1, \ldots, N$.

(iii) Suppose that J contains a Jordan block of order 2, corresponding to the eigenvalue λ_{i_0} of C. Then the matrix $\tilde{F}$ contains the block matrix of the form

$$\mathcal{A} = \begin{bmatrix} A_{i_0} + \lambda_{i_0} H & H \\ 0 & A_{i_0+1} + \lambda_{i_0+1} H \end{bmatrix} \tag{5}$$

It follows easily that, $e_{i_0+1} \otimes \xi_{i_0+1j}^k, k = 1, 2, \ldots, \gamma_{i_0+1j}$ are eigenvectors of $\tilde{F}$ corresponding to the eigenvalues $\mu_{i_0+1}^j, j = 1, 2, \ldots, q_{i_0+1}$. If $\xi_{i_0 j_0}^k H = 0$ for all $k = 1, 2, \ldots, \gamma_{i_0 j_0}$, then $e_{i_0} \otimes \xi_{i_0 j_0}^k, k = 1, 2, \ldots, \gamma_{i_0 j_0}$ are left eigenvectors of $\tilde{F}$ corresponding to the eigenvalue $\mu_{i_0}^{j_0}$. Now suppose that J contains a Jordan block of order $l \geq 2$ for some eigenvalue λ_{i_0} of C, then again we can consider $(l-1)$ block matrices of form (5) and by using the fact that $\xi_{ij}^k H = 0$ for all $i = i_0, i_0+1, \ldots, i_0+l-1, j = 1, 2, \ldots, q_i, k = 1, 2, \ldots, \gamma_{ij}$ we get $e_i \otimes \xi_{ij}^k, k = 1, 2, \ldots, \gamma_{ij}$ are left eigenvectors of $\tilde{F}$ corresponding to the eigenvalue $\mu_i^j, i = 1, 2, \ldots, N, j = 1, 2, \ldots, q_i$. We will prove that these are the only eigenvectors of $\tilde{F}$. Suppose that $\tilde{F}$ does not have any Jordan blocks and let $\xi = \begin{bmatrix} \xi_1 \ \xi_2 \ \ldots \ \xi_N \end{bmatrix} \in \mathbb{R}^{Nn}$ be a left eigenvector of $\tilde{F}$ corresponding to the eigenvalue μ, where $\xi_1, \xi_2, \ldots, \xi_N \in \mathbb{R}^n$. Then $\xi^T \tilde{F} = \mu \xi^T$ implies that

$$\begin{bmatrix} \xi_1 (A_1 + \lambda_1 H) \\ \xi_2 (A_2 + \lambda_2 H) \\ \vdots \\ \xi_N (A_N + \lambda_N H) \end{bmatrix}^T = \mu \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_N \end{bmatrix}^T$$

This in turn implies that μ is an eigenvalue of $A_i + \lambda_i H$ for all i with ξ_i as an eigenvector. Suppose that $\tilde{F}$ has a block of type (5). Then $\xi^T \tilde{F} = \mu \xi^T$ implies that

$$\begin{bmatrix} \xi_1 (A_1 + \lambda_1 H) \\ \vdots \\ \xi_i (A_i + \lambda_i H) \\ \xi_i H + \xi_{i+1} H (A_2 + \lambda_2 H) \\ \vdots \\ \xi_1 (A_N + \lambda_N H) \end{bmatrix}^T = \mu \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_i \\ \xi_{i+1} \\ \vdots \\ \xi_N \end{bmatrix}^T$$

As $\xi_i (A_i + \lambda_i H) = \mu \xi_i$, ξ_i is a left eigenvector of $A_i + \lambda_i H$. Then by our hypothesis, $\xi_i H = 0$. Hence μ is an eigenvalue of $A_i + \lambda_i H$ for all i with ξ as an eigenvector. Thus, if $A_i + \lambda_i H, i = 1, 2, \ldots, N$ does not have a common eigenvalue, then the left eigenvectors of $\tilde{F}$ are of the form $e_i \otimes \xi$, where ξ is a left eigenvector of $A_i + \lambda_i H$ for some i. If they have a common eigenvalue, the eigenvectors are either of the form $e_i \otimes \xi$, where ξ is a left eigenvector of $A_i + \lambda_i H$ for some i or of the form $\sum_{\alpha=1}^{r} e_{i_\alpha} \otimes \xi_{i_\alpha}$, where $A_i + \lambda_i H, i \in \{i_1, i_2, \ldots, i_r\}$ have a common eigenvalue μ with eigenvector ξ_{i_α} for each $i_1, i_2, \ldots, i_r$.

Thus in both cases, by Lemma 2(i) and Lemma 1, $\left(e_i \otimes \xi_{ij}^k\right)(T \otimes I) = e_i T \otimes \xi_{ij}^k (k = 1, \ldots, \gamma_{ij})$ are the left eigenvectors of F corresponding to $\mu_i^j, j = 1, \ldots, q_i, i = 1, \ldots, N$. □

Using the above result, we will prove the following necessary and sufficient conditions for controllability of heterogeneous networked system (3).

