Mathematics of Control, Signals, and Systems (2023) 35:77–123
https://doi.org/10.1007/s00498-022-00331-4

ORIGINAL ARTICLE

Exact boundary controllability of 1D semilinear wave equations through a constructive approach

Kuntal Bhandari[1] · Jérôme Lemoine[1] · Arnaud Münch[1]

Received: 9 March 2022 / Accepted: 7 September 2022 / Published online: 15 October 2022

Abstract

The exact controllability of the semilinear wave equation $y_{tt} - y_{xx} + f(y) = 0$, $x \in (0, 1)$ assuming that f is locally Lipschitz continuous and satisfies the growth condition $\limsup_{|r|\to\infty} |f(r)|/(|r| \ln^p |r|) \leq \beta$ for some β small enough and $p = 2$ has been obtained by Zuazua (Ann Inst H Poincaré Anal Non Linéaire 10(1):109–129, 1993). The proof based on a non-constructive fixed point arguments makes use of precise estimates of the observability constant for a linearized wave equation. Under the above asymptotic assumption with $p = 3/2$, by introducing a different fixed point application, we present a simpler proof of the exact boundary controllability which is not based on the cost of observability of the wave equation with respect to potentials. Then, assuming that f is locally Lipschitz continuous and satisfies the growth condition $\limsup_{|r|\to\infty} |f'(r)|/\ln^{3/2} |r| \leq \beta$ for some β small enough, we show that the above fixed point application is contracting yielding a constructive method to approximate the controls for the semilinear equation. Numerical experiments illustrate the results. The results can be extended to the multi-dimensional case and for nonlinearities involving the gradient of the solution.

Keywords Semilinear wave equation · Exact boundary controllability · Carleman estimates · Fixed point

Mathematics Subject Classification 35L71 · 93B05

Dedicated to the memory of Roland Glowinski

✉ Arnaud Münch
arnaud.munch@uca.fr

Kuntal Bhandari
kuntal.bhandari@uca.fr

Jérôme Lemoine
jerome.lemoine@uca.fr

[1] Université Clermont Auvergne, CNRS, LMBP, 63000 Clermont-Ferrand, France

1 Introduction and main results

Let $\Omega := (0, 1)$ and let $T > 0$. We set $Q_T := \Omega \times (0, T)$. We consider the semilinear 1D wave equation

$$\begin{cases} y_{tt} - y_{xx} + f(y) = 0 & \text{in } Q_T, \\ y(0, \cdot) = 0, \quad y(1, \cdot) = v & \text{in } (0, T), \\ (y(\cdot, 0), y_t(\cdot, 0)) = (u_0, u_1) & \text{in } \Omega, \end{cases} \tag{1}$$

where $(u_0, u_1) \in \boldsymbol{V} := H_0^1(\Omega) \times L^2(\Omega)$ is a given initial state, $v \in H_0^1(0, T)$ is a control function and $f \in \mathcal{C}^0(\mathbb{R})$ is a nonlinear function such that $|f(r)| \le C(1 + |r|)\ln^2(2 + |r|)$ for every $r \in \mathbb{R}$ and some $C > 0$. Then, (1) admits a unique weak solution in $\mathcal{C}^0([0, T]; H^1(\Omega)) \times \mathcal{C}^1([0, T]; L^2(\Omega))$ (see [9]).

The system (1) is said to be *exactly controllable* at time $T > 0$ if for any initial state $(u_0, u_1) \in \boldsymbol{V}$ and target data $(z_0, z_1) \in \boldsymbol{V}$, there exists a control function $v \in H_0^1(0, T)$ such that the associated solution to (1) satisfies $(y(\cdot, T), y_t(\cdot, T)) = (z_0, z_1)$. The controllability time $T > 0$ needs to be large enough in view of the finite speed of propagation of the solutions

Literature- The exact controllability for the linear wave equations is by now well-understood, see for instance the pioneer works by Russell [34], Lions [23, 24], Lagnese and Lions [25]; we also refer [5].

The first work concerning the controllability of finite dimensional nonlinear wave equations has been done by Markus [30] by the way of an implicit function theorem. Later on, this approach has been adapted for the local exact controllability of nonlinear wave equations by Chewning [10], Fattorini [19]. Global exact controllability for the semilinear wave equations in any space dimension has first been obtained by Zuazua [37–39] assuming that the nonlinear functions are globally Lipschitz and asymptotically linear, i.e., assuming that $\limsup_{|r|\to\infty} |f(r)|/|r| < \infty$. For the boundary controllability case, this asymptotic assumption has been removed in [40] in the framework of the HUM method introduced by Lions coupled with a fixed point argument.

Theorem 1 [40, Theorem 2.1] *Assume that $T > 2$. Then, for every globally Lipschitz continuous function f such that $f' \in L^\infty(\mathbb{R})$ and $\gamma \in (0, 1)$, $\gamma \neq \frac{1}{2}$, the system* (1) *is exactly controllable in $H_0^\gamma(0, 1) \times H^{\gamma-1}(0, 1)$ with a control $v \in H_0^\gamma(0, T)$.*

Later on, this result (actually proved in a multidimensional situation) was recovered by I. Lasiecka and R. Triggiani, [28], using a global inversion theorem. The authors improved some regularity of their boundary control still assuming globally Lipschitz nonlinearity.

Then, in the framework of the distributed controllability with a control support $\omega \subset (0, 1)$, the assumption $f' \in L^\infty(\mathbb{R})$ has been relaxed by Zuazua.

Theorem 2 [41, Theorem 1] *Let $\omega = (l_1, l_2)$ with $0 < l_1 < l_2 < 1$. Assume that $T > 2\max\{l_1, 1 - l_2\}$, that f is locally Lipschitz continuous and satisfies*

$(\mathbf{H_1})$ $\limsup_{|r|\to\infty} \frac{|f(r)|}{|r|\ln_+^2|r|} \le \beta$

for some β small enough depending only on ω and T. Then, for any $(u_0, u_1), (z_0, z_1) \in \boldsymbol{V}$, the system

$$\begin{cases} y_{tt} - y_{xx} + f(y) = v\,1_\omega & \text{in } Q_T, \\ y(0,\cdot) = y(1,\cdot) = 0 & \text{in } (0,T), \\ (y(0,\cdot), y_t(0,\cdot)) = (u_0, u_1) & \text{in } \Omega, \end{cases} \tag{2}$$

is exactly controllable with control in $L^2(\omega \times (0,T))$: there exists $v \in L^2(\omega \times (0,T))$ such that $(y(\cdot,T), y_t(\cdot,T)) = (z_0, z_1)$.

Here and in the sequel, we note

$$\ln_+ |r| = \begin{cases} 0 \text{ if } |r| \le 1 \\ \ln |r| \text{ else.} \end{cases}$$

Moreover, it is proved in [41, Theorem 2] that, if f behaves like $-s\ln^p(|s|)$ with $p > 2$ as $|s| \to +\infty$, then the system is not exactly controllable in any time $T > 0$, due to an uncontrollable blow-up phenomenon. Theorem 1 has been slightly improved in [12], weakening the condition $(\mathbf{H_1})$ into

$$\limsup_{|r|\to+\infty} \left|\int_0^r f(r')\,dr'\right| \left(|r| \prod_{k=1}^{+\infty} \ln^{[k]}(e_k + r^2)\right)^{-2} < +\infty$$

where $\ln^{[k]}$ denotes the k^{th} iterate of ln and $e_k > 0$ is such that $\ln^{[k]}(e_k) = 1$. This growth condition is optimal since the solution of (2) may blow up whenever f grows faster at infinity and has the bad sign. The multi-dimensional case in which Ω is a bounded domain of $\mathbb{R}^d$, $d > 1$, with a $\mathcal{C}^{1,1}$ boundary has been addressed in [29]; assuming that the support ω of the control function is a neighborhood of $\partial\Omega$ and that $T > \text{diam}(\Omega\backslash\omega)$, the exact controllability of (2) is proved under the growth condition $\limsup_{|r|\to+\infty} \frac{|f(r)|}{|r|\ln_+^{1/2}|r|} < \beta$ for some β small enough. For control domains ω satisfying the classical multiplier assumption (see [24]), exact controllability has been proved in [36] assuming that f is globally Lipschitz continuous. We also mention [14] where a positive boundary controllability result is proved for steady-state initial and final data and for T large enough by a quasi-static deformation approach. We also mention the work by Dehman et al. [16] which is concerned with the controllability and stabilizability of some subcritical semilinear wave equations in $\Omega' \subset \mathbb{R}^3$. Assuming that the nonlinearity $f \in \mathcal{C}^3(\mathbb{R})$ satisfies

$$f(0) = 0, \quad rf(r) \ge 0, \quad |f^{(j)}(r)| \le C(1+|r|)^{p-j}, \quad j = 1,2,3; \quad 1 \le p < 5$$

the exact internal controllability of the semilinear wave equations at time $T := T(u_0, u_1) > 0$ that depends on the size of the initial data $(u_0, u_1) \in H_0^1(\Omega') \times L^2(\Omega')$. See also [15] achieving the same result in a uniform time under smallness assumption on the initial data. The sign condition has been weakened in [21] to an asymptotic sign assumption leading to a semi-global internal controllability result in the sense that the target data are prescribed in a precise subset of $H_0^1(\Omega') \times L^2(\Omega')$.

The above results devoted to internal controllability, notably Theorem 2, can be employed together with the domain extension method to get indirectly boundary controllability results for system (1) of interest in the present work.

The proof of Theorem 2 is based on a Leray Schauder fixed point argument applies to the operator $\Lambda : L^\infty(Q_T) \to L^\infty(Q_T)$, where $y := \Lambda(z)$ is a controlled solution with the control function v of the linear boundary value problem

$$\begin{cases} y_{tt} - y_{xx} + \widehat{f}(z)\, y = -f(0) + v 1_\omega & \text{in } Q_T, \\ y(0,\cdot) = y(1,\cdot) = 0 & \text{in } (0,T), \\ (y(\cdot,0), y_t(\cdot,0)) = (u_0, u_1) & \text{in } \Omega, \end{cases} \qquad \widehat{f}(r) := \begin{cases} \dfrac{f(r)-f(0)}{r} & \text{if } r \neq 0 \\ f'(0) & \text{if } r = 0 \end{cases} \tag{3}$$

satisfying $(y(\cdot,T), y_t(\cdot,T)) = (z_0, z_1)$. The stability of the operator Λ in $L^\infty(Q_T)$ is based on a precise estimate of the cost of the control v in terms of the potential $\widehat{f}$ and data (u_0, u_1), (z_0, z_1).

Objective - The general goal addressed in this work is the approximation of the controllability problem associated with (1), that is to construct an explicit sequence $(y_k, v_k)_{k\in\mathbb{N}}$ converging strongly toward a control-state pair solution (y, v) for (1). Although almost sharp with respect to the nonlinearity, Theorem 2 is not constructive as it does not provide any convergent sequences $(y_k)_{k\in\mathbb{N}}$ to a fixed point of Λ, i.e., to a controlled solution y of (2). This is due to the fact that the operator Λ is not contracting in general.

Assuming slightly stronger assumptions on f, a constructive convergent sequence has been proposed by the third author and E. Trélat in [32] using a least-squares approach coupled with a Newton type linearization.

Theorem 3 [32, Theorem 2.3] *Let* $\omega = (l_1, l_2)$ *with* $0 < l_1 < l_2 < 1$. *Assume that* $T > 2\max\{l_1, 1 - l_2\}$ *and that* $f \in \mathcal{C}^1(\mathbb{R})$ *satisfies*

$(\mathbf{H_1})'$ $\exists \alpha > 0$, *s.t.* $|f'(r)| \leq \alpha + \beta \ln_+^2 |r|, \quad \forall r \in \mathbb{R}$

for some $\beta > 0$ *small enough depending only on* ω *and* T *and*

$(\overline{\mathbf{H}}_\mathbf{p})$ $\exists p \in (0,1]$ *such that* $\displaystyle\sup_{\substack{a,b\in\mathbb{R}\\ a\neq b}} \frac{|f'(a) - f'(b)|}{|a-b|^p} < +\infty.$

Then, for any $(u_0, u_1), (z_0, z_1) \in \boldsymbol{V}$, *one can construct a sequence* $(y_k, v_k)_{k\in\mathbb{N}}$ *converging strongly to a controlled pair for* (2) *satisfying* $(y(\cdot,T), y_t(\cdot,T)) = (z_0, z_1)$. *Moreover, after a finite number of iterations, the convergence is of order at least* $1 + p$.

The hypothesis on f is stronger here than in Theorem 1: it should be noted, however, that the function $f(r) = a + br + \beta r \ln(1 + |r|)^2$, $a, b \in \mathbb{R}$ which is somehow the limit case in $(\mathbf{H_1})$ satisfies $(\mathbf{H_1})'$ and $(\overline{\mathbf{H}}_\mathbf{1})$. On the other hand, Theorem 3 is constructive, contrary to Theorem 1. The construction makes appear the operator

$\Lambda_1 : L^\infty(Q_T) \to L^\infty(Q_T)$ where $y := \Lambda_1(z)$ is a controlled solution with the control function v of the linear boundary value problem

$$\begin{cases} y_{tt} - y_{xx} + f'(z)y = v1_\omega + f'(z)z - f(z) & \text{in } Q_T, \\ y(0,\cdot) = y(1,\cdot) = 0 & \text{in } (0,T), \\ (y(\cdot,0), y_t(\cdot,0)) = (u_0,u_1) & \text{in } \Omega. \end{cases} \tag{4}$$

Theorem 3 is extended to a multidimensional case, i.e., $\Omega \subset \mathbb{R}^d$ with $d \leq 3$ in [4] under the same condition on f except that the exponent 2 in $(\mathbf{H_1})'$ is replaced by an exponent $1/2$.

Main result of the present work - In this paper, we prove the following result, directly in the framework of the boundary controllability.

Theorem 4 *Assume $T > 2$. Let $s > 0$ large enough.*

- *There exists $\beta^\star > 0$ such that if $f \in \mathcal{C}^0(\mathbb{R})$ satisfies*

$$(\mathbf{H_2}) \ \exists \alpha_1, \alpha_2 > 0, \ s.t. \ |f(r)| \leq \alpha_1 + |r| \left(\alpha_2 + \beta^\star \ln_+^{3/2} |r|\right), \quad \forall r \in \mathbb{R}$$

then system (1) is exactly controllable at time T for initial data in $\boldsymbol{V}$ with controls in $H_0^1(0,T)$.

- *There exists $\beta^\star > 0$ such that if f is locally Lipschitz and satisfies*

$$(\mathbf{H_2'}) \ \exists \alpha > 0, \ s.t. \ |f'(r)| \leq \alpha + \beta^\star \ln_+^{3/2} |r|, \quad \forall r \in \mathbb{R}$$

then, for any initial state (u_0,u_1) and final state (z_0,z_1) in $\boldsymbol{V}$, one can construct a sequence $(y_k,v_k)_{k\in\mathbb{N}^*}$ that converges strongly to a controlled pair (y,v) in $\left(\mathcal{C}^0([0,T];H^1(\Omega)) \cap \mathcal{C}^1([0,T];L^2(\Omega))\right) \times H_0^1(0,T)$ for the system (1). Moreover, the convergence of (y_k,v_k) holds at least with a linear rate for the norm $\|\rho(s)\cdot\|_{L^2(Q_T)} + \|\rho_1(s)\cdot\|_{L^2(0,T)}$ where ρ, ρ_1 are defined in (8) and s is chosen sufficiently large depending on $\|(u_0,u_1)\|_{\boldsymbol{V}}$ and $\|(z_0,z_1)\|_{\boldsymbol{V}}$.

To our knowledge, this result is the first proposing a constructive approximation of boundary controls for the semilinear wave equation without the assumption that f is globally Lipschitz. Under smallness assumptions on the data, we mention the recent works [7] and [33]. As in Theorem 2, the parameter $\beta^\star$ have to be small enough depending only Ω and T through the constant appearing in a Carleman estimates (we refer to Remark 5). Moreover, the lower bound of the Carleman parameter s depends logarithmically on the size $\|(u_0,u_1)\|_{\boldsymbol{V}}$ and $\|(z_0,z_1)\|_{\boldsymbol{V}}$ of the data (we refer to Remark 6).

Concerning the first part of the theorem, if we compare with Theorem 2 (leading indirectly to boundary controllability result by the extension method), the assumption on the asymptotic behavior of f is slightly stronger with an exponent $\frac{3}{2}$ instead of 2. This is due to the fact the method in [41] based on explicit computation (using the d'Alembert formula) is genuinely one dimensional, while the present work is based on Carleman estimates valid in any space dimension. On the other hand, this first part relaxed the regularity assumption to $f \in \mathcal{C}^0(\mathbb{R})$ instead of f locally Lipschitz continuous. Moreover, this first part of Theorem 4 differs from Theorem 2 on the functional spaces as it is based on a different fixed point application leading to a simpler proof. In particular, it is not based on the analysis of the cost of observability of the wave equation with potential. Concerning the second part of the theorem, it relaxes the Hölder assumption ($\overline{\mathbf{H}}_\mathbf{p}$) on f' appearing in Theorem 3 but still leads to a constructive method. As we shall see, this is related to an appropriate choice of the parameter s related to the norm of the initial condition. Again, to our knowledge, this is the first result leading to a convergent approximation of boundary controls for superlinear nonlinearities without smallness assumption notably on the initial condition and target (contrary to the recent works [7] and [33]).

Theorem 4 is obtained by adapting the recent work [17] devoted to a semilinear heat equation. We introduce the following linearized controllability problem: for $\widehat{y}$ in a suitable class $\mathcal{C}_R(s)$ depending on a free parameter $s \geq 1$, find the control v such that the solution y of

$$\begin{cases} y_{tt} - y_{xx} = -f(\widehat{y}) & \text{in } Q_T, \\ y(0,\cdot) = 0, \;\; y(1,\cdot) = v & \text{in } (0,T), \\ (y(\cdot,0), y_t(\cdot,0)) = (u_0, u_1) & \text{in } \Omega, \end{cases} \tag{5}$$

satisfies $(y(\cdot,T), y_t(\cdot,T)) = (z_0, z_1)$ in Ω, and (y, v) corresponds to the minimizer of a functional J_s depending on s and involving Carleman weight functions (see Remark 3).

This will define an operator $\Lambda_s : \widehat{y} \mapsto y$ from some suitable class $\mathcal{C}_R(s)$ into itself, on which we can use fixed point theorems for s sufficiently large depending on $\|(u_0, u_1)\|_{\boldsymbol{V}}$, namely Schauder fixed point theorem for the first item of Theorem 4, and Banach-Picard fixed point theorem for the second item, allowing to exhibit a simple sequence of convergent approximations of the control and controlled trajectory. The analysis of the fixed point operator is based on Carleman estimates as they allow to get precise estimates on the control and controlled trajectories in terms of the parameter s. Choosing the Carleman parameter large allows to limit the influence of lowers order terms and get suitable contracting properties. Such tricks have already been used in the context of inverse problems reformulated through a least-squares functional in [1] and [22].

With respect to the heat equation considered in [17, 27], the Carleman weights are not singular with respect to the time variable, avoiding technicalities. On the other hand, the regularity issue is more delicate for the hyperbolic case which does not enjoys regularizing property. This is *a fortiori* true for boundary control : precisely, in order to get L^∞ estimate for the controlled trajectories solution of (1), the boundary control v needs to be more regular than $L^2(0,T)$. Hopefully, it turns out that the optimal state-control pair for the functional J_s (see Remark 3) involving L^2 norms enjoys suitable regularity property as soon as the initial and final data belongs to $\boldsymbol{V}$

and satisfy compatibility condition at $x = 0$ and $x = 1$. This point is crucial in our analysis.

Outline- The paper is organized as follows. In Sect. 2, we derive a controllability result for the linear wave equation with precise estimates in term of the right hand side, the initial data and the Carleman parameter s large enough. In particular, we prove that the optimal control for the $L^2(0, T)$ norm belongs actually to $H^1(0, T)$: this result stated in Theorem 7 is proven in Appendix A. Then, in Sect. 3, we prove, for any time $T > 2$ the uniform null controllability of (1) assuming that f is continuous and satisfies the condition $(\mathbf{H_2})$. Then, in Sect. 3.5, assuming that f' is continuous and satisfies the condition $(\mathbf{H_1})'$, we show that the operator Λ_s is contracting, yielding the convergence of the Picard iterates $y_{k+1} = \Lambda_s(y_k)$. Section 5 illustrates the result with some numerical experiments, while Sect. 6 concludes with some remarks.
Notations- In this article, C denote generic constants depending on Ω and T, which may change from line to line, but are independent of the Carleman parameter s.

2 Controllability results for the linear wave equation

This section is devoted to a controllability result for a linear wave equation with a right hand side $B \in L^2(Q_T)$ and initial data $(u_0, u_1) \in \boldsymbol{V}$. Precisely, for any $(z_0, z_1) \in \boldsymbol{V}$ and $T > 0$ large enough, we are interested by the existence of a control function $v \in H_0^1(0, T)$ such that the solution y of

$$\begin{cases} y_{tt} - y_{xx} = B & \text{in } Q_T, \\ y(0, \cdot) = 0, \;\; y(1, \cdot) = v & \text{in } (0, T), \\ (y(\cdot, 0), y_t(\cdot, 0)) = (u_0, u_1) & \text{in } \Omega, \end{cases} \tag{6}$$

satisfies $(y(\cdot, T), y_t(\cdot, T)) = (z_0, z_1)$. Though this linear control is by now standard, we aim to get precise weighted estimate of a state-control pair in a given functional space in the data, which will be crucial to handle the nonlinear system (1). We employ Carleman estimates as fundamental tool (see [6]).

2.1 A global Carleman estimate

For any $\beta \in (0, 1)$ and $x_0 < 0$, we define the auxiliary function

$$\psi(x, t) = |x - x_0|^2 - \beta\left(t - \frac{T}{2}\right)^2 + M_0 \quad \text{in } Q_T, \tag{7}$$

where $M_0 > 0$ is chosen in such a way that ψ is strictly positive. Then, for any $\lambda > 0$, we define $\phi(x, t) = e^{\lambda\psi(x,t)}$. For all $s \geq s_0$, let us now define the following weight functions

$$\rho(s; x, t) := e^{-s\phi(x,t)}, \quad \rho_1(s; t) = \rho(s; 1, t), \quad \forall (x, t) \in Q_T. \tag{8}$$

Remark that

$$e^{-cs} \le \rho(s; x, t) \le e^{-s}, \quad e^{-cs} \le \rho_1(s; t) \le e^{-s} \quad \text{in } Q_T$$

with $c := \|\phi\|_{L^\infty(Q_T)}$, that $\rho^{-1}, \rho \in \mathcal{C}^\infty(\overline{Q_T})$ and that $\rho_1, \rho_1^{-1} \in \mathcal{C}^\infty([0, T])$. In short, we shall write $\rho(s)$ and $\rho_1(s)$ to denote the above weight functions.

