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ORIGINAL ARTICLE

Structure theory for ensemble controllability, observability, and duality

Xudong Chen[1]

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Abstract

Ensemble control deals with the problem of using a finite number of control inputs to simultaneously steer a large population (in the limit, a continuum) of control systems. Dual to the ensemble control problem, ensemble estimation deals with the problem of using a finite number of measurement outputs to estimate the initial state of every individual system in the ensemble. We introduce in the paper a novel class of ensemble systems, termed *distinguished ensemble systems*, and establish sufficient conditions for controllability and observability of such systems. Every distinguished ensemble system has two key components, namely a set of *distinguished control vector fields* and a set of *codistinguished observation functions*. Roughly speaking, a set of vector fields is distinguished if it is closed (up to scaling) under Lie bracket, and moreover, every vector field in the set can be obtained by a Lie bracket of two vector fields in the same set. Similarly, a set of functions is codistinguished to a set of vector fields if the Lie derivatives of the functions along the given vector fields yield (up to scaling) the same set of functions. We demonstrate in the paper that the structure of a distinguished ensemble system can significantly simplify the analysis of ensemble controllability and observability. Moreover, such a structure can be used as a guiding principle for ensemble system design. We further address in the paper the problem about existence of a distinguished ensemble system for a given manifold. We provide an affirmative answer for the case where the manifold is a connected semi-simple Lie group. Specifically, we show that every such Lie group admits a set of distinguished vector fields, together with a set of codistinguished functions. The proof is constructive, leveraging the structure theory of semi-simple real Lie algebras and representation theory. Examples will be provided along the presentation of the paper illustrating key definitions and main results.

✉ Xudong Chen
xudong.chen@colorado.edu

[1] College of Engineering and Applied Science, University of Colorado Boulder, 425 UCB #1B55, Boulder, CO 80309, USA

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1 Introduction

We address in the paper controllability and observability of a continuum ensemble of control systems. Roughly speaking, ensemble control deals with the problem of using a *finite* number of control inputs to simultaneously steer a large population (in the limit, a *continuum*) of control systems. These individual control systems may be structurally identical, but show variations in their tuning parameters. Dual to ensemble control, ensemble estimation deals with the problem of estimating the state of *every* individual control system in the ensemble using only a *finite* number of measurement outputs. We refer the reader to Fig. 1 for an illustration of a continuum ensemble of control systems indexed by a parameter of a two-dimensional surface. Note that any finite ensemble of control systems can be viewed as a proper subsystem of the continuum ensemble. Controllability (or observability) of the continuum ensemble will guarantee the controllability (or observability) of any such finite subsystem of it.

The framework of ensemble control and estimation naturally has many applications across various disciplines in engineering and science. The individual control systems in the ensemble can be used to model, for example, spin dynamics that are controlled by a magnetic field [14], molecules that respond to external stimuli such as light [35] and heat [33], or micro-robotics that are steered by a broadcast control signal [3]. We further note that an individual control system does not necessarily have only one single physical entity, but rather it can comprise multiple interacting components (or agents). In this case, every individual control system is itself a networked control system (or a multi-agent system). For example, a mathematical model for a continuum ensemble of multi-agent formation systems has recently been proposed and investigated in [5].

Many existing ensemble control and estimation theories deal only with linear ensembles (i.e., ensembles of linear control systems). For nonlinear ensembles, the

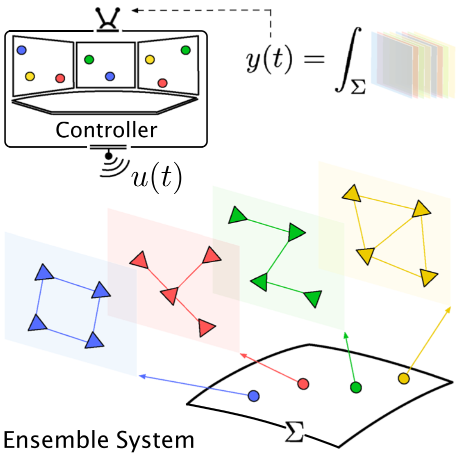

Fig. 1 A continuum ensemble of systems indexed by a parameter of a surface Σ. A controller broadcasts a signal $u(t)$ as a common control input to steer every individual system in the ensemble. Meanwhile, it receives a measurement output $y(t)$ integrating the information of individual states of all the systems

literature is relatively sparse on controllability, and much less on observability. There is also a lack of methodologies for designing nonlinear dynamics of individual control systems so that an ensemble of such systems is controllable and observable. To address the above issues, we introduce in the paper a novel class of nonholonomic ensemble systems, termed *distinguished ensembles*. Every such system has two key components: a set of finely structured control vector fields, termed *distinguished vector fields*, and a set of costructured observations functions, termed *codistinguished functions*. Details about the structure of a distinguished ensemble will be provided below. We will demonstrate that controllability and/or observability of a distinguished ensemble system can be easily fulfilled under some mild assumption. The first half of the paper is devoted to establishing the fact. For the second half, we will investigate the problem about existence of a distinguished ensemble. We focus on the case where the state space of every individual system is a Lie group or its homogeneous space. We leverage existing results [6] and structure theory of Lie algebras to construct explicitly distinguished vector fields and codistinguished functions.

1.1 Mathematical models for ensemble control and estimation

The model of an ensemble system considered in the paper comprises two parts, namely ensemble control and ensemble estimation. We introduce these two parts subsequently.

Model for ensemble control We consider a continuum ensemble of control systems indexed by a parameter $\sigma \in \Sigma$, where Σ is the **parameterization space**. We assume in the paper that Σ is compact, real analytic, and path-connected. We allow Σ to have boundary. If an individual control system in the ensemble is associated with index σ, then we call it **system**-σ. The state space of each individual system is the same, which we denote by M. We assume that M is real analytic. Further, let $x_\sigma(t) \in M$ be the state of system-σ at time t. Then, in general, the control model of an ensemble system can be described by the following differential equation:

$$\dot{x}_\sigma(t) := \frac{\partial x_\sigma(t)}{\partial t} = f(x_\sigma(t), \sigma, u(t)), \quad x_\sigma \in M \text{ for all } \sigma \in \Sigma, \tag{1}$$

where $u(t)$ is a finite-dimensional control input common to *all* of the individual control systems and f is an analytic vector field. Let

$$x_\Sigma(t) := \{x_\sigma(t) \mid \sigma \in \Sigma\}$$

be the collection of system states. One can treat $x_\Sigma(t)$ as a function from Σ to M. We call $x_\Sigma(t)$ a **profile**. Let $\mathrm{C}^\omega(\Sigma, M)$ be the space of real analytic functions from Σ to M. We assume that for any given t, the profile $x_\Sigma(t)$ belongs to $\mathrm{C}^\omega(\Sigma, M)$. We call $\mathrm{C}^\omega(\Sigma, M)$ the **profile space**.

We focus in the paper on a special class of ensemble systems, namely systems such that the vector fields f are separable in state x, the parameter σ, and the control input u. Specifically, we consider the following type of ensemble system:

$$\dot{x}_\sigma(t) = f_0(x_\sigma(t), \sigma) + \sum_{i=1}^{m}\sum_{s=1}^{r} u_{i,s}(t)\rho_s(\sigma) f_i(x_\sigma(t)), \quad x_\sigma \in M \text{ and } \sigma \in \Sigma, \quad (2)$$

where f_0 is a drifting term, the f_i's are **control vector fields** depending only on $x_\sigma(t)$, the ρ_i's are **parameterization functions** defined on Σ, and the $u_{i,s}$'s are scalar control inputs. We assume in the paper that all the vector fields and parameterization functions are analytic in their variables. All the control inputs are integrable functions over any finite time interval. For convenience, we let $u(t)$ be the collection of all the $u_{i,s}(t)$'s.

Model for ensemble estimation We assume that there are l (scalar) measurement outputs $y^j(t)$, for $j = 1, \ldots, l$, at our disposal. Each $y^j(t)$ is a certain average of an observation function $\phi^j(x_\sigma(t))$ over the parameterization space Σ. Specifically, we first let Σ be equipped with a strictly positive Borel measure, i.e., $\int_U \mathrm{d}\sigma > 0$ for any nonempty open subset U of Σ. Next, let each ϕ^j, for $j = 1, \ldots, l$, be an analytic function defined on M. Then, the measurement outputs $\{y^j(t)\}_{j=1}^l$ are described by

$$y^j(t) = \int_\Sigma \phi^j(x_\sigma(t))\mathrm{d}\sigma, \quad j = 1, \ldots, l. \quad (3)$$

For convenience, let $y(t)$ be the collection of the $y^j(t)$'s.

Model for an ensemble system Combining (2) and (3), we arrive at the following mathematical model of an ensemble system:

$$\begin{cases} \dot{x}_\sigma(t) = f_0(x_\sigma(t), \sigma) + \sum_{i=1}^{m}\sum_{s=1}^{r} u_{i,s}(t)\rho_s(\sigma) f_i(x_\sigma(t)), & \forall \sigma \in \Sigma, \\ y^j(t) = \int_\Sigma \phi^j(x_\sigma(t))\mathrm{d}\sigma, & \forall j = 1, \ldots, l. \end{cases} \quad (4)$$

Examples of the above system will be given along the presentation.

1.2 Distinguished structure and examples

A major contribution of the paper is to introduce a novel class of nonholonomic ensemble systems (4), termed *distinguished ensembles*. Every such ensemble system has two key components: a set of distinguished control vector fields $\{f_i\}_{i=1}^m$ and a set of codistinguished observation functions $\{\phi^j\}_{j=1}^l$. Roughly speaking, a set of vector fields $\{f_i\}_{i=1}^m$ is said to be *distinguished* if the Lie bracket of any two vector fields f_i and f_j is, up to scaling, another vector field f_k, i.e., $[f_i, f_j] = \lambda f_k$ for λ a constant, and conversely, any vector field f_k in the set can be obtained in this way. Such a structure is motivated by Li and Khaneja [26] for their earlier study on ensemble control of Bloch equations. Similarly, a set of functions $\{\phi^j\}_{j=1}^l$ is said to be *codistinguished* to the vector fields $\{f_i\}_{i=1}^m$ if the Lie derivative of any ϕ^j along any f_i is, up to scaling, another function ϕ^k, i.e., $f_i\phi^j = \lambda\phi^k$ for λ a constant, and conversely, any function ϕ^k in the set can be obtained in this way (see Definitions 2 and 3, Sect. 3.1 for details).

We note here that although the notion of a "distinguished set" of a Lie algebra appears to be new, such set arises naturally in different areas. Here are a few examples:

(1) When dealing with the rigid motions of a three-dimensional object with a fixed center, we have that the infinitesimal motions of rotations around three axes of an orthonormal frame $\Theta \in SO(3)$ are given by

$$f_1(\Theta) = \Theta\Omega_{23}, \qquad f_2(\Theta) := \Theta\Omega_{31}, \qquad f_3(\Theta) := \Theta\Omega_{12},$$

where each Ω_{ij} is a skew-symmetric matrix with 1 on the ijth entry, -1 on the jith entry, and 0 elsewhere. By computation, $[f_i, f_j] = f_k$ where (i, j, k) is any cyclic rotation of $(1, 2, 3)$. Thus, the above vector fields form a distinguished set.

(2) In quantum mechanics, the Pauli spin matrices are used to represent angular momentum operators. We recall that they are given by

$$\sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 := \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} \quad \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

where i is the imaginary unit. Similarly, if (i, j, k) is a cyclic rotation of $(1, 2, 3)$, then $[\sigma_i, \sigma_j] = 2\mathrm{i}\sigma_k$. Although the constant 2i is not real, one can multiple all the three matrices by i so that the new set $\{\mathrm{i}\sigma_i\}_{i=1}^3$ now satisfies $[\mathrm{i}\sigma_i, \mathrm{i}\sigma_j] = -2\mathrm{i}\sigma_k$. Note that the set $\{\mathrm{i}\sigma_i\}_{i=1}^3$ belongs to $\mathfrak{su}(2)$ i.e., the special unitary Lie algebra. However, we shall note that $\mathfrak{su}(2)$ is isomorphic to $\mathfrak{so}(3)$.

(3) We also note that the ladder operators represented by the following matrices in the special linear Lie algebra $\mathfrak{sl}(2, \mathbb{R})$:

$$H := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad X := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad Y := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

satisfy the desired property: $[H, X] = 2X$, $[H, Y] = -2Y$, and $[X, Y] = H$.

The examples given above demonstrate the existence of distinguished sets in Lie algebras $\mathfrak{so}(3) \approx \mathfrak{su}(2)$ and $\mathfrak{sl}(2, \mathbb{R})$. In fact, we have shown in [6] that every semi-simple *real* Lie algebra has a distinguished set. We review such a fact in Sect. 4.1.

1.3 Literature review

Among related works about controllability of nonlinear ensembles, we first mention [25,26] by Li and Khaneja in which the authors establish the controllability of an ensemble of Bloch equations parameterized by a pair of scalar parameters (σ_1, σ_2) over a square $\Sigma := [a_1, b_1] \times [a_2, b_2]$ in $\mathbb{R}^2$:

$$\dot{x}(t) = (\sigma_1\Omega_{12} + u_1(t)\sigma_2\Omega_{13} + u_2(t)\sigma_2\Omega_{23})x(t).$$

Ensemble control of Bloch equations has also been addressed in [2] using tools from functional analysis. We further note that the controllability of a general ensemble of control-affine systems has been recently addressed in [1], in which the authors established an ensemble version of Rachevsky–Chow theorem via a Lie algebraic method. We do not to intend to reproduce in the paper the results established there, but rather our contribution related to ensemble controllability is to demonstrate that if the set of control vector fields $\{f_i\}_{i=1}^m$ is distinguished, then the ensemble version of Rachevsky-Chow criterion can be easily verified in analysis and fulfilled in system design. For ensemble control of linear systems, we refer the reader to [19,24], [11, Ch. 12] and references therein. We further refer the reader to [4,7,8] for optimal control of probability distributions evolving along linear systems.

Observability of a continuum ensemble system has been mostly addressed within the class of linear systems. We first refer the reader to [11, Ch. 12] where the following ensemble of linear systems is investigated:

$$\dot{x}_\sigma(t) = A(\sigma)x_\sigma(t) \in \mathbb{R}^n, \quad y(t) = \int_\Sigma C(\sigma)x_\sigma(t)\mathrm{d}\sigma \in \mathbb{R}^l.$$

The authors addressed the observability of the above ensemble system using the duality between controllability and observability of infinite-dimensional linear systems [9]. We also refer the reader to [36] for a related observability problem about estimating the probability distribution of the initial state. Specifically, the authors there considered a *single* time-invariant linear system: $\dot{x}(t) = Ax(t) + Bu(t)$ and $y(t) = Cx(t)$. An initial probability distribution p_0 of $x \in \mathbb{R}^n$ induces a distribution $\bar{p}_t$ of $y(t)$ for a given control input $u(t)$. The observability problem addressed there is whether one is able to estimate p_0 given that the entire distributions $\bar{p}_t$ (which are *infinite*-dimensional), for all $t \geq 0$, are known. A key difference between our model (4) and theirs is that we only allow a *finite*-dimensional measurement output $y(t)$. We further refer the reader to [10,12,13,20,34] for the study of observability of a *single* nonlinear system using the so-called observability codistribution.

1.4 Outline of contribution and organization of the paper

The technical contribution of the paper is twofold: (1) We establish a structure theory for controllability and observability of a distinguished ensemble system. (2) We prove the existence of distinguished ensemble systems over semi-simple Lie groups.

Structure theory We establish in Sect. 3 a sufficient condition for controllability and observability of a distinguished (and pre-distinguished) ensemble system. In particular, we demonstrate how distinguished vector fields and codistinguished functions can simplify the analysis and lead to ensemble controllability and observability. The structure theory established in the paper also provides a solution to the problem of ensemble system design—i.e., the problem of codesigning the control vector fields f_i's, the observations functions ϕ^j's, and the parameterization functions ρ_s's so that system (4) is controllable and/or observable. In particular, it divides the problem into two independent subproblems—one is about finding a set of distinguished vector fields $\{f_i\}_{i=1}^m$ and a set of codistinguished function $\{\phi^j\}_{j=1}^l$ over the given manifold M while the other is about finding a set of parameterization functions $\{\rho_s\}_{s=1}^r$ that separates points of the parameterization space Σ.

Existence of distinguished ensembles We prove in Sect. 4 that every semi-simple Lie group G admits a set of distinguished vector fields, together with a set of codistinguished functions. The proof of the existence result is constructive: (1) For distinguished vector fields, we leverage the result established in [6] where we have shown how to construct a distinguished set on the Lie algebra level. We then identify the distinguished set with the corresponding set of left- (or right-) invariant vector fields over the group G. (2) For codistinguished functions, we show how to generate these functions using representation theory. In particular, we show in Sect. 4.2 that a selected set of matrix coefficients associated with a finite-dimensional Lie group representation could be used as a set of codistinguished functions (with respect to a set of left-invariant vector fields). Then, in Sect. 4.3, we focus on a special representation, namely the adjoint representation. We show, in this case, that there indeed exists a set of matrix coefficients as codistinguished functions. In particular, if G is a matrix Lie group, then these matrix coefficients are simply given by $\phi^{ij}(g) = \mathrm{tr}(gX_j g^{-1} X_i^\top)$ where X_i and X_j are selected matrices out of the Lie algebra $\mathfrak{g}$ of G. We further address, in Sect. 4.5, the existence problem for homogeneous spaces.

We provide key definitions and notations in Sect. 2 and conclusions at the end.

2 Definitions and notations

(1) Manifolds Let M be a real analytic manifold. For a point $x \in M$, let $T_x M$ be the tangent space and $T_x^* M$ be the cotangent space of M at x. Let $TM := \cup_{x\in M} T_x M$ be the tangent bundle and $T^*M := \cup_{x\in M} T_x^* M$ be the cotangent bundle.