Theorem 3 *Let T be a nonsingular matrix triangularizing matrix C such that $T \otimes I$ commutes with A. If J contains a Jordan block of order $l \geq 2$ corresponding to the eigenvalue λ_{i_0} of C, then assume that $\xi_{ij}^k H = 0$ for all $i = i_0, i_0+1, \ldots, i_0+l-1, j = 1, 2, \ldots, q_i, k = 1, 2, \ldots, \gamma_{ij}$, where $\xi_{ij}^k, i = 1, 2, \ldots, N, j = 1, 2, \ldots, q_i, k = 1, 2, \ldots, \gamma_{ij}$ are the left eigenvectors of $A_i + \lambda_i H$ corresponding to the eigenvalues $\mu_i^j, i = 1, 2, \ldots, N, j = 1, 2, \ldots, q_i$. Then networked system (3) is controllable if and only if*

(i) *$e_i T D \neq 0$ for all $i = 1, \ldots, N$*
(ii) *$(A_i + \lambda_i H, B)$ is controllable, for $i = 1, 2, \ldots, N$; and*
(iii) *If matrices $A_{i_1} + \lambda_{i_1} H, A_{i_2} + \lambda_{i_2} H, \ldots, A_{i_p} + \lambda_{i_p} H (\lambda_{i_k} \in \sigma(C), k = 1, \ldots, p$, where $p > 1)$ have a common eigenvalue σ, then $(e_{i_1} T D) \otimes (\xi_{i_1}^1 B), \ldots, (e_{i_1} T D) \otimes (\xi_{i_1}^{\gamma_{i_1}} B), \ldots, (e_{i_p} T D) \otimes (\xi_{i_p}^1 B), \ldots, (e_{i_p} T D) \otimes (\xi_{i_p}^{\gamma_{i_p}} B)$ are linearly independent vectors, where $\gamma_{i_k} \geq 1$ is the geometric multiplicity of σ for $A_{i_k} + \lambda_{i_k} H$ and $\xi_{i_k}^l (l = 1, \ldots, \gamma_{i_k})$ are the left eigenvectors of $A_{i_k} + \lambda_{i_k} H$ corresponding to $\sigma, k = 1, \ldots, p$.*

Proof **(Necessary part)** From Theorem 2 it follows that $e_i T \otimes \xi_{ij}^k (k = 1, \ldots, \gamma_{ij})$ are left eigenvectors of F corresponding to $\mu_i^j, j = 1, \ldots, q_i, i = 1, \ldots, N$. If networked system (3) is controllable, then

$$(e_i T \otimes \xi_{ij}^l)(D \otimes B) \neq 0, \quad \text{for} \quad l = 1, \ldots, \gamma_{ij}, j = 1, \ldots, q_i, i = 1, \ldots, N$$

which implies that

$$e_i T D \neq 0, \quad i = 1, \ldots, N,$$

and

$$\xi_{ij}^l B \neq 0, \quad \text{for} \quad l = 1, \ldots, \gamma_{ij}, j = 1, \ldots, q_i, i = 1, \ldots, N$$

Since ξ_{ij}^l is an arbitrary left eigenvector of $A_i + \lambda_i H$, the controllability of $(A_i + \lambda_i H, B)$, for $i = 1, \ldots, N$ follows.

Assume that the matrices $A_{i_1} + \lambda_{i_1} H, \ldots, A_{i_p} + \lambda_{i_p} H (\lambda_{i_k} \in \sigma(C), k = 1, \ldots, p,$ where $p > 1)$ have a common eigenvalue σ. Then all the left eigenvectors of F corresponding to σ can be expressed in the form of $\sum_{k=1}^{p} \sum_{l=1}^{\gamma_{i_k}} \alpha_{kl}(e_{i_k} T \otimes \xi_{i_k}^l)$, where $\alpha_{kl} \in \mathbb{R}(k = 1, \ldots, p, l = 1, \ldots, \gamma_{i_k})$ are scalars, not all are zero and $\xi_{i_k}^1, \ldots, \xi_{i_k}^{\gamma_{i_k}}$, are the eigenvectors of $A_{i_k} + \lambda_{i_k} H$ corresponding to the eigenvalue σ, where $k = 1, \ldots, p$. If the networked system is controllable, then

$$\left[\sum_{k=1}^{p} \sum_{l=1}^{\gamma_{i_k}} \alpha_{kl}(e_{i_k} T \otimes \xi_{i_k}^l) \right] (D \otimes B) \neq 0$$

Consequently, we have

$$\sum_{k=1}^{p} \sum_{l=1}^{\gamma_{ik}} \alpha_{kl}(e_{i_k} T D) \otimes (\xi_{ik}^l B) \neq 0$$

for any scalars $\alpha_{kl} \in \mathbb{R}(k = 1, \ldots, p, l = 1, \ldots, \gamma_{ik})$, not all of them are zero. Therefore, $(e_{i1} T D) \otimes (\xi_{i1}^1 B), \ldots, (e_{i1} T D) \otimes (\xi_{i1}^{\gamma_{i1}} B), \ldots, (e_{ip} T D) \otimes (\xi_{ip}^1 B), \ldots, (e_{ip} T D) \otimes (\xi_{ip}^{\gamma_{ip}} B)$ are linearly independent vectors in $\mathbb{R}^{Nn}$.