Then, for any $\delta > 0$ such that $T - 2\delta > 2\sup_{\overline{\Omega}} |x - x_0|$, we introduce a cut-off function $\eta \in \mathcal{C}_c^\infty(\mathbb{R})$ satisfying the following properties:

$$\begin{cases} 0 \le \eta(t) \le 1 & \text{in } (\delta, T - \delta), \\ \eta(t) = 1 & \text{in } [2\delta, T - 2\delta], \\ \eta(t) = 0 & \text{in } (-\infty, \delta] \cup [T - \delta, +\infty). \end{cases} \tag{9}$$

Let

$$P := \Big\{ w \in \mathcal{C}^0([0, T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0, T]; L^2(\Omega)), w_{tt} - w_{xx} \in L^2(Q_T) \Big\}.$$

Recall that $w_x(1, \cdot) \in L^2(0, T)$ for every $w \in P$ (see [23, Theorem 4.1]).

The controllability property for the linear system (6) is based on the following Carleman estimate with boundary observation at $x = 1$.

Theorem 5 *Assume $T > 2$. There exists $s_0 > 0$, $\lambda > 0$ and $C > 0$, such that for any $s \ge s_0$, we have the following Carleman inequality*

$$\begin{aligned} & s \int_{Q_T} \rho^{-2}(s)(|w_t|^2 + |w_x|^2) + s^3 \int_{Q_T} \rho^{-2}(s)|w|^2 \\ & \quad + s \int_\Omega \rho^{-2}(s; \cdot, 0)(|w_t(\cdot, 0)|^2 + |w_x(\cdot, 0)|^2) + s^3 \int_\Omega \rho^{-2}(s; \cdot, 0)|w(\cdot, 0)|^2 \\ & \quad + s \int_\Omega \rho^{-2}(s; \cdot, T)(|w_t(\cdot, T)|^2 + |w_x(\cdot, T)|^2)\,\mathrm{d}x + s^3 \int_\Omega \rho^{-2}(s; \cdot, T)|w(\cdot, T)|^2 \\ & \le C \int_{Q_T} \rho^{-2}(s)|w_{tt} - w_{xx}|^2 + Cs \int_0^T \eta^2(t)\rho_1^{-2}(s)|w_x(1, \cdot)|^2 \end{aligned} \tag{10}$$

for every $w \in P$.

Proof We refer to [8, Lemma 2.3] using [1, Remark 2.9 and Theorem 2.5]. Remark that the occurrence of the terms at $t = T$ on the left hand side are due to the fact that $\rho(\cdot, t) = \rho(\cdot, T - t)$ and the reversibility of the wave operator. □

2.2 Application to controllability

In a standard way, Theorem 5 allows to deduce some controllability results for the system (6). For any $s \ge s_0$, we define the bilinear form

$$(w, z)_{P,s} := \int_{Q_T} \rho^{-2}(s) Lw Lz + s \int_0^T \eta^2(t)\rho_1^{-2}(s) w_x(1, t) z_x(1, t), \tag{11}$$

for any $w, z \in P$. Here and in what follows, we use the notation $Lw := w_{tt} - w_{xx}$. It is easily seen that (11) defines a scalar product in P and if P_s denotes P endowed with this scalar product, then P_s is an Hilbert space.

We can state the main result of this section, devoted without loss of generality to the null controllability case, for which $(z_0, z_1) = (0, 0)$ in Ω.

Theorem 6 *Assume that $T > 2$ and let $\eta \in \mathcal{C}^\infty(\mathbb{R})$ be a cut-off function satisfying (9). For $s \geq s_0$, $B \in L^2(Q_T)$ and $(u_0, u_1) \in \boldsymbol{V}$, there exists unique $w_s \in P_s$, depending only on B, u_0, u_1 such that*

$$(w_s, z)_{P,s} = \int_\Omega u_1\, z(\cdot, 0)\, \mathrm{d}x - \int_\Omega u_0\, z_t(\cdot, 0)\, \mathrm{d}x + \int_{Q_T} Bz, \quad \forall z \in P_s. \tag{12}$$

Then, $v_s(t) = s\eta^2(t)\rho_1^{-2}(s)(w_s)_x(1, t)$ is a control function for (6) where $y_s = \rho^{-2}(s)Lw_s$ is the associated controlled trajectory, that is $y_s(x, T) = (y_s)_t(x, T) = 0$ for all $x \in \Omega$ and the operator defined by

$$\Lambda_s^0 : (B, u_0, u_1) \mapsto y_s \tag{13}$$

is linear, continuous from $L^2(Q_T) \times H_0^1(\Omega) \times L^2(\Omega)$ to $L^2(Q_T)$.

Moreover, we have the following estimates for y_s and v_s for some constant $C > 0$ independent of s:

$$\begin{aligned} &\|\rho(s)y_s\|_{L^2(Q_T)} + s^{-1/2}\left\|\frac{\rho_1(s)}{\eta}v_s\right\|_{L^2(\delta, T-\delta)} \\ &\quad \leq C\left(s^{-3/2}\|\rho(s)B\|_{L^2(Q_T)} + s^{-1/2}e^{-s}\|u_0\|_{L^2(\Omega)} + s^{-3/2}e^{-s}\|u_1\|_{L^2(\Omega)}\right). \end{aligned} \tag{14}$$

Before going to the proof, we make the following remarks.

Remark 1 It is well-known that the boundary controllability of (6) with $L^2(0, T)$ controls holds true with initial data (u_0, u_1) only in $L^2(\Omega) \times H^{-1}(\Omega)$. We start directly with $(u_0, u_1) \in \boldsymbol{V}$ since the application of some fixed point theorem to deal with the semilinear case shall require regularity on the state-control pair (see Sect. 2.3).

Remark 2 In the framework of exact controllability with no vanishing target $(z_0, z_1) \in \boldsymbol{V}$, the right hand side of estimate (14) contains the extra quantities

$$s^{-1/2}e^{-s}\|z_0\|_{L^2(\Omega)} + s^{-3/2}e^{-s}\|z_1\|_{L^2(\Omega)}.$$

The point here to be noted is that the coefficients (powers of s or exponentials associated with s) in front of the norms of u_0, u_1 and z_0, z_1 are the same, and this would hold for any subsequent estimates.

This is why, there is no loss of generality to choose $(z_0, z_1) = (0, 0)$ which will make the computations shorter and simpler.

Proof We first ensure the solvability of the variational Eq. (12). Since $(\cdot,\cdot)_{P,s}$ is a scalar product on P_s, we only need to check that the right hand side of (12) is a linear continuous form on P_s.

- For all $z \in P_s$: since $\rho(s)B \in L^2(Q_T)$, we have

$$\left|\int_{Q_T} Bz\right| \leq \left(\int_{Q_T} |\rho(s)B|^2\right)^{1/2} \left(\int_{Q_T} |\rho^{-1}(s)z|^2\right)^{1/2}.$$

Now, since z enjoys the Carleman inequality (10), one has $\|\rho^{-1}(s)z\|_{L^2(Q_T)} \leq Cs^{-3/2}\|z\|_{P_s}$ (recall the definition of the inner product (11) on P_s). Thus, we have

$$\left|\int_{Q_T} Bz\right| \leq Cs^{-3/2}\|\rho(s)B\|_{L^2(Q_T)}\|z\|_{P_s}.$$

- Next, we observe that

$$\begin{aligned}\left|\int_\Omega u_0 z_t(\cdot,0)\,\mathrm{d}x\right| &\leq \|\rho(s;\cdot,0)u_0\|_{L^2(\Omega)}\|\rho^{-1}(s;\cdot,0)z_t(\cdot,0)\|_{L^2(\Omega)}\\ &\leq Cs^{-1/2}e^{-s}\|u_0\|_{L^2(\Omega)}\|z\|_{P_s},\end{aligned}$$

using the Carleman inequality (10) and that $|\rho(s;x,0)| = |e^{-s\phi(x,0)}| \leq e^{-s}$ (since $\phi \geq 1$).
- Similarly, we get

$$\begin{aligned}\left|\int_\Omega u_1 z(\cdot,0)\,\mathrm{d}x\right| &\leq \|\rho(s;\cdot,0)u_1\|_{L^2(\Omega)}\|\rho^{-1}(s;\cdot,0)z(\cdot,0)\|_{L^2(\Omega)}\\ &\leq Cs^{-3/2}e^{-s}\|u_1\|_{L^2(\Omega)}\|z\|_{P_s}.\end{aligned}$$

Combining the above three items, the right hand side of (12) corresponds to a linear functional on P_s. The Riesz representation theorem implies the existence of a unique $w_s \in P_s$ satisfying the formulation (12) which additionally satisfies

$$\|w_s\|_{P_s} \leq C\left(s^{-3/2}\|\rho(s)B\|_{L^2(Q_T)} + s^{-1/2}e^{-s}\|u_0\|_{L^2(\Omega)} + s^{-3/2}e^{-s}\|u_1\|_{L^2(\Omega)}\right), \tag{15}$$

where the constant $C > 0$ is independent of $s \geq s_0$.

Then, set $y_s = \rho^{-2}(s)Lw_s$ and $v_s = s\eta^2\rho^{-2}(s)(w_s)_x(1,\cdot)$. From the equality (12), the pair (y_s, v_s) satisfies

$$\begin{aligned}\int_{Q_T} y_s Lz\,\mathrm{d}x\mathrm{d}t + \int_0^T v_s z_x(1,\cdot)\mathrm{d}t = \int_\Omega u_1\, z(\cdot,0)\,\mathrm{d}x - \int_\Omega u_0\, z_t(\cdot,0)\,\mathrm{d}x\\ + \int_{Q_T} Bz, \quad \forall z \in P_s,\end{aligned}$$

meaning that $y_s \in L^2(Q_T)$ is a solution to the linear system (6) associated with the function $v_s \in L^2(0,T)$ in the sense of transposition. By uniqueness, y_s indeed solves (6) in a weak sense. Eventually, using the estimate (15) for w_s, we get that $\rho(s)y_s = \rho^{-1}(s)Lw_s \in L^2(Q_T)$ and $s^{-1/2}\rho_1(s)v_s = s^{1/2}\eta^2\rho_1^{-1}(s)(w_s)_x(1,\cdot) \in L^2(0,T)$ and deduce the weighted estimate (14). □

Remark 3 The functions y_s and v_s introduced by Theorem 6 can be characterized as the unique minimizer of the following functional

$$J_s(y,v) := s\int_{Q_T} \rho^2(s)y^2 + \int_\delta^{T-\delta} \eta^{-2}\rho_1^2(s)v^2 \tag{16}$$

over the set $\Big\{(y,v) : y\in L^2(Q_T),\, \eta^{-1}v\in L^2(\delta,T-\delta)$ solution of (6) with $y(\cdot,T) = y_t(\cdot,T) = 0$ in $\Omega\Big\}$. We refer to [8, Section 2] for the details.

Remark 4 The controlled state $y_s = \rho^{-2}Lw_s$ satisfies

$$\begin{cases} Ly_s = B & \text{in } Q_T,\\ y_s(0,\cdot) = 0, \quad y_s(1,\cdot) = s\eta^2\rho_1^{-2}(s)(w_s)_x(1,\cdot) & \text{in } (0,T),\\ (y_s(\cdot,0),(y_s)_t(\cdot,0)) = (u_0,u_1), & \text{in } \Omega, \end{cases} \tag{17}$$

implying (by standard regularity results for the wave equation) that $y_s \in \mathcal{C}^0([0,T]; L^2(\Omega)) \cap \mathcal{C}^1([0,T]; H^{-1}(\Omega))$. On the other hand, the function w_s uniquely satisfies the equation

$$\begin{cases} Lw_s = \rho^2 y_s & \text{in } Q_T,\\ w_s(0,\cdot) = w_s(1,\cdot) = 0 & \text{in } (0,T), \end{cases} \tag{18}$$

implying that $(w_s(\cdot,0), \partial_t w_s(\cdot,0)) \in \boldsymbol{V}$ (see estimate (10)) and $w_s \in \mathcal{C}^0([0,T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega))$.

2.3 Estimate for the state-control pair in $\mathcal{C}^0([0,T];H^1(\Omega)) \times H^1(0,T)$

In this section, we prove that the state-control pair (y_s, v_s) given by Theorem 6 enjoys additional regularity property, under the assumption $(u_0,u_1) \in \boldsymbol{V}$ and the introduction of the cut-off function η with respect to the time variable. In particular, we obtain that $v_s \in H_0^1(0,T)$ and $y_s \in L^\infty(Q_T)$. This gain is crucial for the analysis of the semilinear case.

Theorem 7 *Let any $(u_0,u_1) \in \boldsymbol{V}$ and $B \in L^2(Q_T)$ be given. Then, the solution (y_s,v_s) of (6) defined in Theorem 6 satisfies $v_s \in H^1(0,T)$, $y_s \in \mathcal{C}^0([0,T]; H^1(\Omega))\cap$*

$\mathcal{C}^1([0,T];L^2(\Omega))$ *and the following estimate :*

$$\begin{aligned}
&\|\rho(s)(y_s)_t\|_{L^2(Q_T)} + s^{-1/2}\|\rho_1(s)(v_s)_t\|_{L^2(0,T)} \\
&\quad\le C\left(s^{-1/2}\|\rho(s)B\|_{L^2(Q_T)} + s^{-1/2}e^{-s}\|u_1\|_{L^2(\Omega)}\right. \\
&\qquad\left.+s^{1/2}e^{-s}\|u_0\|_{L^2(\Omega)} + s^{-1/2}e^{-s}\|(u_0)_x\|_{L^2(\Omega)}\right).
\end{aligned} \tag{19}$$

We refer to [18, Theorem 5.4] where a similar gain of regularity is proved in the simpler case of control of minimal $L^2(0,T)$-norm, i.e., J_s in (16) is replaced by $J(y,v) = \|v\|^2_{L^2(0,T)}$. We also refer to [15] for internal control by introducing a cut-off function in space. The proof of Theorem 7 is long and requires several steps. It is done in Appendix A.

Let us prescribe the following regularity estimate for the controlled trajectory y_s.

Lemma 1 *Let us recall the controlled trajectory y_s and the control v_s for the linear system (6), defined by Theorem 6. Then, y_s satisfies the following bound*

$$\begin{aligned}
&\|y_s\|_{\mathcal{C}^0([0,T];H^1(\Omega))} + \|(y_s)_t\|_{\mathcal{C}^0([0,T];L^2(\Omega))} \\
&\quad\le C\Big(\|B\|_{L^2(Q_T)} + e^{cs}\|\rho(s)B\|_{L^2(Q_T)} + e^{(c-1)s}\|(u_0)_x\|_{L^2(\Omega)} \\
&\qquad+se^{(c-1)s}\|u_0\|_{L^2(\Omega)} + s^{1/2}e^{(c-1)s}\|u_1\|_{L^2(\Omega)}\Big),
\end{aligned} \tag{20}$$

where $C > 0$ is a constant that does not depend on $s \ge s_0$, and $c = \|\phi\|_{L^\infty(Q_T)}$.

Proof It is well-known that for given data $(u_0, u_1) \in \boldsymbol{V}$ and $B \in L^2(Q_T)$, we have

$$\begin{aligned}
&\|y_s\|_{\mathcal{C}^0([0,T];H^1(\Omega))} + \|(y_s)_t\|_{\mathcal{C}^0([0,T];L^2(\Omega))} \\
&\quad\le C\left(\|B\|_{L^2(Q_T)} + \|(v_s)_t\|_{L^2(0,T)} + \|(u_0)_x\|_{L^2(\Omega)} + \|u_1\|_{L^2(\Omega)}\right).
\end{aligned} \tag{21}$$

Then, using the estimate (19) from Theorem 7, we obtain (20). □

3 Controllability result for the semilinear problem with $f \in \mathcal{C}^0(\mathbb{R})$: a Schauder fixed point argument

For any $s \ge s_0$ and $R > 0$, we introduce the class $\mathcal{C}_R(s)$, defined as the closed convex subset of $L^\infty(Q_T)$

$$\mathcal{C}_R(s) := \left\{\widehat{y} \in L^\infty(Q_T) : \ \|\widehat{y}\|_{L^\infty(Q_T)} \le R, \ \ \|\rho(s)\widehat{y}\|_{L^2(Q_T)} \le R^{1/2}\right\} \tag{22}$$

and assume that that the nonlinear function $f \in \mathcal{C}^0(\mathbb{R})$ in (1) satisfies the growth assumption $(\mathbf{H_2})$ for some $\beta^\star$ positive precisely chosen later.

Then, for $T > 2$, $s \geq s_0$ (to be fixed later) and for all $\widehat{y} \in \mathcal{C}_R(s)$, we first solve the linearized boundary control problem, given by

$$\begin{cases} y_{tt} - y_{xx} = -f(\widehat{y}) & \text{in } Q_T, \\ y(0,\cdot) = 0, \quad y(1,\cdot) = v & \text{in } (0,T), \\ (y(\cdot,0), y_t(\cdot,0)) = (u_0,u_1) & \text{in } \Omega \end{cases} \tag{23}$$

with v such that $(y(\cdot,T), y_t(\cdot,T)) = (0,0)$ in Ω. The existence of a controlled trajectory $y \in L^\infty(Q_T)$ is guaranteed by Theorem 7 with the source term $-f(\widehat{y}) \in L^2(Q_T)$. Now, our aim is to prove that there exists a fixed point of the following operator

$$\Lambda_s : L^\infty(Q_T) \mapsto L^\infty(Q_T), \qquad \Lambda_s(\widehat{y}) = y. \tag{24}$$

Note that, $\Lambda_s(\widehat{y}) = \Lambda_s^0(-f(\widehat{y}), u_0, u_1)$, as per the definition (13).

Claim: We are going to show that

1. for $\beta^\star > 0$ small enough, there exist $R > 0$ large enough and $s \geq s_0$ such that $\mathcal{C}_R(s)$ is stable under the map Λ_s; see Sect. 3.2;
2. $\Lambda_s(\mathcal{C}_R(s))$ is relatively compact subset of $\mathcal{C}_R(s)$ for the norm $\|\cdot\|_{L^\infty(Q_T)}$; see Sect. 3.3;
3. Λ_s is a continuous map in $\mathcal{C}_R(s)$ for the topology induced by the norm $\|\cdot\|_{L^\infty(Q_T)}$; see Sect. 3.4.

Accordingly, by the Schauder fixed point theorem, there will exist a fixed point of Λ_s, denote by y, which will be the controlled trajectory for our semilinear problem (1).

3.1 Estimate of $\|\Lambda_s(\widehat{y})\|_{L^\infty(Q_T)}$

We begin with the following lemma.

Lemma 2 *Assume $T > 2$ and that there exists $\beta^\star \geq 0$ such that $f \in \mathcal{C}^0(\mathbb{R})$ satisfies* ($\mathbf{H_2}$). *For any $s \geq s_0$ and $\widehat{y} \in L^\infty(Q_T)$, the quantity $f(\widehat{y})$ satisfies the following estimates:*

$$\|f(\widehat{y})\|_{L^2(Q_T)} \leq \alpha_1 T + \left(\alpha_2 + \beta^\star \ln_+^{3/2} \|\widehat{y}\|_{L^\infty(Q_T)}\right) e^{cs} \|\rho(s)\widehat{y}\|_{L^2(Q_T)},$$
$$\|\rho(s) f(\widehat{y})\|_{L^2(Q_T)} \leq \alpha_1 T e^{-s} + \left(\alpha_2 + \beta^\star \ln_+^{3/2} \|\widehat{y}\|_{L^\infty(Q_T)}\right) \|\rho(s)\widehat{y}\|_{L^2(Q_T)},$$

with $c = \|\phi\|_{L^\infty(Q_T)}$.

Proof The proof of above lemma follows from the growth assumption ($\mathbf{H_2}$) on f. Observe that

$$\begin{aligned}\left(\int_{Q_T} |f(\widehat{y})|^2\right)^{1/2} &\leq \alpha_1 T + \|\rho^{-1}(s)\|_\infty \left(\int_{Q_T} |\rho(s)\widehat{y}|^2 \left(\alpha_2 + \beta^\star \ln_+^{3/2} |\widehat{y}|\right)^2\right)^{1/2} \\ &\leq \alpha_1 T + \left(\alpha_2 + \beta^\star \ln_+^{3/2} \|\widehat{y}\|_{L^\infty(Q_T)}\right) e^{cs} \|\rho(s)\widehat{y}\|_{L^2(Q_T)},\end{aligned}$$

where we have used that $\|\rho^{-1}(s)\|_{L^\infty(Q_T)} = \|e^{s\phi}\|_{L^\infty(Q_T)} \leq e^{cs}$. The other estimate is obtained in a similar way. □

Proposition 1 *Under the assumptions of Lemma 2, for $s \geq s_0$ and for all $\widehat{y} \in L^\infty(Q_T)$, the solution $y = \Lambda_s(\widehat{y})$ to the linearized system (23) satisfies the following estimates:*

$$\begin{aligned}
&\|\rho(s)y\|_{L^2(Q_T)} + s^{-1/2}\|\rho_1(s)v\|_{L^2(0,T)} \\
&\qquad \leq Cs^{-3/2}\Big(\alpha_2 + \beta^\star \ln_+^{3/2} \|\widehat{y}\|_{L^\infty(Q_T)}\Big)\|\rho(s)\widehat{y}\|_{L^2(Q_T)} \\
&\qquad\quad + Cs^{-3/2}\alpha_1 T e^{-s} + Cs^{-1/2}e^{-s}\|u_0\|_{L^2(\Omega)} + Cs^{-3/2}e^{-s}\|u_1\|_{L^2(\Omega)}, \qquad (25)\\
&\|\rho(s)y_t\|_{L^2(Q_T)} + s^{-1/2}\|\rho_1(s)v_t\|_{L^2(0,T)} \\
&\qquad \leq Cs^{-1/2}\Big(\alpha_2 + \beta^\star \ln_+^{3/2} \|\widehat{y}\|_{L^\infty(Q_T)}\Big)\|\rho(s)\widehat{y}\|_{L^2(Q_T)} \\
&\qquad\quad + Cs^{-1/2}\alpha_1 T e^{-s} + Cs^{-1/2}e^{-s}\|(u_0)_x\|_{L^2(\Omega)} + Cs^{1/2}e^{-s}\|u_0\|_{L^2(\Omega)} \\
&\qquad\quad + Ce^{-s}\|u_1\|_{L^2(\Omega)}. \qquad (26)
\end{aligned}$$

Moreover, $y \in \mathcal{C}^0([0,T]; H^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega))$ and

$$\begin{aligned}
\|y\|_{L^\infty(Q_T)} &\leq C\alpha_1 T + C\alpha_1 T e^{(c-1)s} \\
&\quad + C\Big(\alpha_2 + \beta^\star \ln_+^{3/2} \|\widehat{y}\|_{L^\infty(Q_T)}\Big)e^{cs}\|\rho(s)\widehat{y}\|_{L^2(Q_T)} + Ce^{(c-1)s}\|(u_0)_x\|_{L^2(\Omega)} \\
&\quad + Cse^{(c-1)s}\|u_0\|_{L^2(\Omega)} + Cs^{1/2}e^{(c-1)s}\|u_1\|_{L^2(\Omega)}. \qquad (27)
\end{aligned}$$

Proof Put $B = -f(\widehat{y})$ in the linear model (6). Then, the proof is followed as a consequence of Theorems 6, 7, Lemma 1 and Theorem 2. □

3.2 Stability of the class $\mathcal{C}_R(s)$ for suitable choices of parameters

We express the result in terms of the following lemma. We hereby recall the set $\mathcal{C}_R(s)$ defined in (22).