Let $\mathrm{C}^\omega(M)$ be the set of real analytic functions on M. Denote by $\mathbf{1}_M \in \mathrm{C}^\omega(M)$ the constant function whose value is 1 everywhere. Let $\mathfrak{X}(M)$ be the set of real analytic vector fields over M. Let $\phi \in \mathrm{C}^\omega(M)$ and $f \in \mathfrak{X}(M)$. Denote by $f\phi \in \mathrm{C}^\omega(M)$ the Lie derivative of ϕ along f. If we embed M into a Euclidean space, then $f\phi$ is simply given by

$$(f\phi)(x) := \lim_{\epsilon\to 0} \frac{\phi(x + \epsilon f(x)) - \phi(x)}{\epsilon}, \quad \forall x \in M.$$

For any $\phi \in \mathrm{C}^\omega(M)$, we let $\mathrm{d}\phi \in T^*M$ be a one-form defined as follows: Let $\mathrm{d}\phi_x \in T_x^*M$ be the evaluation of $\mathrm{d}\phi$ at x. Then, for any $f \in \mathfrak{X}(M)$, we have that $\mathrm{d}\phi_x(f(x)) = (f\phi)(x)$. For two vector fields $f_i, f_j \in \mathfrak{X}(M)$, we let $[f_i, f_j]$ be the Lie bracket, which is defined such that $[f_i, f_j]\phi = f_i f_j \phi - f_j f_i \phi$ for all $\phi \in \mathrm{C}^\omega(M)$.

Let $\{f_i\}_{i=1}^m$ be a subset of $\mathfrak{X}(M)$. Let $\mathrm{w} = w_1 \cdots w_k$ be a *word* over the alphabet $\{1, \ldots, m\}$ of length k. For a function $\phi \in \mathrm{C}^\omega(M)$, we define $f_\mathrm{w}\phi := f_{w_1} \cdots f_{w_k}\phi$. If $\mathrm{w} = \varnothing$, i.e., an empty word (of zero length), then we set $f_\mathrm{w}\phi := \phi$.

Let $\eta : M \to N$ be a diffeomorphism. Denote by $\eta_* : TM \to TN$ the derivative of η. For a vector field $f \in \mathfrak{X}(M)$, let $\eta_* f \in \mathfrak{X}(N)$ be the *pushforward* defined as $(\eta_* f)(y) := \eta_*(f(\eta^{-1} y))$ for all $y \in N$. For a function $\phi \in \mathrm{C}^\omega(N)$, let $\eta^*\phi \in \mathrm{C}^\omega(M)$ be the *pullback* defined as $(\eta^*\phi)(x) := \phi(\eta(x))$ for all $x \in M$.

(2) Algebra of functions Let Σ be an analytic, compact manifold and $\{\rho_s\}_{s=1}^r$ be a set of real-valued functions on Σ. For any $k \geq 0$, let $\rho_s^k(\sigma) := \rho_s(\sigma)^k$. Note, in particular, that $\rho_s^0 = \mathbf{1}_\Sigma$. If ρ_s is everywhere nonzero, then ρ_s^k is defined for all $k \in \mathbb{Z}$. We call $\prod_{s=1}^r \rho_s^{k_s}$, for $k_s \geq 0$, a *monomial*. Its degree is defined by $k := \sum_{s=1}^r k_s$. Let $\mathcal{P}$ be the collection of all monomials. We decompose $\mathcal{P} = \sqcup_{k\geq 0}\mathcal{P}(k)$, where $\mathcal{P}(k)$ is comprised of monomials of degree k. Denote by $\mathcal{S}$ the subalgebra generated by the set of functions $\{\rho_s\}_{s=1}^r$. It is defined such that if $\mathrm{p} \in \mathcal{S}$, then p can be expressed as a linear combination of a finite number of monomials with real coefficients.

(3) Lie groups and Lie algebras Let G be a Lie group with e the identity element. Let $\mathfrak{g}$ be the associated Lie algebra, and $[\cdot,\cdot]$ be the Lie bracket. We identify each element $X \in \mathfrak{g}$ with a *left-invariant vector field* L_X over G, i.e., $L_X(g) = gX$ for any $g \in G$. Thus, $L_{[X,Y]} = [L_X, L_Y]$. Note that to each $X \in \mathfrak{g}$, there also corresponds a right-invariant vector field R_X. For any $X, Y \in \mathfrak{g}$, we have $R_{[X,Y]} = -[R_X, R_Y]$.

A *subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$ is a vector subspace closed under Lie bracket, i.e., $[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{h}$. An *ideal* $\mathfrak{i}$ of $\mathfrak{g}$ is a subalgebra such that $[\mathfrak{i},\mathfrak{g}] \subseteq \mathfrak{i}$. We say that $\mathfrak{g}$ is *simple* if it is not abelian and, moreover, the only ideals of $\mathfrak{g}$ are 0 and itself. Simple real Lie algebras have been completely classified (up to isomorphism) by Élie Cartan. A complete list of (non-complex) simple real Lie algebras can be found in [22, Thm. 6.105]. A *semi-simple* Lie algebra is a direct sum of simple Lie algebras. A *Cartan subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$ is maximal among the abelian subalgebras $\mathfrak{h}'$ of $\mathfrak{g}$ such that the adjoint representation $\mathrm{ad}(X)(\cdot) := [X,\cdot]$ is simultaneously diagonalizable (over $\mathbb{C}$) for all $X \in \mathfrak{h}'$.

(4) Representation Let V be a finite-dimensional vector space over $\mathbb{R}$. Let $\mathrm{Aut}(V)$ and $\mathrm{End}(V)$ be the sets of automorphisms and endomorphisms of V, respectively. A *representation* π of G on V is a group homomorphism $\pi : G \to \mathrm{Aut}(V)$, i.e., $\pi(e)$ is the identity map and $\pi(gh) = \pi(g)\pi(h)$.

Let $\langle\cdot,\cdot\rangle$ be an inner product on V. We say that the representation π is C^k (i.e., kth continuously differentiable) if the map $\pi : (g, v) \in G \times V \mapsto \pi(g)v \in V$ is C^k. A *matrix coefficient* is any C^k-function on G defined as $\langle v_i, \pi(g)v_j\rangle$ where v_i, v_j belong to V. In particular, if the v_i's form an orthonormal basis of V, then $\langle v_i, \pi(g)v_j\rangle$ is exactly the ijth entry of the matrix $\pi(g)$ with respect to the given basis.

A group representation π induces a Lie algebra homomorphism $\pi_* : \mathfrak{g} \to \mathrm{End}(V)$, where π_* is the derivative of π at the identity $e \in G$. It satisfies the following condition:

$$\pi_*([X,Y]) = \pi_*(X)\pi_*(Y) - \pi_*(Y)\pi_*(X), \quad \forall X, Y \in \mathfrak{g}.$$

We call π_* a *representation* of $\mathfrak{g}$ on V, or simply a Lie algebra representation.

Let $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ be the *adjoint representation*, i.e., for each $g \in G$, $\mathrm{Ad}(g) : T_eG \to T_eG$ is the derivative of the conjugation $h \in G \mapsto ghg^{-1} \in G$ at the identity e. Denote by $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ the induced Lie algebra representation of Ad, which is given by $\mathrm{ad}(X)(\cdot) = [X,\cdot]$ for all $X \in \mathfrak{g}$.

(5) Lie products Let $A := \{X_1, \ldots, X_k\}$ be a set of free generators. Let $\mathcal{L}_A$ be the collection of formal Lie products of the X_i's in A. For a given element $\xi \in \mathcal{L}_A$, we let $\mathrm{dep}(\xi)$ be the *depth* of ξ defined as the number Lie brackets in ξ. For example, the depth of $[X_{i_1}, [X_{i_2}, X_{i_3}]]$ is 2. We further decompose $\mathcal{L}_A = \sqcup_{k\geq 0}\mathcal{L}_A(k)$ where $\mathcal{L}_A(k)$ is comprised of Lie products of depth k.

(6) Miscellaneous Let $\{e_i\}_{i=1}^n$ be the standard basis of $\mathbb{R}^n$. We denote by $\det(e_{i_1}, \ldots, e_{i_n})$ the determinant of a matrix whose jth column is e_{i_j} for $i_j \in \{1, \ldots, n\}$.

Let V be a vector space over $\mathbb{R}$. We denote by V^* the dual space, i.e., it is the collection of all linear functions from V to $\mathbb{R}$.

The following definition will be frequently used throughout the paper:

Definition 1 Two subsets V' and V'' of a real vector space V are said to be **projectively identical** if for any $v' \in V'$, there exists a $v'' \in V''$ and a constant $c \in \mathbb{R}$ such that $v' = cv''$, and vice versa. We write $V' \equiv V''$ to indicate such equivalence relation.

Let S be an arbitrary set with an operation "$*$" defined so that $s_1 * s_2$ belongs to S for all $s_1, s_2 \in S$. For any two subsets S' and S'' of S, we let $S' * S''$ be the subset of S comprised of the elements $s' * s''$ for all $s' \in S'$ and $s'' \in S''$. Here are two examples in which such a notation will be used: (i) If S is a vector space and "$*$" is the addition "$+$", then we write $S' + S''$. (ii) If S is the commutative algebra of analytic functions $C^\omega(\Sigma)$ and "$*$" is the pointwise multiplication, then we simply write $S'S''$.

However, we note that the above notation does not apply to $[\mathfrak{g}_1, \mathfrak{g}_2]$ for $\mathfrak{g}_1$ and $\mathfrak{g}_2$ two subsets of a Lie algebra $\mathfrak{g}$. By convention, $[\mathfrak{g}_1, \mathfrak{g}_2]$ is the *linear span* of all $[X_1, X_2]$ with $X_1 \in \mathfrak{g}_1$ and $X_2 \in \mathfrak{g}_2$. We adopt such a convention in the paper as well.

For a general control system $\dot{x}(t) = f(x(t), u(t))$, we denote by $u[0, T]$ the control input $u(t)$ over the time interval $[0, T]$ for $T > 0$. Correspondingly, we let $x[0, T]$ be the trajectory of the control system generated by $u[0, T]$.

3 Distinguished ensemble systems

3.1 Distinguished vector fields and codistinguished functions

We introduce in the section the class of (pre-)distinguished ensemble systems and establish controllability and observability of any such ensemble system. We start by introducing two key components of the system, namely distinguished vector fields and codistinguished functions. We first have the following definition:

Definition 2 (*Distinguished vector fields*) A set of vector fields $\{f_i\}_{i=1}^m$ over an analytic manifold M is **distinguished** if the following hold:

(1) For any $x \in M$, the set $\{f_i(x)\}_{i=1}^m$ spans $T_x M$.
(2) For any two f_i and f_j, there exist an f_k and a real number λ such that

$$[f_i, f_j] = \lambda f_k; \tag{5}$$

conversely, for any f_k, there exist f_i and f_j and a *nonzero* λ such that (5) holds.

Recall that $\mathfrak{X}(M)$ is the Lie algebra of analytic vector fields over M, which is infinite-dimensional. However, if $F := \{f_i\}_{i=1}^m$ is distinguished, then by item 2 of Definition 2, the $\mathbb{R}$-span of the f_i's, which we denote by $\mathbb{L}_F$, is a *finite-dimensional* subalgebra of $\mathfrak{X}(M)$. We note here that $\mathbb{L}_F$ is *perfect*, i.e., $[\mathbb{L}_F, \mathbb{L}_F] = \mathbb{L}_F$.

Let N be any manifold diffeomorphic to M, and $\eta : M \to N$ be the diffeomorphism. Recall that for a vector field f over M, we denote by $\eta_* f$ the pushforward of f as a vector field over N. We have the following fact:

Lemma 1 *If* $\{f_i\}_{i=1}^m$ *is distinguished over* M*, then* $\{\eta_* f_i\}_{i=1}^m$ *is distinguished over* N.

Proof If $[f_i, f_j] = \lambda f_k$, then $[\eta_* f_i, \eta_* f_j] = \eta_*[f_i, f_j] = \lambda \eta_* f_k$. □

We next introduce the definition of codistinguished functions:

Definition 3 (*Codistinguished functions*) A set of functions $\{\phi^j\}_{j=1}^l$ on M is **codistinguished** to a set of vector fields $\{f_i\}_{i=1}^m$ if the following hold:

(1) For any $x \in M$, the set of (exact) one-forms $\{\mathrm{d}\phi_x^j\}$ spans $T_x^* M$.
(2) For any f_i and any ϕ^j, there exist a ϕ^k and a real number λ such that

$$f_i \phi^j = \lambda \phi^k; \tag{6}$$

conversely, for any ϕ^k, there exist f_i, ϕ^j, and a *nonzero* λ such that (6) holds.
(3) For $x, x' \in M$, if $\phi^j(x) = \phi^j(x')$ for all $j = 1, \ldots, l$, then $x = x'$.

If $\{\phi^j\}_{j=1}^l$ satisfies only (1) and (2), then it is **weakly codistinguished** to $\{f_i\}_{i=1}^m$.

Let $\tilde{\eta} : N \to M$ be a diffeomorphism. Recall that for a function ϕ on M, we denote by $\tilde{\eta}^*\phi$ the pullback of ϕ as a function on N. We have the following fact:

Lemma 2 *If* $\{\phi^j\}_{j=1}^l$ *on* M *is codistinguished to* $\{f_i\}_{i=1}^m$*, then* $\{\tilde{\eta}^*\phi^j\}_{j=1}^l$ *on* N *is codistinguished to* $\{\tilde{\eta}_*^{-1} f_i\}_{i=1}^m$.

Proof If $f_i \phi^j = \lambda \phi^k$, then $(\tilde{\eta}_*^{-1} f_i)(\tilde{\eta}^*\phi^j) = \tilde{\eta}^*(f_i \phi^j) = \lambda \tilde{\eta}^* \phi^k$. □

We say that a set of vector fields $F := \{f_i\}_{i=1}^m$ and a set of functions $\Phi := \{\phi^j\}_{j=1}^l$ are *(weakly) jointly distinguished* if F is distinguished and Φ is (weakly) codistinguished to F. Note that Lemmas 1 and 2 imply that the property of having a set of (weakly) jointly distinguished pair (F, Φ) is topologically invariant. Let F and Φ be (weakly) disjoined. Recall that $\mathbb{L}_F$ is a finite-dimensional Lie algebra spanned by F (since F is distinguished). Let $\mathbb{L}_\Phi$ be the $\mathbb{R}$-span of Φ. Then, by the second item of Definition 3, the following map:

$$(f, \phi) \in \mathbb{L}_F \times \mathbb{L}_\Phi \mapsto f\phi \in \mathbb{L}_\Phi$$

is a finite-dimensional Lie algebra representation of $\mathbb{L}_F$ on $\mathbb{L}_\Phi$.

For the remainder of the subsection, we provide an example about jointly distinguished vector fields and functions on SO(3). These vector fields and functions will be further generalized in Sect. 4 so that they exist on *any* semi-simple Lie group.

Example 1 Let SO(3) be the matrix Lie group of 3×3 special orthogonal matrices, and $\mathfrak{so}(3)$ be the associated Lie algebra. We define a basis $\{X_i\}_{i=1}^3$ of $\mathfrak{so}(3)$ as follows:

$$X_i := e_j e_k^\top - e_k e_j^\top \quad \text{where } \det(e_i, e_j, e_k) = 1, \quad \forall i = 1, 2, 3.$$

Let $\{L_{X_i}\}_{i=1}^3$ be the corresponding left-invariant vector fields. By computation,

$$[L_{X_i}, L_{X_j}] = -\det(e_i, e_j, e_k) L_{X_k}, \quad \forall i \neq j. \tag{7}$$

It follows that $\{L_{X_i}\}_{i=1}^3$ is distinguished.

Denote by $\mathrm{tr}(\cdot)$ the trace of a square matrix. We next define functions $\{\phi^{ij}\}_{i,j=1}^3$ on SO(3) as follows:

$$\phi^{ij}(g) := \mathrm{tr}(gX_j g^\top X_i^\top), \quad 1 \le i, j \le 3.$$

We show below that $\{\phi^{ij}\}_{i,j=1}^3$ is codistinguished to $\{L_{X_i}\}_{i=1}^3$. First, for any left-invariant vector field L_X with $X \in \mathfrak{so}(3)$, we obtain by computation that

$$\mathrm{d}\phi_g^{ij}(L_X(g)) = (L_X\phi^{ij})(g) = \mathrm{tr}(g[X, X_j]g^\top X_i^\top). \tag{8}$$

We now prove that the three items of Definition 3 are satisfied for $\{\phi^{ij}\}_{i,j=1}^3$ and $\{L_{X_i}\}_{i=1}^3$:

(1) We fix an arbitrary group element $g \in \mathrm{SO}(3)$ and show that $\{\mathrm{d}\phi_g^{ij}\}_{i,j=1}^3$ spans $T_g^*\mathrm{SO}(3)$. For convenience, let $\hat{X}_{ij} := [X_j, g^\top X_i^\top g]$. Then, by (8), we obtain that

$$\mathrm{d}\phi_g^{ij}(L_X(g)) = \mathrm{tr}(X[X_j, g^\top X_i^\top g]) = \mathrm{tr}(X\hat{X}_{ij}).$$

Note that $\mathrm{tr}(\cdot,\cdot)$ is negative definite on $\mathfrak{so}(3)$. Thus, $\{\mathrm{d}\phi_g^{ij}\}_{i,j=1}^3$ spans $T_g^*\mathrm{SO}(3)$ if and only if $\{\hat{X}_{ij}\}_{i,j=1}^3$ spans $\mathfrak{so}(3)$. It now suffices to show that $\{\hat{X}_{ij}\}_{i,j=1}^3$ spans $\mathfrak{so}(3)$. But, this holds because both $\{X_j\}_{j=1}^3$ and $\{g^\top X_i^\top g\}_{i=1}^3$ span $\mathfrak{so}(3)$. Moreover, $\mathfrak{so}(3)$ is simple so that $[\mathfrak{so}(3), \mathfrak{so}(3)] = \mathfrak{so}(3)$.