(Sufficiency part) Suppose that the networked system is uncontrollable, then we will prove that at least one condition in Theorem 1 does not hold. If the networked system is not controllable, then there exists a left eigenpair of F, denoted as $(\tilde{\mu}, \tilde{v})$, such that $\tilde{v} G = 0$.

If $\tilde{\mu} \in \sigma(A_{i_0} + \lambda_{i_0} H)$ and $\tilde{\mu} \notin \sigma(A_1 + \lambda_1 H) \cup \ldots \cup \sigma(A_{i_0 - 1} + \lambda_{i_0 - 1} H) \cup \sigma(A_{i_0 + 1} + \lambda_{i_0 + 1} H) \cup \ldots \cup \sigma(A_{i_N} + \lambda_{i_N} H)$. Again $\tilde{v}$ can be written as a linear combination, $\tilde{v} = \sum_{l=1}^{\gamma_{i_0 j_0}} \alpha_0^l (e_{i_0} T \otimes \xi_{i_0 j_0}^l)$, where $\xi_{i_0 j_0}^1, \ldots, \xi_{i_0 j_0}^{\gamma_{i_0 j_0}}$ of left eigenvectors of $A_{i_0} + \lambda_{i_0} H$ corresponding to $\tilde{\mu}$, where $\left[\alpha_0^1, \ldots, \alpha_0^{\gamma_{i_0 j_0}}\right]$ is some nonzero vector. Now $\tilde{v} G = 0$ implies

$$\begin{aligned} \sum_{l=1}^{\gamma_{i_0 j_0}} \alpha_0^l (e_{i_0} T \otimes \xi_{i_0 j_0}^l)(D \otimes B) &= \sum_{l=1}^{\gamma_{i_0 j_0}} \alpha_0^l (e_{i_0} T D) \otimes (\xi_{i_0 j_0}^l B) \\ &= (e_{i_0} T D) \otimes \left(\sum_{l=1}^{\gamma_{i_0 j_0}} \alpha_0^l \xi_{i_0 j_0}^l B \right) = 0 \end{aligned}$$

This implies that $e_{i_0} T D = 0$ or $\sum_{l=1}^{\gamma_{i_0 j_0}} \alpha_0^l \xi_{i_0 j_0}^l B = 0$. If $\sum_{l=1}^{\gamma_{i_0 j_0}} \alpha_0^l \xi_{i_0 j_0}^l B = 0$, then $\left(A_{i_0} + \lambda_{i_0} H, B\right)$ is uncontrollable as $\sum_{l=1}^{\gamma_{i_0 j_0}} \alpha_0^l \xi_{i_0 j_0}^l$ is a left eigenvector of $A_{i_0} + \lambda_{i_0} H$. Thus, if the networked system is uncontrollable, then either there exists $\lambda_{i_0} \in \sigma(C)$ such that $\left(A_{i_0} + \lambda_{i_0} H, B\right)$ is uncontrollable or $e_{i_0} T D = 0$ for some i_0.

Let $\tilde{\mu}$ be the common eigenvalue of the matrices $A_{i_1} + \lambda_{i_1} H, \ldots, A_{i_p} + \lambda_{i_p} H (\lambda_{i_k} \in \sigma(C),$ for $k = 1, \ldots, p, p > 1)$ and the corresponding eigenvectors of $A_{i_k} + \lambda_{i_k}$

are $\xi_{i_k}^1, \ldots, \xi_{i_k}^{\gamma_{i_k}}$, where $k = 1, \ldots, p$. Since $\tilde{v}$ can be expressed in the form $\sum_{k=1}^{p} \sum_{l=1}^{\gamma_{i_k}} \alpha_0^{kl} \left(e_{i_k} T \otimes \xi_{ik}^l\right)$, where $\alpha_0^{kl} (l = 1, \ldots, \gamma_{i_k}, k = 1, \ldots, p)$ are some scalars, which are not all zero. Then $\tilde{v}G = 0$ implies that there exists a nonzero vector $\left[\alpha_0^{11}, \ldots, \alpha_0^{1\gamma_{i_1}}, \ldots, \alpha_0^{p1}, \ldots, \alpha_0^{p\gamma_{i_p}}\right]$ such that

$$\left[\sum_{k=1}^{p} \sum_{l=1}^{\gamma_{i_k}} \alpha_0^{kl} \left(e_{i_k} T \otimes \xi_{i_k}^l\right)\right] (D \otimes B) = \sum_{k=1}^{p} \sum_{l=1}^{\gamma_{i_k}} \alpha_0^{kl} \left[(e_{i_k} T D) \otimes (\xi_{i_k}^l B)\right] = 0$$