Lemma 3 *Under the assumptions of Lemma 2, if $\beta^\star$ in $(\mathbf{H_2})$ is small enough, there exists an s and $R > 0$ large enough, such that we have*

$$\Lambda_s\big(\mathcal{C}_R(s)\big) \subset \mathcal{C}_R(s)$$

where $\mathcal{C}_R(s)$ is the class given in (22).

Proof We start with any $\widehat{y} \in \mathcal{C}_R(s)$ for $s \geq s_0 \geq 1$ and we look for the bounds of the solution $y = \Lambda_s(\widehat{y})$ (to (23)) with respect to the associated norms. Since $\widehat{y} \in \mathcal{C}_R(s)$, one has $\|\rho(s)\widehat{y}\|_{L^2(Q_T)} \leq R^{1/2}$ and $\|\widehat{y}\|_{L^\infty(Q_T)} \leq R$. Therefore, the estimate (25) yields

$$\|\rho(s)y\|_{L^2(Q_T)} \leq Cs^{-3/2}\Big(\alpha_2 + \beta^\star \ln_+^{3/2} R\Big)R^{1/2} + Cs^{-3/2}\alpha_1 T e^{-s}$$

$$+Cs^{-1/2}e^{-s}\|u_0\|_{L^2(\Omega)} + Cs^{-3/2}e^{-s}\|u_1\|_{L^2(\Omega)}. \tag{28}$$

Similarly, estimate (27) implies

$$\|y\|_{L^\infty(Q_T)} \leq C\alpha_1 T + C\alpha_1 T e^{(c-1)s} + C\Big(\alpha_2 + \beta^\star \ln_+^{3/2} R\Big)e^{cs}R^{1/2}$$
$$+Ce^{(c-1)s}\|(u_0)_x\|_{L^2(\Omega)} + Cse^{(c-1)s}\|u_0\|_{L^2(\Omega)} + Cs^{1/2}e^{(c-1)s}\|u_1\|_{L^2(\Omega)}. \tag{29}$$

We now fix the parameter s in terms of R as follows :

$$s = \frac{1}{32c}\ln_+ R, \quad \text{with } c = \|\phi\|_{L^\infty(Q_T)} > 1, \tag{30}$$

where $R > 0$ is chosen large enough to ensure $s \geq s_0 \geq 1$. With this choice of s, the solution $y = \Lambda_s(\widehat{y})$ satisfies, in view of (28) and the fact that $\hat{y}$ belongs to $\mathcal{C}_R(s)$,

$$\|\rho(s)y\|_{L^2(Q_T)} \leq \frac{C(32c)^{3/2}}{\ln_+^{3/2} R}\big(\alpha_2 + \beta^\star \ln_+^{3/2} R\big)R^{1/2} + \frac{C\alpha_1 T(32c)^{3/2}}{R^{1/32c}\ln_+^{3/2} R}$$
$$+\frac{C\sqrt{32c}}{R^{1/32c}\ln_+^{1/2} R}\|u_0\|_{L^2(\Omega)} + \frac{C(32c)^{3/2}}{R^{1/32c}\ln_+^{3/2} R}\|u_1\|_{L^2(\Omega)}. \tag{31}$$

Thus, if $\beta^\star > 0$ is small enough such that

$$C(32c)^{3/2}\beta^\star < 1/4, \tag{32}$$

it can be guaranteed for large enough $R > 0$ that

$$\begin{cases} \dfrac{C(32c)^{3/2}}{\ln_+^{3/2} R}\big(\alpha_2 + \beta^\star \ln_+^{3/2} R\big)R^{1/2} + \dfrac{C\alpha_1 T(32c)^{3/2}}{R^{1/32c}\ln_+^{3/2} R} \leq \dfrac{1}{3}R^{1/2}, \\ \dfrac{C\sqrt{32c}}{R^{1/32c}\ln_+^{1/2} R}\|u_0\|_{L^2(\Omega)} \leq \dfrac{1}{3}R^{1/2}, \\ \dfrac{C(32c)^{3/2}}{R^{1/32c}\ln_+^{3/2} R}\|u_1\|_{L^2(\Omega)} \leq \dfrac{1}{3}R^{1/2} \end{cases} \tag{33}$$

involving, in view of 31 that $\|\rho(s)y\|_{L^2(Q_T)} \leq R^{1/2}$.

Similarly, in view of (29) and the fact that $\widehat{y}$ belongs to $\mathcal{C}_R(s)$,we infer that

$$\|y\|_{L^\infty(Q_T)} \leq C\alpha_1 T + C\alpha_1 T R^{\left(\frac{1}{32}-\frac{1}{32c}\right)} + C\big(\alpha_2 + \beta^\star \ln_+^{3/2} R\big)R^{1/32}R^{1/2}$$
$$+\frac{C}{32c}(\ln_+ R)R^{\left(\frac{1}{32}-\frac{1}{32c}\right)}\|u_0\|_{L^2(\Omega)} + CR^{\left(\frac{1}{32}-\frac{1}{32c}\right)}\|(u_0)_x\|_{L^2(\Omega)}$$
$$+\frac{C}{\sqrt{32c}}(\ln_+^{1/2} R)R^{\left(\frac{1}{32}-\frac{1}{32c}\right)}\|u_1\|_{L^2(\Omega)}. \tag{34}$$

Taking $\beta^\star > 0$ as before and R large enough, we infer that (recall that $c = \|\phi\|_{L^\infty(Q_T)} > 1$ so that $0 < \frac{1}{32} - \frac{1}{32c} < 1$)

$$\begin{cases} C\big(\alpha_2 + \beta^\star \ln_+^{3/2} R\big) R^{1/32} R^{1/2} \le R/5, \\ \dfrac{C}{32c} (\ln_+ R) R^{\left(\frac{1}{32} - \frac{1}{32c}\right)} \|u_0\|_{L^2(\Omega)} \le R/5, \\ C R^{\left(\frac{1}{32} - \frac{1}{32c}\right)} \|(u_0)_x\|_{L^2(\Omega)} \le R/5, \\ \dfrac{C}{\sqrt{32c}} (\ln_+^{1/2} R) R^{\left(\frac{1}{32} - \frac{1}{32c}\right)} \|u_1\|_{L^2(\Omega)} \le R/5, \\ C\alpha_1 T + C\alpha_1 T R^{\left(\frac{1}{32} - \frac{1}{32c}\right)} \le R/5 \end{cases} \tag{35}$$

implying from (34) that $\|y\|_{L^\infty(Q_T)} \le R$. It follows that $y = \Lambda_s(\widehat{y}) \in \mathcal{C}_R(s)$. This concludes the proof. □

Remark 5 The smallness condition on $\beta^\star$ is explicit:

$$\beta^\star < \frac{1}{4C(32c)^{3/2}}, \tag{36}$$

where C is the constant appearing in Proposition 1.

Remark 6 Provided we impose the relation (30), the above proof shows that $\mathcal{C}_R(s)$ is stable for Λ_s for any $R \ge R_0$ (equivalently $s \ge s_0$) for a suitably large R_0 (equivalently s_0). With the above choices, in view of (33)–(35), the lower bound R_0 depends on $\|(u_0, u_1)\|_V$ as a power of $\|(u_0, u_1)\|_V$, so that the lower bound s_0 can be chosen as depending logarithmically on $\|(u_0, u_1)\|_V$. Note also that there is no upper bound for R so that the parameter s (appearing notably in the definition of the weights) can be taken arbitrarily large.

3.3 Relative compactness of the set $\Lambda_s(\mathcal{C}_R(s))$

Proposition 2 *Under the assumptions of Lemma 3, $\Lambda_s(\mathcal{C}_R(s))$ is a relatively compact subset of $\mathcal{C}_R(s)$ for the $L^\infty(Q_T)$ norm.*

Proof Let $(y_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\Lambda_s(\mathcal{C}_R(s))$. We prove that there exists a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ that converges strongly to some $y \in \mathcal{C}_R(s)$ with respect to the $L^\infty(Q_T)$ norm.

Thanks to Lemma 3, $(y_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\mathcal{C}_R(s)$ and so, there exists a subsequence $y_{n_k} \in \mathcal{C}_R(s)$ and $y \in \mathcal{C}_R(s)$ such that

$$y_{n_k} \rightharpoonup^\star y \ \text{ weakly}^\star \text{ in } L^\infty(Q_T), \text{ as } k \to +\infty. \tag{37}$$

Now, since $(y_{n_k})_{k\in\mathbb{N}} \subset \Lambda_s(\mathcal{C}_R(s))$, there is a sequence $(\widehat{y}_{n_k})_{k\in\mathbb{N}} \subset \mathcal{C}_R(s)$ such that $y_{n_k} = \Lambda_s(\widehat{y}_{n_k})$, $\forall k \in \mathbb{N}$. More precisely, there exists a sequence $(v_{n_k})_{k\in\mathbb{N}} \in H_0^1(0, T)$

such that, for all $k \in \mathbb{N}$, y_{n_k} satisfies

$$\begin{cases} (y_{n_k})_{tt} - (y_{n_k})_{xx} = -f(\widehat{y}_{n_k}) & \text{in } Q_T, \\ y_{n_k}(0,t) = 0, \quad y_{n_k}(1,t) = v_{n_k}(t) & \text{in } (0,T), \\ (y_{n_k}(\cdot,0), (y_{n_k})_t(\cdot,0)) = (u_0, u_1) & \text{in } \Omega. \end{cases}$$

Moreover, for some $C_1 > 0$, we have

$$\|y_{n_k}\|_{\mathcal{C}^0([0,T];H_0^1(\Omega))} \leq C_1 R, \quad \|(y_{n_k})_t\|_{\mathcal{C}^0([0,T];L^2(\Omega))} \leq C_1 R,$$

thanks to their estimates in (20) for $B = -f(\widehat{y}_{n_k})$ and the analysis in Lemma 3. Since the embedding $\{y \in L^\infty(0,T;H_0^1(\Omega)) \mid y_t \in L^\infty(0,T;L^2(\Omega))\} \hookrightarrow \mathcal{C}^0(\overline{Q_T})$ is compact (see [35, Corollary 8 p. 90 and Lemma 12 p. 91]), this ensures the strong convergence of $(y_{n_k})_{k\in\mathbb{N}}$ in $\mathcal{C}^0(\overline{Q_T})$ as $k \to +\infty$. □

3.4 Continuity of the map Λ_s in $\mathcal{C}_R(s)$

We prove the following result.

Proposition 3 *Under the assumptions and result of Lemma 3, the map $\Lambda_s : \mathcal{C}_R(s) \to \mathcal{C}_R(s)$ is continuous with respect to the $L^\infty(Q_T)$ norm.*

Proof Let $(\widehat{y}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}_R(s)$ such that $\widehat{y}_n \to \widehat{y}$ as $n \to +\infty$ w.r.t. the $L^\infty(Q_T)$ norm for some $\widehat{y} \in \mathcal{C}_R(s)$.

Let $y_n = \Lambda_s(\widehat{y}_n)$ and prove that $y_n \to y := \Lambda_s(\widehat{y})$ as $n \to +\infty$ w.r.t. the same norm. Since $f \in \mathcal{C}^0(\mathbb{R})$, f is uniformly continuous in $[-R,R]$ implying that

$$f(\widehat{y}_n) \to f(\widehat{y}) \ \text{ in } \ L^\infty(Q_T), \quad \text{as } \ n \to +\infty \tag{38}$$

and thus $f(\widehat{y}_n) \to f(\widehat{y})$ in $L^2(Q_T)$ as $n \to +\infty$.

Now (as mentioned in Theorem 6), $\Lambda_s(\widehat{y}_n) = \Lambda_s^0(-f(\widehat{y}_n), u_0, u_1)$ is linear continuous map from $L^2(Q_T) \times H_0^1(\Omega) \times L^2(\Omega)$ to $L^2(Q_T)$. Thus, combining the estimates (14) and (20) with $B = f(\widehat{y}_n) - f(\widehat{y})$, we get that

$$\|y_n - y\|_{L^\infty(Q_T)} \leq C_1 \|\rho(s)(f(\widehat{y}_n) - f(\widehat{y}))\|_{L^2(Q_T)} \leq C_1 e^{-s} \|f(\widehat{y}_n) - f(\widehat{y})\|_{L^2(Q_T)}$$

where the constant C_1 depends only on R and $y = \Lambda_s(\widehat{y})$. Consequently, $y_n \to y$ as $n \to +\infty$ in $L^\infty(Q_T)$. □

3.5 Proof of the first item of Theorem 4

Taking $\beta^\star$ small enough (see (32)) so that Lemma 3 applies, with s and R given by (30), we can apply Schauder fixed point theorem to Λ_s on $\mathcal{C}_R(s)$: there exists $y_s \in \mathcal{C}_R(s) \subset L^\infty(Q_T)$ such that $y_s = \Lambda_s(y_s)$. By construction of Λ_s, there exists a function $v \in H_0^1(0,T)$ such that y_s is the solution of the null controllability problem

(23) with $\hat{y} = y_s$: it follows that this element y_s is a controlled solution of the semilinear wave Eq. (1).

4 Construction of control by Banach fixed point approach with $f \in \mathcal{C}^1(\mathbb{R})$: proof of the second item of Theorem 4

In this section, we assume that f is locally Lipschitz continuous and that f' satisfies $(\mathbf{H_2'})$ with $\beta^\star$ small as before. Remark that condition $(\mathbf{H_2'})$ implies the condition $(\mathbf{H_2})$ used in the previous section.

We endow the convex set $\mathcal{C}_R(s)$ with the distance d defined by $d(y,z) = \|\rho(s)(y-z)\|_{L^2(Q_T)}$. We easily check that $(\mathcal{C}_R(s), d)$ is a complete space. In the next proposition, we prove that the operator $\Lambda_s : \mathcal{C}_R(s) \to \mathcal{C}_R(s)$ is a contracting mapping leading to constructive method to find its fixed point.

Proposition 4 *Assume that f satisfies $(\mathbf{H_2'})$ with $\beta^\star$ satisfying (36), R and s as chosen in Lemma 3. Then, for any $\widehat{y}_1, \widehat{y}_2 \in \mathcal{C}_R(s)$,*

$$d(\Lambda_s(\widehat{y}_2), \Lambda_s(\widehat{y}_1)) \leq \frac{1}{2} d(\widehat{y}_2, \widehat{y}_1). \tag{39}$$

In particular, Λ_s is a contraction mapping from $\mathcal{C}_R(s)$ into itself.

Proof Let $\widehat{y}_1, \widehat{y}_2 \in \mathcal{C}_R(s)$. From (14), we get that

$$\|\rho(s)(\Lambda_s(\widehat{y}_2) - \Lambda_s(\widehat{y}_1))\|_{L^2(Q_T)} \leq C s^{-3/2} \|\rho(s)(f(\widehat{y}_2) - f(\widehat{y}_1))\|_{L^2(Q_T)}.$$

Then, we can use $(\mathbf{H_2'})$ to deduce

$$\begin{aligned} &\|\rho(s)(\Lambda_s(\widehat{y}_2) - \Lambda_s(\widehat{y}_1))\|_{L^2(Q_T)} \\ &\quad \leq C(32c)^{3/2} \left(\frac{\alpha}{\ln_+^{3/2} R} + \beta^\star \right) \|\rho(s)(\widehat{y}_2 - \widehat{y}_1)\|_{L^2(Q_T)}, \end{aligned} \tag{40}$$

since s is given by (30). Since $C(32c)^{3/2}\beta^\star \leq 1/4$, the result follows as soon as R is large enough. □

As as corollary of the previous result and the classical Banach-Picard's fixed point theorem, the contraction property of the operator Λ_s for $\beta^\star$ small enough given in (32) and s and R given by (30) allows to define a convergent sequence $(y_k, v_k)_{k\in\mathbb{N}}$ to a controlled pair for (1) and prove the following precise version of the second item of Theorem 4.

Theorem 8 *Let $(u_0, u_1) \in \boldsymbol{V}$. Assume that f is locally Lipschitz continuous and satisfies $(\mathbf{H_2'})$ with $\beta^\star$ satisfying (36), s and R as chosen in Lemma 3. Then, for any $y_0 \in \mathcal{C}_R(s)$, the sequence $(y_k)_{k\in\mathbb{N}^\star} \subset \mathcal{C}_R(s)$ given by*

$$y_{k+1} = \Lambda_s(y_k), \quad k \geq 0,$$

(where Λ_s is defined by (24)) together with the associated sequence of controls $(v_k)_{k\in\mathbb{N}^\star} \subset H_0^1(0,T)$ strongly converges in $L^2(Q_T)\times L^2(0,T)$ to a controlled solution for (1). Moreover, the convergence is at least linear with respect to the distance d.

Proof The convergence of the sequence $(y_k)_{k\in\mathbb{N}}$ toward $y=\Lambda_s(y)\in\mathcal{C}_R(s)$ with linear rate follows from the contraction property of Λ_s:

$$\begin{aligned}\|\rho(s)(y-y_k)\|_{L^2(Q_T)} &= \|\rho(s)(\Lambda_s(y)-\Lambda_s(y_{k-1}))\|_{L^2(Q_T)}\\ &\le \frac{1}{2^k}\|\rho(s)(y-y_0)\|_{L^2(Q_T)} \le \frac{1}{2^k}\left(R^{1/2}+e^{-s}\|y_0\|_{L^2(Q_T)}\right).\end{aligned}$$

Let now $v\in H_0^1(0,T)$ be associated with y so that $y-y_k$ satisfies, for every $k\in\mathbb{N}^\star$

$$\begin{cases}(y-y_k)_{tt}-(y-y_k)_{xx}=-\big(f(y)-f(y_{k-1})\big) & \text{in } Q_T,\\ (y-y_k)(0,\cdot)=0,\quad (y-y_k)(1,\cdot)=(v-v_k) & \text{in } (0,T),\\ ((y-y_k)(\cdot,0),(y-y_k)_t(\cdot,0))=(0,0) & \text{in } \Omega,\\ ((y-y_k)(\cdot,T),(y-y_k)_t(\cdot,T))=(0,0) & \text{in } \Omega.\end{cases}$$

Estimate (14) then implies (recall $s=\frac{1}{32c}\ln_+ R$)

$$\begin{aligned}\|\rho_1(s)(v-v_k)\|_{L^2(0,T)} &\le Cs^{-1}\|\rho(s)\big(f(y)-f(y_{k-1})\big)\|_{L^2(Q_T)}\\ &\le C\frac{32c}{\ln_+ R}\big(\alpha+\beta^\star\ln_+^{3/2}R\big)\|\rho(s)(y-y_{k-1})\|_{L^2(Q_T)}\end{aligned}$$

and therefore the convergence is at a linear rate of the sequence $(v_k)_{k\in\mathbb{N}^\star}$ toward an exact control for (1). □

Remark 7 It can be observed from (40) that the constant appearing in front of $\|\rho(s)(\widehat{y}_2-\widehat{y}_1)\|_{L^2(Q_T)}$ is getting smaller as R (consequently s) getting larger. In particular, if f satisfies

$$\lim_{|r|\to+\infty}\frac{|f'(r)|}{\ln_+^{3/2}|r|}=0, \tag{41}$$

then, for any given $\epsilon>0$ (however, small), the map Λ_s is ϵ-contractive for large enough $s\ge s_0$. In other words, the speed of convergence of the sequence $(y_k)_{k\ge1}$ introduced by Theorem 8 increases with s.

5 Numerical illustrations

We present some numerical illustrations of the convergence result given by Theorem 8 and emphasize the influence of the parameter s. More precisely, for s large enough, we compute the sequence $(y_k, v_k)_{k\in\mathbb{N}}$ solution to

$$\begin{cases} y_{k,tt} - y_{k,xx} = -f(y_{k-1}) & \text{in } Q_T, \\ y_k(0,\cdot) = 0, \quad y_k(1,\cdot) = v_k & \text{in } (0,T), \\ (y_k(\cdot,0), (y_k)_t(\cdot,0)) = (u_0, u_1) & \text{in } \Omega, \\ (y_k(\cdot,T), (y_k)_t(\cdot,T)) = (0,0) & \text{in } \Omega, \end{cases} \tag{42}$$

obtained through the variational formulation (12) with the source term $B = -f(y_{k-1})$. We first sketch the algorithm and then discuss some numerical experiments obtained with the software FreeFem++ (see [20]).