(2) For the second item, we combine (7) and (8) to obtain the following:

$$L_{X_i}\phi^{i'j} = \begin{cases} -\det(e_i, e_j, e_k)\phi^{i'k}, & \text{if } i \ne j, \\ 0, & \text{otherwise.} \end{cases}$$

(3) Finally, let g and g' be such that $\phi^{ij}(g) = \phi^{ij}(g')$ for all $1 \le i, j \le 3$:

$$\mathrm{tr}(gX_j g^\top X_i^\top) = \mathrm{tr}(g'X_j g'^\top X_i^\top), \quad \forall i = 1, 2, 3.$$

Because $\{X_i\}_{i=1}^3$ spans $\mathfrak{so}(3)$ and $\mathrm{tr}(\cdot,\cdot)$ is negative definite on $\mathfrak{so}(3)$, we have that $gX_jg^\top = g'X_jg'^\top$. Since this holds for all X_j, it follows that $g^\top g'$ belongs to the center of SO(3). But the center is trivial. We thus conclude that $g = g'$. □

3.2 Controllability and observability of distinguished ensemble system

We establish in the subsection a sufficient condition for controllability and observability of ensemble system (4). For convenience, we reproduce below the mathematical model of the ensemble system introduced in Sect. 1:

$$\begin{cases} \dot{x}_\sigma(t) = f_0(x_\sigma(t), \sigma) + \sum_{i=1}^m \sum_{s=1}^r u_{i,s}(t)\rho_s(\sigma) f_i(x_\sigma(t)), & \forall \sigma \in \Sigma, \\ y^j(t) = \int_\Sigma \phi^j(x_\sigma(t))\mathrm{d}\sigma, & \forall j = 1, \dots, l. \end{cases} \tag{9}$$

The common state space M is an analytic manifold, equipped with a Riemannian metric. We denote by $\mathrm{d}_M(x_1, x_2)$ the distance between two points x_1 and x_2 in M. The parameterization space Σ is analytic, compact, and path-connected. It is equipped with a strictly positive measure. All vector fields and parameterization functions are analytic. For any $T > 0$, the control inputs $u_{i,s} : [0, T] \to \mathbb{R}$ are integrable functions. We denote by $u(t)$ (resp. $y(t)$) the collection of $u_{i,s}(t)$ (resp. $y^j(t)$).

We recall that $x_\Sigma(t)$ is the profile of system (9) at time t, which can be viewed as an analytic function Σ to M. We also recall that $\mathrm{C}^\omega(\Sigma, M)$ is the profile space. Now, let $x_\Sigma[0, T]$ be the collection of trajectories of individual systems:

$$x_\Sigma[0, T] := \{x_\sigma[0, T] \mid \sigma \in \Sigma\}.$$

We call $x_\Sigma[0, T]$ a *trajectory of profiles*. We assume in the paper that $x_\Sigma[0, T]$ is continuous in time t. We now have the following definition for ensemble controllability.

Definition 4 (*Ensemble controllability*) System (9) is **approximately ensemble path-controllable** if for any initial profile $x_\Sigma(0)$, any target trajectory of profiles $\hat{x}_\Sigma[0, T]$ of class C^1 with $\hat{x}_\Sigma(0) = x_\Sigma(0)$, and any error tolerance $\epsilon > 0$, there is a control input $u(t)$ such that the trajectory $x_\Sigma[0, T]$ generated by $u(t)$ satisfies

$$\mathrm{d}_M(x_\sigma(t), \hat{x}_\sigma(t)) < \epsilon, \quad \forall (t, \sigma) \in [0, T] \times \Sigma.$$

If, further, the control input $u(t)$ can always be of class C^k, then system (9) is approximately ensemble path-controllable under C^k-inputs.

Remark 1 The continuity of $\hat{x}_\Sigma[0, T]$ implies that any two profiles $\hat{x}_\Sigma(t_1)$ and $\hat{x}_\Sigma(t_2)$, for $t_1, t_2 \in [0, T]$, are homotopic. Thus, the above definition concerns about capability of approximating a target trajectory of profiles within a homotopy class. In general, there may exist multiple homotopy classes. For example, if $\Sigma = S^n$, then all the homotopy classes of continuous functions from S^n to M form the so-called nth homotopy group [17]. If, further, $M = S^n$, then the group is known to be $\mathbb{Z}$. The above arguments imply that given an initial profile $x_\Sigma(0)$ and a target profile $\hat{x}_\Sigma(T)$, there may not exist a continuous trajectory of profiles that connects $x_\Sigma(0)$ and $\hat{x}_\Sigma(T)$. □

We next introduce the definition for ensemble observability. To proceed, we first have the following one about output equivalence, which straightforwardly generalizes the notion for a *single* nonlinear control system (see, for example, [20]):

Definition 5 (*Output equivalence*) Two initial profiles $x_\Sigma(0)$ and $\bar{x}_\Sigma(0)$ of system (9) are **output equivalent**, which we denote by $x_\Sigma(0) \sim \bar{x}_\Sigma(0)$, if for any $T > 0$ and any integrable function $u : [0, T] \to \mathbb{R}^m$ as a control input, the following holds:

$$\int_\Sigma \phi^j(x_\sigma(t))\mathrm{d}\sigma = \int_\Sigma \phi^j(\bar{x}_\sigma(t))\mathrm{d}\sigma,$$

for all $t \in [0, T]$ and for all $j = 1, \dots, l$.

For a given $x_\Sigma(0)$, we let $O(x_\Sigma(0))$ be the collection of all initial profiles in $\mathrm{C}^\omega(\Sigma, M)$ that are output equivalent to $x_\Sigma(0)$, i.e.,

$$O(x_\Sigma(0)) := \{\bar{x}_\Sigma(0) \mid \bar{x}_\Sigma(0) \sim x_\Sigma(0)\}. \quad (10)$$

The set $O(x_\Sigma(0))$ can be viewed as a "measure of ambiguity" for the ensemble estimation problem. With the above definition of output equivalence, we now introduce the definition of ensemble observability.

Definition 6 (*Ensemble observability*) System (9) is **weakly ensemble observable** if for any profile $x_\Sigma(0)$, there is an $\epsilon > 0$ such that if $\bar{x}_\Sigma(0) \sim x_\Sigma(0)$ and $\bar{x}_\Sigma(0) \neq x_\Sigma(0)$, then $\mathrm{d}_M(x_\sigma(0), \bar{x}_\sigma(0)) \geq \epsilon$ for all $\sigma \in \Sigma$. Further, system (9) is **ensemble observable** if for any profile $x_\Sigma(0)$, the set $O(x_\Sigma(0)) = \{x_\Sigma(0)\}$ is a singleton.

We establish below a sufficient condition for ensemble controllability and observability of system (9). To state the condition, we need a few more preliminaries.

First, we say that the set of parameterization functions $\{\rho_s\}_{s=1}^r$ defined on Σ is a *separating set* if for any two distinct points $\sigma, \sigma' \in \Sigma$, there exists a function ρ_s, for some $s \in \{1, \ldots, r\}$, such that $\rho_s(\sigma) \neq \rho_s(\sigma')$. Note that by Stone–Weierstrass theorem [30, Chp. 7], if $\{\rho_s\}_{s=1}^r$ separates point and contains an everywhere nonzero function, then the subalgebra generated by $\{\rho_s\}_{s=1}^r$ is dense in the space $\mathrm{C}^0(\Sigma)$ of continuous functions on Σ.

Next, for convenience, we let $\phi := (\phi^1, \ldots, \phi^l)$ be a vector-valued function on M. For a given $x \in M$, we let $[x]_\phi$ be the pre-image of $\phi(x)$, i.e., $[x]_\phi$ is the collection of all points x' in M such that $\phi(x') = \phi(x)$. Note that if the set of one-forms $\{\mathrm{d}\phi_x^j\}_{j=1}^l$ spans T_x^*M for all $x \in M$, then $[x]_\phi$ is a discrete set. Let χ_ϕ be defined as follows:

$$\chi_\phi := \sup_{x \in M} \left|[x]_\phi\right|.$$

If χ_ϕ is unbounded, then we set $\chi_\phi := \infty$. We have the following fact:

Lemma 3 *If M is compact and the one-forms $\{\mathrm{d}\phi_x^j\}_{j=1}^l$ span T_x^*M for all $x \in M$, then χ_ϕ is a finite number.*

Proof First, note that for any $x \in M$, $|[x]_\phi|$ is a finite number because otherwise $[x]_\phi$ contains an accumulation point x_* and the one-forms $\{\mathrm{d}\phi_{x_*}^j\}_{j=1}^l$ *cannot* span $T_{x_*}^*M$. In fact, since the one-forms $\{\mathrm{d}\phi_x^j\}_{j=1}^l$ span T_x^*M for all $x \in M$, there is an open ball $B_{\epsilon(x)}(x)$ centered at x with radius $\epsilon(x)$ such that $|[x']_\phi| = |[x]_\phi|$ for all $x' \in B_{\epsilon(x)}(x)$. The collection of open balls $\{B_{\epsilon(x)}(x)\}_{x \in M}$ is an open cover of M. Since M is compact, there is a finite subcover $\{B_{\epsilon(x_i)}(x_i)\}_{i=1}^N$. It then follows that $\chi_\phi := \max_{i=1}^N |[x_i]_\phi|$. ☐

We are now in a position to state the first main result of the paper. The result establishes connections between the "distinguished" structure introduced in the previous subsection and ensemble controllability/observability of system (9):

Theorem 1 *Consider ensemble system (9). Suppose that $\{\rho_s\}_{s=1}^r$ is a separating set and contains an everywhere nonzero function; then, the following hold:*

(1) *If the set of control vector fields $\{f_i\}_{i=1}^m$ is distinguished, then system (9) is approximately ensemble path-controllable under C^1-inputs.*

(2) *If the set of observation functions $\{\phi^j\}_{j=1}^l$ is (weakly) codistinguished to $\{f_i\}_{i=1}^m$, then system (9) is (weakly) ensemble observable. If, further, M is compact, then for any $x_\Sigma(0)$, the set $O(x_\Sigma(0))$ defined in (10) is finite and $|O(x_\Sigma(0))| \leq \chi_\phi$.*

Following the above theorem, we introduce the following definition:

Definition 7 An ensemble system (9) is **distinguished** if (1) the set of parameterization functions $\{\rho_s\}_{s=1}^r$ separates points and contains an everywhere nonzero function, and (2) the set of control vector fields $\{f_i\}_{i=1}^m$ and the set of observation functions $\{\phi^j\}_{j=1}^l$ are (weakly) jointly distinguished.

By Theorem 1, a distinguished ensemble system is approximately ensemble path-controllable and (weakly) ensemble observable. We provide below an example of a distinguished ensemble system:

Example 2 Recall that in Example 1, we have introduced jointly distinguished left-invariant vector fields $\{L_{X_i}\}_{i=1}^3$ and functions $\{\mathrm{tr}(gX_j g^\top X_i^\top)\}_{i,j=1}^3$ on SO(3). Now, consider a continuum ensemble of control systems defined on SO(3), parameterized by a scalar parameter σ over a closed interval $[a, b]$ with $0 < a < b$. Let $\rho(\sigma) := \sigma$ be the parameterization function. The singleton $\{\rho\}$ is a separating set and ρ is everywhere nonzero. Thus, the following ensemble system is distinguished:

$$\begin{cases} \dot{g}_\sigma(t) = f_0(g_\sigma(t), \sigma) + \sum_{i=1}^3 u_i(t)\sigma L_{X_i}(g_\sigma(t)), & \sigma \in [a, b], \\ y^{ij}(t) = \int_\Sigma \mathrm{tr}(g_\sigma(t) X_j g_\sigma^\top(t) X_i^\top)\mathrm{d}\sigma, & 1 \leq i, j \leq 3. \end{cases}$$

Thus, it is approximately ensemble path-controllable and ensemble observable. □

We have the following remark on the set of parameterization functions:

Remark 2 For any analytic manifold Σ, there exists a set of separating set. By the Nash embedding theorem [15,29], the manifold Σ can be isometrically embedded into a Euclidean space $\mathbb{R}^N$. We write $\sigma = (\sigma_1, \ldots, \sigma_N)$ as the coordinate of a point $\sigma \in \Sigma$. Now, let $\rho_s(\sigma) := \sigma_s$, for $s = 1, \ldots, N$, be the standard coordinate functions (more precisely, the restrictions of the coordinate functions to Σ). Further, let $\rho_{N+1} := \mathbf{1}_\Sigma$ be the unit function. Then, $\{\rho_s\}_{s=1}^{N+1}$ satisfies the assumption of Theorem 1. □

We establish Theorem 1. The proof will be divided into two parts: We deal with ensemble controllability and ensemble observability separately. The proofs will be given in Sects. 3.3 and 3.4, respectively.

3.3 Proof of approximate ensemble path controllability

We establish here the first item of Theorem 1. The proof relies on the use of the technique of Lie extension, the structure of distinguished vector fields, and the Stone-Weierstrass theorem. We provide details below.

3.3.1 On the use of Lie extension and distinguished vector fields

Recall that for an arbitrary *single* control-affine system:

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t)), \tag{11}$$

the first-order Lie extension of the system is a new control-affine system given by

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t)) + \sum_{i,j=1}^{m} u_{ij}(t)[f_i, f_j](x(t)).$$

By repeatedly applying Lie extensions, we obtain a family of control-affine systems with an increasing number of control vector fields. All of these control vector fields can be expressed as Lie products involving the f_i's in (11). We make the statement precise below. First, for the given set of vector fields $F := \{f_i\}_{i=1}^m$, we use $\mathcal{L}_F$ to denote the collection of Lie products generated by F in which the f_i's are treated as if they were "free" generators. For ease of notation, we will simply write $\mathcal{L}$ by omitting the subindex F. Decompose $\mathcal{L} := \sqcup_{k\geq 0}\mathcal{L}(k)$ where each $\mathcal{L}(k)$ is comprised of Lie products of depth k. Then, the kth-order Lie extension of (11) is a control-affine system given by

$$\dot{x}(t) = f_0(x(t)) + \sum_{l=0}^{k} \sum_{\xi\in\mathcal{L}(l)} u_\xi(t)\xi(x(t)). \tag{12}$$

By increasing the order k, we obtain an infinite family of Lie extended systems. Lie extension has been used in [23,28,32] for nonholonomic motion planning.

It is known that original control-affine system (11) is approximately path-controllable if and only if any of its Lie extended systems is. Specifically, we let $u^*(t)$ be the collection of control inputs $u_\xi(t)$ of Lie extended system (12). The following fact is established in [27,32] by Sussmann and Liu:

Lemma 4 *Given any order k of Lie extension and any control input $u^*[0, T]$ of class C^1 for system (12), there exist a sequence of control inputs $\{u^{(j)}[0, T]\}_{j=1}^\infty$ of class C^1 for original system (11) such that the trajectory generated by $u^{(j)}$ converges uniformly to the trajectory of system (12) generated by u^* over $[0, T]$.*

Remark 3 We note here that the above result is "formal" in a sense that the control sequence $\{u^{(j)}\}_{j=1}^\infty$ depends only on u^* but *not* on the vector fields f_i [27,32]—if one replaces f_i with any other sufficiently smooth vector fields g_i, then the *same* control sequence $\{u^{(j)}[0, T]\}_{j=1}^\infty$ can still be used to obtain the convergence result. □

We now apply the technique of Lie extension to ensemble system (9). For convenience, we reproduce below the control part of the system:

$$\dot{x}_\sigma(t) = f_0(x_\sigma(t), \sigma) + \sum_{i=1}^{m}\sum_{s=1}^{r} u_{i,s}(t)\rho_s(\sigma) f_i(x_\sigma(t)), \quad \forall\sigma \in \Sigma. \tag{13}$$

In this case, we have that for any individual system-σ, the control vector fields are $\rho_s(\sigma) f_i(x_\sigma)$, for $1 \leq s \leq r$ and $1 \leq i \leq m$. Note that the Lie bracket of any two of these control vector fields is given by $[\rho_s(\sigma) f_i, \rho_{s'}(\sigma) f_j] = \rho_s(\sigma)\rho_{s'}(\sigma)[f_i, f_j]$. Thus, the first-order Lie extension of (13) is given by

$$\begin{aligned} \dot{x}_\sigma(t) = f_0(x_\sigma(t), \sigma) &+ \sum_{i=1}^{m}\sum_{s=1}^{r} u_{i,s}(t)\rho_s(\sigma) f_i(x_\sigma(t)) \\ &+ \sum_{i,j=1}^{m}\sum_{s,s'=1}^{r} u_{ij,ss'}(t)\rho_s(\sigma)\rho_{s'}(\sigma)[f_i, f_j](x_\sigma(t)), \quad \forall \sigma \in \Sigma. \end{aligned}$$

The last term of the above expression can be simplified as follows:

$$\sum_{\xi \in \mathcal{L}(1)} \sum_{\mathrm{p} \in \mathcal{P}(2)} u_{\xi,\mathrm{p}}(t)\mathrm{p}(\sigma)\xi(x_\sigma(t)),$$

where $\mathcal{P}(2)$ is the collection of monomials $\rho_s\rho_{s'}$ of degree 2. In general, we obtain the following kth-order Lie extension of (13):

$$\dot{x}_\sigma(t) = f_0(x_\sigma(t), \sigma) + \sum_{l=0}^{k} \sum_{\xi \in \mathcal{L}(l)} \sum_{\mathrm{p} \in \mathcal{P}(l+1)} u_{\xi,\mathrm{p}}(t)\mathrm{p}(\sigma)\xi(x_\sigma(t)), \quad \forall \sigma \in \Sigma. \tag{14}$$

Recall that two arbitrary sets of vector fields $\{f_i\}_{i=1}^m$ and $\{f'_{i'}\}_{i'=1}^{m'}$ over M are said to be projectively identical, which we denote by $\{f_i\}_{i=1}^m \equiv \{f'_{i'}\}_{i'=1}^{m'}$, if for any f_i, there exist an $f'_{i'}$ and a real number λ such that $f_i = \lambda f'_{i'}$, and vice versa. We will use such an equivalence relation in the following way: In original ensemble control system (13), the set of control vector fields $\{f_i\}_{i=1}^m$ is, by assumption, distinguished. Thus, by the second item of Definition 2, if we evaluate the Lie products in each $\mathcal{L}(k)$, then

$$\mathcal{L}(k) \equiv \{f_i\}_{i=1}^m, \quad \forall k \geq 0. \tag{15}$$

Since every control vector field f in (14) is obtained by evaluating a Lie product involving the f_i's, by using the above fact, we can simplify Lie extended system (14) as follows:

$$\dot{x}_\sigma(t) = f_0(x_\sigma(t), \sigma) + \sum_{i=1}^{m}\sum_{l=0}^{k} \sum_{\mathrm{p} \in \mathcal{P}(l+1)} \big(u_{i,\mathrm{p}}(t)\mathrm{p}(\sigma)\big) f_i(x_\sigma(t)), \quad \forall \sigma \in \Sigma. \tag{16}$$