This implies that $(e_{i_1} T D) \otimes (\xi_{i_1}^1 B), \ldots, (e_{i_1} T D) \otimes (\xi_{i_1}^{\gamma_{i_1}} B), \ldots$
$, (e_{i_p} T D) \otimes (\xi_{i_p}^1 B), \ldots, (e_{i_p} T D) \otimes (\xi_{i_p}^{\gamma_{i_p}} B)$ are linearly dependent.

Therefore, if the networked system is uncontrollable, then at least one condition in Theorem 3 does not hold, true. □

The following examples demonstrate the application of the result for testing controllability of heterogeneous networked systems.

Example 1 Consider a heterogeneous networked system as shown in Fig. 1 composed of 3 nodes in which two nodes are identical. The state matrices of each node (A_1, A_2, A_3), control matrix B, inner coupling matrix H and the network topology matrix C are given by

$$A_1 = A_3 = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, H = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$
$$C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \tag{6}$$

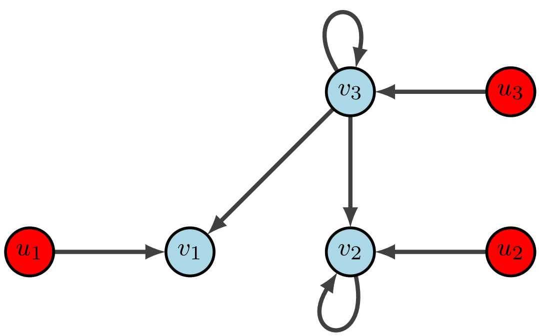

Fig. 1 Controllable heterogeneous networked system with triangularizable network topology C and node dynamics as given in (6)

As all the nodes have control input, the external control input matrix, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

For the network topology matrix C, there exists a nonsingular matrix $T = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ such that $TCT^{-1} = J$, where $J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Clearly, $T \otimes I$ commutes with A. J contains a Jordan block of order 2 and the eigenvalues of C are $\lambda_1 = 0$, $\lambda_2 = 1$ and $\lambda_3 = 1$. Observe that $\xi_2^1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ is the only left eigenvector corresponding to the eigenvalue 1 of matrix $A_2 + H$ and it satisfies $\xi_2^1 H = 0$. Then, we can easily verify the following:

(i) As $TD = T = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $e_i TD \neq 0$ for all $i = 1, 2, 3$.
(ii) (A_1, B), $(A_2 + H, B)$ and $(A_3 + H, B)$ are controllable.
(iii) $\sigma = 1$ is a common eigenvalue of the matrices $A_2 + H$ and $A_3 + H$ have with left eigenvectors $\xi_2^1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ and $\xi_3^1 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$, respectively. Also, the vectors $e_2 TD \otimes \xi_2^1 B = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$ and $e_3 TD \otimes \xi_3^1 B = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}$ are linearly independent vectors.

As all the conditions (i)–(iii) of Theorem 3 are verified, the heterogeneous networked system is controllable.

Example 2 Consider a heterogeneous networked system shown in Fig. 2, which is composed of 3 nodes in which two nodes are identical. Let

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix}, A_2 = A_3 = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & 2 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, H = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \, and \, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7}$$

.

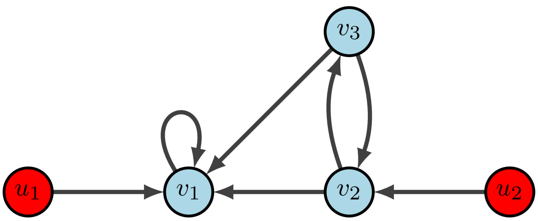

Fig. 2 Controllable heterogeneous networked system with triangularizable network topology C and node dynamics as in (7)

We observe that, for the network topology matrix C, there exists a triangularizing nonsingular matrix $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ such that $TCT^{-1} = J$, where $J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. J contains a Jordan block of order 2 and $T \otimes I$ commutes with A. The eigenvalues of C are, $\lambda_1 = 1, \lambda_2 = 1$ and $\lambda_3 = -1$. Observe that $\xi_{11}^1 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$ is the only left eigenvector corresponding to the matrix $A_1 + H$ and $\xi_{11}^1 H = 0$. Further, we can verify that

(i) $e_i TD \neq 0$ for all $i = 1, 2, 3$.
(ii) $(A_1 + H, B)$, $(A_2 + H, B)$ and $(A_3 - H, B)$ are controllable.
(iii) As the matrices $A_1 + H$,$A_2 + H$ and $A_3 - H$ do not have a common eigenvalue, the condition (iii) in Theorem 3 is satisfied.