5.1 Construction of the sequence $(y_k, v_k)_{k\geq 1}$

Starting with some suitable initial guess $y_0 \in \mathcal{C}_R(s)$, we can obtain the solution y_k to (42) with a control v_k based on Theorem 6. Assume that the value of the Carleman parameter s satisfies Lemma 3. Then, for each $k \geq 1$, we define the unique solution $w_k \in P_s$ (see Theorem 6) of

$$(w_k, z)_{P,s} = \int_\Omega u_1 z(\cdot,0)\,\mathrm{d}x - \int_\Omega u_0 z_t(\cdot,0)\,\mathrm{d}x - \int_{Q_T} f(y_{k-1}) z\,\mathrm{d}x\mathrm{d}t \quad \forall z \in P_s, \tag{43}$$

then we set $y_k = \rho^{-2}(s)Lw_k$ in Q_T and $v_k = s\eta^2\rho_1^{-2}(s)(w_k)_x(1,\cdot)$ in $(0,T)$.

The numerical approximation of the variational formulation (43) has been addressed in [8, 13] and more recently in [2]. A conformal finite dimensional approximations, say $P_{s,h}$ of P_s, leads to a strong convergent approximation $w_{k,h}$ of w_k for the P_s norm as the discretization parameter h goes to 0. Then, from $w_{k,h}$, we can define the approximated controlled solution $y_{k,h} := \rho^{-2}(s)Lw_{k,h}$ and $v_{k,h} := s\eta^2\rho_1^{-2}(s)(w_{k,h})_x(1,\cdot)$. In our semilinear setting, we shall employ an equivalent but different formulation, more appropriate for numerical purposes. First, in order to avoid the possible numerical blow up—for s large—of the terms $\rho^{-2}(s)$ and $\rho_1^{-2}(s)$ appearing in the formulation and of order $\mathcal{O}(e^{2s})$, we introduce a change of variable. Second, in order to avoid second differentiation in order to compute y_k from the definition $y_k = \rho^{-2}(s)(w_{k,tt} - w_{k,xx})$, we incorporate directly the controlled state solution in the formulation. We refer to [31, Section 3.2] where this normalization procedure has been employed in the context of the heat equation. Precisely, we introduce the variables

$$m_k = \rho_1^{-1}(s)w_k, \quad p_k = \rho^{-1}(s)Lw_k \quad \text{in} \quad Q_T \tag{44}$$

so that $p_k = \rho^{-1}(s)L(\rho_1(s)m_k)$ and $y_k = \rho^{-1}(s)p_k$ and then replace the formulation (43) by the equivalent and well-posed following mixed formulation: find $(m_k, p_k, \lambda_k) \in \rho^{-1}(s)P_s \times L^2(Q_T) \times L^2(Q_T)$ solution of

$$\begin{cases} \int_{Q_T} p_k \overline{p}\,\mathrm{d}x\mathrm{d}t + s\int_0^T \eta^2(t)(m_k)_x(1,t)\overline{m}_x(1,t)\mathrm{d}t \\ \quad + \int_{Q_T} \lambda_k \left(\overline{p} - \rho^{-1}(s)L(\rho_1(s)\overline{m})\right)\mathrm{d}x\mathrm{d}t \\ \quad = \int_0^1 u_1\rho_1(s,0)\overline{m}(\cdot,0)\,\mathrm{d}x - \int_0^1 u_0(x)\left[\rho_1(0;s)\overline{m}_t(0,x)\right. \\ \quad \left. +(\partial_t\rho_1)(0;s)\overline{m}(0,x)\right]\mathrm{d}x - \int_{Q_T} f(\rho^{-1}(s)p_{k-1})\rho_1(s)\overline{m}\,\mathrm{d}x\mathrm{d}t, \\ \int_{Q_T} \overline{\lambda}\left(p_k - \rho^{-1}(s)L(\rho_1(s)m_k)\right)\mathrm{d}x\mathrm{d}t = 0, \end{cases} \tag{45}$$

for all $(\overline{m},\overline{p},\overline{\lambda}) \in \rho^{-1}(s)P_s \times L^2(Q_T) \times L^2(Q_T)$. The variable λ_k stands as a Lagrange multiplier for the constraint $p_k - \rho^{-1}(s)L^\star(\rho_1(s)m_k) = 0$ in Q_T. We check the following inequality

$$\begin{aligned} &\rho^{-1}(s)L^\star(\rho_1(s)m_k) \\ &\quad = \rho^{-1}(s)\rho_1(s)Lm_k + \rho^{-1}(s)\partial_{tt}\rho_1(s)m_k + 2\rho^{-1}(s)\partial_t\rho_1(s)(m_k)_t \\ &\quad = A_1Lm_k + A_2m_k + A_3(m_k)_t, \end{aligned}$$

with

$$\begin{cases} \partial_t\rho_1(s;t) = s\lambda\beta(2t-T)\phi(1,t)\rho_1(s;t), \\ \partial_{tt}\rho_1(s;t) = 2s\lambda\beta\phi(1,t)\rho_1(s;t) - s\lambda^2\beta^2(2t-T)^2\phi(1,t)\rho_1(s;t) \\ \qquad\qquad + s^2\lambda^2\beta^2(2t-T)^2\phi^2(1,t)\rho_1(s;t), \\ A_1 = \rho^{-1}(s)\rho_1(s), \quad A_2 = \rho^{-1}(s)\partial_{tt}\rho_1(s), \quad A_3 = 2\rho^{-1}(s)\partial_t\rho_1(s) \end{cases}$$

and we observe that the functions A_i do not contain exponential with positive arguments. For instance, we get

$$\rho^{-1}(s)\rho_1(s) = e^{-s(\phi(1,t)-\phi(x,t))},$$

and recall that $\phi(1,t)-\phi(x,t) \geq 0$ in Q_T. Eventually, from the solution (m_k, p_k, λ_k), the controlled pair (y_k, v_k) can be retrieved using the formula

$$y_k = \rho^{-1}(s)p_k, \quad v_k = s\eta^2\rho_1^{-1}(s)(m_k)_x(1,\cdot). \tag{46}$$

The sequence $(y_k, v_k)_{k\geq 1}$ is initialized with $(y_0, v_0) = (0,0)$ so that the iteration (y_1, v_1) is the solution to the linear system (42) with the right hand side $B = -f(y_0) = -f(0)$. We perform the iterations until the following criterion (based on Proposition 4) is fulfilled

$$\frac{\|\rho(s)y_{k+1} - \rho(s)y_k\|_{L^2(Q_T)}}{\|\rho(s)y_k\|_{L^2(Q_T)}} \leq 10^{-6}. \tag{47}$$

We denote by $k^\star$ the smallest integer k such that (47) holds. Remark, since the weight $\rho(s)$ are uniformly positive and bounded in Q_T, the convergence of the sequence $\{\rho(s)y_k\}_{k\in\mathbb{N}}$ stated in Proposition 4) also implies the convergence of the sequence $\{y_k\}_{k\in\mathbb{N}}$.

Eventually, concerning the approximation of the formulation (45), we use a conformal space-time finite element method (as addressed in [13]). We introduce a regular triangulation $\mathcal{T}_h$ of Q_T such that $\overline{Q}_T = \bigcup_{K\in\mathcal{T}_h} K$. We assume that $\{\mathcal{T}_h\}_{h>0}$ is a regular family, where the index h is such that $h = \max_{K\in\mathcal{T}_h} diam\,(K)$. We then approximate of the variables p_k and λ_k in the space $P_h := \{p_h \in \mathcal{C}^0(\overline{Q}_T) : p_h|_K \in \mathbb{P}_1(K),\ \forall K \in \mathcal{T}_h\} \subset L^2(Q_T)$, where $\mathbb{P}_1(K)$ denotes the space of affine functions both in x and t.

On the other hand, the variable m_k is approximated with the space $V_h := \{v_h \in \mathcal{C}^1(\overline{Q}_T) : v_h|_K \in \mathbb{P}(K),\ \forall K \in \mathcal{T}_h\} \subset \rho_1^{-1}(s)P_s$, where $\mathbb{P}(K)$ denotes the composite Hsieh–Clough–Tocher $\mathcal{C}^1$ element defined for triangles. We refer to [11, p. 356] and [3] where the implementation has been discussed. We refer to [2] for the numerical analysis of the formulation (45).

5.2 Experiments

In what follows, we take $T = 2.5$ and $(u_0(x), u_1(x)) = c_{u_0}(\sin(\pi x), 0)$ with $x \in \Omega = (0, 1)$ parametrized by the real c_{u_0}. Then, we define the various weight functions appearing in the Carleman inequality (10) in Sect. 2.1 as follows: we take

$$\psi(x,t) = (x+0.02)^2 - 0.9\,(t - T/2)^2 + 2, \quad \phi(x,t) = e^{\frac{1}{2}\psi(x,t)} \qquad \text{in } Q_T,$$

so that $\psi \geq 0.5$ in Q_T. The weights ρ and ρ_1 are then defined by (8). The cut-off function η is chosen as follows:

$$\eta(t) = e^{-\frac{1}{(t+10^{-6})(T-t+10^{-6})}}, \quad \forall t \in (0, T).$$

Eventually, we employ a regular space-time mesh composed of 25600 triangles and 13041 vertices corresponding to the discretization parameter $h \approx 1.25 \times 10^{-2}$.

5.2.1 Nonlinear functions with growth $r\ln^{3/2}(2+|r|)$

In the mixed formulation (45), let us first consider the semilinear function

$$f(r) = c_f r(\alpha_2 + \beta^\star \ln^{3/2}(2+|r|)), \quad \forall r \in \mathbb{R} \tag{48}$$

with $\alpha_2 = \beta^\star = 1$ and some $c_f \in \mathbb{R}^*$, so that $f(0) = 0$. We check that f satisfies $(\mathbf{H_2})$ and $(\mathbf{H_2'})$. In this case, the source term in (45) can be rigorously written as

$$\rho_1(s) f(\rho^{-1}(s)p_{k-1}) = c_f \rho_1(s)\rho^{-1}(s)p_{k-1}\Big(\alpha_2 + \beta^\star \ln^{3/2}(2+|y_{k-1}|)\Big).$$

Table 1 $c_{u_0} = 20$; $c_f = 5$; $f(r) = c_f r(1 + \ln^{3/2}(2 + |r|))$; Norms of $(y_{k^\star}, v_{k^\star})$ w.r.t. s

s	$\|y_{k^\star}\|_{L^2(Q_T)}$	$\|\rho(s)y_{k^\star}\|_{L^2(Q_T)}$	$\|v_{k^\star}\|_{L^2(0,T)}$	$\|\rho_1(s)v_{k^\star}\|_{L^2(0,T)}$	$\|v_{k^\star}\|_{H_0^1(0,T)}$	$\|v_{k^\star}\|_{L^\infty(0,T)}$	$k^\star$
2	59.75	1.99×10^{-1}	215.16	2.801×10^{-2}	798.40	313.01	20
3	58.54	3.92×10^{-2}	210.26	3.725×10^{-4}	781.82	306.16	16
4	50.89	8.25×10^{-3}	179.03	1.06×10^{-5}	675.97	262.19	13
5	43.34	1.79×10^{-3}	148.81	5.624×10^{-7}	575.22	219.94	12
6	37.92	3.99×10^{-4}	130.51	3.096×10^{-8}	523.09	196.31	11
7	37.66	9.01×10^{-5}	144.45	1.709×10^{-9}	610.47	224.47	10
8	49.55	2.06×10^{-5}	207.90	9.607×10^{-11}	874.74	318.43	9

I. Experiments for fixed (c_f, c_{u_0}) w.r.t. the parameter s. Let us make the following experiments given by Table 1, 2, for some fixed parameters c_f (associated with the nonlinear function) and c_{u_0} (associated with the initial data). For some large parameters $c_f = 5$ and $c_{u_0} = 20$, it has been checked that the value $s = 1$ is not large enough to imply the Banach contraction property (ensuring the convergence of the algorithm w.r.t. the criterion (47)). Then, by choosing $s \geq 2$, we recover the required convergence criterion (47). We provide the results in Table 1.

Figure 1-left depicts the evolution of the relative error $\frac{\|\rho(s)y_{k+1}-\rho(s)y_k\|_{L^2(Q_T)}}{\|\rho(s)y_k\|_{L^2(Q_T)}}$ w.r.t. to the iteration number k for $s \in \{1, 2, 3, 4, 5\}$. In agreement with Remark 7, we observe that the decay of the error is amplified with larger values of s. We observe that the weighted norm $\|\rho(s)y_{k^\star}\|_{L^2(Q_T)}$ and $\|\rho_1(s)y_{k^\star}\|_{L^2(0,T)}$ decrease with respect to s in agreement with the fact that the weight $\rho(s)$ and $\rho_1(s)$ decreases with s (they behave like e^{-s}). We observe on the contrary that the norms $\|y_{k^\star}\|_{L^2(Q_T)}$ and $\|v_{k^\star}\|_{L^2(0,T)}$ of the control-state pair are not monotonous with respect to s; this is due to the fact that the weights $s\rho^2(s)$ and $\rho_1^2(s)$ appearing in the cost J_s (see (16)) are of the same order for the values of s considered. Remark that the influence of the parameter is much stronger in the parabolic situations as it makes appear unbounded weights; we refer to [17, Section 5] for numerical experiments emphasizing this phenomenon. The stopping criterion (47) used here involves the weight $\rho(s)$ as Proposition 4 guarantees as soon as s is large enough the convergence of $\{\|\rho(s)y_k\|_{L^2(Q_T)}\}_{k\in\mathbb{N}}$. This implies notably the convergence of the sequence $\{\|y_k\|_{L^2(Q_T)}\}_{k\in\mathbb{N}}$.

If we do not incorporate the weight $\rho(s)$ in the criterion, we still observe the convergence of the corresponding sequence $\{y_k\}_{k\in\mathbb{N}}$ for the $L^2(Q_T)$-norm with again an amplification of the rate as s increases (see Fig. 1-right): a precise dependence of the rate with respect to s is, however, unknown in that case.

Table 2 reports experiments in the unfavorable situation for which $c_f < 0$. We checked that for $c_f = -5$ ($c_{u_0} = 20$ as previous), the convergence is observed from $s = 3$. It is noticeable that the L^2 norms of the solutions and the associated controls are relatively larger compare to the case of positive c_f given by Table 1. The number of iterations $k^\star$ to reach convergence is also larger : for instance, with $s = 3$, we get $\|y_{k^\star}\|_{L^2(Q_T)} \approx 59.75$ and $k^\star = 20$ for $c_f = 5$, while we get $\|y_{k^\star}\|_{L^2(Q_T)} \approx 3846.94$ and $k^\star = 32$ for $c_f = -5$.

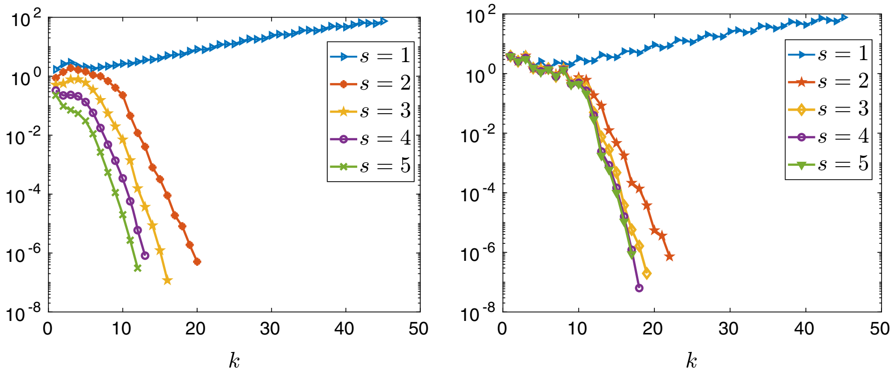

Fig. 1 $\frac{\|\rho(s)y_{k+1}-\rho(s)y_k\|_{L^2(Q_T)}}{\|\rho(s)y_k\|_{L^2(Q_T)}}$ (left) and $\frac{\|y_{k+1}-y_k\|_{L^2(Q_T)}}{\|y_k\|_{L^2(Q_T)}}$ (right) w.r.t. iterations k for $(c_f, c_{u_0}) = (5, 20)$

Table 2 $c_{u_0} = 20$; $c_f = -5$; $f(r) = c_f r(1 + \ln^{3/2}(2 + |r|))$; Norms of $(y_{k^\star}, v_{k^\star})$ w.r.t. s

s	$\|y_{k^\star}\|_{L^2(Q_T)}$	$\|\rho(s)y_{k^\star}\|_{L^2(Q_T)}$	$\|v_{k^\star}\|_{L^2(0,T)}$	$\|\rho_1(s)v_{k^\star}\|_{L^2(0,T)}$	$\|v_{k^\star}\|_{H_0^1(0,T)}$	$\|v_{k^\star}\|_{L^\infty(0,T)}$	$k^\star$
3	3846.94	6.768×10^{-1}	3905.78	1.298×10^{-1}	57298.8	4576.07	32
4	2621.55	3.154×10^{-2}	2888.41	4.064×10^{-3}	35910.6	2919.18	23
5	2120.91	2.711×10^{-3}	2393.94	1.851×10^{-4}	29619.4	2373.23	20
6	1842.65	4.677×10^{-4}	2088.93	9.851×10^{-6}	26358.5	2093.53	17
7	1666.94	9.993×10^{-5}	1910.54	5.657×10^{-7}	24108.5	1902.18	15
8	1539.51	2.231×10^{-5}	1680.7	3.396×10^{-8}	22176.5	1815.9	14

II. Experiments for fixed (s, c_{u_0}) w.r.t. c_f. Hereafter, for fixed a Carleman parameter and initial data, we consider several values of c_f to study the number of iterations for which the pair the solution $y_{k^\star}$ satisfies the criterion (47); Table 3 reports some values corresponding to $s = 3$ and $c_{u_0} = 20$; As expected, larger negative values of c_f lead to larger norms of the state-control pair; we also observe that the required number of iterations $k^\star$ increases with $|c_f|$, including in the *a priori* more favorable case for which $c_f > 0$. For larger values of c_f, for instance $c_f = 8$, the algorithm fails to converge, somehow in agreement with the smallness assumption on $\beta^\star$ in our Theorem 4. For $c_f = 8$, the convergence is recovered by taking a larger value of s, for instance $s = 4$.

III. Experiments for fixed (s, c_f) w.r.t. the parameter c_{u_0}.

We now fix the parameters s and c_f and then vary the size of the initial data u_0 in terms of the parameter c_{u_0}. We give some results in Tables 4 and 5 for $(s, c_f) = (3, -2)$ and $(s, c_f) = (3, 2)$, respectively. One can observe that for large c_{u_0} also, the algorithm converges. The quantity $C(y, v)$ defined by (following the estimates in (6) or Proposition 1)

$$C(y, v) = \frac{\|\rho(s)y\|_{L^2(Q_T)} + s^{-1/2}\|\rho_1(s)v\|_{L^2(0,T)}}{s^{-3/2}\|\rho(s)f(y)\|_{L^2(Q_T)} + s^{-1/2}e^{-s}\|u_0\|_{L^2(\Omega)}} \tag{49}$$

Table 3 $c_{u_0} = 20$; $s = 3$; $f(r) = c_f r(1 + \ln^{3/2}(2 + |r|))$; Norms of $(y_{k^\star}, v_{k^\star})$ w.r.t. c_f

c_f	$\|y_{k^\star}\|_{L^2(Q_T)}$	$\|\rho(s)y_{k^\star}\|_{L^2(Q_T)}$	$\|v_{k^\star}\|_{L^2(0,T)}$	$\|\rho_1(s)v_{k^\star}\|_{L^2(0,T)}$	$\|v_{k^\star}\|_{L^\infty(0,T)}$	$k^\star$
−5	3846.94	6.768×10^{-1}	3905.78	1.29×10^{-1}	4576.07	32
−4	600.95	1.243×10^{-1}	558.07	2.07×10^{-2}	662.12	21
−3	107.61	5.635×10^{-2}	81.84	4.41×10^{-3}	132.96	14
−2	21.07	4.807×10^{-2}	19.08	6.15×10^{-4}	20.45	10
−1	9.87	4.528×10^{-2}	7.14	1.24×10^{-4}	8.43	7
0	11.80	4.352×10^{-2}	11.33	1.94×10^{-4}	13.79	1
1	15.49	4.223×10^{-2}	24.21	1.45×10^{-4}	30.46	7
2	16.79	4.123×10^{-2}	32.56	2.00×10^{-4}	43.48	9
3	16.52	4.042×10^{-2}	31.94	2.88×10^{-4}	38.57	10
4	28.20	3.977×10^{-2}	80.49	3.11×10^{-4}	115.36	13
5	58.54	3.922×10^{-2}	210.26	3.725×10^{-4}	306.16	16
6	94.03	3.875×10^{-2}	376.43	6.47×10^{-4}	564.80	19
7	113.32	3.835×10^{-2}	482.02	1.03×10^{-3}	753.92	25

Table 4 $(s, c_f) = (3, -2)$; $f(r) = c_f r(1 + \ln^{3/2}(2 + |r|))$; Norms of $(y_{k^\star}, v_{k^\star})$ w.r.t. c_{u_0}

c_{u_0}	$\|y_{k^\star}\|_{L^2(Q_T)}$	$\|\rho(s)y_{k^\star}\|_{L^2(Q_T)}$	$\|v_{k^\star}\|_{L^2(0,T)}$	$\|\rho_1(s)v_{k^\star}\|_{L^2(0,T)}$	$\|v_{k^*}\|_{L^\infty(0,T)}$	$C(y_{k^*}, v_{k^*})$	$k^\star$
10	6.30	2.32×10^{-2}	2.66	5.57×10^{-5}	3.05	9.55×10^{-2}	8
50	1.68×10^{2}	1.31×10^{-1}	1.73×10^{2}	6.73×10^{-3}	1.99×10^{2}	9.19×10^{-2}	12
100	7.80×10^{2}	3.14×10^{-1}	6.07×10^{2}	3.15×10^{-2}	9.57×10^{2}	9.01×10^{-2}	14
200	3.58×10^{3}	9.19×10^{-1}	4.08×10^{3}	1.35×10^{-1}	4.25×10^{3}	8.47×10^{-2}	16
500	2.64×10^{4}	5.15	2.61×10^{4}	8.97×10^{-1}	2.91×10^{4}	7.33×10^{-2}	19
1000	1.13×10^{5}	2.09×10^{2}	9.26×10^{4}	3.76	1.23×10^{5}	6.39×10^{-2}	21
2000	4.77×10^{5}	8.66×10^{2}	4.94×10^{5}	1.58×10^{1}	5.13×10^{5}	5.54×10^{-2}	23
5000	3.18×10^{6}	5.63×10^{2}	3.49×10^{6}	1.06×10^{2}	3.60×10^{6}	4.62×10^{-2}	28
10000	1.31×10^{7}	2.30×10^{3}	1.19×10^{7}	4.50×10^{2}	1.71×10^{7}	4.08×10^{-2}	31
15000	2.97×10^{7}	5.25×10^{3}	2.53×10^{7}	1.04×10^{3}	4.12×10^{7}	3.81×10^{-2}	33
20000	5.30×10^{7}	9.42×10^{3}	4.61×10^{7}	1.91×10^{3}	7.55×10^{7}	3.64×10^{-2}	35

is uniform with respect to the quantity c_{u_0} in agreement with our theoretical results.