The control inputs $u_{i,\mathrm{p}}(t)$ in the above expression are defined such that

$$u_{i,\mathrm{p}}(t) := \sum_{\xi} \lambda_\xi \, u_{\xi,\mathrm{p}}(t),$$

where the summation is over Lie products ξ of depth $(\deg(\mathrm{p})-1)$ such that $\xi = \lambda_\xi f_i$. We now have the following fact (see, also, similar results in [1]):

Lemma 5 *Original system* (13) *is approximately ensemble path-controllable under* C^1*-inputs if and only if any of its Lie extended system* (16) *is.*

Proof By Lemma 4 and Remark 3, we know that for any control input $u^*[0,T]$ of class C^1 for Lie extended system (16), there is a sequence of control inputs $\{u^{(j)}[0,T]\}_{j=1}^{\infty}$ of class C^1 for each individual system-σ such that the trajectory $x_\sigma^{(j)}[0,T]$ generated by $u^{(j)}[0,T]$ converges uniformly to the trajectory $x_\sigma^*[0,T]$ of system (16) generated by $u^*[0,T]$. We now fix an arbitrary $\epsilon > 0$ and show that there exists an integer j_Σ such that if $j \geq j_\Sigma$, then

$$\mathrm{d}_M(x_\sigma^{(j)}(t), x_\sigma^*(t)) < \epsilon, \quad \forall (t,\sigma) \in [0,T] \times \Sigma. \tag{17}$$

To establish the fact, we first note that the initial profile $x_\Sigma(0)$ is analytic in σ. We next note that the drifting vector field f_0 and the monomials p are analytic functions. It follows that for any $\sigma \in \Sigma$, there exist an integer j_σ and an open neighborhood U_σ of σ such that $\mathrm{d}_M(x_{\sigma'}^{(j')}(t), x_{\sigma'}^*(t)) < \epsilon$ for any $j' \geq j_\sigma$, any $\sigma' \in U_\sigma$, and any $t \in [0,T]$. All such U_σ form an open cover of Σ. Since Σ is compact, there is a finite subcover $\{U_{\sigma_i}\}_{i=1}^N$. It then suffices to set $j_\Sigma := \max_{i=1}^N \{j_{\sigma_i}\}$ so that (17) holds. □

3.3.2 On the use of Stone–Weierstrass theorem

By Lemma 5, it now suffices to establish controllability of system (16) for a certain order k with C^1-control inputs $u_{i,\mathrm{p}}[0,T]$. We prove the fact below. Let $\hat{x}_\Sigma[0,T]$ be an arbitrary target trajectory of profiles. By the first item of Definition 2, we have that the set $\{f_i(x)\}_{i=1}^m$ spans T_xM for all $x \in M$. This, in particular, implies that there are functions $c_i(t,\sigma)$ continuous in both t and σ, for $i = 1,\dots,m$, such that

$$\frac{\partial \hat{x}_\sigma(t)}{\partial t} - f_0(\hat{x}_\sigma(t),\sigma) = \sum_{i=1}^m c_i(t,\sigma) f_i(\hat{x}_\sigma(t)), \quad \forall (t,\sigma) \in [0,T] \times \Sigma. \tag{18}$$

To see this, we first note that for any given (t,σ), there is an open neighborhood U of (t,σ) in $[0,T] \times \Sigma$ such that local existence of such continuous functions $c_i^U(t,\sigma)$ is guaranteed over U. All such open neighborhoods U form an open cover of $[0,T]\times\Sigma$. Since $[0,T]\times\Sigma$ is compact, there is a finite subcover $\{U_j\}_{j=1}^N$. Let $\{h_j\}_{j=1}^N$ be a partition of unity [31] subordinate to $\{U_j\}_{j=1}^N$. We then define $c_i := \sum_{j=1}^N h_j c_i^{U_j}$.

Comparing (16) with (18), we see that if there exist an order $k \geq 0$ and a set of control inputs $u_{i,\mathrm{p}}$, for $i = 1,\dots,m$ and for p a monomial with $1 \leq \deg(\mathrm{p}) \leq k+1$, such that the following holds:

$$c_i(t,\sigma) = \sum_{l=0}^{k} \sum_{\mathrm{p}\in\mathcal{P}(l+1)} u_{i,\mathrm{p}}(t)\mathrm{p}(\sigma), \quad \forall (t,\sigma) \in [0,T] \times \Sigma \text{ and } \forall i = 1,\dots,m, \tag{19}$$

then the trajectory of profiles $x_\Sigma[0, T]$ generated by system (16), with $x_\Sigma(0) = \hat{x}_\Sigma(0)$, will be exactly $\hat{x}_\Sigma[0, T]$. Said in another way, if (19) holds, then one can steer the kth-order Lie extended system (16) to follow the trajectory $\hat{x}_\Sigma[0, T]$.

But, in general, equality (19) cannot be satisfied by a finite sum. Nevertheless, we show below that the two sides of the expression can be made arbitrarily close to each other provided that k is sufficiently large, i.e.,

$$\left|\sum_{l=0}^{k}\sum_{\mathrm{p}\in\mathcal{P}(l+1)} u_{i,\mathrm{p}}(t)\mathrm{p}(\sigma) - c_i(t,\sigma)\right| < \delta, \tag{20}$$

for all $(t, \sigma) \in [0, T] \times \Sigma$ and for all $i = 1, \ldots, m$. This essentially follows from the Stone-Weierstrass theorem. We provide details below. Note that if (20) holds for any given $\delta > 0$, then one can apply Grönwall type inequalities [16] to show that the distance $\|x_\sigma(t) - \hat{x}_\sigma(t)\|$ or, in general, $\mathrm{d}_M(x_\sigma(t), \hat{x}_\sigma(t))$ can be made uniformly and arbitrarily small for all $(t, \sigma) \in [0, T] \times \Sigma$.

We now establish (20) for any given $\delta > 0$. By the assumption of Theorem 1, the set $\{\rho_s\}_{s=1}^r$ is a separating set and contains an everywhere nonzero function. Without loss of generality, we let ρ_1 be such a function, i.e., $\rho_1(\sigma) \neq 0$ for all $\sigma \in \Sigma$. It follows from the Stone-Weierstrass theorem that the subalgebra generated by the set $\{\rho_s\}_{s=1}^r$ is dense in $\mathrm{C}^0(\Sigma)$. In particular, we have the following fact: For any given $\delta' > 0$, there exist an integer $k \geq 0$ and a set of smooth functions $u'_{i,\mathrm{p}'} : [0, T] \to \mathbb{R}$, for $i = 1, \ldots, m$ and for p' a monomial with $0 \leq \deg(\mathrm{p}') \leq k$, such that

$$\left|\sum_{l=0}^{k}\sum_{\mathrm{p}'\in\mathcal{P}(l)} u'_{i,\mathrm{p}'}(t)\mathrm{p}'(\sigma) - \rho_1^{-1}(\sigma)c_i(t,\sigma)\right| < \delta', \tag{21}$$

for all $(t, \sigma) \in [0, T] \times \Sigma$ and for all $i = 1, \ldots, m$. To see this, we first note that for any given $t \in [0, T]$, there is an open neighborhood $\mathcal{I}$ of t such that the local existence of such functions $u'_{i,\mathrm{p}'} : \mathcal{I} \to \mathbb{R}$ is guaranteed by the Stone-Weierstrass theorem. Then, by applying smooth partition of unity for the closed interval $[0, T]$, we obtain desired functions $u'_{i,\mathrm{p}'}$ defined globally over the entire $[0, T]$.

Let $\gamma := \max\{|\rho_1^{-1}(\sigma)| \mid \sigma \in \Sigma\} > 0$. Note that γ exists because ρ_1 is everywhere nonzero and Σ is compact. Now, given an arbitrary $\delta > 0$, we define $\delta' := \delta/\gamma$ and let inequality (21) be satisfied. By the definition of γ, we have that

$$\left|\sum_{l=0}^{k}\sum_{\mathrm{p}'\in\mathcal{P}(l)} u'_{i,\mathrm{p}'}(t)(\rho_1(\sigma)\mathrm{p}'(\sigma)) - c_i(t,\sigma)\right| < \gamma\delta' = \delta,$$

for all $(t, \sigma) \in [0, T] \times \Sigma$ and for all $i = 1, \ldots, m$. Note that each $\rho_1\mathrm{p}'$ in the above expression is a monomial and $1 \leq \deg(\rho_1\mathrm{p}') \leq k + 1$. Next, for any $i = 1, \ldots, m$

and any monomial p with $1 \le \deg(\mathrm{p}) \le k+1$, we let the corresponding control input $u_{i,\mathrm{p}}(t)$ be defined such that for any $t \in [0, T]$,

$$u_{i,\mathrm{p}}(t) := \begin{cases} u'_{i,\mathrm{p}'}(t) & \text{if } \mathrm{p} = \rho_1 \mathrm{p}' \text{ with } 0 \le \deg(\mathrm{p}') \le k, \\ 0 & \text{otherwise.} \end{cases}$$

With the above-defined control inputs $u_{i,\mathrm{p}}(t)$, we conclude that (20) is satisfied. □

3.4 Proof of ensemble observability

We will now establish the second item of Theorem 1. Let a profile $\bar{x}_\Sigma(0)$ be chosen such that it is output equivalent to $x_\Sigma(0)$. The majority of effort will be devoted to proving the following fact: If $\{\phi^j\}_{j=1}^l$ is weakly codistinguished to $\{f_i\}_{i=1}^m$, then there is an open neighborhood U of $x_\Sigma(0)$ in $\mathrm{C}^\omega(\Sigma, M)$ such that if $\bar{x}_\Sigma(0)$ intersects U, then $\bar{x}_\Sigma(0) = x_\Sigma(0)$. The proof relies on the use of a special class of control inputs, namely piecewise constant control inputs and the structure of codistinguished functions.

3.4.1 On the use of piecewise constant control inputs

We first introduce a few key notations that will be used in the proof. For an arbitrary differential equation $\dot{x}(t) = f(x(t))$, we denote by $e^{tf}x(0)$ the solution of the equation at time t with initial condition $x(0)$. We will use such a notation to denote a solution $x_\sigma(t)$, for any $\sigma \in \Sigma$, of system (9). Next, we recall that $u(t)$ is the collection of the control inputs $u_{i,s}(t)$, for $1 \le i \le m$ and $1 \le s \le r$, in system (9). We introduce a notation for a piecewise constant control input $u(t)$ over $[0, T]$ as follows:

$$u[0, T] := (i_1, s_1, \nu_1, t_1) \cdots (i_k, s_k, \nu_k, t_k), \tag{22}$$

where $0 < t_1 < \cdots < t_k = T$ is an increasing sequence of switching times, ν_p's are real numbers, and (i_p, s_p)'s are pairs of indices chosen out of $\{1, \dots, m\} \times \{1, \dots, r\}$. The piecewise constant control input $u[0, T]$ is defined such that if $t \in [t_{p-1}, t_p)$, then

$$u_{i,s}(t) = \begin{cases} \nu_p, & \text{if } (i, s) = (i_p, s_p), \\ 0, & \text{otherwise.} \end{cases}$$

Note, in particular, that at any time $t \in [0, T]$, there is at most one nonzero scalar control input $u_{i,s}(t)$ in $u(t)$.

We will now apply piecewise constant control input (22) to excite system (9). For convenience, we define

$$\tau_p := t_p - t_{p-1}, \quad \forall p = 1, \dots, k,$$

with $t_0 := 0$. We further define a set of vector fields $\{\tilde{f}_p\}_{p=1}^k$ as follows:

$$\tilde{f}_p := \nu_p \rho_{s_p} f_{i_p} + f_0, \quad \forall p = 1, \dots, k,$$

where we have omitted all the arguments in the expression. Since $x_\Sigma(0) \sim \bar{x}_\Sigma(0)$, we have that for all $j = 1, \ldots, l$,

$$\int_\Sigma \phi^j \left(e^{\tau_k \tilde{f}_k} \cdots e^{\tau_1 \tilde{f}_1} x_\sigma(0)\right) \mathrm{d}\sigma = \int_\Sigma \phi^j \left(e^{\tau_k \tilde{f}_k} \cdots e^{\tau_1 \tilde{f}_1} \bar{x}_\sigma(0)\right) \mathrm{d}\sigma.$$

Moreover, the above equality holds for any τ_p and ν_p, with $p = 1, \ldots, k$.

We next take the partial derivative $\partial^k/\partial\tau_1\cdots\partial\tau_k$ on both sides of the above expression and evaluate the derivatives at $\tau_1 = \cdots = \tau_k = 0$. By computation, we obtain that

$$\int_\Sigma \left(\tilde{f}_1 \cdots \tilde{f}_k \phi^j\right)(x_\sigma(0))\mathrm{d}\sigma = \int_\Sigma \left(\tilde{f}_1 \cdots \tilde{f}_k \phi^j\right)(\bar{x}_\sigma(0))\mathrm{d}\sigma.$$

We further take the partial derivative $\partial^k/\partial\nu_1\cdots\partial\nu_k$ and evaluate at $\nu_1 = \cdots = \nu_k = 0$. By computation, we obtain that

$$\int_\Sigma (f_{\mathrm{w}}\phi^j)(x_\sigma(0))\mathrm{p}(\sigma)\mathrm{d}\sigma = \int_\Sigma (f_{\mathrm{w}}\phi^j)(\bar{x}_\sigma(0))\mathrm{p}(\sigma)\mathrm{d}\sigma, \tag{23}$$

where $\mathrm{w} := i_1 \cdots i_k$ is a word and $\mathrm{p} := \rho_{s_1} \cdots \rho_{s_k}$ is a monomial.

3.4.2 On the use of codistinguished functions

Note that $\{\phi^j\}_{j=1}^l$ is (weakly) codistinguished to $\{f_i\}_{i=1}^m$. By the second item of Definition 3, we have that for any $j' = 1, \ldots, l$, there exist a word w over the alphabet $\{1, \ldots, m\}$ of length k, a function ϕ^j, and a *nonzero* λ such that $f_{\mathrm{w}}\phi^j = \lambda\phi^{j'}$. Since (23) holds for all words w of length k for k arbitrary, we obtain that

$$\int_\Sigma \phi^j(x_\sigma(0))\mathrm{p}(\sigma)\mathrm{d}\sigma = \int_\Sigma \phi^j(\bar{x}_\sigma(0))\mathrm{p}(\sigma)\mathrm{d}\sigma, \tag{24}$$

for all $j = 1, \ldots, l$ and for all monomials $\mathrm{p} \in \mathcal{P}$.

We now let $\mathrm{L}^2(\Sigma)$ be the Hilbert space of all square-integrable functions on Σ, where the inner product is defined as follows:

$$\langle \mathrm{q}_1, \mathrm{q}_2\rangle_{\mathrm{L}^2} := \int_\Sigma \mathrm{q}_1(\sigma)\mathrm{q}_2(\sigma)\mathrm{d}\sigma, \quad \forall \mathrm{q}_1, \mathrm{q}_2 \in \mathrm{L}^2(\Sigma).$$

Note that Σ is compact. By the assumption of Theorem 1, the set of parameterization functions $\{\rho_s\}_{s=1}^r$ separates points and contains an everywhere nonzero function, so the subalgebra generated by the set is dense in $\mathrm{L}^2(\Sigma)$. Thus, if there is a function $\mathrm{q} \in \mathrm{L}^2(\Sigma)$ such that $\langle \mathrm{q}, \mathrm{p}\rangle_{\mathrm{L}^2} = 0$ for all monomials $\mathrm{p} \in \mathcal{P}$, then q is zero almost everywhere (it differs from the identically zero function over a set of measure zero). In the case here, we define for each $j = 1, \ldots, l$ the following function:

$$\mathrm{q}^j(\sigma) := \phi^j(x_\sigma(0)) - \phi^j(\bar{x}_\sigma(0)).$$

Then, one can rewrite (24) as follows:

$$\langle \mathrm{q}^j, \mathrm{p}\rangle_{\mathrm{L}^2} = 0, \quad \forall \mathrm{p} \in \mathcal{P} \text{ and } \forall j = 1, \ldots, l.$$

Because $x_\sigma(0), \bar{x}_\sigma(0)$ are analytic in σ and each $\phi^j(x)$ is analytic in x, we have that each $\mathrm{q}^j(\sigma)$ is analytic in σ. Furthermore, since Σ is equipped with a strictly positive Borel measure, we have that each q^j is identically zero, i.e.,

$$\phi^j(x_\sigma(0)) = \phi^j(\bar{x}_\sigma(0)), \quad \forall \sigma \in \Sigma \text{ and } \forall j = 1, \ldots, l. \tag{25}$$

Since $\{\phi^j\}_{j=1}^l$ is (weakly) codistinguished to $\{f_i\}_{i=1}^m$, by the first item of Definition 3, the set of one-forms $\{\mathrm{d}\phi_x^j\}_{j=1}^l$ spans the cotangent space T_x^*M for all $x \in M$. It follows that for any $x \in M$, there is an open ball $B_{\epsilon(x)}(x)$ centered at x with radius $\epsilon(x) > 0$ such that if $\bar{x} \in B_{\epsilon(x)}(x)$ and $\phi^j(x) = \phi^j(\bar{x})$ for all $j = 1, \ldots, l$, then $\bar{x} = x$. Furthermore, since each ϕ^j is analytic, for any fixed $x \in M$, the radius $\epsilon(x)$ of the open ball can be chosen such that it is locally continuous around x. Since the initial profile $x_\Sigma(0)$ is analytic in σ, the above arguments have the following implication: For each $\sigma \in \Sigma$, there is an open neighborhood V_σ of σ in Σ and a positive number ϵ_σ such that if $\sigma' \in V_\sigma$ and $\bar{x}_{\sigma'}(0)$ belongs to the open ball $B_{\epsilon_\sigma}(x_{\sigma'}(0))$ with $\phi^j(x_{\sigma'}(0)) = \phi^j(\bar{x}_{\sigma'}(0))$ for all $j = 1, \ldots, l$, then $\bar{x}_{\sigma'}(0) = x_{\sigma'}(0)$.