Thus all the conditions (i)–(iii) of Theorem 3 are verified. Hence, the heterogeneous system is controllable.

Now, we give an example of a controllable networked system having heterogeneous dynamics with diagonalizable network topology matrix.

Example 3 Consider a heterogeneous network system composed of 3 nodes in which two nodes are identical, where $A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $A_2 = A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. There exists a nonsingular matrix $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$ such that $TCT^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = J$. J has no Jordan block of order ≥ 2 and $T \otimes I$ commutes with A. $\lambda_1 = 1, \lambda_2 = 0$ and $\lambda_3 = 1$ are the eigenvalues of C. Also,

(i) $e_i TD \neq 0$ for all $i = 1, 2, 3$.
(ii) $(A_1 + H, B)$, (A_2, B) and $(A_3 + H, B)$ are controllable.
(iii) The matrices $A_1 + H$ and $A_3 + H$ have a common eigenvalue 1 with left eigenvectors $\xi_1^1 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$ and $\xi_3^1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, respectively. Further, $e_1 TD \otimes \xi_1^1 B = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}$ and $e_3 TD \otimes \xi_3^1 B = \begin{bmatrix} 0 & \sqrt{2} & 0 \end{bmatrix}$ are linearly independent vectors.

Thus, all the conditions (i)–(iii) of Theorem 3 are verified. Hence, the heterogeneous network system is controllable.

This approach enables as to find the nodes to which a control can be applied to make an uncontrollable system to a controllable system.

Remark 1 If $e_i TD = 0$ for some $i = 1, 2, \ldots, N$, then the given system is not controllable. For, we have, $e_i T \otimes \xi_{ij}^k (k = 1, \ldots, \gamma_{ij})$ are left eigenvectors of F corresponding to μ_i^j, $j = 1, \ldots, q_i$, $i = 1, \ldots, N$. If $e_i TD = 0$ for some i, say i_0, then $(e_{i_0} T \otimes \xi_{i_0 j}^k)(D \otimes B) = (e_{i_0} TD \otimes \xi_{i_0 j}^k B) = 0$ for all $j = 1, 2, \ldots, q_{i_0}$, $k = 1, 2, \ldots, \gamma_{i_0 j}$. Then by Lemma 3, the given system is not controllable.

Now we may be able to modify the external input matrix D, so that $e_i T D \neq 0$, $i = 1, \ldots, N$ as shown in the following example.

Example 4 Consider a homogeneous network system composed of 3 nodes, where

$$A_1 = A_2 = A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$
$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8}$$

There exists a nonsingular matrix $T = \begin{bmatrix} 0 & 1 & -1 \\ \frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix}$ such that $TCT^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = J$. From Corollary 1, it is easy to verify that the networked system is not controllable as $e_2 T D = 0$. Here $TD = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Observe that either δ_1 or δ_3 must be 1 so that $e_i T D \neq 0$ for all $i = 1, 2, 3$. Modify D as $\tilde{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. In other words, either node v_1 or node v_3 is supplied with a control input. Then $e_i T \tilde{D} \neq 0$ for all $i = 1, 2, 3$.

For the modified network system, we can verify the conditions (ii) and (iii) of Theorem 3. The eigenvalues of C are $\lambda_1 = 0, \lambda_2 = 1$ and $\lambda_3 = -1$. Clearly, $(A, B), (A + H, B), (A - H, B)$ are controllable and these matrices does not have a

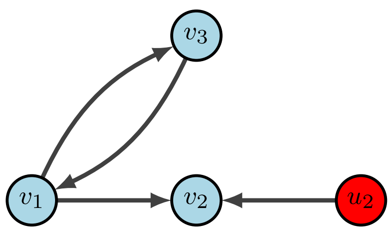

Fig. 3 Heterogeneous networked system which is not controllable with a triangularizable network topology C and node dynamics given in (8)

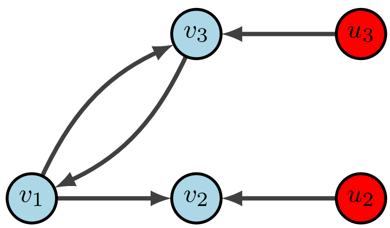

Fig. 4 The networked system becomes controllable with node dynamics as in (8), if the external control input matrix is $\tilde{D}$

common eigenvalue. Thus, all the conditions of Theorem 3 are satisfied and hence the heterogeneous system is controllable.

The condition that the matrix $T \otimes I$ commutes with A in Theorem 3 is satisfied when the networked system is homogeneous as we can see in the following proposition.