IV. Evolution of the controlled solutions. In this paragraph, we present some figures of the controlled solutions and the associated controls for our semilinear system. We fix $c_f = -3$ and $c_{u_0} = 10$. Figure 2 depicts the controlled solutions y_{k^*} for $s = 1$, 5 and 9, respectively. The corresponding optimal control is given in Fig. 3-Left for $s \in \{1, 3, 5, 9\}$. The evolution of the $L^2(\Omega)$ norm w.r.t. $t \in (0, T)$ is depicted in Fig. 3-Right. Figures 4 and 5 are concerned with the case $c_f = 3$, leading to control-state pairs with lower norms. Observe also that the optimal controls are pointwise less sensitive with respect to s when c_f is positive.

Table 5 $(s, c_f) = (3, 2)$; $f(r) = c_f r(1 + \ln^{3/2}(2 + |r|))$; Norms of $(y_{k^\star}, v_{k^\star})$ w.r.t. c_{u_0}

c_{u_0}	$\|y_{k^\star}\|_{L^2(Q_T)}$	$\|\rho(s)y_{k^\star}\|_{L^2(Q_T)}$	$\|v_{k^\star}\|_{L^2(0,T)}$	$\|\rho_1(s)v_{k^\star}\|_{L^2(0,T)}$	$\|v_{k^*}\|_{L^\infty(0,T)}$	$C(y_{k^*}, v_{k^*})$	$k^\star$
10	8.302	2.084×10^{-2}	15.062	7.91×10^{-5}	19.481	8.75×10^{-2}	8
50	40.582	1.015×10^{-1}	77.126	6.83×10^{-4}	1.01×10^{2}	7.66×10^{-2}	10
100	99.183	2.005×10^{-1}	2.23×10^{2}	1.51×10^{-3}	3.02×10^{2}	7.18×10^{-2}	11
200	331.403	3.963×10^{-1}	9.74×10^{2}	3.04×10^{-3}	1.36×10^{3}	6.72×10^{-2}	13
500	1476.63	9.765×10^{-1}	5.04×10^{3}	9.20×10^{-3}	7.09×10^{3}	6.16×10^{-2}	14
1000	3667.51	1.933	1.32×10^{4}	2.73×10^{-2}	1.89×10^{4}	5.79×10^{-2}	16
2000	7261.36	3.830	2.67×10^{4}	7.79×10^{-2}	3.99×10^{5}	5.45×10^{-2}	17
5000	17146.3	9.467	6.25×10^{4}	2.72×10^{-1}	8.01×10^{5}	5.05×10^{-2}	19
10000	76904.8	18.79	3.14×10^{5}	7.30×10^{-1}	4.81×10^{5}	4.78×10^{-2}	21
15000	182361	28.073	7.75×10^{5}	1.37	1.1725×10^{6}	4.65×10^{-2}	22
20000	317709	37.334	1.37×10^{6}	2.19	2.07×10^{6}	4.56×10^{-2}	23

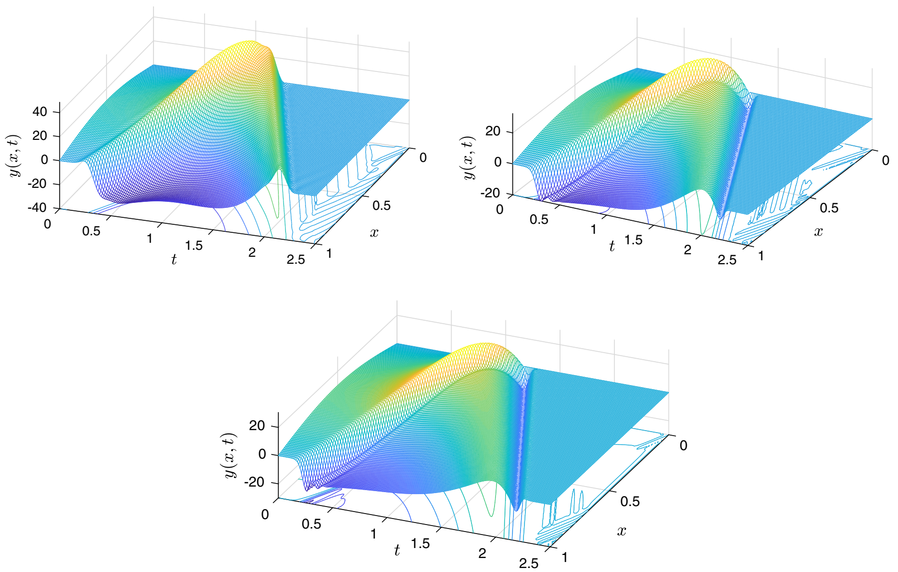

Fig. 2 Controlled solution y_{k^*} for $c_f = -3$, $c_{u_0} = 10$ and $f(r) = c_f r(1 + \ln^{3/2}(2 + |r|))$; $s \in \{1, 5, 9\}$

5.3 Nonlinear functions with growth $r|\cos(r^2)|\ln^{3/2}(2 + |r|)$

In order to enhance the importance of the assumption $(\mathbf{H_2'})$ on the first derivative of f, let us consider the following nonlinear function

$$f(r) = c_f r|\cos(r^2)|\big(\alpha_2 + \beta^\star \ln^{3/2}(2 + |r|)\big), \quad \forall r \in \mathbb{R} \tag{50}$$

with some $c_f \in \mathbb{R}^*$ and $\alpha_2 = \beta^\star = 1$. It satisfies the assumption $(\mathbf{H_2})$ but not $(\mathbf{H_2'})$. We check that for small values of c_f, the algorithm converges for $s = 1$. For instance,

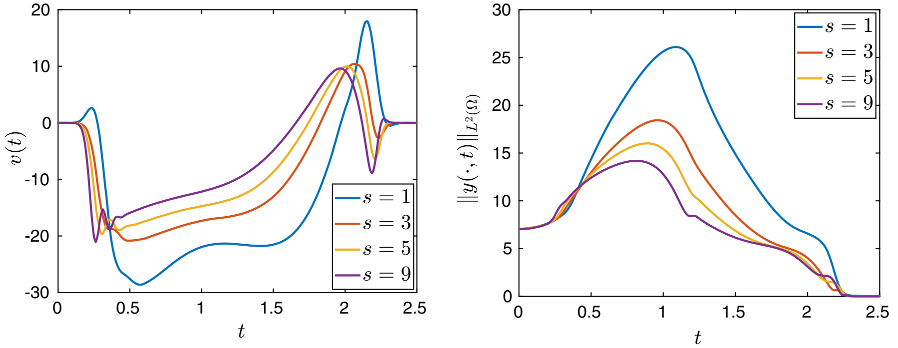

Fig. 3 $c_f = -3, c_{u_0} = 10$ and $f(r) = c_f r(1+\ln^{3/2}(2+|r|))$; Left: Control v_{k^*} w.r.t. s; Right: Evolution of $\|y_{k^*}(\cdot,t)\|_{L^2(\Omega)}$ w.r.t. time t

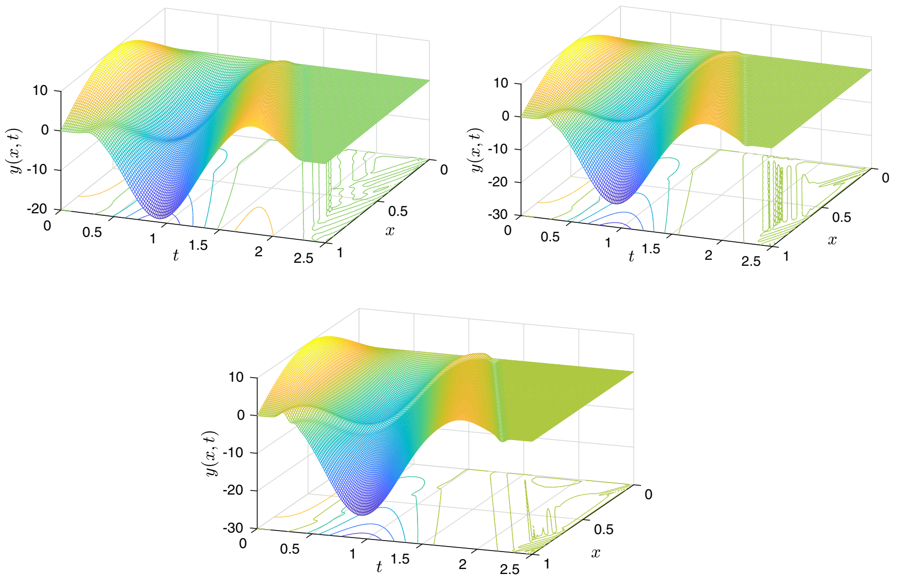

Fig. 4 Controlled solution y_{k^*} for $c_f = 3, c_{u_0} = 10$ and $f(r) = c_f r(1+\ln^{3/2}(2+|r|))$; $s \in \{1, 5, 9\}$

in Table 6 we give some experiments for $c_f = -1$, $c_{u_0} = 20$ and $s \in \{1, 2, 3, 4, 5\}$. On the other hand, with $c_f = -2$, the method fails to converge for $s \in \{1, 2, 3, 4, 5\}$, meaning that the contraction property is lost. The convergence is recovered for s larger or equal than 6, see Table 7. Moreover, the number of iterations $k^\star$ to fulfill the convergence criterion (47) increases significantly from $c_f = -1$ to $c_f = -2$, suggesting that the amplitude of the derivative of the nonlinearity is crucial in the contracting property of the operator Λ_s.

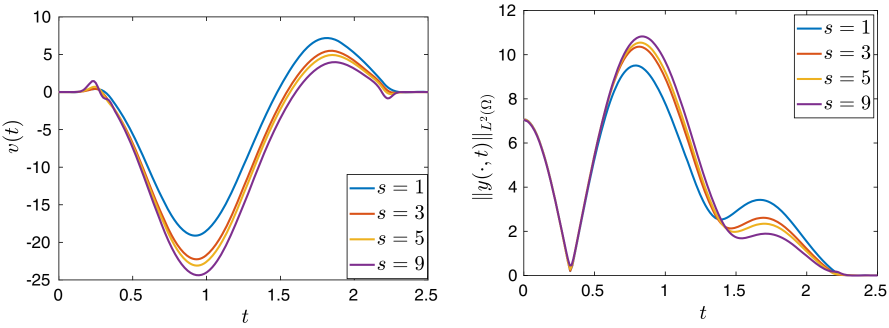

Fig. 5 $c_f = 3$, $c_{u_0} = 10$ and $f(r) = c_f r(1 + \ln^{3/2}(2 + |r|))$; Left: Control v_{k*} w.r.t. t; Right: Evolution of $\|y_{k*}(\cdot, t)\|_{L^2(\Omega)}$ w.r.t. t

Table 6 $c_{u_0} = 20$; $c_f = -1$; $f(r) = c_f r|\cos(r^2)|(1 + \ln^{3/2}(2 + |r|))$; Norms of $(y_{k^\star}, v_{k^\star})$ w.r.t. s

s	$\|y_{k^\star}\|_{L^2(Q_T)}$	$\|\rho(s) y_{k^\star}\|_{L^2(Q_T)}$	$\|v_{k^\star}\|_{L^2(0,T)}$	$\|\rho_1(s) v_{k^\star}\|_{L^2(0,T)}$	$\|v_{k^\star}\|_{H_0^1(0,T)}$	$\|v_{k^\star}\|_{L^\infty(0,T)}$	$k^\star$
1	10.41	1.31	7.89	1.63×10^{-1}	33.08	7.97	24
2	10.21	2.29×10^{-1}	7.92	4.54×10^{-3}	33.39	8.19	20
3	9.98	4.45×10^{-2}	8.01	1.75×10^{-4}	34.30	8.62	18
4	9.79	9.12×10^{-3}	8.08	8.96×10^{-6}	35.61	8.97	14
5	9.65	1.94×10^{-3}	8.19	4.97×10^{-7}	37.01	9.32	14

Table 7 $c_{u_0} = 20$; $c_f = -2$; Norms of $(y_{k^\star}, v_{k^\star})$ w.r.t. s; when $f = c_f r|\cos(r^2)|(1 + \ln^{3/2}(2 + |r|))$

s	$\|y_{k^\star}\|_{L^2(Q_T)}$	$\|\rho(s) y_{k^\star}\|_{L^2(Q_T)}$	$\|v_{k^\star}\|_{L^2(0,T)}$	$\|\rho_1(s) v_{k^\star}\|_{L^2(0,T)}$	$\|v_{k^\star}\|_{H_0^1(0,T)}$	$\|v_{k^\star}\|_{L^\infty(0,T)}$	$k^\star$
6	10.65	4.27×10^{-4}	6.45	1.01×10^{-8}	24.19	7.47	87
7	10.49	9.51×10^{-5}	6.46	6.43×10^{-10}	25.25	7.64	118
8	10.51	2.15×10^{-5}	6.46	4.57×10^{-11}	26.73	7.46	90
9	10.64	4.94×10^{-6}	6.54	3.66×10^{-12}	29.27	7.77	45

5.4 Nonlinear functions with growth $r \ln^p(2 + |r|)$ for $p \geq 2$

In this section, we first consider the following form of the nonlinear function:

$$f(r) = c_f r(\alpha_2 + \beta^\star \ln^2(2 + |r|)) \quad \forall r \in \mathbb{R} \tag{51}$$

which satisfies $(\mathbf{H_1})$ but not $(\mathbf{H_2})$ nor $(\mathbf{H_2'})$. For $(c_f, c_{u_0}) = (4, 10)$, we have checked that the algorithm does not converge for the Carleman parameters $s = 1$ and $s = 2$. For the experiments, we need at least $s > 2$ to fulfill the convergence criterion (47). We present some results in Table 8.

But, as soon as we increase the L^2-norm of the initial data u_0, the result is getting worse, even if we keep the value of $c_f = 4$. For instance, considering $c_{u_0} = 20$ is giving the convergence for $s > 4$, see Table 9. In other words, the algorithm does not really fit w.r.t. large values of norms for the initial data.

Table 8 $c_{u_0} = 10$; $c_f = 4$; $f(r) = c_f r(1 + \ln^2(2 + |r|))$; Norms of $(y_{k^\star}, v_{k^\star})$ w.r.t. s

s	$\|y_{k^\star}\|_{L^2(Q_T)}$	$\|\rho(s)y_{k^\star}\|_{L^2(Q_T)}$	$\|v_{k^\star}\|_{L^2(0,T)}$	$\|\rho_1(s)v_{k^\star}\|_{L^2(0,T)}$	$\|v_{k^\star}\|_{H_0^1(0,T)}$	$\|v_{k^\star}\|_{L^\infty(0,T)}$	$k^\star$
3	18.16	1.98×10^{-2}	61.87	1.68×10^{-4}	247.79	93.91	17
4	14.99	4.16×10^{-3}	47.83	5.92×10^{-6}	196.93	72.47	14
5	12.56	9.05×10^{-4}	37.03	2.86×10^{-7}	158.61	55.24	13
6	11.79	2.01×10^{-4}	38.38	1.47×10^{-8}	176.91	59.85	11
7	12.74	4.53×10^{-5}	46.64	7.77×10^{-10}	221.24	75.80	10

Table 9 $c_{u_0} = 20$; $c_f = 4$; $f(r) = c_f r(1 + \ln^2(2 + |r|))$; Norms of $(y_{k^\star}, v_{k^\star})$ w.r.t. s

s	$\|y_{k^\star}\|_{L^2(Q_T)}$	$\|\rho(s)y_{k^\star}\|_{L^2(Q_T)}$	$\|v_{k^\star}\|_{L^2(0,T)}$	$\|\rho_1(s)v_{k^\star}\|_{L^2(0,T)}$	$\|v_{k^\star}\|_{H_0^1(0,T)}$	$\|v_{k^\star}\|_{L^\infty(0,T)}$	$k^\star$
5	131.60	1.77×10^{-3}	616.05	4.78×10^{-7}	2668.36	1004.82	36
6	119.54	3.95×10^{-4}	559.19	2.59×10^{-8}	2470.46	933.93	25

Table 10 $c_{u_0} = 10$; $s = 3$; $f(r) = c_f r(1 + \ln^2(2 + |r|))$; Norms of $(y_{k^\star}, v_{k^\star})$ w.r.t. c_f

c_f	$\|y_{k^\star}\|_{L^2(Q_T)}$	$\|\rho(s)y_{k^\star}\|_{L^2(Q_T)}$	$\|v_{k^\star}\|_{L^2(0,T)}$	$\|\rho_1(s)v_{k^\star}\|_{L^2(0,T)}$	$\|v_{k^\star}\|_{L^\infty(0,T)}$	$k^\star$
−3	433.899	8.057×10^{-2}	473.795	1.517×10^{-2}	503.043	36
−2	16.512	2.446×10^{-2}	19.926	6.372×10^{-4}	20.044	12
−1	5.0298	2.271×10^{-2}	3.587	5.838×10^{-5}	4.268	7
0	5.902	2.176×10^{-2}	5.6657	9.726×10^{-5}	6.895	1
1	7.819	2.108×10^{-2}	12.589	7.245×10^{-5}	15.975	7
2	8.536	2.056×10^{-2}	17.4798	1.067×10^{-4}	23.876	9
3	8.425	2.014×10^{-2}	17.138	1.519×10^{-4}	20.768	11
4	18.16	1.981×10^{-2}	61.872	1.687×10^{-4}	93.918	17

We also perform some experiments for $s = 3$, $c_{u_0} = 10$ to see how the algorithm behaves with respect to different values of c_f. In Table 10, we see that the algorithm converges for the values of $c_f \in [-3, 4]$. On the other hand, for the same quantities $(s, c_{u_0}) = (3, 10)$, we have the divergence of our method when the nonlinear parameter $c_f \leq -4$ or $c_f \geq 5$.

Next, we make some experiments for the nonlinearities f that behave like $r \ln^p |r|$ at infinity when $p > 2$ and therefore does not satisfy $(\mathbf{H_1})$. Below, we consider the nonlinear function

$$f_p(r) = c_f r\big(1 + \ln^p(2 + |r|)\big), \quad \text{for } p > 2, \quad \forall r \in \mathbb{R}. \tag{52}$$

We refer to Table 11 for some results associated with $c_f = -2$, $c_{u_0} = 10$ and $s = 3$ for $p \in \{2, 2.3\}$. As expected, the value $k^\star$ increases with the value of p. Moreover, for $p \geq 2.4$, the algorithm does not converge anymore; more precisely, the norms of the state-control pair $(y_k, v_k)_{k \in \mathbb{N}}$ blow up with respect to k, in agreement with the

Table 11 $c_f = -2$, $s = 3$, $c_{u_0} = 10$; Norms of $(y_{k^\star}, v_{k^\star})$ w.r.t. f_p given by (52)

	$\Vert y_{k^\star}\Vert_{L^2(Q_T)}$	$\Vert\rho(s)y_{k^\star}\Vert_{L^2(Q_T)}$	$\Vert v_{k^\star}\Vert_{L^2(0,T)}$	$\Vert\rho_1(s)v_{k^\star}\Vert_{L^2(0,T)}$	$\Vert v_{k^\star}\Vert_{L^\infty(0,T)}$	$k^\star$
$p = 2$	16.512	2.446×10^{-2}	19.926	6.372×10^{-4}	20.044	12
$p = 2.05$	20.581	2.472×10^{-2}	23.047	8.334×10^{-4}	25.093	12
$p = 2.1$	26.176	2.508×10^{-2}	24.076	1.099×10^{-3}	32.173	13
$p = 2.15$	34.134	2.564×10^{-2}	26.570	1.468×10^{-3}	44.653	15
$p = 2.2$	47.681	2.662×10^{-2}	49.532	2.011×10^{-3}	63.213	17
$p = 2.25$	76.081	2.883×10^{-2}	84.956	2.938×10^{-3}	95.777	20
$p = 2.3$	136.668	3.542×10^{-2}	121.539	5.045×10^{-3}	156.929	26

result in Theorem 2 that the operator Λ_s does not enjoy a stability result for c_{u_0} and c_f large enough soon as p is larger than 2.

6 Concluding remarks

By introducing a functional in the Carleman setting different than in the seminal paper of Zuazua [41], we have derived, under similar assumptions, a somehow simpler proof of the boundary controllability of a semilinear wave equation of the form $y_{tt} - y_{xx} + f(y) = 0$. Moreover, assuming an additional growth assumption on f', we have constructed a sequence of state-control pairs, solution of a linear boundary controllability problem, converging pointwise and with a linear rate to a solution of the semilinear equation. As in the recent work [17] devoted to the distributed controllability for a semilinear heat equation, the analysis emphasizes the role of the Carleman weights parameterized by the real s. Numerical experiments illustrate that the speed of convergence of the sequence is amplified as the Carleman parameter s is larger.

Our analysis is based on a simple fixed point strategy which consists to see the nonlinear term as a source term. It would be interesting to analyze whether or not the fixed point operator introduced by Zuazua in 1993, involving a potential, is, after reformulation in a functional Carleman setting, contracting for s large enough.

The fixed point argument employed here requires uniform bounds of the controlled trajectories for a linear wave equation: this is achieved by assuming the initial data in $H_0^1(\Omega) \times L^2(\Omega)$ and by imposing that the control satisfies at the initial and final time some compatibility conditions with the solution: this leads to boundary controls in $H_0^1(0, T)$ and then, in our one-dimensional situation, to trajectories in $L^\infty(Q_T)$. Assuming more regularity on the initial conditions, we may extend our results for multi-dimensional situations and for nonlinearities depending on the gradient of the solution. This will be addressed in future works.

Acknowledgements The authors thank the funding by the French government research program “Investissements d’Avenir” through the IDEX-ISITE initiative 16-IDEX-0001 (CAP 20-25).

Appendix: Proof of Theorem 7

In what follows, in order to simplify the notations, we shall just write $\rho = \rho(t)$ and $\rho_1 = \rho_1(t)$ instead of $\rho = \rho(s; x, t)$ and $\rho_1 = \rho_1(s; t)$.