The collection of the above open sets $\{V_\sigma\}_{\sigma\in\Sigma}$ is an open cover of Σ. Since Σ is compact, there exists a finite subcover $\{V_{\sigma_i}\}_{i=1}^N$ of Σ. We then let

$$\epsilon := \min\left\{\epsilon_{\sigma_i} \mid i = 1, \ldots, N\right\} > 0.$$

We show below that if there is a certain $\sigma \in \Sigma$ such that $\mathrm{d}_M(x_\sigma(0), \bar{x}_\sigma(0)) < \epsilon$, then $\bar{x}_\Sigma(0) = x_\Sigma(0)$. This, in particular, implies weak ensemble observability of system (9).

To establish the fact, we first note that by the construction of ϵ, $\bar{x}_\sigma(0) = x_\sigma(0)$. Now, let σ' be any other point of Σ. We need to show that $\bar{x}_{\sigma'}(0) = x_{\sigma'}(0)$. Because Σ is path-connected, there is a continuous path $p : [0, 1] \to \Sigma$ with $p(0) = \sigma$ and $p(1) = \sigma'$. Again, by the definition of ϵ, we have that for any $\lambda \in [0, 1]$, there are only two cases: Either $\bar{x}_{p(\lambda)}(0) = x_{p(\lambda)}(0)$ or $\mathrm{d}_M(x_{p(\lambda)}, \bar{x}_{p(\lambda)}) \geq \epsilon$. On the other hand, the profile $\bar{x}_\Sigma(0)$ is continuous in σ and $p(\lambda)$ is continuous in λ, so $\bar{x}_{p(\lambda)}(0)$ is continuous in λ as well. But then, since $\bar{x}_{p(0)}(0) = x_{p(0)}(0)$, it follows that $\bar{x}_{p(\lambda)}(0) = x_{p(\lambda)}(0)$ for all $\lambda \in [0, 1]$. In particular, $\bar{x}_{\sigma'}(0) = x_{\sigma'}(0)$.

We now show that if, further, M is compact, then $|O(x_\Sigma(0))| \leq \chi_\phi$. Recall that $\phi := (\phi^1, \ldots, \phi^l)$ and $[x]_\phi$ is the pre-image of $\phi(x)$. Because any two different profiles in $O(x_\Sigma(0))$ are completely disjoint, it suffices to show that $|[x_\sigma(0)]_\phi| \leq \chi_\phi$ for some (and, hence, any) $\sigma \in \Sigma$. But, this follows from the definition of χ_ϕ and Lemma 3.

Finally, note that if $\{\phi^j\}_{j=1}^l$ is codistinguished to $\{f_i\}_{i=1}^m$ (and, hence, the third item of Definition 3 is satisfied), then by (25), $x_\sigma(0) = \bar{x}_\sigma(0)$ for all $\sigma \in \Sigma$, i.e., $O(x_\Sigma(0)) = \{x_\Sigma(0)\}$. Thus, system (9) is ensemble observable. This completes the proof. □

3.5 Pre-distinguished ensemble system

We consider in the subsection a scenario where the set of control vector fields $\{f_i(x)\}_{i=1}^m$ (resp. the set of one-forms $\{\mathrm{d}\phi^j(x)\}_{j=1}^l$) in system (9) does not necessarily span the tangent space $T_x M$ (resp. the cotangent space $T_x^* M$). Nevertheless, the two sets $\{f_i\}_{i=1}^m$ and $\{\phi^j\}_{j=1}^l$ together can "generate" (weakly) jointly distinguished vector fields and functions. We make the statement precise below.

To proceed, we first introduce a few definitions and notations. Let $F := \{f_i\}_{i=1}^m$ and $\mathcal{L}_F$ be the collection of Lie products generated by F (the f_i's are treated as "free" generators). We say that $\mathcal{L}_F$ is *projectively finite* if there is a finite set of vector fields $\bar{F} := \{\bar{f}_i\}_{i=1}^{\bar{m}}$ over M such that if one evaluates the Lie products in $\mathcal{L}_F$, then $\mathcal{L}_F \equiv \bar{F}$.

Next, let $\mathcal{W}$ be the collection of all words over the alphabet $\{1, \ldots, m\}$. Recall that for a given word $\mathrm{w} = i_1 \cdots i_k$ and an analytic function ϕ on M, we use $f_{\mathrm{w}}\phi$ to denote $f_{i_1} \cdots f_{i_k}\phi$. If $\mathrm{w} = \varnothing$, then $f_{\mathrm{w}}\phi = \phi$. Given a set function $\Phi := \{\phi^j\}_{j=1}^l$ on M and the set of vector fields F, we define

$$F_{\mathcal{W}}\Phi := \{f_{\mathrm{w}}\phi^j \mid \mathrm{w} \in \mathcal{W} \text{ and } j = 1, \ldots, l\}.$$

Similarly, we say that $F_{\mathcal{W}}\Phi$ is *projectively finite* if there is a finite subset $\bar{\Phi} := \{\bar{\phi}^j\}_{j=1}^{\bar{l}}$ of $\mathrm{C}^\omega(M)$ such that $F_{\mathcal{W}}\Phi \equiv \bar{\Phi}$. Note, in particular, that F and Φ are, up to scaling, subsets of $\bar{F}$ and $\bar{\Phi}$, respectively. We now have the following definition:

Definition 8 A set of vector fields $F := \{f_i\}_{i=1}^m$ over M is **pre-distinguished** if there exists a distinguished set $\bar{F}$ of vector fields such that $\mathcal{L}_F \equiv \bar{F}$. Similarly, a set of functions $\Phi := \{\phi^j\}_{j=1}^l$ on M is **(weakly) pre-codistinguished** to F if there exists a finite set $\bar{\Phi}$ of functions, (weakly) codistinguished to F, such that $F_{\mathcal{W}}\Phi \equiv \bar{\Phi}$.

Note that given a pair of jointly distinguished sets F and Φ, one can look for (proper) subsets $F' \subseteq F$ and $\Phi' \subseteq \Phi$ so that $\mathcal{L}_{F'} \equiv F$ and $F'_{\mathcal{W}}\Phi' \equiv \Phi$, i.e., F' and Φ' are *jointly pre-distinguished*. In particular, we say that (F', Φ') is *minimal* if removal of any element out of F' or Φ' will violate the condition in the above definition. We do not intend to characterize here minimal pairs for a given jointly distinguished pair (F, Φ). But instead, we provide below an example for illustration.

Example 3 We consider again the vector fields $F := \{L_{X_i}\}_{i=1}^3$ and the functions $\Phi := \{\phi_{ij} = \mathrm{tr}(gX_j g^\top X_i^\top)\}_{i,j=1}^3$ introduced in Example 1. We have shown that F and Φ are jointly distinguished on SO(3). Now, we define for each $i = 1, 2, 3$, a subset $F_i := F - \{L_{X_i}\}$ and for each $j = 1, 2, 3$, a subset $\Phi^j := \{\phi^{ij}\}_{i=1}^3$. Recall that we have the following relationships:

$$[L_{X_i}, L_{X_j}] = \det(e_i, e_j, e_k)L_{X_k} \quad \text{and} \quad L_{X_i}\phi^{i'j} = -\det(e_i, e_j, e_k)\phi^{i'k}.$$

It follows that $\mathcal{L}_{F_i} \equiv F$ for all $i = 1, 2, 3$, and $F_{i\mathcal{W}}\Phi^j \equiv \Phi$ for all $1 \leq i, j \leq 3$. Moreover, every such pair (F_i, Φ^j) is minimal. □

With the above definition, we state the following fact which generalizes Theorem 1:

Theorem 2 *Consider ensemble system (9). Suppose that $\{\rho_s^2\}_{s=1}^r$ is a separating set and contains an everywhere nonzero function; then, the following hold:*

(1) *If the set of control vector fields $\{f_i\}_{i=1}^m$ is pre-distinguished, then system (9) is approximately ensemble path-controllable under C^1-inputs.*
(2) *If the set of observation functions $\{\phi^j\}_{j=1}^l$ is (weakly) pre-codistinguished to $\{f_i\}_{i=1}^m$, then system (9) is (weakly) ensemble observable. If, further, M is compact, then for any initial profile $x_\Sigma(0)$, the set $O(x_\Sigma(0))$ is finite and $|O(x_\Sigma(0))| \le \chi_\phi$.*

We establish Theorem 2 in the following subsection. Similar to Definition 7, we have the following definition:

Definition 9 An ensemble system (9) is a **pre-distinguished** if (1) the set $\{\rho_s^2\}_{s=1}^r$ separates points and contains an everywhere nonzero function, and (2) the set of control vector fields $\{f_i\}_{i=1}^m$ and the set of observation functions $\{\phi^j\}_{j=1}^l$ are (weakly) jointly pre-distinguished.

It follows from Theorem 2 that if a system is pre-distinguished, then it is approximately ensemble path-controllable and (weakly) ensemble observable. We next have the following remark on the existence of a desired set of parameterization functions that satisfies the assumption of Theorem 2 (compared to Remark 2):

Remark 4 We first note that if $\{\rho_s^2\}_{s=1}^r$ is a separating set, then, for any positive integer k, $\{\rho_s^k\}_{s=1}^r$ is also a separating set. Conversely, if $\{\rho_s\}_{s=1}^r$ is a separating set and each ρ_s is nonnegative (i.e., $\rho_s(\sigma) \ge 0$ for all $\sigma \in \Sigma$), then $\{\rho_s^2\}_{s=1}^r$ will be a separating set. Such a set $\{\rho_s\}_{s=1}^r$ exists for any analytic, compact manifold Σ. To see this, we again embed Σ into a Euclidean space $\mathbb{R}^N$. Since Σ is compact, one can translate the coordinates, if necessary, such that Σ is embedded in the positive orthant of $\mathbb{R}^N$. Then, by restricting the coordinate functions of $\mathbb{R}^N$ to Σ, we obtain a separating set $\{\rho_i(\sigma) := \sigma_i\}_{i=1}^N$ comprised of all positive functions. □

3.6 Analysis and proof of Theorem 2

3.6.1 Indicator sequences

Let $\bar{F} = \{\bar{f}_i\}_{i=1}^{\bar{m}}$ be such that $\bar{F} \equiv \mathcal{L}_F$. Decompose $\mathcal{L}_F := \sqcup_{k\ge 0}\mathcal{L}_F(k)$ where $\mathcal{L}_F(k)$ is comprised of Lie products of depth k. In contrast to (15), we do not necessarily have that $\mathcal{L}(k) \equiv \bar{F}$ for all $k \ge 0$. It is possible that each $\mathcal{L}_F(k)$ is, up to scaling, a proper subset of $\bar{F}$ (see Example 4). To tackle the issue, we first introduce the following definitions:

Definition 10 Let $\mathcal{L}_F$ be projectively finite and $\bar{F} = \{\bar{f}_i\}_{i=1}^{\bar{m}}$ be such that $\bar{F} \equiv \mathcal{L}_F$. For each $i = 1, \dots, \bar{m}$, define a sequence of natural numbers $\mathbb{N}_i$ as follows: If $k \in \mathbb{N}_i$, then there exist a Lie product $\xi \in \mathcal{L}_F(k)$ and a real number λ such that by evaluating ξ, we have $\bar{f}_i = \lambda\xi$. We call every such sequence $\mathbb{N}_i$ an **indicator sequence for** $\bar{f}_i$.

Similarly, we have the following counterpart of the above definition:

Definition 11 Let $F_{\mathcal{W}}\Phi$ be projectively finite and $\bar{\Phi} = \{\bar{\phi}^j\}_{j=1}^{\bar{l}}$ be such that $\bar{\Phi} \equiv F_{\mathcal{W}}\Phi$. For each $j = 1, \dots, \bar{l}$, define a sequence of natural numbers $\mathbb{N}^j$ as follows: If $k \in \mathbb{N}^j$, then there exist a word w of length k over the alphabet $\{1, \dots, m\}$, a function $\phi^{j'} \in \Phi$, and a real number λ such that $\bar{\phi}^j = \lambda f_{\mathrm{w}}\phi^{j'}$. We call every such sequence $\mathbb{N}^j$ an **indicator sequence for** $\bar{\phi}^j$.

Note that if F and Φ are (weakly) jointly distinguished, then $\mathbb{N}_i = \mathbb{N}^j = \mathbb{N}$ for all $i = 1, \dots, m(= \bar{m})$ and for all $j = 1, \dots, l(= \bar{l})$.

Example 4 Consider the subsets $F_1 = F - \{L_{X_1}\}$ and $\Phi^1 = \{\phi^{i1}\}_{i=1}^3$ introduced in Example 3. We have that $\mathcal{L}_{F_1} \equiv F$ and $F_{1_{\mathcal{W}}}\Phi^1 \equiv \Phi$. By computation (with details omitted), the indicator sequences $\mathbb{N}_i$ for L_{X_i} are given by $\mathbb{N}_1 = \{2k + 1\}_{k\geq 0}$ and $\mathbb{N}_2 = \mathbb{N}_3 = \{2k\}_{k\geq 0}$. The indicator sequences $\mathbb{N}^{ij}$ for ϕ^{ij} are given by $\mathbb{N}^{i1} = \{2k\}_{k\geq 0}$ and $\mathbb{N}^{i2} = \mathbb{N}^{i3} = \{2k + 1\}_{k\geq 0}$ for all $i = 1, 2, 3$. □

A sequence $\{n_k\}_{k=0}^{\infty}$ is said to be an *arithmetic sequence* if there is a δ such that $n_{k+1} - n_k = \delta$ for all $k \geq 0$. We now establish the following fact:

Proposition 1 *Every indicator sequence $\mathbb{N}_i$ for $\bar{f}_i$ (or $\mathbb{N}^j$ for $\bar{\phi}^j$) contains an infinite arithmetic sequence as a subsequence.*

Proof We establish the proposition for $\mathbb{N}_i$ and $\mathbb{N}^j$ subsequently.

Proof for $\mathbb{N}^i$. We fix an $i = 1, \dots, \bar{m}$ and prove that $\mathbb{N}_i$ contains an arithmetic sequence. Because F is pre-distinguished, there exists a Lie product $\xi_1 \in \mathcal{L}_F$, with $\mathrm{dep}(\xi_1) \geq 1$, and a real number λ_1 such that $\lambda_1\xi_1 = \bar{f}_i$. Denote by $f_{i_1} \in F$ the first element that shows up in ξ_1 (e.g., $\xi_1 = [f_{i_1}, [f_{i_1'}, f_{i_1''}]]$). Applying the same argument, but with $\bar{f}_i$ replaced by f_{i_1}, we obtain that $\lambda_2\xi_2 = f_{i_1}$ for some $\xi_2 \in \mathcal{L}_F$ with $\mathrm{dep}(\xi_2) \geq 1$ and some $\lambda_2 \in \mathbb{R}$.

Next, we let $\xi_1 \lhd \xi_2$ be a Lie product in $\mathcal{L}_F$ defined by replacing the first element f_{i_1} in ξ_1 with the Lie product ξ_2. For example, if $\xi_1 = [f_{i_1}, [f_{i_1'}, f_{i_1''}]]$, then $\xi_1 \lhd \xi_2 = [\xi_2, [f_{i_1'}, f_{i_1''}]]$. It should be clear that

$$\lambda_1\lambda_2\xi_1 \lhd \xi_2 = f_i, \quad \text{with } \mathrm{dep}(\xi_1 \lhd \xi_2) = \mathrm{dep}(\xi_1) + \mathrm{dep}(\xi_2).$$

By repeating the above procedure, we obtain (1) a sequence of Lie products $\{\xi_k\}_{k\geq 1}$, (2) a sequence of vector fields $\{f_{i_k}\}_{k\geq 1}$ with $f_{i_k} \in F$, and (3) a sequence of real numbers $\{\lambda_k\}_{k\geq 1}$ such that the first element in ξ_k is f_{i_k} and $\lambda_k\xi_k = f_{i_{k-1}}$. It then follows that

$$\alpha_k\xi_1 \lhd \cdots \lhd \xi_k = f_i \;\; \text{where } \alpha_k := \prod_{l=1}^{k} \lambda_l, \quad \forall k \geq 1.$$

Note that $\xi_1 \lhd \cdots \lhd \xi_k$ is well defined because the operator "$\lhd$" is associative.

Since each f_{i_k} belongs to the finite set F, there is a repetition in the sequence. Without loss of generality, we assume that $f_{i_k} = f_{i_{k'}}$ for some $k' > k \geq 1$. We then define a Lie product ξ as follows:

$$\xi := \xi_{k+1} \lhd \cdots \lhd \xi_{k'} \quad \text{and} \quad \delta := \text{dep}(\xi) = \sum_{l=k+1}^{k'} \text{dep}(\xi_l).$$

Note that the first element in ξ is f_{i_k} and ${}^{\alpha_{k'}}/_{\alpha_k}\xi = f_{i_k}$. In fact, the statement can be strengthened: For any given $N \geq 0$, we define

$$\xi^N := \xi \lhd \cdots \lhd \xi$$

where the number of copies of ξ in the expression is N. If $N = 0$, then we let $\xi^0 := f_{i_k}$. It should be clear that for any $N \geq 0$, the first element in ξ^N is f_{i_k} and, moreover, ${}^{\alpha_{k'}^N}/_{\alpha_k^N}\xi^N = f_{i_k}$. We further define a Lie product ξ_0 as follows:

$$\xi_0 := \xi_1 \lhd \cdots \lhd \xi_k \quad \text{and} \quad \delta_0 := \text{dep}(\xi_0) = \sum_{l=1}^{k} \text{dep}(\xi_l).$$

It then follows that for any $N \geq 0$.

$$\left({}^{\alpha_{k'}^N}/_{\alpha_k^{N-1}}\right) \xi_0 \lhd \xi^N = \bar{f}_i,$$

which implies that $\mathbb{N}_i$ contains $\{\delta_0 + N\delta\}_{N\geq 0}$ as a subsequence.