Proposition 1 *If networked system (1) is a homogeneous system, that is, $A_i = \tilde{A}$ for $i = 1, 2, \ldots, N$, then $T \otimes I$ commutes with A.*

Proof If the given system is a homogeneous system, then it can be represented in the compact form

$$\dot{X}(t) = FX(t) + GU(t)$$

where $F = A + C \otimes H$ and $G = D \otimes B$. From Eq. (4),

$$A = blockdiag\{A_1, A_2, \ldots, A_N\} = blockdiag\{\tilde{A}, \tilde{A}, \ldots, \tilde{A}\} = I \otimes \tilde{A}$$

Clearly,

$$(T \otimes I)A = (T \otimes I)(I \otimes \tilde{A}) = T \otimes \tilde{A} = (I \otimes \tilde{A})(T \otimes I) = A(T \otimes I)$$

Thus, $T \otimes I$ commutes with A. □

Consequently, for a homogeneous networked system, we have the following result.

Theorem 4 *Suppose that networked system (1) is a homogeneous system, that is, $A_i = \tilde{A}$ for all $i = 1, \ldots, N$ with*

(a) *a triangularizable network topology. That is, $TCT^{-1} = J = uppertriang\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$, where J is the Jordan Canonical Form of C; and*
(b) *if J contains a Jordan block of order $l \geq 2$ corresponding to the eigenvalue λ_{i_0} of C and $\xi_{ij}^k H = 0$ for all $i = i_0, i_0 + 1, \ldots, i_0 + l - 1$, $j = 1, 2, \ldots, q_i$, $k = 1, 2, \ldots, \gamma_{ij}$, where ξ_{ij}^k, $i = 1, 2, \ldots, k = 1, 2, \ldots, \gamma_{ij}$ are the left eigenvectors of $A_i + \lambda_i H$ corresponding to the eigenvalues μ_i^j and $\gamma_{ij} \geq 1$ represents the geometric multiplicity of μ_i^j.*

Then networked system (3) is controllable if and only if the following conditions are satisfied.

(i) *$e_i TD \neq 0$ for all $i = 1, \ldots, N$, where $\{e_i\}$ is the canonical basis for $\mathbb{R}^N$.*
(ii) *$(\tilde{A} + \lambda_i H, B)$ is controllable, for $i = 1, 2, \ldots, N$; and*
(iii) *If matrices $\tilde{A} + \lambda_{i_1} H, \ldots, \tilde{A} + \lambda_{i_p} H (\lambda_{i_k} \in \sigma(C),\ for\ k = 1, \ldots, p,\ p > 1)$ have a common eigenvalue σ, then $(e_{i_1} TD) \otimes (\xi_{i_1}^1 B), \ldots, (e_{i_1} TD) \otimes (\xi_{i_1}^{\gamma_{i_1}} B), \ldots, (e_{i_p} TD) \otimes (\xi_{i_p}^1 B), \ldots, (e_{i_p} TD) \otimes (\xi_{i_p}^{\gamma_{i_p}} B)$ are linearly independent vectors where $\gamma_{i_k} \geq 1$ is the geometric multiplicity of σ for $\tilde{A} + \lambda_{i_k} H$ and $\xi_{i_k}^l (l = 1, \ldots, \gamma_{i_k})$ are the left eigenvectors of $\tilde{A} + \lambda_{i_k} H$ corresponding to σ, $k = 1, \ldots, p$.*

In the following example, we verify the conditions of (i)–(iii) Theorem 4 to obtain the controllability of a homogeneous networked system.

Example 5 Consider a networked system with two identical nodes, $A_1 = A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then, there exists a nonsingular matrix $T = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ such that $TCT^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Here, $\lambda_1 = -1$ and $\lambda_2 = 1$ are the eigenvalues of C. Observe that

(i) $e_i TD \neq 0$ for all $i = 1, 2$.
(ii) $(A_1 - H, B)$, $(A_2 + H, B)$ are controllable. As the matrices $A_1 - H$ and $A_2 + H$ do not have a common eigenvalue, condition (iii) does not apply.

Thus, all the conditions of Theorem 4 are verified. Hence, the system is controllable.

Remark 2 Verification of following conditions restricts the application of Theorem 3 to a general heterogeneous networked system.

(i) $T \otimes I$ commutes with A.
(ii) If the network topology is triangularizable, the condition that $\xi_{ij}^k H = 0$ for all $i = i_0, i_0 + 1, \ldots, i_0 + l - 1$, $j = 1, 2, \ldots, q_i$, $k = 1, 2, \ldots, \gamma_{ij}$ if J contains a Jordan block of order $l \geq 2$ corresponding to the eigenvalue λ_{i_0} of C.

However, condition (i) is trivially satisfied in the case for a homogeneous networked system and condition (ii) does not apply when the network topology is diagonalizable. The network topology being triangularizable is an advantage over the existing results as the available results are only for systems with a diagonalizable network topology. If a triangularizable network topology is applied to a homogeneous system, Hao et al.'s [11] result does not ensure controllability of the system as the network topology matrix is not diagonalizable. But, we have seen in Example 3 that our result can be applied to a homogeneous networked system with nondiagonalizable network topology. Also, as seen in Examples 1 and 2, our result can be applied to heterogeneous networked systems with triangularizable network topology matrix. Another advantage is that, as shown in Example 4, we can identify nodes of an uncontrollable system in which one can apply control to a node to make the modified networked system controllable.