Preliminary to the proof and following [18], for all $f \in \mathcal{C}^0(\mathbb{R}; E)$ (where E is a Banach space) and any $\tau > 0$, we define $\delta_\tau f := f\left(t + \frac{\tau}{2}\right) - f\left(t - \frac{\tau}{2}\right)$ and

$$\mathcal{T}_\tau f := \frac{1}{\tau}\delta_\tau\left(\frac{\delta_\tau f}{\tau}\right) = \frac{f(t+\tau) - 2f(t) + f(t-\tau)}{\tau^2}.$$

Let now $w_s \in P_s$ and the solution $y_s \in L^2(Q_T)$ be given by 6. Then, z defined by $z = \mathcal{T}_\tau w_s$ belongs to P_s, where w_s as well as y_s can be extended uniquely on $(-\infty, 0)$ and $(T, +\infty)$. Indeed, in the interval $(-\infty, 0)$ the solution y_s satisfies the following set of equations

$$\begin{cases} Ly_s = 0 & \text{in } \Omega \times (-\infty, 0), \\ y_s(0,t) = y_s(1,t) = 0 & \text{for } t \in (-\infty, 0), \\ (y_s(\cdot, 0), \partial_t y_s(\cdot, 0)) = (u_0, u_1) & \text{in } \Omega, \end{cases} \tag{53}$$

where the source term $B \in L^2(Q_T)$ is assumed to be extendable by 0 outside $(0, T)$. Recall that the boundary condition $y_s(1,t) = 0$ holds outside $(0, T)$ since $\eta = 0$ (appearing in the formula of v_s) vanishes outside $(\delta, T - \delta)$.

Similarly, in $(T, +\infty)$ we can define the solution y_s uniquely, and $y_s(t) = 0$ for all $t \geq T$. It follows that the solution y_s satisfies $y_s \in \mathcal{C}^0(\mathbb{R}; L^2(\Omega)) \cap \mathcal{C}^1(\mathbb{R}; H^{-1}(\Omega))$ and $y_s \in \mathcal{C}^0((-\infty, \delta]; H^1(\Omega)) \cap \mathcal{C}^1((-\infty, \delta]; L^2(\Omega))$ and $y_s \in \mathcal{C}^0([T-\delta, +\infty); H^1(\Omega)) \cap \mathcal{C}^1([T-\delta, +\infty); L^2(\Omega))$ (see [26]). We extend as well the weights ρ and ρ_1 in $\Omega \times \mathbb{R}$ so that it preserves smoothness and positivity properties.

This ensures the extension of the solution w_s which satisfies the following set of equations in $\mathbb{R}$

$$\begin{cases} Lw_s = \rho^2 y_s & \text{in } \Omega \times \mathbb{R}, \\ w_s(0,t) = w_s(1,t) = 0, & \text{in } \mathbb{R}. \end{cases} \tag{54}$$

Moreover, it can be seen that $Lw_s = 0$ in $[T, +\infty)$, since y_s is a controlled solution to (53).

We now proceed to the proof of Theorem 7, done in three steps.

Step 1 : We suppose first that $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_1 \in H_0^1(\Omega)$ and $B \in \mathcal{D}(0, T; L^2(\Omega))$ and prove that $v_s \in H^1(0, T)$ and $(y_s)_t \in L^2(Q_T)$.

We start by considering the variational formulation (12) by choosing $z = \mathcal{T}_\tau w_s$ as test function. Since $w_s \in \mathcal{C}^0(\mathbb{R}; H_0^1(\Omega)) \cap \mathcal{C}^1(\mathbb{R}; L^2(\Omega))$ solves (54), it is clear that $\mathcal{T}_\tau w_s \in \mathcal{C}^0([0, T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0, T]; L^2(\Omega))$, $(\mathcal{T}_\tau w_s)_x \in L^2(0, T)$. With this z, the formulation reads

$$\begin{aligned}&\int_{Q_T} \rho^{-2} L w_s L \mathcal{T}_\tau w_s \,\mathrm{d}x\mathrm{d}t + s\int_0^T \eta^2(t)\rho_1^{-2}(w_s)_x(1,t)\mathcal{T}_\tau(w_s)_x(1,t)\mathrm{d}t\\ &= \int_\Omega u_1(x)\mathcal{T}_\tau w_s(x,0)\,\mathrm{d}x - \int_\Omega u_0(x)\mathcal{T}_\tau(w_s)_t(x,0)\,\mathrm{d}x + \int_{Q_T} B\mathcal{T}_\tau w_s \,\mathrm{d}x\mathrm{d}t. \qquad (55)\end{aligned}$$

Sub-step 1. Let us start with the first integral in the left hand side of (55). We have

$$\begin{aligned}&\int_{Q_T} \rho^{-2}(t) L w_s(t) L\mathcal{T}_\tau w_s(t)\,\mathrm{d}x\mathrm{d}t\\
&\quad= \frac{1}{\tau}\int_{Q_T} \rho^{-2}(t) L w_s(t)\frac{L w_s(t+\tau) - L w_s(t)}{\tau}\,\mathrm{d}x\mathrm{d}t\\
&\qquad - \frac{1}{\tau}\int_{Q_T} \rho^{-2}(t) L w_s(t)\frac{L w_s(t) - L w_s(t-\tau)}{\tau}\,\mathrm{d}x\mathrm{d}t\\
&\quad= \frac{1}{\tau}\int_{Q_T} \rho^{-2}(t) L w_s(t)\frac{L w_s(t+\tau) - L w_s(t)}{\tau}\,\mathrm{d}x\mathrm{d}t\\
&\qquad - \frac{1}{\tau}\int_{-\tau}^{T-\tau}\int_\Omega \rho^{-2}(t+\tau) L w_s(t+\tau)\frac{L w_s(t+\tau) - L w_s(t)}{\tau}\,\mathrm{d}x\mathrm{d}t\\
&\quad= \int_{Q_T} \left(\frac{\rho^{-2}(t) L w_s(t) - \rho^{-2}(t+\tau) L w_s(t+\tau)}{\tau}\right)\left(\frac{L w_s(t+\tau) - L w_s(t)}{\tau}\right)\mathrm{d}x\mathrm{d}t\\
&\qquad - \frac{1}{\tau}\int_{-\tau}^{0}\int_\Omega \rho^{-2}(t+\tau) L w_s(t+\tau)\left(\frac{L w_s(t+\tau) - L w_s(t)}{\tau}\right)\mathrm{d}x\mathrm{d}t\\
&\qquad - \frac{1}{\tau}\int_{T}^{T-\tau}\int_\Omega \rho^{-2}(t+\tau) L w_s(t+\tau)\left(\frac{L w_s(t+\tau) - L w_s(t)}{\tau}\right)\mathrm{d}x\mathrm{d}t.\end{aligned} \qquad (56)$$

Now, observe that

$$\begin{aligned}&\int_{Q_T} \left(\frac{\rho^{-2}(t) L w_s(t) - \rho^{-2}(t+\tau) L w_s(t+\tau)}{\tau}\right)\left(\frac{L w_s(t+\tau) - L w_s(t)}{\tau}\right)\mathrm{d}x\mathrm{d}t\\
&\quad= \int_{Q_T} \rho^2(t)\left(\frac{\rho^{-2}(t) L w_s(t) - \rho^{-2}(t+\tau) L w_s(t+\tau)}{\tau}\right)\rho^{-2}(t)\\
&\quad\times\left(\frac{L w_s(t+\tau) - L w_s(t)}{\tau}\right)\mathrm{d}x\mathrm{d}t\\
&\quad= -\int_{Q_T} \rho^2(t)\left|\frac{\rho^{-2}(t) L w_s(t) - \rho^{-2}(t+\tau) L w_s(t+\tau)}{\tau}\right|^2\mathrm{d}x\mathrm{d}t\\
&\qquad + \int_{Q_T} \rho^2(t)\left(\frac{\rho^{-2}(t) L w_s(t) - \rho^{-2}(t+\tau) L w_s(t+\tau)}{\tau}\right)\\
&\quad\times\left(\frac{\rho^{-2}(t) - \rho^{-2}(t+\tau)}{\tau}\right) L w_s(t+\tau)\,\mathrm{d}x\mathrm{d}t.\end{aligned} \qquad (57)$$

The equality (56) then reads

$$
\begin{aligned}
&\int_{Q_T} \rho^{-2} Lw_s(t) L\mathcal{T}_\tau w_s(t)\,\mathrm{d}x\mathrm{d}t\\
&= -\int_{Q_T} \rho^2(t) \left|\frac{\rho^{-2}(t)Lw_s(t) - \rho^{-2}(t+\tau)Lw_s(t+\tau)}{\tau}\right|^2 \mathrm{d}x\mathrm{d}t\\
&\quad + \int_{Q_T} \rho^2(t) \left(\frac{\rho^{-2}(t)Lw_s(t) - \rho^{-2}(t+\tau)Lw_s(t+\tau)}{\tau}\right)\\
&\qquad \times \left(\frac{\rho^{-2}(t) - \rho^{-2}(t+\tau)}{\tau}\right) Lw_s(t+\tau)\,\mathrm{d}x\mathrm{d}t\\
&\qquad - \frac{1}{\tau}\int_{-\tau}^{0}\int_\Omega \rho^{-2}(t+\tau)Lw_s(t+\tau)\left(\frac{Lw_s(t+\tau) - Lw_s(t)}{\tau}\right)\mathrm{d}x\mathrm{d}t\\
&\qquad - \frac{1}{\tau}\int_{T}^{T-\tau}\int_\Omega \rho^{-2}(t+\tau)Lw_s(t+\tau)\left(\frac{Lw_s(t+\tau) - Lw_s(t)}{\tau}\right)\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{58}
$$

Next, we shall look into the second term in the left hand side of (55). First, recall the smooth function η given by (9) satisfies $\eta = 0$ in $(-\infty, \delta] \cup [T-\delta, +\infty)$ (with $\delta > 0$ given in (9)). Then, in a similar way that have lead to (56), we have assuming $|\tau| \le \delta$:

$$
\begin{aligned}
&\int_0^T \eta^2(t)\rho_1^{-2}(t)(w_s)_x(1,t)\mathcal{T}_\tau(w_s)_x(1,t)\mathrm{d}t\\
&\quad = \int_0^T \left(\frac{\eta^2(t)\rho_1^{-2}(t)(w_s)_x(1,t) - \eta^2(t+\tau)\rho_1^{-2}(t+\tau)(w_s)_x(1,t+\tau)}{\tau}\right)\\
&\qquad \times \left(\frac{(w_s)_x(1,t+\tau) - (w_s)_x(1,t)}{\tau}\right)\mathrm{d}t.
\end{aligned} \tag{59}
$$

Then, using the identity

$$
ad - bc = \frac{(a-c)(b+d) - (a+c)(b-d)}{2}, \qquad \forall (a,b,c,d) \in \mathbb{R}^4 \tag{60}
$$

with $a = \eta^2(t)\rho_1^{-2}(t)$, $b = (w_s)_x(1, t+\tau)$, $c = \eta^2(t+\tau)\rho_1^{-2}(t+\tau)$ and $d = (w_s)_x(1,t)$, we obtain from (59)

$$
\begin{aligned}
&\int_0^T \eta^2(t)\rho_1^{-2}(t)(w_s)_x(1,t)\mathcal{T}_\tau(w_s)_x(1,t)\mathrm{d}t\\
&= \int_0^T \frac{\left(\eta^2(t)\rho_1^{-2}(t) - \eta^2(t+\tau)\rho_1^{-2}(t+\tau)\right)((w_s)_x(1,t) + (w_s)_x(1,t+\tau))}{2\tau}\\
&\quad \times \left(\frac{(w_s)_x(1,t+\tau) - (w_s)_x(1,t)}{\tau}\right)\mathrm{d}t
\end{aligned}
$$

$$-\int_0^T \frac{\left(\eta^2(t)\rho_1^{-2}(t)+\eta^2(t+\tau)\rho_1^{-2}(t+\tau)\right)}{2}\left|\frac{(w_s)_x(1,t+\tau)-(w_s)_x(1,t)}{\tau}\right|^2 \mathrm{d}t. \tag{61}$$

Now, using (58) and (61) in the formulation (55), we have

$$\begin{aligned}
&\int_{Q_T} \rho^2(t)\left|\frac{\rho^{-2}(t)Lw_s(t)-\rho^{-2}(t+\tau)Lw_s(t+\tau)}{\tau}\right|^2 \mathrm{d}x\mathrm{d}t\\
&+s\int_0^T \frac{\left(\eta^2(t)\rho_1^{-2}(t)+\eta^2(t+\tau)\rho_1^{-2}(t+\tau)\right)}{2}\left|\frac{(w_s)_x(1,t+\tau)-(w_s)_x(1,t)}{\tau}\right|^2 \mathrm{d}t\\
=&\int_{Q_T} \rho^2(t)\left(\frac{\rho^{-2}(t)Lw_s(t)-\rho^{-2}(t+\tau)Lw_s(t+\tau)}{\tau}\right)\\
&\quad\times\left(\frac{\rho^{-2}(t)-\rho^{-2}(t+\tau)}{\tau}\right)Lw_s(t+\tau)\,\mathrm{d}x\mathrm{d}t\\
&\qquad-\frac{1}{\tau}\int_{-\tau}^0\int_\Omega \rho^{-2}(t+\tau)Lw_s(t+\tau)\left(\frac{Lw_s(t+\tau)-Lw_s(t)}{\tau}\right)\mathrm{d}x\mathrm{d}t\\
&\qquad-\frac{1}{\tau}\int_T^{T-\tau}\int_\Omega \rho^{-2}(t+\tau)Lw_s(t+\tau)\left(\frac{Lw_s(t+\tau)-Lw_s(t)}{\tau}\right)\mathrm{d}x\mathrm{d}t\\
&+s\int_0^T \frac{\left(\eta^2(t)\rho_1^{-2}(t)-\eta^2(t+\tau)\rho_1^{-2}(t+\tau)\right)((w_s)_x(1,t)+(w_s)_x(1,t+\tau))}{2\tau}\\
&\qquad\qquad\times\left(\frac{(w_s)_x(1,t+\tau)-(w_s)_x(1,t)}{\tau}\right)\mathrm{d}t\\
&-\int_{Q_T} B\mathcal{T}_\tau w_s\,\mathrm{d}x\mathrm{d}t-\int_\Omega u_1\mathcal{T}_\tau w_s(\cdot,0)\,\mathrm{d}x+\int_\Omega u_0\mathcal{T}_\tau(w_s)_t(\cdot,0)\,\mathrm{d}x\\
:=&I_1+I_2+I_3+I_4+I_5+I_6+I_7.
\end{aligned} \tag{62}$$

Sub-step 2. In this step, we obtain precise estimates for the terms I_1 and I_4 and then an estimate of the left hand side of (62).

(i) *Estimate of* I_1. Young's inequality leads to

$$\begin{aligned}
|I_1|\le&\frac{1}{2}\int_{Q_T}\rho^2(t)\left|\frac{\rho^{-2}(t)Lw_s(t)-\rho^{-2}(t+\tau)Lw_s(t+\tau)}{\tau}\right|^2 \mathrm{d}x\mathrm{d}t\\
&+\frac{1}{2}\int_{Q_T}\rho^2(t)\left|\left(\frac{\rho^{-2}(t)-\rho^{-2}(t+\tau)}{\tau}\right)Lw_s(t+\tau)\right|^2 \mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{63}$$

(ii) *Estimate of* I_4. We have

$$\begin{aligned}
|I_4| &= s\left|\int_0^T \frac{\left(\eta(t)\rho_1^{-1}(t)-\eta(t+\tau)\rho_1^{-1}(t+\tau)\right)((w_s)_x(1,t)+(w_s)_x(1,t+\tau))}{2\tau}\right.\\
&\qquad\left.\times\frac{\left(\eta(t)\rho_1^{-1}(t)+\eta(t+\tau)\rho_1^{-1}(t+\tau)\right)((w_s)_x(1,t+\tau)-(w_s)_x(1,t))}{\tau}\mathrm{d}t\right|\\
&\le 2s\int_0^T \left|\frac{\left(\eta(t)\rho_1^{-1}(t)-\eta(t+\tau)\rho_1^{-1}(t+\tau)\right)((w_s)_x(1,t)+(w_s)_x(1,t+\tau))}{2\tau}\right|^2\mathrm{d}t\\
&\quad+\frac{s}{8}\int_0^T \left|\eta(t)\rho_1^{-1}(t)+\eta(t+\tau)\rho_1^{-1}(t+\tau)\right|^2\left|\frac{(w_s)_x(1,t+\tau)-(w_s)_x(1,t)}{\tau}\right|^2\mathrm{d}t\\
&\le 2s\int_0^T \left|\frac{\left(\eta(t)\rho_1^{-1}(t)-\eta(t+\tau)\rho_1^{-1}(t+\tau)\right)((w_s)_x(1,t)+(w_s)_x(1,t+\tau))}{2\tau}\right|^2\mathrm{d}t\\
&\quad+\frac{s}{2}\int_0^T \frac{\left(\eta^2(t)\rho_1^{-2}(t)+\eta^2(t+\tau)\rho_1^{-2}(t+\tau)\right)}{2}\left|\frac{(w_s)_x(1,t+\tau)-(w_s)_x(1,t)}{\tau}\right|^2\mathrm{d}t.
\end{aligned} \tag{64}$$

(iii) *A first estimate of the left hand side of* (62). The previous estimates and (62) give

$$\begin{aligned}
&\frac{1}{2}\int_{Q_T}\rho^2(t)\left|\frac{\rho^{-2}(t)Lw_s(t)-\rho^{-2}(t+\tau)Lw_s(t+\tau)}{\tau}\right|^2\mathrm{d}x\mathrm{d}t\\
&+\frac{s}{2}\int_0^T \frac{\left(\eta^2(t)\rho_1^{-2}(t)+\eta^2(t+\tau)\rho_1^{-2}(t+\tau)\right)}{2}\left|\frac{(w_s)_x(1,t)-(w_s)_x(1,t+\tau)}{\tau}\right|^2\mathrm{d}t\\
&\le\frac{1}{2}\int_{Q_T}\rho^2(t)\left|\left(\frac{\rho^{-2}(t)-\rho^{-2}(t+\tau)}{\tau}\right)Lw_s(t+\tau)\right|^2\mathrm{d}x\mathrm{d}t\\
&+\left|\frac{1}{\tau}\int_{-\tau}^0\int_\Omega \rho^{-2}(t+\tau)Lw_s(t+\tau)\left(\frac{Lw_s(t+\tau)-Lw_s(t)}{\tau}\right)\mathrm{d}x\mathrm{d}t\right|\\
&+\left|\frac{1}{\tau}\int_T^{T-\tau}\int_\Omega \rho^{-2}(t+\tau)Lw_s(t+\tau)\left(\frac{Lw_s(t+\tau)-Lw_s(t)}{\tau}\right)\mathrm{d}x\mathrm{d}t\right|\\
&+2s\int_0^T\left|\frac{\left(\eta(t)\rho_1^{-1}(t)-\eta(t+\tau)\rho_1^{-1}(t+\tau)\right)((w_s)_x(1,t)+(w_s)_x(1,t+\tau))}{2\tau}\right|^2\mathrm{d}t\\
&+\left|\int_{Q_T}B\mathcal{T}_\tau w_s\,\mathrm{d}x\mathrm{d}t\right|+\left|\int_\Omega u_0\mathcal{T}_\tau(w_s)_t(\cdot,0)\,\mathrm{d}x\right|+\left|\int_\Omega u_1\mathcal{T}_\tau w_s(\cdot,0)\,\mathrm{d}x\right|\\
&:=J_1+J_2+J_3+J_4+J_5+J_6+J_7.
\end{aligned} \tag{65}$$

Sub-step 3 : We prove that the left hand side of (65) is bounded uniformly with respect to $|\tau| \in [0, \delta]$.

(i) J_1 is bounded. Since $\rho^{-2} \in \mathcal{C}^\infty(\mathbb{R} \times \overline{\Omega})$, $(\rho^{-1})_t = -2s\lambda\beta(t - \frac{T}{2})\phi\rho^{-1}$ and $Lw_s \in \mathcal{C}^0(\mathbb{R}; L^2(\Omega))$:

$$\int_{Q_T} \rho^2(t) \left| \left(\frac{\rho^{-2}(t) - \rho^{-2}(t+\tau)}{\tau} \right) Lw_s(t+\tau) \right|^2 \mathrm{d}x\mathrm{d}t \\ \to 4s^2\lambda^2\beta^2 \int_{Q_T} \left(t - \frac{T}{2}\right)^2 \phi^2(t)\rho^2(t) y_s^2 \,\mathrm{d}x\mathrm{d}t$$

as $\tau \to 0$ and thus J_1 is bounded.

(ii) J_2 is bounded. Since $\rho^{-2} Lw_s = y_s \in \mathcal{C}^0(\mathbb{R}; L^2(\Omega))$, $\rho^{-2} \in \mathcal{C}^\infty(\mathbb{R} \times \overline{\Omega})$ and $Lw_s = \rho^2 y_s \in \mathcal{C}^1((-\infty, \delta]; L^2(\Omega))$ we have, as $\tau \to 0$

$$\left| \frac{1}{\tau} \int_{-\tau}^{0} \int_\Omega \rho^{-2}(t+\tau) Lw_s(t+\tau) \left(\frac{Lw_s(t+\tau) - Lw_s(t)}{\tau} \right) \mathrm{d}x\mathrm{d}t \right| \\ \to \left| \int_\Omega y_s(0)(\rho^2 y_s)_t(0) \right|$$

and thus J_2 is bounded.

(iii) J_3 is bounded. Since $\rho^{-2} Lw_s = y_s \in \mathcal{C}^0(\mathbb{R}; L^2(\Omega))$, $\rho^{-2} \in \mathcal{C}^\infty(\mathbb{R} \times \overline{\Omega})$ and $Lw_s = \rho^2 y_s \in \mathcal{C}^1([T - \delta, +\infty); L^2(\Omega))$ we have, as $\tau \to 0$

$$\left| \frac{1}{\tau} \int_{T}^{T-\tau} \int_\Omega \rho^{-2}(t+\tau) Lw_s(t+\tau) \left(\frac{Lw_s(t+\tau) - Lw_s(t)}{\tau} \right) \mathrm{d}x\mathrm{d}t \right| \\ \to \left| \int_\Omega y_s(T)(\rho^2 y_s)_t(T) \right| = 0$$

and thus J_3 is bounded.