Proof for $\mathbb{N}^j$. The arguments will be similar to the ones used above. We fix a $j = 1, \ldots, \bar{l}$, and prove that $\mathbb{N}^j$ contains an arithmetic sequence. Since Φ is pre-codistinguished to F, there exist a word w_1 of positive length, a function ϕ^{j_1} out of Φ, and a real number μ_1 such that $\mu_1 f_{\mathrm{w}_1}\phi^{j_1} = \bar{\phi}^j$. Applying the same argument, but with $\bar{\phi}^j$ replaced by ϕ^{j_1}, we obtain that $\mu_2 f_{\mathrm{w}_2}\phi^{j_2} = \phi^{j_1}$ for some word w_2 of positive length, some function ϕ^{j_2} out of Φ, and some real number μ_2. Note, in particular, that

$$\mu_1\mu_2 f_{\mathrm{w}_1} f_{\mathrm{w}_2}\phi^{j_2} = \bar{\phi}^j.$$

By repeating the procedure, we obtain (1) a sequence of functions $\{\phi^{j_k}\}_{k\geq 1}$ where each ϕ^{j_k} belongs to Φ, (2) a sequence of words $\{\mathrm{w}_k\}_{k\geq 1}$ of positive lengths, and (3) a sequence of real numbers $\{\mu_k\}_{k\geq 1}$ such that $\mu_k f_{\mathrm{w}_k}\phi^{j_k} = \phi^{j_{k-1}}$. It then follows that

$$\beta_k f_{\mathrm{w}_1} \cdots f_{\mathrm{w}_k}\phi^{j_k} = \bar{\phi}^j \quad \text{where } \beta_k := \prod_{l=1}^{k} \mu_l, \quad \forall k \geq 1.$$

Since each ϕ^{j_k} belongs to the finite set Φ, there is a repetition in the sequence, say $\phi^{j_k} = \phi^{j_{k'}}$ for some $k' > k \geq 1$. It then implies that ${}^{\beta_{k'}}/_{\beta_k} f_{\mathrm{w}}\phi^{j_k} = \phi^{j_k}$ where $\mathrm{w} := \mathrm{w}_{k+1} \cdots \mathrm{w}_{k'}$ is obtained by concatenation. Denote by δ the length w. For a nonnegative integer N, we let w^N be a word obtained by concatenating N copies of w.

If $N = 0$, then $\mathrm{w}^N = \varnothing$. We further let $\mathrm{w}_0 := \mathrm{w}_1 \cdots \mathrm{w}_k$ and δ_0 be the length of w_0. It then follows that for any $N \geq 0$,

$$\left(\beta_{k'}^N / \beta_k^{N-1}\right) f_{\mathrm{w}_0} f_{\mathrm{w}^N} \phi^{j_k} = \bar{\phi}^j,$$

which implies that $\mathbb{N}^j$ contains $\{\delta_0 + N\delta\}_{N\geq 0}$ as a subsequence. □

3.6.2 Proof of Theorem 2

The arguments we will use for proving the theorem will be similar to those for Theorem 1. We elaborate below only on the difference.

We first establish item 1 of Theorem 2. By repeatedly applying Lie extensions of system (9), we obtain the following formal expression:

$$\dot{x}_\sigma(t) = f_0(x_\sigma(t), \sigma) + \sum_{l\geq 0} \sum_{\xi \in \mathcal{L}_F(l)} \sum_{\mathrm{p}\in\mathcal{P}(l+1)} u_{\xi,\mathrm{p}}(t)\mathrm{p}(\sigma)\xi(x_\sigma(t)), \quad \forall \sigma \in \Sigma.$$

One obtains a kth-order Lie extended system by truncating the infinite summation over l and keeping only the terms with $l \leq k$. Because $F = \{f_i\}_{i=1}^m$ is pre-distinguished, we let $\bar{F} = \{\bar{f}_i\}_{i=1}^{\bar{m}}$ be such that $\bar{F} \equiv \mathcal{L}_F$. Then, by the definition of indicator sequence $\mathbb{N}_i$ for $\bar{f}_i$, the above equation can be simplified as follows:

$$\dot{x}_\sigma(t) = f_0(x_\sigma(t), \sigma) + \sum_{i=1}^{\bar{m}} \sum_{l\in\mathbb{N}_i} \sum_{\mathrm{p}\in\mathcal{P}(l+1)} u_{i,\mathrm{p}}(t)\mathrm{p}(\sigma)\bar{f}_i(x_\sigma(t)), \quad \forall \sigma \in \Sigma.$$

To establish ensemble controllability of the above system (or more precisely, a truncated version after a certain order), it suffices to show that for any $i = 1, \ldots, \bar{m}$, the $\mathbb{R}$-span of monomials in $\sqcup_{l\in\mathbb{N}_i}\mathcal{P}(l+1)$ is dense in $\mathrm{C}^0(\Sigma)$. We prove this fact below.

We fix an $i = 1, \ldots, \bar{m}$. By Proposition 1, the indicator sequence $\mathbb{N}_i$ contains an infinite arithmetic sequence, which we denote by $\{n_k\}_{k\geq 0}$ with $\delta := n_{k+1} - n_k > 0$ for all $k \geq 0$. We next define functions on Σ as follows:

$$\bar{\rho}_s := \rho_s^\delta, \quad \forall s = 1, \ldots, r.$$

By the assumption of Theorem 2, the set $\{\rho_s^2\}_{s=1}^r$ is a separating set and contains an everywhere nonzero function, say ρ_1. It follows that $\{\bar{\rho}_s\}_{s=1}^r$ is also a separating set with $\bar{\rho}_1$ an everywhere nonzero function. Thus, the subalgebra generated by $\{\bar{\rho}_s\}_{s=1}^r$ is dense in $\mathrm{C}^0(\Sigma)$. Denote the subalgebra by $\bar{\mathcal{S}}$. Since ρ_1 is everywhere nonzero, the following set:

$$\rho_1^{n_0+1}\bar{\mathcal{S}} := \left\{\rho_1^{n_0+1}\mathrm{p} \mid \mathrm{p} \in \bar{\mathcal{S}}\right\}$$

is dense in $\mathrm{C}^0(\Sigma)$ as well. On the other hand, the $\mathbb{R}$-span of $\sqcup_{l\in\mathbb{N}_i}\mathcal{P}(l+1)$ contains $\rho_1^{n_0+1}\bar{\mathcal{S}}$ as a subset; indeed, if p is a monomial that can be expressed as

$$\mathrm{p} = \rho_1^{n_0+1}\prod_{s=1}^{r}\bar{\rho}_s^{k_s}$$

with $k_s \geq 0$, then $\mathrm{p} \in \mathcal{P}(n_k+1)$ where $k := \sum_{s=1}^{r} k_s$. We have thus shown that the $\mathbb{R}$-span of $\sqcup_{l\in\mathbb{N}_i}\mathcal{P}(l+1)$ is dense in $\mathrm{C}^0(\Sigma)$.

We now establish item 2 of Theorem 2. Let $\bar{x}_\Sigma(0)$ and $x_\Sigma(0)$ two initial profiles that are output equivalent. The same arguments in Sect. 3.4 can be used here to obtain the following fact: Let $k \geq 0$ be an arbitrary integer. Let w be any word of length k and p be any monomial of degree k. Then, for any $j = 1, \ldots, l$, we have

$$\int_\Sigma (f_{\mathrm{w}}\phi^j)(x_\sigma(0))\mathrm{p}(\sigma)\mathrm{d}\sigma = \int_\Sigma (f_{\mathrm{w}}\phi^j)(\bar{x}_\sigma(0))\mathrm{p}(\sigma)\mathrm{d}\sigma. \tag{26}$$

Because $\Phi = \{\phi^j\}_{j=1}^{l}$ is (weakly) pre-codistinguished to F, we let $\bar{\Phi} = \{\bar{\phi}^j\}_{j=1}^{\bar{l}}$ be such that $F_{\mathcal{W}}\Phi = \bar{\Phi}$. For each $j = 1, \ldots, \bar{l}$, we define a function q^j on Σ as follows:

$$\mathrm{q}^j(\sigma) := \bar{\phi}^j(x_\sigma(0)) - \bar{\phi}^j(\bar{x}_\sigma(0)).$$

By the definition of indicator sequence $\mathbb{N}^j$ for $\bar{\phi}^j$, we can simplify (26) as follows:

$$\langle \mathrm{q}^j, \mathrm{p}\rangle_{\mathrm{L}^2} = 0, \quad \forall \mathrm{p} \in \sqcup_{l\in\mathbb{N}^j}\mathcal{P}(l).$$

Note that the above expression holds for all $j = 1, \ldots, \bar{l}$. It now suffices to show that the $\mathbb{R}$-span of $\sqcup_{l\in\mathbb{N}^j}\mathcal{P}(l)$ is dense in $\mathrm{L}^2(\Sigma)$. This, again, follows from Proposition 1; indeed, since $\mathbb{N}^j$ contains an infinite arithmetic sequence, it follows by the same arguments (for $\mathbb{N}_i$) that the $\mathbb{R}$-span of $\sqcup_{l\in\mathbb{N}^j}\mathcal{P}(l)$ is dense in $\mathrm{C}^0(\Sigma)$. Because Σ is compact, $\mathrm{C}^0(\Sigma)$ is dense in $\mathrm{L}^2(\Sigma)$. This completes the proof. □

4 Existence of distinguished ensemble systems

We have shown in the previous section that (weakly) jointly distinguished vector fields $\{f_i\}_{i=1}^m$ and functions $\{\phi^j\}_{j=1}^l$ are key ingredients for an ensemble system to be approximately ensemble path-controllable and (weakly) ensemble observable. We address in the section the issue about the existence of these finely structured vector fields and functions for a given manifold M. Among other things, we provide an affirmative answer for the case where M is a connected, semi-simple Lie group:

Theorem 3 *For any connected semi-simple Lie group G, there exist weakly jointly distinguished vector fields $\{f_i\}_{i=1}^m$ and functions $\{\phi^j\}_{j=1}^l$ on G. Moreover, if G has a trivial center, then $\{f_i\}_{i=1}^m$ and $\{\phi^j\}_{j=1}^l$ are jointly distinguished.*

4.1 Distinguished sets of semi-simple real Lie algebras

Let G be a semi-simple Lie group and $\mathfrak{g}$ be its Lie algebra. We address in the subsection the existence of distinguished vector fields over G. These vector fields will be certain left- (or right-) invariant vector fields. We can thus address the existence issue on the Lie algebra level. To proceed, we first have the following definition [6]:

Definition 12 Let $\mathfrak{g}$ be a semi-simple real Lie algebra. A spanning set $\{X_i\}_{i=1}^m$ of $\mathfrak{g}$ is **distinguished** if for any X_i and X_j, there exist an X_k and a real number λ such that

$$[X_i, X_j] = \lambda X_k. \tag{27}$$

Conversely, for any X_k, there exist X_i, X_j, and a *nonzero* λ such that (27) holds.

Note that the cardinality of a distinguished set $\{X_i\}_{i=1}^m$ is, in general, greater than the dimension of $\mathfrak{g}$, i.e., the spanning set $\{X_i\}_{i=1}^m$ may contain a basis of $\mathfrak{g}$ as its proper subset. We have established in [6] the following result:

Proposition 2 *Every semi-simple real Lie algebra admits a distinguished set.*

The proposition then implies that every semi-simple Lie group admits a set of distinguished left- (or right-) invariant vector fields. Since the proposition will be of great use in the paper, we outline below a constructive approach for generating a desired distinguished set. A complete proof can be found in [6]. The proof leverages the structure theory of semi-simple real Lie algebras. A reader not interested in the proof can skip the remainder of the subsection.

Sketch of proof Recall that $\mathrm{ad}(X)(\cdot) := [X, \cdot]$ is the adjoint representation. Denote by $B(X, Y) := \mathrm{tr}(\mathrm{ad}_X\, \mathrm{ad}_Y)$ the Killing form. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{g}^{\mathbb{C}}$ (resp. $\mathfrak{h}^{\mathbb{C}}$) be the complexification of $\mathfrak{g}$ (resp. $\mathfrak{h}$). We let Δ be the set of roots. For each $\alpha \in \Delta$, we let $h_\alpha \in \mathfrak{h}^{\mathbb{C}}$ be such that $\alpha(H) = B(h_\alpha, H)$ for all $H \in \mathfrak{h}^{\mathbb{C}}$. Denote by $\langle \alpha, \beta \rangle := B(h_\alpha, h_\beta)$, which is an inner product defined over the $\mathbb{R}$-span of Δ. We denote by $|\alpha| := \sqrt{\langle \alpha, \alpha \rangle}$ the length of α. Let $H_\alpha := 2h_\alpha/|\alpha|^2$. For a root $\alpha \in \Delta$, let $\mathfrak{g}_\alpha$ be the corresponding root space (as a one-dimensional subspace of $\mathfrak{g}^{\mathbb{C}}$ over $\mathbb{C}$). □

Suppose, for the moment, that one aims to obtain a distinguished set for the semi-simple *complex* Lie algebra $\mathfrak{g}^{\mathbb{C}}$; then, with slight modification, such a set can be obtained via the Chevalley basis [21, Chapter VII], which we recall below:

Lemma 6 *There are $X_\alpha \in \mathfrak{g}^{\mathbb{C}}_\alpha$, for $\alpha \in \Delta$, such that the following hold:*

(1) *For any $\alpha \in \Delta$, we have $[X_\alpha, X_{-\alpha}] = H_\alpha$.*
(2) *For any two non-proportional roots α, β, we let $\beta + n\alpha$, with $-q \le n \le p$, be the α-string that contains β. Then,*

$$[X_\alpha, X_\beta] = \begin{cases} c_{\alpha,\beta} X_{\alpha+\beta}, & \text{if } \alpha + \beta \in \Delta, \\ 0, & \text{otherwise}, \end{cases}$$

where $c_{\alpha,\beta} \in \mathbb{Z}$ with $c_{\alpha,\beta}^2 = (q+1)^2$.

We also note that for any $\alpha, \beta \in \Delta$, $[H_\alpha, X_\beta] = {}^{2\langle\alpha,\beta\rangle}/_{|\alpha|^2} X_\beta$ and, moreover, ${}^{2\langle\alpha,\beta\rangle}/_{|\alpha|^2} \in \mathbb{Z}$. It thus follows from Lemma 6 that

$$A := \{H_\alpha, X_\alpha, X_{-\alpha} \mid \alpha \in \Delta\}$$

is a distinguished set of $\mathfrak{g}^{\mathbb{C}}$. The above arguments have the following implications:

(1) A semi-simple complex Lie algebra can also be viewed as a Lie algebra over $\mathbb{R}$. We call any such real Lie algebra *complex* [22, Chapter VI]. In particular, if the real Lie algebra $\mathfrak{g}$ is complex, then the $\mathbb{R}$-span of $A \cup \mathrm{i}A$, with A defined above, is $\mathfrak{g}$. Moreover, since the coefficients ${}^{2\langle\alpha,\beta\rangle}/_{|\alpha|^2}$ and $c_{\alpha,\beta}$ are all integers (and hence real), the set $A \cup \mathrm{i}A$ is a distinguished set of $\mathfrak{g}$.
(2) If the Lie algebra $\mathfrak{g}$ is obtained as the $\mathbb{R}$-span of A (i.e., $\mathfrak{g}$ is a *split real form* of $\mathfrak{g}^{\mathbb{C}}$), then A is a distinguished set of $\mathfrak{g}$.

Thus, the technical difficulty for establishing Proposition 2 lies in the case where $\mathfrak{g}$ is neither complex nor a split real form of $\mathfrak{g}^{\mathbb{C}}$. We have dealt with such a case in [6]. We reproduce below a key result established in that paper.

First, recall that a *Cartan involution* $\theta : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra automorphism, with $\theta^2 = \mathrm{id}$. Moreover, the symmetric bilinear form B_θ, defined as

$$B_\theta(X, Y) := -B(X, \theta Y),$$

is positive definite on $\mathfrak{g}$. One can extend θ to $\mathfrak{g}^{\mathbb{C}}$ by $\theta(X + \mathrm{i}Y) = \theta X + \mathrm{i}\theta Y$.

Next, for a subset $S \subset \mathfrak{g}$, we let $\mathcal{L}_S$ be the collection of Lie products generated by S. Similarly, we say that $\mathcal{L}_S$ is projectively finite if there exists a finite subset $\bar{S}$ of $\mathfrak{g}$ such that $\mathcal{L}_S \equiv \bar{S}$. Further, we say that the set S is *pre-distinguished* if $\bar{S}$ is a distinguished set of $\mathfrak{g}$ (compared with Definition 8). We now have the following fact:

Proposition 3 *Let $\mathfrak{g}$ be a simple real Lie algebra, which is neither complex nor a split real form of $\mathfrak{g}^{\mathbb{C}}$. Then, there exist a Cartan involution θ and elements $X_\alpha \in \mathfrak{g}^{\mathbb{C}}_\alpha$, for $\alpha \in \Delta$, such that the items of Lemma 6 are satisfied and the following set belongs to $\mathfrak{g}$:*

$$S := \{Y_\alpha := X_\alpha - \theta X_{-\alpha}, \quad Z_\alpha := \mathrm{i}(X_\alpha + \theta X_{-\alpha}) \mid \alpha \in \Delta\}.$$

Furthermore, the following hold:

(1) *If the underlying root system of $\mathfrak{g}$ is not G_2, then the set S is pre-distinguished.*
(2) *If the underlying root system of $\mathfrak{g}$ is G_2, then $\mathfrak{g}$ is the compact real form of $\mathfrak{g}^{\mathbb{C}}$. Decompose $\Delta = \Delta_{\text{short}} \cup \Delta_{\text{long}}$ where Δ_{short} (resp. Δ_{long}) is comprised of short (resp. long) roots. Then, the following set is pre-distinguished:*

$$\bigcup_{\gamma \in \Delta_{\text{long}}} \{Y_\gamma, Z_\gamma\} \cup \bigcup_{\substack{\alpha, \beta \in \Delta_{\text{short}} \\ \text{and } \alpha \neq \pm\beta}} \left\{[Y_\alpha, Y_\beta], [Y_\alpha, Z_\beta], [Z_\alpha, Y_\beta], [Z_\alpha, Z_\beta]\right\}.$$

We refer the reader to [6] for a complete proof. It follows from Proposition 3 that every semi-simple real Lie algebra admits a distinguished set. This establishes Proposition 2.

4.2 Matrix coefficients as codistinguished functions

Let $\{X_i\}_{i=1}^m$ be a distinguished set of $\mathfrak{g}$. We address in the subsection the existence of (weakly) codistinguished functions on G to the set of left- (resp., right-) invariant vector fields $\{L_{X_i}\}_{i=1}^m$ (resp. $\{R_{X_i}\}_{i=1}^m$). Because of the symmetry, the focus will be mostly on the functions codistinguished to the *left-invariant* vector fields. We provide a remark at the end of the subsection to address the existence of codistinguished functions to the right-invariant vector fields.