Hao et al. [11] have proved Theorem 1 as a necessary and sufficient condition for the controllability of a homogeneous networked system with a diagonalizable network topology matrix. With the help of the following proposition, we now show that Theorem 3 is a generalization of Theorem 1 of Hao et al. [11].

Proposition 2 *Suppose that the network topology matrix C is diagonalizable. That is, there exists a matrix T such that $TCT^{-1} = J$, where $J = diag\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$. Then (C, D) is controllable if and only if $e_i TD \neq 0$, $i = 1, 2, \ldots, N$.*

Proof Given that there exists a matrix T such that $TCT^{-1} = J$, where $J = diag\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$. Now,

$$TCT^{-1} = J \Rightarrow TC = JT$$

$$\Rightarrow e_i TC = e_i JT \quad \forall\, i = 1, 2, \ldots, N$$
$$\Rightarrow (e_i T)C = \lambda_i (e_i T) \quad \forall\, i = 1, 2, \ldots, N$$

That is, $e_i T$ is a left eigenvector of C corresponding to the eigenvalue λ_i, $i = 1, 2, \ldots, N$. Then by Lemma 3, (C, D) is controllable if and only if $e_i T D \neq 0$, $i = 1, 2, \ldots, N$. □

Thus by Proposition 2, we can now deduce the necessary and sufficient condition for the controllability of a homogeneous networked system with a diagonalizable network topology matrix C, established by Hao et al. [11] as a corollary of Theorem 4 as follows.

Corollary 1 *Consider a homogeneous networked system, that is, $A_i = \tilde{A}$ for all $i = 1, \ldots, N$ with a diagonalizable network topology matrix C. Let $\sigma(C) = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$. Then networked system (1) is controllable if and only if the following conditions are satisfied:*

(i) *(C, D) is controllable;*
(ii) *$(\tilde{A} + \lambda_i H, B)$ is controllable, for $i = 1, 2, \ldots, N$; and*
(iii) *If matrices $\tilde{A} + \lambda_{i_1} H, \ldots, \tilde{A} + \lambda_{i_p} H$ $\left(\lambda_{i_k} \in \sigma(C),\ for\ k = 1, \ldots, p,\ p > 1\right)$ have a common eigenvalue ρ, then $(t_{i_1} D) \otimes (\xi_{i_1}^1 B), \ldots, (t_{i_1} D) \otimes (\xi_{i_1}^{\gamma_{i_1}} B), \ldots, (t_{i_p} D) \otimes (\xi_{i_p}^1 B), \ldots, (t_{i_p} D) \otimes (\xi_{i_p}^{\gamma_{i_p}} B)$ are linearly independent, where t_{i_k} is the left eigenvector of C corresponding to the eigenvalue λ_{i_k}; $\gamma_{i_k} \geq 1$ is the geometric multiplicity of ρ for $A + \lambda_{i_k} H$; $\xi_{i_k}^l (l = 1, \ldots, \gamma_{i_k})$ are the left eigenvectors of $A + \lambda_{i_k} H$ corresponding to ρ, $k = 1, \ldots, p$.*

In view of Proposition 2, the condition (i) of Theorem 4 and condition (i) of Corollary 1 are equivalent. The condition (ii) in Theorem 4 and Corollary 1 coincides. As per the result of Hao et al. [11], if (λ_i, t_i) and (μ, ξ) are the left eigenpairs of C and $A + \lambda_i H$, respectively, then $(\mu, \xi(t_i \otimes I_n))$ is a left eigenpair of $F = I_N \otimes \tilde{A} + C \otimes H$. This in turn implies that the condition (iii) in Theorem 4 is equivalent to condition 3 in Corollary 1.

Remark 3 The existence of the matrix T satisfying all the required conditions is crucial in applying the theorem. If the given system is such that $A_i \neq A_j$ for all $i \neq j$, then for $A = blockdiag\{A_1, A_2, \ldots, A_N\}$ to commute with $(T \otimes I)$, T must be a diagonal matrix. If $A_i = A_j$ for some $i \neq j$, then T_{ij} and T_{ji} are the only possible nonzero elements along with the diagonal entries.

4.2 Necessary conditions for controllability in special network topologies

Now we obtain some controllability results over some specific network topologies. In case there exists a node v_j with no incoming edge, we obtain a necessary condition for controllability of heterogeneous networked system (3) as shown below.