(iv) J_4 is bounded. Since $(w_s)_x(1, \cdot) \in L^2(0, T)$ and $\eta\rho_1^{-1} \in \mathcal{C}^1(\mathbb{R})$ we have

$$2s \int_0^T \left| \frac{\left(\eta(t)\rho_1^{-1}(t) - \eta(t+\tau)\rho_1^{-1}(t+\tau)\right) ((w_s)_x(1,t) + (w_s)_x(1,t+\tau))}{2\tau} \right|^2 \mathrm{d}t \\ \to 2s \int_0^T \left| (\eta\rho_1^{-1})_t(t)(w_s)_x(1,t) \right|^2 \mathrm{d}t$$

as $\tau \to 0$ and thus J_4 is bounded.

(v) J_5 is bounded. For τ small enough, since $B \in \mathcal{D}(\mathbb{R}; L^2(\Omega))$ and $w_s \in \mathcal{C}^0(\mathbb{R}; L^2(\Omega))$, we have

$$\left| \int_{Q_T} B\mathcal{T}_\tau w_s \,\mathrm{d}x\mathrm{d}t \right| = \left| \int_{-\tau}^{T+\tau} \int_\Omega \mathcal{T}_\tau B w_s \,\mathrm{d}x\mathrm{d}t \right| \to \left| \int_{Q_T} B_{tt} w_s \,\mathrm{d}x\mathrm{d}t \right| \tag{66}$$

as $\tau \to 0$ and thus J_5 is bounded.

(vi) J_6 is bounded. We have $Lw_s = \rho^2 y_s \in \mathcal{C}^0(\mathbb{R}; L^2(\Omega))$ and $w_s \in \mathcal{C}^0(\mathbb{R}; H_0^1(\Omega))$, thus $(w_s)_{tt} = Lw_s + (w_s)_{xx} \in \mathcal{C}(\mathbb{R}; H^{-1}(\Omega))$. We then have, for all $t \in \mathbb{R}$:

$$(w_s)_t(t) - (w_s)_t(0) = \int_0^t Lw_s(\xi)\,\mathrm{d}\xi + \int_0^t (w_s)_{xx}(\xi)\,\mathrm{d}\xi.$$

This yields

$$\begin{aligned}
\mathcal{T}_\tau(w_s)_t(0) &= \frac{(w_s)_t(\tau) - 2(w_s)_t(0) + (w_s)_t(-\tau)}{\tau^2} \\
&= \frac{1}{\tau^2}\bigg(\int_0^\tau Lw_s(\xi)\,\mathrm{d}\xi + \int_0^{-\tau} Lw_s(\xi)\,\mathrm{d}\xi \\
&\quad + \int_0^\tau (w_s)_{xx}(\xi)\,\mathrm{d}\xi + \int_0^{-\tau} (w_s)_{xx}(\xi)\,\mathrm{d}\xi\bigg) \\
&= \frac{2}{\tau}\int_0^\tau \frac{\xi}{\tau}\left(\frac{Lw_s(\xi) - Lw_s(-\xi)}{2\xi}\right)\mathrm{d}\xi \\
&\quad + \frac{2}{\tau}\int_0^\tau \frac{\xi}{\tau}\left(\frac{(w_s)_{xx}(\xi) - (w_s)_{xx}(-\xi)}{2\xi}\right)\mathrm{d}\xi.
\end{aligned}$$

Now, since $Lw_s = \rho^2 y_s \in \mathcal{C}^1((-\infty, \delta]; L^2(\Omega))$, we write that

$$\begin{aligned}
&\int_\Omega u_0 \frac{2}{\tau}\int_0^\tau \frac{\xi}{\tau}\left(\frac{Lw_s(\xi) - Lw_s(-\xi)}{2\xi}\right)\mathrm{d}\xi\,\mathrm{d}x \\
&\to \int_\Omega u_0(\rho^2 y_s)_t(0)\,\mathrm{d}x = -2s\lambda\beta T\int_\Omega \phi(0)\rho^2(0)u_0^2\,\mathrm{d}x + \int_\Omega \rho^2(0)u_0u_1\,\mathrm{d}x \quad \text{as } \tau \to 0.
\end{aligned} \tag{67}$$

On the other hand, since $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $w_s \in \mathcal{C}^0(\mathbb{R}; H_0^1(\Omega))$:

$$\begin{aligned}
&\frac{2}{\tau}\int_0^\tau \left\langle \frac{\xi}{\tau}\left(\frac{(w_s)_{xx}(\xi) - (w_s)_{xx}(-\xi)}{2\xi}\right), u_0\right\rangle_{H^{-1},H_0^1}\,\mathrm{d}\xi \\
&= \frac{2}{\tau}\int_0^\tau \frac{\xi}{\tau}\int_\Omega (u_0)_{xx}\frac{(w_s)(\xi) - (w_s)(-\xi)}{2\xi}\,\mathrm{d}x\,\mathrm{d}\xi \\
&\to \int_\Omega (u_0)_{xx}(w_s)_t(0)\,\mathrm{d}x \quad \text{as } \tau \to 0
\end{aligned} \tag{68}$$

since moreover $w_s \in \mathcal{C}^1(\mathbb{R}; L^2(\Omega))$. Thus

$$\int_\Omega u_0\mathcal{T}_\tau(w_s)_t(\cdot, 0)\,\mathrm{d}x \to -2s\lambda\beta T\int_\Omega \phi(0)\rho^2(0)u_0^2 + \int_\Omega \rho^2(0)u_0u_1$$

$$+\int_\Omega (u_0)_{xx}(w_s)_t(0)\,\mathrm{d}x \tag{69}$$

as $\tau \to 0$ and thus J_6 is bounded.

(vii) J_7 is bounded. We have $Lw_s = \rho^2 y_s \in \mathcal{C}^0(\mathbb{R}; L^2(\Omega))$ and $w_s \in \mathcal{C}^1(\mathbb{R}; H_0^1(\Omega))$, thus $(w_s)_{tt} = \rho^2 y_s + (w_s)_{xx} \in \mathcal{C}^0(\mathbb{R}; H^{-1}(\Omega))$. Therefore

$$\mathcal{T}_\tau w(0) = \frac{w(\tau) - 2w(0) + w(-\tau)}{\tau^2} \to (w_s)_{tt}(0) = (w_s)_{xx}(0) + \rho^2(0)u_0 \text{ in } H^{-1}(\Omega)$$

as $\tau \to 0$ and thus

$$\begin{aligned}\int_\Omega u_1 \mathcal{T}_\tau w(\cdot,0)\,\mathrm{d}x \to & \langle (w_s)_{tt}(0), u_1\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}\\ &= \langle (w_s)_{xx}(0), u_1\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} + \int_\Omega \rho^2(0)u_1u_0\,\mathrm{d}x \\ &= -\int_\Omega (w_s)_x(\cdot,0)(u_1)_x\,\mathrm{d}x + \int_\Omega \rho^2(0)u_1u_0\,\mathrm{d}x.\end{aligned} \tag{70}$$

as $\tau \to 0$.

$J_7 = \left|\int_\Omega u_1 \mathcal{T}_\tau w(\cdot,0)\,\mathrm{d}x\right|$ is therefore bounded.

(viii) Then we can conclude, from (65), that the terms

$$\int_{Q_T} \rho^2(t)\left|\frac{\rho^{-2}(t)Lw_s(t) - \rho^{-2}(t+\tau)Lw_s(t+\tau)}{\tau}\right|^2 \mathrm{d}x\mathrm{d}t$$

and

$$\int_0^T \frac{\left(\eta^2(t)\rho_1^{-2}(t) + \eta^2(t+\tau)\rho_1^{-2}(t+\tau)\right)}{2}\left|\frac{(w_s)_x(1,t) - (w_s)_x(1,t+\tau)}{\tau}\right|^2 \mathrm{d}t$$

are bounded. Remark that this implies that the two terms $\int_{Q_T}\rho^{-2}(t)|L(\frac{\delta_\tau w_s}{\tau})|^2\,\mathrm{d}x\mathrm{d}t$ and $\int_0^T \eta^2(t)\rho_1^{-2}(t)|(\frac{\delta_\tau w_s}{\tau})_x(1,t)|^2\mathrm{d}t$ are bounded; indeed,

$$\begin{aligned}&\int_0^T \eta^2(t)\rho_1^{-2}(t)|(\frac{\delta_\tau w_s}{\tau})_x(1,t)|^2\mathrm{d}t\\ &\le 2\int_0^T \frac{\left(\eta^2(t)\rho_1^{-2}(t) + \eta^2(t+\tau)\rho_1^{-2}(t+\tau)\right)}{2}\left|\frac{(w_s)_x(1,t) - (w_s)_x(1,t+\tau)}{\tau}\right|^2 \mathrm{d}t.\end{aligned}$$

We also have

$$\int_{Q_T} \rho^{-2}(t)|L(\frac{\delta_\tau w_s}{\tau})|^2 \,\mathrm{d}x\mathrm{d}t$$
$$\leq 2\int_{Q_T} \rho^2(t)\left|\frac{\rho^{-2}(t)Lw_s(t)-\rho^{-2}(t+\tau)Lw_s(t+\tau)}{\tau}\right|^2 \mathrm{d}x\mathrm{d}t$$
$$+2\int_{Q_T} \rho^2(t)\left|\frac{\rho^{-2}(t)-\rho^{-2}(t+\tau)}{\tau}Lw_s(t+\tau)\right|^2 \mathrm{d}x\mathrm{d}t$$

and each term of the right hand side is bounded.

Sub-step 4. In this step, we prove that $v_s \in H^1(0,T)$ and $y_s \in \mathcal{C}^0([0,T]; H^1(\Omega))\cap \mathcal{C}^1([0,T]; L^2(\Omega))$.

Since $\frac{\delta_\tau w_s}{\tau} \in \mathcal{C}^0([0,T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega))$ and satisfies $(\frac{\delta_\tau w_s}{\tau})_x(1,\cdot) \in L^2(0,T)$ then the Carleman estimates (10) gives

$$s\int_{Q_T} \rho^{-2}(t)\left(|(\frac{\delta_\tau w_s}{\tau})_t|^2+|(\frac{\delta_\tau w_s}{\tau})_x|^2\right)\mathrm{d}x\mathrm{d}t$$
$$+s^3\int_{Q_T} \rho^{-2}(t)|\frac{\delta_\tau w_s}{\tau}|^2 \,\mathrm{d}x\mathrm{d}t$$
$$+s\int_{\Omega} \rho^{-2}(0)\left(|(\frac{\delta_\tau w_s}{\tau})_t(x,0)|^2+|(\frac{\delta_\tau w_s}{\tau})_x(x,0)|^2\right)\mathrm{d}x$$
$$+s^3\int_{\Omega} \rho^{-2}(0)|\frac{\delta_\tau w_s}{\tau}(x,0)|^2 \,\mathrm{d}x$$
$$\leq C\int_{Q_T} \rho^{-2}(t)|L(\frac{\delta_\tau w_s}{\tau})|^2 \,\mathrm{d}x\mathrm{d}t + Cs\int_0^T \eta^2(t)\rho_1^{-2}(t)|(\frac{\delta_\tau w_s}{\tau})_x(1,t)|^2\mathrm{d}t.$$

Therefore, since the right hand side is bounded, $(\frac{\delta_\tau w_s}{\tau})_t$ and $(\frac{\delta_\tau w_s}{\tau})_x$ are bounded in $L^2(Q_T)$ and thus $(w_s)_{tt} \in L^2(Q_T)$ and $(w_s)_t \in L^2(0,T; H_0^1(\Omega))$. Moreover, $\frac{\delta_\tau w_s}{\tau}(\cdot,0)$ is bounded in $H_0^1(\Omega)$ thus $(w_s)_t(\cdot,0) \in H_0^1(\Omega)$. We also have $L(\frac{\delta_\tau w_s}{\tau})$ bounded in $L^2(Q_T)$ so $L(w_s)_t \in L^2(Q_T)$. Thus $(w_s)_t$ satisfies

$$\begin{cases} L(w_s)_t \in L^2(Q_T), \\ (w_s)_t(0,t) = (w_s)_t(1,t) = 0, \quad t\in(0,T) \\ ((w_s)_t(0),(w_s)_{tt}(0)) \in H_0^1(\Omega)\times L^2(\Omega) \end{cases} \tag{71}$$

and thus $(w_s)_t \in \mathcal{C}^0([0,T]; H_0^1(\Omega))\cap\mathcal{C}^1([0,T]; L^2(\Omega))$ and $(w_s)_{tx}(1,\cdot) \in L^2(0,T)$. Therefore from the definition of v_s, $v_s \in H^1(0,T)$ while from the equation satisfied by (y_s, v_s) (see (17)), $y_s \in \mathcal{C}^0([0,T]; H^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega))$.

Remark 1 e then have $w_s \in \mathcal{C}^1([0,T]; H_0^1(\Omega)) \cap \mathcal{C}^2([0,T]; L^2(\Omega))$ and from the equation satisfied by w_s, since $Lw_s \in \mathcal{C}^1([0,T]; L^2(\Omega))$ we deduce that $(w_s)_{xx} = (w_s)_{tt} - Lw_s \in \mathcal{C}^0([0,T]; L^2(\Omega))$ and thus that $(w_s)_{xx}(\cdot,0) \in L^2(\Omega)$.

Step 2 : In this step, we give estimates on $(v_s)_t$ and $(y_s)_t$.

First of all, since $(w_s)_t \in \mathcal{C}^0([0;T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega))$, $L(w_s)_t \in L^2(Q_T)$ and $(w_s)_{tx}(1,\cdot) \in L^2(0,T)$, we can write the Carleman estimate (10) for $(w_s)_t$ leading to

$$
\begin{aligned}
& s\int_{Q_T} \rho^{-2}(t)(|(w_s)_{tt}|^2 + |(w_s)_{tx}|^2)\,\mathrm{d}x\mathrm{d}t + s^3\int_{Q_T} \rho^{-2}(t)|(w_s)_t|^2\,\mathrm{d}x\mathrm{d}t \\
& \quad + s\int_\Omega \rho^{-2}(0)(|(w_s)_{tt}(x,0)|^2 + (w_s)_{tx}(x,0)|^2)\,\mathrm{d}x + s^3\int_\Omega \rho^{-2}(0)|(w_s)_t(x,0)|^2\,\mathrm{d}x \\
& \quad \le C\int_{Q_T} \rho^{-2}(t)|(Lw_s)_t|^2\,\mathrm{d}x\mathrm{d}t + Cs\int_0^T \eta^2(t)\rho_1^{-2}(t)|(w_s)_{tx}(1,t)|^2\mathrm{d}t. \qquad (72)
\end{aligned}
$$

Sub-step 1 : In this step, we pass to the limit when $\tau \to 0$ in equation (62). We have, since $y_s = \rho^{-2}Lw_s \in \mathcal{C}^1(\mathbb{R}; L^2(\Omega))$:

$$
\int_{Q_T} \rho^2(t)\left|\frac{\rho^{-2}(t)Lw_s(t) - \rho^{-2}(t+\tau)Lw_s(t+\tau)}{\tau}\right|^2\mathrm{d}x\mathrm{d}t \to \int_{Q_T} \rho^2(t)\,|(y_s)_t|^2\,\mathrm{d}x\mathrm{d}t
$$

as $\tau \to 0$ and since $(w_s)_{tx}(1,\cdot) \in L^2(-\delta, T+\delta)$ and $\eta\rho_1^{-1} \in \mathcal{C}(\mathbb{R})$:

$$
\begin{aligned}
& \int_0^T \frac{\left(\eta^2(t)\rho_1^{-2}(t) + \eta^2(t+\tau)\rho_1^{-2}(t+\tau)\right)}{2}\left|\frac{(w_s)_x(1,t+\tau) - (w_s)_x(1,t)}{\tau}\right|^2\mathrm{d}t \\
& \qquad \to \int_0^T \eta^2(t)\rho_1^{-2}(t)\,|(w_s)_{tx}(1,t)|^2\,\mathrm{d}t
\end{aligned}
$$

as $\tau \to 0$. Since $y_s = \rho^{-2}Lw_s \in \mathcal{C}^1(\mathbb{R}; L^2(\Omega))$, $Lw_s \in \mathcal{C}^1(\mathbb{R}; L^2(\Omega))$ and $(\rho^{-1})_t = -2s\lambda\beta(t - \frac{T}{2})\phi\rho^{-1}$ in Q_T, we infer that

$$
\begin{aligned}
& \int_{Q_T} \rho^2(t)\left(\frac{\rho^{-2}(t)Lw_s(t) - \rho^{-2}(t+\tau)Lw_s(t+\tau)}{\tau}\right) \\
& \quad \times \left(\frac{\rho^{-2}(t) - \rho^{-2}(t+\tau)}{\tau}\right)Lw_s(t+\tau)\,\mathrm{d}x\mathrm{d}t \\
& \quad \to -2s\lambda\beta\int_{Q_T}(t - \frac{T}{2})\phi(t)\rho^2(t)(y_s)_t y_s\,\mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

$$
\begin{aligned}
& \frac{1}{\tau}\int_{-\tau}^0\int_\Omega \rho^{-2}(t+\tau)Lw_s(t+\tau)\left(\frac{Lw_s(t+\tau) - Lw_s(t)}{\tau}\right)\mathrm{d}x\mathrm{d}t \\
& \quad \to \int_\Omega y_s(0)(\rho^2 y_s)_t(0)\,\mathrm{d}x
\end{aligned}
$$

and

$$\frac{1}{\tau}\int_T^{T-\tau}\int_\Omega \rho^{-2}(t+\tau)Lw_s(t+\tau)\left(\frac{Lw_s(t+\tau)-Lw_s(t)}{\tau}\right)\mathrm{d}x\mathrm{d}t \to 0$$

as $\tau \to 0$.

Similarly, since $w_s \in \mathcal{C}^2(\mathbb{R}; L^2(\Omega))$ and $(w_s)_{tx}(1,\cdot) \in L^2(-\delta, T+\delta)$,

$$\int_0^T \frac{\left(\eta^2(t)\rho_1^{-2}(t)-\eta^2(t+\tau)\rho_1^{-2}(t+\tau)\right)((w_s)_x(1,t)+(w_s)_x(1,t+\tau))}{2\tau} \times \left(\frac{(w_s)_x(1,t+\tau)-(w_s)_x(1,t)}{\tau}\right)\mathrm{d}t$$
$$\to \int_0^T (\eta^2\rho_1^{-2})_t(w_s)_x(1,t)(w_s)_{tx}(1,t)\mathrm{d}t$$

and

$$\int_{Q_T} B\mathcal{T}_\tau w_s\,\mathrm{d}x\mathrm{d}t \to \int_{Q_T} B(w_s)_{tt}\,\mathrm{d}x\mathrm{d}t$$

as $\tau \to 0$. Since $(w_s)_{tt}(\cdot,0) \in L^2(\Omega)$, the convergence (70) reads

$$\int_\Omega u_1\mathcal{T}_\tau w_s(\cdot,0)\,\mathrm{d}x \to \int_\Omega (w_s)_{tt}(\cdot,0)u_1\,\mathrm{d}x.$$

Similarly, since $(w_s)_t(\cdot,0) \in H_0^1(\Omega)$, (69) reads

$$\int_\Omega u_0\mathcal{T}_\tau(w_s)_t(\cdot,0)\,\mathrm{d}x \to -2s\lambda\beta T\int_\Omega \phi(0)\rho^2(0)u_0^2\,\mathrm{d}x + \int_\Omega \rho^2(0)u_0u_1\,\mathrm{d}x$$
$$-\int_\Omega (u_0)_x(w_s)_{tx}(\cdot,0)\,\mathrm{d}x.$$

We conclude that the limit with respect to $\tau \to 0$ in (62) leads to the following equality

$$\int_{Q_T}\rho^2(t)\,|(y_s)_t|^2\,\mathrm{d}x\mathrm{d}t + s\int_0^T \eta^2(t)\rho_1^{-2}(t)\,|(w_s)_{tx}(1,t)|^2\,\mathrm{d}t$$
$$= -2s\lambda\beta\int_{Q_T}(t-\frac{T}{2})\phi(t)\rho^2(t)(y_s)_t y_s\,\mathrm{d}x\mathrm{d}t - \int_\Omega y_s(0)(\rho^2 y_s)_t(0)$$
$$+s\int_0^T (\eta^2\rho_1^{-2})_t(t)(w_s)_x(1,t)(w_s)_{tx}(1,t)\mathrm{d}t$$
$$-\int_{Q_T} B(w_s)_{tt}\,\mathrm{d}x\mathrm{d}t - \int_\Omega (w_s)_{tt}(\cdot,0)u_1\,\mathrm{d}x$$

$$
\begin{aligned}
&-2s\lambda\beta T \int_\Omega \phi(0)\rho^2(0)u_0^2\,\mathrm{d}x + \int_\Omega \rho^2(0)u_1u_0\,\mathrm{d}x \\
&-\int_\Omega (u_0)_x(w_s)_{tx}(\cdot,0)\,\mathrm{d}x \\
:=\ & K_1+K_2+K_3+K_4+K_5+K_6+K_7+K_8.
\end{aligned}
\tag{73}
$$

Sub-step 2 : In this step, we estimate each term K_i, $i=1,\cdots,8$.
(i) We get that, there exists $C>0$ only depending on T such that

$$
|K_1| \le \frac{1}{8}\int_{Q_T} \rho^2(t)|(y_s)_t|^2\,\mathrm{d}x\mathrm{d}t + Cs^2\int_{Q_T}\rho^2(t)|y_s|^2\,\mathrm{d}x\mathrm{d}t.
$$

(ii) Similarly, recalling that $y_s(\cdot,0)=u_0$ and $(y_s)_t(\cdot,0)=u_1$, there exists $C>0$ such that

$$
|K_2| \le Cs\|\rho(0)u_0\|^2_{L^2(\Omega)} + \|\rho(0)u_0\|_{L^2(\Omega)}\|\rho(0)u_1\|_{L^2(\Omega)}.
$$