To proceed, we first recall that the so-called *right-regular representation* of G on $\mathrm{C}^\omega(G)$, denoted by $r : G \times \mathrm{C}^\omega(G) \to \mathrm{C}^\omega(G)$, is defined by

$$(x, \phi) \in G \times \mathrm{C}^\omega(G) \mapsto (r(x)\phi)(g) := \phi(gx).$$

Correspondingly, the induced Lie algebra representation r_* is the Lie derivative along a left-invariant vector field, i.e., $r_*(X)\phi = L_X\phi$. Note, in particular, that if $\Phi = \{\phi^j\}_{j=1}^l$ is codistinguished to $\{L_{X_i}\}_{i=1}^m$, then $r_*|_{\mathbb{L}_\Phi}$ is a finite-dimensional representation of $\mathfrak{g}$ on $\mathbb{L}_\Phi$; indeed, we have that

$$\begin{aligned} r_*([X_i, X_{i'}])\phi^j &= r_*(X_i)r_*(X_{i'})\phi^j - r_*(X_{i'})r_*(X_i)\phi^j \\ &= L_{X_i}L_{X_{i'}}\phi^j - L_{X_{i'}}L_{X_i}\phi^j = L_{[X_i, X_{i'}]}\phi^j \in \mathbb{L}_\Phi. \end{aligned}$$

Thus, in order to find a set of codistinguished functions to $\{L_{X_i}\}_{i=1}^m$, our strategy is comprised of two steps as outlined below:

(1) Construct a finite-dimensional subspace $\mathbb{L}$ of $\mathrm{C}^\omega(G)$ such that it is closed under r so that $r_*|_{\mathbb{L}}$ will be a Lie algebra representation of $\mathfrak{g}$ on $\mathbb{L}$;
(2) Find a finite subset $\Phi = \{\phi^j\}_{j=1}^l$ out of the space $\mathbb{L}$ such that it is codistinguished to a certain set of left-invariant vector fields $\{L_{X_i}\}_{i=1}^m$.

We now address, one by one, the above two steps.

Our approach for the first step about constructing a finite-dimensional subspace $\mathbb{L}$ of $\mathrm{C}^\omega(G)$ is to use matrix coefficients associated with a Lie group representation. Specifically, we consider an arbitrary analytic representation π of G on a finite-dimensional inner-product space $(V, \langle\cdot,\cdot\rangle)$. Let $\{v_i\}_{i=1}^p$ be any spanning subset of V. We next define a set of matrix coefficients as follows:

$$\pi^{ij}(g) := \langle v_i, \pi(g)v_j\rangle \in \mathrm{C}^\omega(G), \quad 1 \le i, j \le p. \tag{28}$$

Then, we let $\mathbb{L}_\pi$ be a finite-dimensional subspace of $\mathrm{C}^\omega(G)$ spanned by π^{ij}:

$$\mathbb{L}_\pi := \left\{ \sum_{i,j=1}^p c_{ij}\pi^{ij} \mid c_{ij} \in \mathbb{R} \right\}.$$

The following fact is certainly known in the literature. But, for completeness of presentation, we provide a proof after the statement:

Lemma 7 *The vector space $\mathbb{L}_\pi$ is closed under $r(x)$ for all $x \in G$, i.e., for any $\phi \in \mathbb{L}_\pi$, $r(x)\phi \in \mathbb{L}_\pi$. Thus, $r|_{\mathbb{L}_\pi}$ (resp. $r_*|_{\mathbb{L}_\pi}$) is a representation of G (resp. $\mathfrak{g}$) on $\mathbb{L}_\pi$.*

Proof The lemma follows directly from computation. For any $x \in G$ and any $g \in G$,

$$(r(x)\pi^{ij})(g) = \pi^{ij}(gx) = \langle v_i, \pi(gx)v_j\rangle = \langle v_i, \pi(g)\pi(x)v_j\rangle.$$

Since $\{v_1, \ldots, v_p\}$ spans V, there exist real coefficients c_{lk}'s such that

$$\pi(x)v_j = \sum_{l,k=1}^{p} c_{lk}\langle v_l, \pi(x)v_j\rangle v_k = \sum_{l,k=1}^{p} c_{lk}\pi^{lj}(x)v_k.$$

It then follows that

$$(r(x)\pi^{ij})(g) = \sum_{l,k=1}^{p} \left(c_{lk}\pi^{lj}(x)\right)\pi^{ik}(g),$$

which implies that $r(x)\pi^{ij}$ is a linear combination of π^{ik} for $k = 1, \ldots, p$. □

We now address the second step of our strategy about finding a finite subset $\{\phi^j\}_{j=1}^{l}$ out of $\mathbb{L}_\pi$ so that it is codistinguished to a given set of left-invariant vector fields $\{L_{X_i}\}_{i=1}^{m}$. To proceed, we first have the following definition as a dual to Definition 12:

Definition 13 Let π be a finite-dimensional representation of G on V, and π_* be the corresponding Lie algebra representation. A spanning set $\{v_j\}_{j=1}^{p}$ of V is **codistinguished** to a subset $\{X_i\}_{i=1}^{m}$ of $\mathfrak{g}$ if it satisfies the following properties:

(1) The set of one-forms $\{\mathrm{d}\pi_e^{ij}\}_{i,j=1}^{p}$ spans $T_e^*G \approx \mathfrak{g}^*$.
(2) For any X_i and v_j, there exist a v_k and a real number λ such that

$$\pi_*(X_i)v_j = \lambda v_k; \tag{29}$$

conversely, for any v_k, there exist X_i, v_j, and a *nonzero* λ such that (29) holds.
(3) For any $g, g' \in G$, if $\pi^{ij}(g) = \pi^{ij}(g')$ for all $1 \le i, j \le p$, then $g = g'$.

If only (1) and (2) hold, then $\{v_j\}_{j=1}^{p}$ is **weakly codistinguished** to $\{X_i\}_{i=1}^{m}$.

With the above definition, we now have the following fact:

Lemma 8 *If $\{v_j\}_{j=1}^{p}$ is codistinguished to $\{X_i\}_{i=1}^{m}$, then the set of matrix coefficients $\{\pi^{ij}\}_{i,j=1}^{p}$ is codistinguished to the set of left-invariant vector fields $\{L_{X_i}\}_{i=1}^{m}$.*

Proof We show below that if $\{v_j\}_{j=1}^{p}$ is codistinguished to $\{X_i\}_{i=1}^{m}$, then the three items of Definition 3 are satisfied.

(1) For the first item of Definition 3, we show that for any $g \in G$, the one-forms $\{d\pi_g^{ij}\}_{i,j=1}^p$ span T_g^*G. With slight abuse of notation, we write

$$d\pi_g^{ij}(X) := d\pi_g^{ij}(gX) = \langle v_i, \pi(g)\pi_*(X)v_j\rangle, \quad \forall X \in \mathfrak{g}.$$

In this way, each one-forms $d\pi_g^{ij}$ can be viewed as an element in $\mathfrak{g}^*$. But then, the two subspaces of $\mathfrak{g}^*$: $\text{span}\{d\pi_e^{ij}\}_{i,j=1}^p$ and $\text{span}\{d\pi_g^{ij}\}_{i,j=1}^p$ are isomorphic:

$$\sum_{i,j=1}^p c_{ij}\langle v_i, \pi_*(\cdot)v_j\rangle \underset{\pi(g^{-1})}{\overset{\pi(g)}{\rightleftarrows}} \sum_{i,j=1}^p c_{ij}\langle v_i, \pi(g)\pi_*(\cdot)v_j\rangle.$$

The first item of Definition 3 then follows from the first item of Definition 13.

(2) For the second item of Definition 3, it suffices to show that if $\pi_*(X_i)v_j = \lambda v_k$, then $L_{X_i}\pi^{qj} = \lambda\pi^{qk}$ for any $q = 1, \ldots, p$. This holds because

$$(L_{X_i}\pi^{qj})(g) = \langle v_q, \pi(g)\pi_*(X_i)v_j\rangle = \lambda\langle v_q, \pi(g)v_k\rangle = \lambda\pi^{qk}(g).$$

(3) The third item of Definition 3 directly follows from the third item of Definition 13.

□

We have so far provided an approach for generating a set of matrix coefficients that is (weakly) codistinguished to a given set of left-invariant vector fields. The same approach can be slightly modified to generate a set of functions codistinguished to a set of *right-invariant* vector fields. We provide details in the following remark:

Remark 5 We first recall that the *left-regular representation* of G is given by

$$(x, \phi) \in G \times C^\omega(G) \mapsto (l(x)\phi)(g) := \phi(x^{-1}g),$$

The corresponding Lie algebra representation is given by $l_*(X)\phi = -R_X\phi$. We again let π be a representation of G on a finite-dimensional inner-product space $(V, \langle\cdot,\cdot\rangle)$, and $\{v_i\}_{i=1}^p$ be a spanning set of V. We next define functions on G as follows:

$$\tilde{\pi}^{ij}(g) := \langle v_i, \pi(g^{-1})v_j\rangle, \quad \forall 1 \le i, j \le p. \tag{30}$$

Let $\mathbb{L}_{\tilde{\pi}}$ be the $\mathbb{R}$-span of these $\tilde{\pi}^{ij}$. The same arguments in the proof of Lemma 7 can be used here to show that $\mathbb{L}_{\tilde{\pi}}$ is closed under $l(x)$ for all $x \in G$. Furthermore, if the set $\{v_j\}_{j=1}^p$ is chosen to be codistinguished to $\{X_i\}_{i=1}^m$, then similar arguments in the proof of Lemma 8 can be used to show that the set of functions $\{\tilde{\pi}^{ij}\}_{i,j=1}^p$ is codistinguished to the set of right-invariant vector fields $\{R_{X_i}\}_{i=1}^m$. □

In summary, we have shown in the subsection that a finite-dimensional representation π of G on an inner-product space V can be used to generate a set of matrix coefficients codistinguished to a given set of left- (or right-) invariant vector fields provided that the assumption of Lemma 8 is satisfied.

4.3 On the adjoint representation

We follow the discussions in the previous subsection, and consider here the adjoint representation of G on $\mathfrak{g}$, i.e., $\pi = \mathrm{Ad}$ and $V = \mathfrak{g}$. We show that in this special case, there indeed exists a set of matrix coefficients (weakly) codistinguished to a distinguished set of left- (or right-) invariant vector fields.

To proceed, we first recall that $B(X, Y) = \mathrm{tr}(\mathrm{ad}_X\, \mathrm{ad}_Y)$ is the Killing form, θ is a Cartan involution of $\mathfrak{g}$, and $B_\theta(X, Y) = -B(X, \theta Y)$ is an inner product on $\mathfrak{g}$. We also recall that by Proposition 2, there exists a distinguished set $\{X_i\}_{i=1}^m$ out of $\mathfrak{g}$. We fix such a set in the sequel. Note, in particular, that by Definition 12, the distinguished set $\{X_i\}_{i=1}^m$ spans $\mathfrak{g}$. Now, we follow the two-step strategy proposed in the previous section and define a set of matrix coefficients $\{\phi^{ij}\}_{i,j=1}^m$ as follows:

$$\phi^{ij}(g) := \mathrm{Ad}^{ij}(g) = B_\theta(\mathrm{Ad}(g)X_j, X_i), \quad 1 \le i, j \le m. \tag{31}$$

This is nothing but specializing (28) to the case of adjoint representation. To further illustrate (31), we take advantage of the following fact [22, Prop. 6.28]:

Lemma 9 *Every semi-simple real Lie algebra $\mathfrak{g}$ is isomorphic to a Lie algebra of real matrices that is closed under transpose, with the Cartan involution θ carried to negative transpose, i.e., $\theta X = -X^\top$ for all $X \in \mathfrak{g}$.*

We note that for a given semi-simple Lie algebra $\mathfrak{g}$ of real matrices, the Killing form $B(X, Y)$ is linearly proportional to $\mathrm{tr}(XY)$, i.e., $B(X, Y) = c\,\mathrm{tr}(XY)$ for a real positive constant c. Now, suppose that G is isomorphic to a matrix Lie group; then, it follows from Lemma 9 that one can rewrite (31) as follows:

$$\phi^{ij}(g) = c\,\mathrm{tr}(gX_j g^{-1} X_i^\top). \tag{32}$$

In particular, it generalizes the functions $\{\phi^{ij}\}_{1\le i,j\le 3}$ on SO(3) introduced in Example 1 to functions on an arbitrary semi-simple matrix Lie group. However, we shall note that not every semi-simple Lie group is isomorphic to a matrix Lie group. Nevertheless, expression (31) is always valid.

Recall that a center $Z(G)$ of a group G is defined such that if $z \in G$, then z commutes with every group element g of G, i.e.,

$$Z(G) := \{z \in G \mid zg = gz, \quad \forall g \in G\}.$$

Let $\phi := (\ldots, \phi^{ij}, \ldots)$ be the collective of ϕ^{ij}. For any group element $g \in G$, we let $[g]_\phi$ be the pre-image of $\phi(g)$. We now have the following result:

Theorem 4 *Let $\{X_i\}_{i=1}^m$ be a distinguished set of $\mathfrak{g}$. Then, the set of matrix coefficients $\{\phi^{ij}\}_{i,j=1}^m$ defined in (31) is weakly codistinguished to $\{L_{X_i}\}_{i=1}^m$. Moreover,*

$$[g]_\phi = \{gz \mid z \in Z(G)\}, \quad \forall g \in G. \tag{33}$$

In particular, $\{\phi^{ij}\}_{i,j=1}^m$ is codistinguished to $\{L_{X_i}\}_{i=1}^m$ if and only if $Z(G)$ is trivial.

Theorem 3 then follows from Proposition 2 and Theorem 4. We establish Theorem 4 in the next subsection.

Remark 6 Note that if one aims to construct a set of functions codistinguished to the right-invariant vector fields $\{R_{X_i}\}_{i=1}^m$; then, by Remark 5, one can simply define functions as follows:

$$\tilde{\phi}^{ij}(g) := B_\theta(\mathrm{Ad}(g^{-1})X_j, X_i), \quad \forall 1 \le i, j \le m. \tag{34}$$

If one replaces in the statement L_{X_i} with R_{X_i} and correspondingly, ϕ^{ij} with $\tilde{\phi}^{ij}$, then Theorem 4 will still hold.

Since $\mathfrak{g}$ is semi-simple, the center $Z(G)$ is discrete. If, further, G is compact, then $Z(G)$ is finite. The centers of a few commonly seen matrix Lie groups are given below:

(1) If $G = \mathrm{SU}(n)$ is the special unitary group, then $Z(G) = \{zI \mid z^n = 1, z \in \mathbb{C}\}$.
(2) If $G = \mathrm{SL}(n, \mathbb{R})$ is the special linear group or if $G = \mathrm{SO}(n)$ is the special orthogonal group, then

$$Z(G) = \begin{cases} \{I\} & \text{if } n \text{ is odd,} \\ \{\pm I\} & \text{if } n \text{ is even.} \end{cases}$$

(3) Similarly, if $G = \mathrm{SO}^+(p, q)$ is the identity component of indefinite orthogonal group $\mathrm{O}(p, q)$ (e.g., the Lorentz group $\mathrm{O}(1, 3)$), then

$$Z(G) = \begin{cases} \{I\} & \text{if } p + q \text{ is odd,} \\ \{\pm I\} & \text{if } p + q \text{ is even.} \end{cases}$$

(4) If $G = \mathrm{Sp}(2n, \mathbb{R})$ is the symplectic group, then $Z(G) = \{\pm I_{2n}\}$.

4.4 Analysis and proof of Theorem 4

We establish in subsection Theorem 4. By Lemma 8, it suffices to show that the subset $\{X_i\}_{i=1}^m$ of $\mathfrak{g}$ is codistinguished to itself with respect to the adjoint representation. This fact will be established after a sequence of lemmas. For convenience, we reproduce below the set of functions $\{\phi^{ij}\}_{i,j=1}^m$:

$$\phi^{ij}(g) := \mathrm{Ad}^{ij}(g) = B_\theta(\mathrm{Ad}(g)X_j, X_i), \quad 1 \le i, j \le m.$$

We show below that the set $\{\phi^{ij}\}_{i,j=1}^m$ satisfies the three items of Definition 13 under the assumption of Theorem 4. The arguments we will use below generalize the ones used in Example 1. For the first item of Definition 13, we have the following fact:

Lemma 10 *The set of one-forms $\{\mathrm{d}\phi_e^{ij}\}_{i,j=1}^m$ spans the cotangent space $T_e^*G \approx \mathfrak{g}^*$.*

Proof First, note that for any $X \in \mathfrak{g}$, we have

$$\mathrm{d}\phi_e^{ij}(X) = B_\theta([X, X_j], X_i) = -B([X, X_j], \theta X_i).$$

Because the Killing form is adjoint-invariant, i.e., $B([X, Y], Z) = B(X, [Y, Z])$ for any $X, Y, Z \in \mathfrak{g}$, it follows that

$$\mathrm{d}\phi_e^{ij}(X) = -B([X, X_j], \theta X_i) = -B(X, [X_j, \theta X_i]) = B_\theta(X, [\theta X_j, X_i]),$$

where the last equality holds because θ is a Lie algebra automorphism with $\theta^2 = \mathrm{id}$ and, hence, $\theta[X_j, \theta X_i] = [\theta X_j, X_i]$.