Theorem 5 *Suppose that there exists a node v_j with no edge from any other nodes. Then, if (A_j, B) is not controllable, then networked system (3) is not controllable.*

Proof If there exists a node v_j with no edge from any other nodes, the network topology matrix C is of the form

$$C = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1N} \\ c_{21} & c_{22} & \dots & c_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ c_{(j-1)1} & c_{(j-1)2} & \dots & c_{(j-1)N} \\ 0 & 0 & \dots & 0 \\ c_{(j+1)1} & c_{(j+1)2} & \dots & c_{(j+1)N} \\ \vdots & \vdots & \vdots & \vdots \\ c_{N1} & c_{N2} & \dots & c_{NN} \end{bmatrix}$$

Suppose that (A_j, B) is not controllable. Then by Lemma 3, there exists a nonzero eigenvector ξ of A_j such that $\xi B = 0$. The state matrix of the networked system F is given by

$$F = \begin{bmatrix} A_1 + c_{11}H & c_{12}H & \dots & \dots & \dots & c_{1N}H \\ c_{21} & A_2 + c_{22}H & \dots & \dots & \dots & c_{2N}H \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & A_j & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{N1}H & c_{N2}H & \dots & \dots & \dots & A_N + c_{NN}H \end{bmatrix}$$

and hence $e_j \otimes \xi$ is a left eigenvector of F. Since $\xi B = 0$, $(e_j \otimes \xi)(D \otimes B) = e_j D \otimes \xi B = 0$. Then the networked system is not controllable. □

We have seen that the controllability of an individual node is necessary when there are no incoming edges to that node. But this is not the case when there are no outgoing edges from a node. Example 6 shows that controllability of an individual node is not necessary for network controllability, even if there are no outgoing edges from that node.

Remark 4 If there exists some node v_j with no edge to other nodes, the controllability of (A_j, B) is not necessary for the controllability of the networked system.

Example 6 Consider a system with two nodes which are nonidentical,

$$A_1 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{9}$$

From Fig 5, it is easy to observe that there is no edge from node 2. Also, (A_2, B) is not controllable. But the networked system is controllable.

Fig. 5 The networked system is controllable with parameters given in (9). Observe that there is no edge from node v_2 to node v_1

The following theorem addresses a situation where the controllability of an individual node with no outgoing edges. Here controllability of the individual node becomes a necessary condition for network controllability.

Theorem 6 *Suppose that there exists a node v_j with no edge to any other nodes. If $\xi H = 0$ for all left eigenvectors ξ of A_j, then the controllability of (A_j, B) is necessary for the controllability of the networked system.*

Proof If there exists some node v_j with no edge to any other nodes, then the network topology matrix C takes the form,

$$C = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1(j-1)} & 0 & c_{1(j+1)} & \dots & c_{1N} \\ c_{21} & c_{22} & \dots & c_{2(j-1)} & 0 & c_{2(j+1)} & \dots & c_{2N} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ c_{N1} & c_{N2} & \dots & c_{N(j-1)} & 0 & c_{N(j+1)} & \dots & c_{NN} \end{bmatrix}$$

The state matrix of the networked system F is given by,

$$F = \begin{bmatrix} A_1 + c_{11}H & c_{12}H & \dots & 0 & \dots & c_{1N}H \\ c_{21}H & A_2 + c_{22}H & \dots & 0 & \dots & c_{2N}H \\ \vdots & \vdots & \ddots & \vdots & \dots & \vdots \\ c_{j1}H & c_{j2}H & \dots & A_j & \dots & c_{jN}H \\ \vdots & \vdots & \dots & \vdots & \ddots & \vdots \\ c_{N1}H & c_{N2}H & \dots & 0 & \dots & A_N + c_{NN}H \end{bmatrix}$$

Suppose that (A_j, B) is not controllable. Then there exists a left eigenvector ξ of A_j such that $\xi B = 0$. Since $\xi H = 0$ for all left eigenvectors of A_j, $e_j \otimes \xi$ is a left eigenvector of F with $(e_j \otimes \xi)(D \otimes B) = e_j D \otimes \xi B = 0$ and hence the networked system is not controllable. □

5 Conclusion and future scope of work

In this paper, a necessary and sufficient condition has been derived for controllability of a class of heterogeneous networked systems under both a directed and weighted topology. Examples are provided to illustrate the theoretical results. Furthermore, our result generalizes the work of Hao et al. [11] on controllability of homogeneous LTI networked systems, allowing us to broaden the scope of study to a larger class of systems. In addition, controllability results have been derived for a networked system over some particular network topologies. Our result is more informative regarding the role of subsystem dynamics, network topology, etc., in the controllability of a networked system than the existing results and is easy to validate. In the present study, the control matrix is uniform in all subsystems. However, in the future, we intend to study the controllability of networked systems with heterogeneous control matrices. Another line of research could be an investigation of the controllability of networked systems with delays and impulses. However, research in this direction is performed for homogeneous networked systems with one-dimensional communication having delays in control. But for heterogeneous networked systems, such an investigation is yet to be performed.

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Declarations

Conflict of interest The authors declare that there is no potential conflict of interest.

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