(iii) Using that $(\rho_1^{-1})_t = -2s\lambda\beta(t-\frac{T}{2})\phi\rho_1^{-1}$, we obtain

$$
\begin{aligned}
|K_3| &= \left| s\int_0^T (\eta^2\rho_1^{-2})_t(w_s)_x(1,t)(w_s)_{tx}(1,t)\mathrm{d}t\right| \\
&\le 2\left(s\int_0^T |(\eta\rho_1^{-1})_t|^2|(w_s)_x(1,t)|^2\mathrm{d}t\right)^{1/2}\left(s\int_0^T \eta^2(t)\rho_1^{-2}(t)|(w_s)_{tx}(1,t)|^2\mathrm{d}t\right)^{1/2} \\
&\le C\left(s^3\int_0^T \eta^2(t)\rho_1^{-2}(t)|(w_s)_x(1,t)|^2\mathrm{d}t + s\int_0^T \rho_1^{-2}(t)|(w_s)_x(1,t)|^2\mathrm{d}t\right)^{1/2} \\
&\qquad\qquad \times\left(s\int_0^T \eta^2(t)\rho_1^{-2}(t)|(w_s)_{tx}(1,t)|^2\mathrm{d}t\right)^{1/2} \\
&\le C\left(s\int_\delta^{T-\delta}\frac{\rho_1^2(t)}{\eta^2(t)}v_s^2\mathrm{d}t + s\int_0^T \rho_1^{-2}(t)|(w_s)_x(1,t)|^2\mathrm{d}t\right) \\
&\quad + \frac{s}{8}\int_0^T \eta^2(t)\rho_1^{-2}|(w_s)_{tx}(1,t)|^2\mathrm{d}t.
\end{aligned}
$$

We now estimate the term $\int_0^T \rho_1^{-2}|(w_s)_x(1,t)|^2\mathrm{d}t$ appearing in the previous inequality: proceeding as in [23, Lemma 3.7] with $q(x,t)=x\rho^{-2}(x,t)$ such that $q(0,t)=0$ and $q(1,t)=\rho_1^{-2}(t)$, we get the equality

$$
\begin{aligned}
&\frac{1}{2}\int_0^T \rho_1^{-2}(t)|(w_s)_x(1,t)|^2 \\
&\quad = 2\int_{Q_T} x\rho^{-1}\rho_t^{-1}w_xw_t + \frac{1}{2}\int_{Q_T}(\rho^{-2}-2x\rho^{-1}\rho_x^{-1})(w_x^2+w_t^2) \\
&\qquad + \int_\Omega [x\rho^{-2}w_tw_x]_0^T.
\end{aligned}
$$

Writing that $|\rho^{-1}\rho_x^{-1}| \leq Cs\rho^{-2}$ and $|\rho^{-1}\rho_t^{-1}| \leq Cs\rho^{-2}$, we obtain (since $s \geq 1$)

$$\begin{aligned}
&\frac{1}{2}\int_0^T \rho_1^{-2}(t)|(w_s)_x(1,t)|^2 \\
&\quad \leq Cs\int_{Q_T} \rho^{-2}(w_x^2 + w_t^2) + Cs\int_\Omega \Big(\rho^{-2}(0)(w_t^2 + w_x^2)(0) \\
&\qquad + \rho^{-2}(T)(w_t^2 + w_x^2)(T)\Big)
\end{aligned}$$

leading, using the Carleman estimate (10), to

$$s\int_0^T \rho_1^{-2}|(w_s)_x(1,t)|^2 \mathrm{d}t \leq Cs\left(\|\rho y_s\|^2_{L^2(Q_T)} + s^{-1}\|\frac{\rho_1}{\eta}v_s\|^2_{L^2(\delta,T-\delta)}\right). \tag{74}$$

Thus,

$$|K_3| \leq Cs\left(\|\rho y_s\|^2_{L^2(Q_T)} + \|\frac{\rho_1}{\eta}v_s\|^2_{L^2(\delta,T-\delta)}\right) + \frac{s}{8}\int_0^T \eta^2(t)\rho_1^{-2}(w_s)_{tx}(1,t)\mathrm{d}t.$$

(iv) Using the Carleman estimate (72) we have

$$\begin{aligned}
|K_4| &= \left|\int_{Q_T} B(w_s)_{tt}\,\mathrm{d}x\mathrm{d}t\right| \leq \left(s^{-1}\int_{Q_T}\rho^2 B^2\right)^{1/2}\left(s\int_{Q_T}\rho^{-2}|(w_s)_{tt}|^2\right)^{1/2} \\
&\leq C\left(s^{-1}\int_{Q_T}\rho^2 B^2\right)^{1/2}\left(\int_{Q_T}\rho^{-2}|(\rho^2 y_s)_t|^2\,\mathrm{d}x\mathrm{d}t + s\int_0^T \eta^2(t)\rho_1^{-2}|(w_s)_{tx}(1,t)|^2\mathrm{d}t\right)^{1/2} \\
&\leq C\left(s^{-1}\int_{Q_T}\rho^2 B^2\right)^{1/2}\left(s^2\int_{Q_T}\rho^2|y_s|^2\,\mathrm{d}x\mathrm{d}t + \int_{Q_T}\rho^2|(y_s)_t|^2\,\mathrm{d}x\mathrm{d}t\right. \\
&\quad \left. + s\int_0^T \eta^2(t)\rho_1^{-2}|(w_s)_{tx}(1,t)|^2\mathrm{d}t\right)^{1/2} \\
&\leq C\left(s^{-1}\int_{Q_T}\rho^2 B^2 + s^2\int_{Q_T}\rho^2|y_s|^2\,\mathrm{d}x\mathrm{d}t\right) + \frac{1}{8}\int_{Q_T}\rho^2|(y_s)_t|^2\,\mathrm{d}x\mathrm{d}t \\
&\quad + \frac{s}{8}\int_0^T \eta^2(t)\rho_1^{-2}|(w_s)_{tx}(1,t)|^2\mathrm{d}t.
\end{aligned}$$

(v) Similarly, using again the Carleman estimate (72) we have

$$\begin{aligned}
|K_5| &= \left|\int_\Omega (w_s)_{tt}(\cdot,0)u_1\,\mathrm{d}x\right| \leq \|\rho(0)u_1\|_{L^2(\Omega)}\|\rho^{-1}(0)(w_s)_{tt}(\cdot,0)\|_{L^2(\Omega)} \\
&\leq Cs^{-1/2}\|\rho(0)u_1\|_{L^2(\Omega)}\left(\int_{Q_T}\rho^{-2}(t)|(\rho^2 y_s)_t|^2\,\mathrm{d}x\mathrm{d}t\right. \\
&\quad \left. + s\int_0^T \eta^2(t)\rho_1^{-2}|(w_s)_{tx}(1,t)|^2\mathrm{d}t\right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\leq Cs^{-1/2}\|\rho(0)u_1\|_{L^2(\Omega)}\left(s^2\int_{Q_T}\rho^2|y_s|^2\,\mathrm{d}x\mathrm{d}t+\int_{Q_T}\rho^2|(y_s)_t|^2\,\mathrm{d}x\mathrm{d}t\right.\\
&\left.+s\int_0^T\eta^2(t)\rho_1^{-2}|(w_s)_{tx}(1,t)|^2\mathrm{d}t\right)^{1/2}\\
&\leq C\left(s^{-1}\|\rho(0)u_1\|^2_{L^2(\Omega)}+s^2\int_{Q_T}\rho^2|y_s|^2\,\mathrm{d}x\mathrm{d}t\right)+\frac{1}{8}\int_{Q_T}\rho^2|(y_s)_t|^2\,\mathrm{d}x\mathrm{d}t\\
&\quad+\frac{s}{8}\int_0^T\eta^2(t)\rho_1^{-2}|(w_s)_{tx}(1,t)|^2\mathrm{d}t.
\end{aligned}
$$

(vi) Simpler, we get

$$|K_6|\leq Cs\|\rho(0)u_0\|^2_{L^2(\Omega)}$$

(vii) and

$$|K_7|\leq\|\rho(0)u_0\|_{L^2(\Omega)}\|\rho(0)u_1\|_{L^2(\Omega)}.$$

(viii) Eventually, (72) leads to

$$
\begin{aligned}
|K_8|&=\left|\int_\Omega(u_0)_x(w_s)_{tx}(\cdot,0)\right|\leq\|\rho(0)(u_0)_x\|_{L^2(\Omega)}\|\rho^{-1}(0)(w_s)_{tx}(\cdot,0)\|_{L^2(\Omega)}\\
&\leq Cs^{-1/2}\|\rho(0)(u_0)_x\|_{L^2(\Omega)}\left(\int_{Q_T}\rho^{-2}(t)|(\rho^2y_s)_t|^2\,\mathrm{d}x\mathrm{d}t\right.\\
&\quad\left.+Cs\int_0^T\eta^2(t)\rho_1^{-2}(t)|(w_s)_{tx}(1,t)|^2\mathrm{d}t\right)^{1/2}\\
&\leq Cs^{-1/2}\|\rho(0)(u_0)_x\|_{L^2(\Omega)}\left(s^2\int_{Q_T}\rho^2(t)|y_s|^2\,\mathrm{d}x\mathrm{d}t+\int_{Q_T}\rho^2(t)|(y_s)_t|^2\,\mathrm{d}x\mathrm{d}t\right.\\
&\quad\left.+\ s\int_0^T\eta^2(t)\rho_1^{-2}(t)|(w_s)_{tx}(1,t)|^2\mathrm{d}t\right)^{1/2}\\
&\leq C\left(s^{-1}\|\rho(0)(u_0)_x\|^2_{L^2(\Omega)}+s^2\int_{Q_T}\rho^2(t)|y_s|^2\,\mathrm{d}x\mathrm{d}t\right)\\
&\quad+\frac{1}{8}\int_{Q_T}\rho^2(t)|(y_s)_t|^2\,\mathrm{d}x\mathrm{d}t\\
&\quad+\frac{s}{8}\int_0^T\eta^2(t)\rho_1^{-2}(t)|(w_s)_{tx}(1,t)|^2\mathrm{d}t.
\end{aligned}
$$

Sub-step 3 : In this step, we give estimates on $(v_s)_t$ and $(y_s)_t$. Collecting the previous estimates, we get from (73)

$$
\begin{aligned}
&\int_{Q_T} \rho^2(t)\, |(y_s)_t|^2 \, \mathrm{d}x\mathrm{d}t + s\int_0^T \eta^2(t)\rho_1^{-2}(t)\, |(w_s)_{tx}(1,t)|^2 \, \mathrm{d}t \\
&\quad \le C\left(s^2\|\rho y_s\|^2_{L^2(Q_T)} + s\|\frac{\rho_1}{\eta}v\|^2_{L^2(\delta,T-\delta)} + s^{-1}\int_{Q_T}\rho^2B^2\right. \\
&\qquad \left. + s\|\rho(0)u_0\|^2_{L^2(\Omega)} + s^{-1}\|\rho(0)(u_0)_x\|^2_{L^2(\Omega)} + s^{-1}\|\rho(0)u_1\|^2_{L^2(\Omega)}\right).
\end{aligned} \tag{75}
$$

We have

$$
\begin{aligned}
&s^{-1}\int_0^T \rho_1^2(t)\, |(v_s)_t|^2 \, dt \\
&\le C\left(s\int_0^T \eta^2(t)\rho_1^{-2}(t)\, |(w_s)_{tx}(1,t)|^2 + s\int_\delta^{T-\delta}\frac{\rho_1^2(t)}{\eta^2(t)}v_s^2(t)\mathrm{d}t\right. \\
&\quad \left. + s\int_0^T \rho_1^{-2}(t)\, |(w_s)_x(1,t)|^2\right)
\end{aligned} \tag{76}
$$

thus using the estimates (74) and (14), (75) implies for $s \ge s_0 \ge 1$ that

$$
\begin{aligned}
&\int_{Q_T} \rho^2(t)\, |(y_s)_t|^2 \, \mathrm{d}x\mathrm{d}t + s^{-1}\int_0^T \rho_1^2(t)\, |(v_s)_t|^2 \, dt \\
&\le C\left(s^{-1}\|\rho B\|^2_{L^2(Q_T)} + se^{-2s}\|u_0\|^2_{L^2(\Omega)} + s^{-1}e^{-2s}\|(u_0)_x\|^2_{L^2(\Omega)}\right. \\
&\quad \left. + s^{-1}e^{-2s}\|u_1\|^2_{L^2(\Omega)}\right)
\end{aligned} \tag{77}
$$

which gives the announced estimate (19) in the case of regular data.

Step 3 : Case where $B \in L^2(Q_T)$ and $(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)$. We proceed by density: there exist $(u_0^n)_{n\in\mathbb{N}} \in H^2(\Omega) \cap H^1_0(\Omega)$, $(u_1^n)_{n\in\mathbb{N}} \in H^1_0(\Omega)$ and $(B^n)_{n\in\mathbb{N}} \in \mathcal{D}(0, T; L^2(\Omega))$ such that $u_0^n \to u_0$ in $H^1_0(\Omega)$, $u_1^n \to u_1$ in $L^2(\Omega)$ and $B^n \to B$ in $L^2(Q_T)$ as $n \to \infty$.

Let (y_s^n, v_s^n) be the solution of (6) given in Theorem 6 associated to (u_0^n, u_1^n, B^n). Then, by linearity, we have for all $(n, m) \in \mathbb{N}^2$, from (14)

$$
\begin{aligned}
&\|\rho(y_s^n - y_s^m)\|_{L^2(Q_T)} + s^{-1/2}\|\rho_1(v_s^n - v_s^m)\|_{L^2(0,T)} \\
&\quad \le \|\rho(y_s^n - y_s^m)\|_{L^2(Q_T)} + s^{-1/2}\left\|\frac{\rho_1}{\eta}(v_s^n - v_s^m)\right\|_{L^2(\delta,T-\delta)} \\
&\quad \le C\left(s^{-3/2}\|\rho(B^n - B^m)\|_{L^2(Q_T)} + s^{-1/2}e^{-s}\|u_0^n - u_0^m\|_{L^2(\Omega)}\right. \\
&\quad \left. + s^{-3/2}e^{-s}\|u_1^n - u_1^m\|_{L^2(\Omega)}\right),
\end{aligned}
$$

while from (19)

$$
\|\rho(y_s^n - y_s^m)_t\|_{L^2(Q_T)} + s^{-1/2}\|\rho_1(v_s^n - v_s^m)_t\|_{L^2(0,T)}
$$

$$\leq \Big(s^{-1/2}\|\rho(B^n - B^m)\|_{L^2(Q_T)} + s^{1/2}e^{-s}\|u_0^n - u_0^m\|_{L^2(\Omega)} + s^{-1/2}e^{-s}\|(u_0^n - u_0^m)_x\|_{L^2(\Omega)} + s^{-1/2}e^{-s}\|u_1^n - u_1^m\|_{L^2(\Omega)}\Big)$$

and from (21).

$$\begin{aligned}&\|(y_s^n - y_s^m)_t\|_{L^\infty(0,T;L^2(\Omega))} + \|(y_s^n - y_s^m)_x\|_{L^\infty(0,T;L^2(\Omega))}\\&\leq C\left(\|B^n - B^m\|_{L^2(Q_T)} + \|u_0^n - u_0^m\|_{L^2(\Omega)}\right.\\&\left.+\|(u_1^n - u_1^m)_x\|_{L^2(\Omega)} + \|v_s^n - v_s^m\|_{H^1(0,T)}\right).\end{aligned}$$

Therefore $v_s^n \to v_s$ in $H^1(0,T)$ and $y_s^n \to y_s \in \mathcal{C}^0([0,T]; H^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega))$ and, passing to the limit in the equation (6) satisfied by (y^n, v^n), we obtain that (y_s, v_s) solves (6). Moreover, passing to the limit in the estimate (19) satisfied by (y_s^n, v_s^n), we deduce that (y_s, v_s) also satisfies (19). Using (10), we easily check that (y_s, v_s) satisfies $v_s = s\eta^2\rho_1^{-2}(w_s)_x(1,\cdot)$ and $y_s = \rho^{-2}Lw_s$ where $w_s \in P_s$ is the unique solution of (12). The proof of Theorem 7 is complete.

References

1. Baudouin L, De Buhan M, Ervedoza S (2013) Global Carleman estimates for waves and applications. Commun Partial Differ Equ 38(5):823–859
2. Burman E, Feizmohammadi A, Münch A, Oksanen L (2021) Space time finite element methods for control problems subject to the wave equation. Preprint hal-03346617
3. Bernadou M, Hassan K (1981) Basis functions for general Hsieh–Clough–Tocher triangles, complete or reduced. Intern J Numer Methods Eng 17(5):784–789
4. Bottois A, Lemoine J, Münch A (2021) Constructive exact control of semilinear multi-dimensional wave equations by a least-squares approach. Preprint hal-03112309
5. Bardos C, Lebeau G, Rauch J (1992) Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. SIAM J Control Optim 30(5):1024–1065
6. Bellassoued M, Yamamoto M (2017) Carleman estimates and applications to inverse problems for hyperbolic systems. Springer Monographs in Mathematics. Springer, Tokyo
7. Calvalcanti M, Cavalcanti V, Carole R, Rosier L (2021) Numerical control of a semilinear wave equation on an interval. Preprint, hal-03369560
8. Cîndea N, Fernández-Cara E, Münch A (2013) Numerical controllability of the wave equation through primal methods and Carleman estimates. ESAIM Control Optim Calc Var 19(4):1076–1108
9. Cazenave T, Haraux A (1980) Équations d'évolution avec non linéarité logarithmique. Ann Fac Sci Toulouse Math (5) 2(1):21–51
10. Chewning WC (1976) Controllability of the nonlinear wave equation in several space variables. SIAM J Control Optim 14(1):19–25
11. Ciarlet PG (2002) The finite element method for elliptic problems, volume 40 of classics in applied mathematics. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA. Reprint of the 1978 original [North-Holland, Amsterdam; MR0520174 (58 #25001)]
12. Cannarsa P, Komornik V, Loreti P (2002) One-sided and internal controllability of semilinear wave equations with infinitely iterated logarithms. Discrete Contin Dyn Syst 8(3):745–756
13. Cîndea N, Münch A (2015) A mixed formulation for the direct approximation of the control of minimal L^2-norm for linear type wave equations. Calcolo 52(3):245–288
14. Coron J-M, Trélat E (2006) Global steady-state stabilization and controllability of 1D semilinear wave equations. Commun Contemp Math 8(4):535–567
15. Dehman B, Lebeau G (2009) Analysis of the HUM control operator and exact controllability for semilinear waves in uniform time. SIAM J Control Optim 48(2):521–550

16. Dehman B, Lebeau G, Zuazua E (2003) Stabilization and control for the subcritical semilinear wave equation. Ann Sci École Norm Sup (4) 36(4):525–551
17. Ervedoza S, Lemoine J, Münch A (2022) Exact controllability of semilinear heat equations through a constructive approach. Evol Equ Control Theory. https://doi.org/10.3934/eect.2022042
18. Ervedoza S, Zuazua E (2010) A systematic method for building smooth controls for smooth data. Discrete Contin Dyn Syst Ser B 14(4):1375–1401
19. Fattorini HO (1975) Local controllability of a nonlinear wave equation. Math Syst Theory 9(1):30–45
20. Hecht F (2012) New development in Freefem++. J Numer Math 20(3–4):251–265
21. Joly R, Laurent C (2014) A note on the semiglobal controllability of the semilinear wave equation. SIAM J Control Optim 52(1):439–450
22. Klibanov MV, Kolesov AE, Nguyen L, Sullivan A (2017) Globally strictly convex cost functional for a 1-D inverse medium scattering problem with experimental data. SIAM J Appl Math 77(5):1733–1755
23. Lions J-L (1988) Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1, volume 8 of Recherches en Mathématiques Appliquées [Research in Applied Mathematics]. Masson, Paris. Contrôlabilité exacte. [Exact controllability], With appendices by E. Zuazua, C. Bardos, G. Lebeau and J. Rauch
24. Lions J-L (1988) Exact controllability, stabilization and perturbations for distributed systems. SIAM Rev 30(1):1–68
25. Lagnese J, Lions J-L (1988) Modelling analysis and control of thin plates, vol 6. Recherches en Mathématiques Appliquées [Research in Applied Mathematics]. Masson, Paris
26. Lasiecka I, Lions J-L, Triggiani R (1986) Nonhomogeneous boundary value problems for second order hyperbolic operators. J Math Pures Appl (9) 65(2):149–192
27. Lemoine J, Münch A (2022) Constructive exact control of semilinear 1d heat equations. Math Control Relat Fields. https://doi.org/10.3934/mcrf.2022001
28. Lasiecka I, Triggiani R (1991) Exact controllability of semilinear abstract systems with application to waves and plates boundary control problems. Appl Math Optim 23(2):109–154
29. Li L, Zhang X (2000) Exact controllability for semilinear wave equations. J Math Anal Appl 250(2):589–597
30. Markus L (1965) Controllability of nonlinear processes. J SIAM Control Ser A 3:78–90
31. Münch A, Souza DA (2016) A mixed formulation for the direct approximation of L^2-weighted controls for the linear heat equation. Adv Comput Math 42(1):85–125
32. Münch A, Trélat E (2022) Constructive exact control of semilinear 1D wave equations by a least-squares approach. SIAM J Control Optim 60(2):652–673
33. Natarajan V, Zhou H-C, Weiss G, Fridman E (2019) Exact controllability of a class of nonlinear distributed parameter systems using back-and-forth iterations. Intern J Control 92(1):145–162
34. Russell DL (1978) Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions. SIAM Rev 20(4):639–739
35. Simon J (1987) Compact sets in the space $L^p(0, T; B)$. Ann Mat Pura Appl 4(146):65–96
36. Zhang X (2000) Explicit observability estimate for the wave equation with potential and its application. R Soc Lond Proc Ser A Math Phys Eng Sci 456(1997):1101–1115
37. Zuazua E (1988) Contrôlabilité exacte de systèmes d'évolution non linéaires. C R Acad Sci Paris Sér I Math 306(3):129–132
38. Zuazua E (1988) Exact controllability for semilinear distributed systems. In: Proceedings of the 27th IEEE conference on decision and control, pp 1265–1268 vol 2
39. Zuazua E (1990) Exact controllability for the semilinear wave equation. J Math Pures Appl 69(1):1–31
40. Zuazua E (1991) Exact boundary controllability for the semilinear wave equation. In: Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. X (Paris, 1987–1988), volume 220 of Pitman Res. Notes Math. Ser. Longman Sci. Tech., Harlow, pp 357–391
41. Zuazua E (1993) Exact controllability for semilinear wave equations in one space dimension. Ann Inst H Poincaré Anal Non Linéaire 10(1):109–129