For convenience, let $Y_j := \theta X_j$ for all $j = 1, \ldots, m$. Since θ is a Lie algebra automorphism and $\{X_i\}_{i=1}^m$ spans $\mathfrak{g}$, the set $\{Y_j\}_{j=1}^m$ spans $\mathfrak{g}$ as well. Next, note that $\mathfrak{g}$ is semi-simple and, hence, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Thus, $\{\hat{X}_{ij} := [Y_j, X_i]\}_{i,j=1}^m$ is a spanning set of $\mathfrak{g}$. It now remains to show that the set of one-forms $\{B_\theta(\cdot, \hat{X}_{ij})\}_{i,j=1}^m$ spans $\mathfrak{g}^*$. But, this follows from the fact that B_θ is positive definite on $\mathfrak{g}$; indeed, any nondegenerate bilinear form induces a linear isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$. Since the set $\{\hat{X}_{ij}\}_{i,j=1}^m$ spans $\mathfrak{g}$, the set of one-forms $\{B_\theta(\cdot, \hat{X}_{ij})\}_{i,j=1}^m$ spans $\mathfrak{g}^*$. □

For the second item of Definition 13, we have the following fact:

Lemma 11 *If $[X_i, X_j] = \lambda X_k$, then $L_{X_i}\phi^{i'j} = \lambda\phi^{i'k}$ for all $i' = 1, \ldots, m$.*

Proof The lemma directly follows from computation:

$$(L_{X_i}\phi^{i'j})(g) = B_\theta(\mathrm{Ad}(g)[X_i, X_j], X_{i'}) = \lambda B_\theta(\mathrm{Ad}(g)X_k, X_{i'}) = \lambda\phi^{i'k}(g),$$

which holds for any $g \in G$. □

Combining Lemmas 10 and 11, we have that the set of functions $\{\phi^{ij}\}_{i,j=1}^m$ is weakly codistinguished to the set of left-invariant vector fields $\{L_{X_i}\}_{i=1}^m$. Finally, for (33), we have the following fact:

Lemma 12 *If $\phi^{ij}(g) = \phi^{ij}(g')$ for all $1 \leq i, j \leq m$, then $g^{-1}g' \in Z(G)$ and vice versa.*

Proof We fix a $j = 1, \ldots, m$ and have the following:

$$\phi^{ij}(g) - \phi^{ij}(g') = B_\theta(\mathrm{Ad}(g)X_j - \mathrm{Ad}(g')X_j, X_i) = 0, \quad \forall i = 1, \ldots, m.$$

Since B_θ is positive definite and $\{X_i\}_{i=1}^m$ spans $\mathfrak{g}$, $\mathrm{Ad}(g)X_j = \mathrm{Ad}(g')X_j$. This holds for all $j = 1, \ldots, m$. Using again the fact that $\{X_j\}_{j=1}^m$ spans $\mathfrak{g}$, we obtain that $\mathrm{Ad}(g^{-1}g')X = X$ for all $X \in \mathfrak{g}$. Thus, $g^{-1}g'$ belongs to the centralizer of the identity component of G. Since G is connected, this holds if and only if $g^{-1}g' \in Z(G)$. □

4.5 On homogeneous spaces

Let a group G act on a manifold M. We say that the group action is *transitive* if for any $x, y \in M$, there exists a group element $g \in G$ such that $gx = y$. Correspondingly, the manifold M is said to be a *homogeneous space* of G. Note that any homogeneous space can be identified with the space G/H of left cosets gH for H a closed Lie subgroup of G. More specifically, we pick an arbitrary point $x \in M$, and let H be the subgroup of G which leaves x fixed (i.e., H is the stabilizer of x). Then, M is diffeomorphic to G/H, and we write $M \approx G/H$. The group action can thus be viewed as a map by sending a pair $(g, g'H)$ to $gg'H$. We also note that the homogeneous space M can be equipped with a unique analytic structure (see [18, Thm. 4.2]).

We address in the subsection the existence of distinguished vector fields and codistinguished functions on homogeneous spaces of a semi-simple Lie group. We provide at the end of the subsection a simple example in which the unit sphere $S^2 \approx \mathrm{SO}(3)/\mathrm{SO}(2)$ is considered.

4.5.1 On distinguished vector fields

There is a canonical way of translating a distinguished set $\{X_i\}_{i=1}^m$ of the Lie algebra $\mathfrak{g}$ to a distinguished set of vector fields over a homogeneous space of G. Precisely, we define a map $\tau : \mathfrak{g} \to \mathfrak{X}(M)$ as follows: Let $\exp : \mathfrak{g} \to G$ be the exponential map. For a given $X \in \mathfrak{g}$, we define a vector field $\tau(X) \in \mathfrak{X}(M)$ such that for any $\phi \in \mathrm{C}^\omega(M)$, the following hold:

$$(\tau(X)\phi)(x) := \lim_{t\to 0} \frac{\phi(\exp(tX)x) - \phi(x)}{t}, \quad \forall x \in M. \tag{35}$$

Let X_i and X_j be any two elements in $\mathfrak{g}$. It is known [18, Chapter 2.3]) that

$$[\tau(X_i), \tau(X_j)] = -\tau([X_i, X_j]), \tag{36}$$

which then leads to the following result:

Proposition 4 *Let G be a semi-simple Lie group with $\mathfrak{g}$ the Lie algebra, and M be a homogeneous space of G. If $\{X_i\}_{i=1}^m$ is a distinguished set of $\mathfrak{g}$, then $\{\tau(X_i)\}_{i=1}^m$ is a distinguished set of vector fields over M.*

Proof It suffices to show that $\{\tau(X_i)(x)\}_{i=1}^m$ spans the tangent space $T_x M$ for all $x \in M$. Let H be the stabilizer of x, and $\mathfrak{h}$ be the corresponding Lie algebra of H. Since $\{X_i\}_{i=1}^m$ spans $\mathfrak{g}$, there must exist a subset of $\{X_i\}_{i=1}^m$, say $\{X_i\}_{i=1}^{m'}$, such that if we let $\mathfrak{m} := \mathrm{span}\{X_i\}_{i=1}^{m'}$, then $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. Moreover, the following map:

$$(a_1, \ldots, a_{m'}) \in \mathbb{R}^{m'} \mapsto \exp\left(\sum_{i=1}^{m'} a_i X_i\right) x \in M$$

is locally a diffeomorphism around $0 \in \mathbb{R}^{m'}$ to an open neighborhood of $x \in M$. This, in particular, implies that $\{\tau(X_i)(x)\}_{i=1}^{m'}$ is a basis of the tangent space $T_x M$. □

4.5.2 On codistinguished functions

We now discuss how to translate a set of codistinguished functions defined on a Lie group G to a set of codistinguished functions on its homogeneous space $M \approx G/H$. We consider below for the case where the closed subgroup H is compact.

We say that a function $\phi \in C^\omega(G)$ is H-*invariant* if for any $g \in G$ and $h \in H$, we have $\phi(gh) = \phi(g)$. In particular, if ϕ is H-invariant, then one can simply define a function ψ on M by $\psi(gH) := \phi(g)$. This is well defined because if $gH = g'H$, then $g^{-1}g'$ belongs to H and, hence, $\phi(g) = \phi(gg^{-1}g') = \phi(g')$. Thus, without any ambiguity, we can treat an H-invariant ϕ as a function defined on M as well.

If a function ϕ is *not* H-invariant, then one can construct an H-invariant function by averaging ϕ over the subgroup H. Since H is compact, we equip H with the normalized Haar measure [22, Ch. VIII], i.e., $\int_H \mathbf{1}_H dh = 1$. We then define a function on G (and on M) by averaging the given function ϕ over H as follows:

$$\bar{\phi}(g) := \int_H \phi(gh)dh. \tag{37}$$

It should be clear that $\bar{\phi}$ is H-invariant; indeed, for any $h' \in H$, we have

$$\bar{\phi}(gh') := \int_H \phi(gh'h)dh = \int_H \phi(gh)d(h'^{-1}h) = \int_H \phi(gh)dh = \bar{\phi}(g).$$

Note that if ϕ itself is H-invariant, then $\bar{\phi} = \phi$. We now have the following fact:

Lemma 13 *Let $\{\phi^j\}_{j=1}^l$ be a set of functions on G codistinguished to a set of right-invariant vector fields $\{R_{X_i}\}_{i=1}^m$. If $R_{X_i}\phi^j = \lambda\phi^k$, then $\tau(X_i)\bar{\phi}^j = \lambda\bar{\phi}^k$.*

Proof The lemma directly follows from computation:

$$(\tau(X_i)\bar{\phi}^j)(gH) = \int_H (R_{X_i}\phi^j)(gh)dh = \lambda \int_H \phi^k(gh)dh = \lambda\bar{\phi}^k(gH),$$

which holds for all $gH \in M \approx G/H$. □

Thus, if the set of one-forms $\{d\bar{\phi}_x^j\}_{j=1}^l$ spans $T_x^* M$, then, by Lemma 13, $\{\bar{\phi}^j\}_{j=1}^l$ is (weakly) codistinguished to $\{\tau(X_i)\}_{i=1}^m$. We provide below an example for illustration.

4.5.3 Example on $S^2 \approx$ SO(3)/SO(2)

Let SO(3) act S^2 by sending $(g, x) \in \mathrm{SO}(3) \times S^2$ to $gx \in S^2$. Let H be a subgroup of G defined as follows:

$$H = \left\{ h(\theta) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix} \mid \theta \in [0, 2\pi) \right\} \approx \mathrm{SO}(2).$$

It follows that H is the stabilizer of the vector $e_1 \in S^2$. Let $\{X_i\}_{i=1}^3$ and $\{\phi^{ij}\}_{i,j=1}^3$ be given in Example 1, i.e.,

$$\begin{cases} X_i = e_j e_k^\top - e_k e_j^\top, & \text{where } \det(e_i, e_j, e_k) = 1, \\ \phi^{ij}(g) = \mathrm{tr}(g X_j g^\top X_i^\top), & 1 \le i, j \le 3. \end{cases}$$

Because the set $\{X_i\}_{i=1}^3$ is distinguished in $\mathfrak{so}(3)$, by Proposition 4, it induces a distinguished set of vector fields $\{\tau(X_i)\}_{i=1}^3$ over S^2 as follows:

$$\tau(X_1)(x) = \begin{bmatrix} 0 \\ x_3 \\ -x_2 \end{bmatrix}, \quad \tau(X_2)(x) = \begin{bmatrix} -x_3 \\ 0 \\ x_1 \end{bmatrix}, \quad \tau(X_3)(x) = \begin{bmatrix} x_2 \\ -x_1 \\ 0 \end{bmatrix}.$$

These vector fields satisfy the following relationship:

$$[\tau(X_i), \tau(X_j)] = \det(e_i, e_j, e_k)\tau(X_k).$$

We next compute the averaged H-invariant functions $\{\bar{\phi}^{ij}\}_{i,j=1}^3$. The normalized Haar measure on H, in this case, is simply given by $dh = \mathrm{d}\theta/2\pi$. It follows that

$$\bar{\phi}^{ij}(gH) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{tr}(g h(\theta) X_j h(\theta)^\top g^\top X_i^\top) \mathrm{d}\theta.$$

To evaluate the above integral, we first have the following computational result:

$$\frac{1}{2\pi} \int_0^{2\pi} h(\theta) X_j h(\theta)^\top \mathrm{d}\theta = \begin{cases} X_1 & \text{if } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the nonzero $\bar{\phi}^{ij}$'s are given by

$$\bar{\phi}^i(gH) := \bar{\phi}^{i1}(gH) = \mathrm{tr}(g X_1 g^\top X_i^\top), \quad \forall i = 1, 2, 3. \tag{38}$$

Each left coset gH corresponds to the point $x = ge_1 \in S^2$. Note that ge_1 is simply the first column of g. We now compute each function $\bar{\phi}^i(x)$ and express the results using only the coordinates x_i of x. First, by computation, we obtain

$$g X_1 g^\top = \begin{bmatrix} 0 & c_{31} & c_{21} \\ -c_{31} & 0 & c_{11} \\ -c_{21} & -c_{11} & 0 \end{bmatrix},$$

where each c_{ij} is the ijth entry of the cofactor matrix $[c_{ij}]$ of $g \in \mathrm{SO}(3)$. Since g is a special orthogonal matrix, $g = [c_{ij}]$. In particular, (c_{11}, c_{21}, c_{31}) is the first column of g, i.e., $(c_{11}, c_{21}, c_{31}) = ge_1 = (x_1, x_2, x_3)$ and, hence,

$$g X_1 g^\top = \begin{bmatrix} 0 & x_3 & x_2 \\ -x_3 & 0 & x_1 \\ -x_2 & -x_1 & 0 \end{bmatrix}.$$

Thus, the functions $\bar{\phi}^i$ in (38), for $i = 1, 2, 3$, are nothing but twice the coordinate functions, i.e.,

$$\bar{\phi}^i(x) = 2x_i.$$

It should be clear that $\{\bar{\phi}^j\}_{j=1}^3$ satisfies items 1 and 3 of Definition 3. For item 2, we have that $\tau(X_i)\bar{\phi}^j = \det(e_i, e_j, e_k)\bar{\phi}^k$. Thus, $\{\bar{\phi}^j\}_{j=1}^3$ is codistinguished to $\{\tau(X_i)\}_{i=1}^3$.

5 Conclusions

We introduced in the paper a novel class of ensemble systems, which we call distinguished ensemble systems. Every such system is comprised of two key components: A set of distinguished vector fields and a set of (weakly) codistinguished functions. We established in Sect. 3 that a distinguished ensemble system is approximately ensemble path controllability and (weakly) ensemble observable. We further extended in Sect. 3.5 the result to a pre-distinguished ensemble system.

We proposed and addressed in Sect. 4 the problem about existence of distinguished vector fields and codistinguished functions on a given manifold M. We provided an affirmative answer for the case where M is a connected, semi-simple Lie group G. Specifically, we showed that every such Lie group G admits a set of distinguished left- (or right-) invariant vector fields, together with a set of matrix coefficients that is (weakly) codistinguished to the set of vector fields. Finally, we discussed in Sect. 4.5 how to translate distinguished vector fields and codistinguished functions from the Lie group G to its homogeneous spaces, yet the problem has not been solved completely and will be addressed in our future work.

References

1. Agrachev A, Baryshnikov Y, Sarychev A (2016) Ensemble controllability by Lie algebraic methods. ESAIM Control Optim Calc Var 22(4):921–938
2. Beauchard K, Coron JM, Rouchon P (2010) Controllability issues for continuous-spectrum systems and ensemble controllability of Bloch equations. Commun Math Phys 296(2):525–557
3. Becker A, Bretl T (2012) Approximate steering of a unicycle under bounded model perturbation using ensemble control. IEEE Trans Robot 28(3):580–591
4. Brockett RW (2007) Optimal control of the Liouville equation. AMS IP Stud Adv Math 39:23
5. Chen X (2018) Controllability of ensemble formation systems over digraphs. Automatica arXiv:1805.11196
6. Chen X, Gharesifard B (2017) Distinguished sets of semi-simple Lie algebras. arXiv:1711.01719
7. Chen Y, Georgiou TT, Pavon M (2016) Optimal steering of a linear stochastic system to a final probability distribution, Part I. IEEE Trans Autom Control 61(5):1158–1169
8. Chen Y, Georgiou TT, Pavon M (2016) Optimal steering of a linear stochastic system to a final probability distribution, Part II. IEEE Trans Autom Control 61(5):1170–1180
9. Curtain RF, Zwart H (2012) An introduction to infinite-dimensional linear systems theory, vol 21. Springer, Berlin
10. De Persis C, Isidori A (2000) On the observability codistributions of a nonlinear system. Syst Control Lett 40(5):297–304
11. Fuhrmann PA, Helmke U (2015) The mathematics of networks of linear systems. Springer, Berlin

12. Gauthier J, Bornard G (1981) Observability for any $u(t)$ of a class of nonlinear systems. IEEE Trans Autom Control 26(4):922–926
13. Gauthier JP, Kupka IA (1994) Observability and observers for nonlinear systems. SIAM J Control Optim 32(4):975–994
14. Glaser SJ, Schulte-Herbrüggen T, Sieveking M, Schedletzky O, Nielsen NC, Sørensen OW, Griesinger C (1998) Unitary control in quantum ensembles: maximizing signal intensity in coherent spectroscopy. Science 280(5362):421–424
15. Greene RE, Jacobowitz H (1971) Analytic isometric embeddings. Ann Math 93(1):189–204
16. Gronwall TH (1919) Note on the derivatives with respect to a parameter of the solutions of a system of differential equations. Ann Math 20(4):292–296
17. Hatcher A (2002) Algebraic topology. Cambridge University Press, Cambridge
18. Helgason S (2001) Differential geometry, Lie groups, and symmetric spaces, vol 34. American Mathematical Society, Providence
19. Helmke U, Schönlein M (2014) Uniform ensemble controllability for one-parameter families of time-invariant linear systems. Syst Control Lett 71:69–77
20. Hermann R, Krener A (1977) Nonlinear controllability and observability. IEEE Trans Autom Control 22(5):728–740
21. Humphreys J (1972) Introduction to Lie algebras and representation theory, vol 9. Springer, Berlin
22. Knapp AW (2013) Lie groups beyond an introduction, vol 140. Springer, Berlin
23. Leonard NE, Krishnaprasad PS (1995) Motion control of drift-free, left-invariant systems on Lie groups. IEEE Trans Autom control 40(9):1539–1554
24. Li JS (2011) Ensemble control of finite-dimensional time-varying linear systems. IEEE Trans Autom Control 56(2):345–357
25. Li JS, Khaneja N (2006) Control of inhomogeneous quantum ensembles. Phys Rev A 73(3):030302
26. Li JS, Khaneja N (2009) Ensemble control of Bloch equations. IEEE Trans Autom Control 54(3):528–536
27. Liu W (1997) An approximation algorithm for nonholonomic systems. SIAM J Control Optim 35(4):1328–1365
28. Murray RM, Sastry SS (1993) Steering nonholonomic control systems using sinusoids. In: Li Z, Canny JF (eds) Nonholonomic motion planning. Springer, Berlin, pp 23–51
29. Nash J (1966) Analyticity of the solutions of implicit function problems with analytic data. Ann Math 84(3):345–355
30. Rudin W (1976) Principles of mathematical analysis. McGraw-Hill, New York
31. Rudin W (2006) Real and complex analysis. Tata McGraw-Hill Education, New York
32. Sussmann HJ, Liu W (1993) Lie bracket extensions and averaging: the single-bracket case. In: Li Z, Canny JF (eds) Nonholonomic motion planning. Springer, Berlin, pp 109–147
33. Taniguchi T, Sugiyama H, Uekusa H, Shiro M, Asahi T, Koshima H (2018) Walking and rolling of crystals induced thermally by phase transition. Nat Commun 9(1):538
34. Van der Schaft A (1982) Observability and controllability for smooth nonlinear systems. SIAM J Control Optim 20(3):338–354
35. Yu Y, Nakano M, Ikeda T (2003) Photomechanics: directed bending of a polymer film by light. Nature 425(6954):145
36. Zeng S, Waldherr S, Ebenbauer C, Allgöwer F (2016) Ensemble observability of linear systems. IEEE Trans Autom Control 61(6):1452–1465