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ORIGINAL ARTICLE

The persistence of impulse controllability

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Abstract
This paper shows that within the space of all LTI systems, equipped with the Zariski topology, the set of impulse controllable systems contains an open dense set of systems; in other words, impulse controllable systems are generic. This genericity persists for many closed subsets of LTI systems of interest, such as the class of singular descriptor systems.

Keywords Impulse controllability · Zariski topology · Robustness · Genericity

Mathematics Subject Classification 93B05 · 93B35 · 93B25

1 Introduction

This paper studies initial value problems for under-determined linear time-invariant systems described by differential-algebraic equations (DAE). The trajectories of the system are defined for $t \geqslant 0$, driven by the initial value at $t = 0^-$. As the system is under-determined, and therefore non-autonomous, there are several possible trajectories satisfying the same initial value, some of which could be impulsive at $t = 0$, i.e. have singular support at $t = 0$. An important control problem for such systems is to determine an input which eliminates such impulses in its response, and to further determine whether a feedback controller could implement such elimination. If this is possible, then the system is said to be impulse controllable.

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These questions were first posed for singular descriptor state space systems described by

$$E\dot{x} = Ax + Bu,$$

for instance, see [3] for an exposition. In this work, Dai's focus is on descriptor systems defined by so-called regular pencils, whereas we include those defined by singular pencils as well. The problem of impulses due to initial values was viewed as an 'inconsistency' in the initial conditions. A comprehensive solution to the problem of eliminating these impulses was provided in [7,12] and then in [2], in terms of necessary and sufficient conditions on the matrices E, A and B. The implementation of eliminating impulses by feedback controllers was also investigated in this work. More about work in this area can be found in the recent survey paper [1].

A vast generalisation of Kalman's state space theory was developed by Willems and summarised in [13]. In this paradigm, a system is defined to be the collection of all its possible trajectories, called its *behaviour*. These trajectories, those that could possibly arise, are exactly the ones that obey the laws of the system. For example, in the case of a descriptor system, the above equations of state are rewritten as $E\dot{x} - Ax - Bu = 0$, and these equations are the laws that govern all the possible evolutions of x and u. All these evolutions, considered together as one object, are the descriptor system, or its behaviour. Additional laws imposed on the system, in the form of additional equations relating x and u, would restrict this set of possible trajectories to a smaller subset, namely those which satisfy these additional equations as well. A careful choice of the additional equations would then result in a desired behaviour of the descriptor system, the undesired trajectories having been ruled out by the imposition of additional laws. This is precisely Willems' behavioural generalisation of the notion of control.

With this interpretation, Willems generalised state space theory further, by considering systems whose laws are more general DAEs, of possibly higher order. In this larger framework, the study of impulsive solutions in higher-order systems, and of their elimination, has been pursued in [5,11]. More recently, in [4], the notion of impulse controllability has also been generalised and conditions deduced for higher-order systems to be impulse controllable.

In this paper, we work in the above general framework. Here we address issues which arise because the equations describing a system involve parameters which can only be approximately determined. For this reason, it is important to study the persistence of system properties due to perturbations of these parameters [10]. These ideas go back to Pontryagin's theory of structural stability for autonomous systems. Thus, we first formulate a notion of perturbation (following [9]) and then determine whether impulse controllability is robust with respect to these perturbations.

We illustrate the nature of the problem with the following elementary example from power electronics.

Example 1.1 Consider the following RLC circuit, which is a substantial simplification of the DC-DC buck-boost converter.

Closing the switch in the circuits in Fig. 1 can cause sparking and damage. This occurs because of a so-called inconsistency in the initial conditions: this is defined later below.

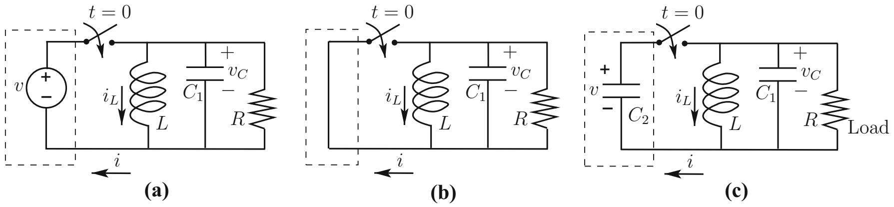

Fig. 1 RLC circuit: a simplified boost converter

The control problem is to prevent this sparking. The system equations for the circuit in Fig. 1a are:

$$\begin{bmatrix} C_1\frac{\mathsf{d}}{\mathsf{dt}}+\frac{1}{R} & 1 & -1 & 0 \\ -1 & L\frac{\mathsf{d}}{\mathsf{dt}} & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} v_C \\ i_L \\ i \\ v \end{bmatrix}=0, \tag{1}$$

whose input is v, often a constant DC source.

In the unforced case, i.e. when the external input $v = 0$, we obtain the circuit in Fig. 1b and this is described by the equations:

$$\begin{bmatrix} C_1\frac{\mathsf{d}}{\mathsf{dt}}+\frac{1}{R} & 1 & -1 \\ -1 & L\frac{\mathsf{d}}{\mathsf{dt}} & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_C \\ i_L \\ i \end{bmatrix}=0. \tag{2}$$

In this case, the system contains impulsive solutions. However, with nonzero inputs, the original system i.e. Fig. 1a, described by Eq. (1), is impulse controllable; namely, for every initial condition, there exists an input v for which the solution is free of impulses.

We next consider the case when the input v is a feedback of the current i, for example the circuit in Fig. 1c with a capacitor C_2 as shown. This system is described by the equations:

$$\begin{bmatrix} C_1\frac{\mathsf{d}}{\mathsf{dt}}+\frac{1}{R} & 1 & -1 & 0 \\ -1 & L\frac{\mathsf{d}}{\mathsf{dt}} & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & C_2\frac{\mathsf{d}}{\mathsf{dt}} \end{bmatrix}\begin{bmatrix} v_C \\ i_L \\ i \\ v \end{bmatrix}=0. \tag{3}$$

This system, now autonomous, still has impulsive solutions and is not impulse controllable. □

The rest of the paper is organised as follows: In Sect. 2, we recall results on impulse controllability. In Sect. 3, we equip the set of all LTI systems with a topology with respect to which we study perturbations of a system. Section 4 contains the main results of this paper, which include robustness results for singular descriptor systems also. Finally, we use the above example to explain these results.

2 Preliminaries

Central to the study of impulse controllability is the space $\mathcal{F}$ in which we locate solutions to a differential equation with initial conditions. We construct it as follows.

For every $\epsilon > 0$, define $\mathcal{F}_\epsilon$ to be the set of distributions on $(-\epsilon, \infty)$ whose singular support is contained in $\{0\}$. Thus, an element in $\mathcal{F}_\epsilon$ is an infinitely differentiable function on $(-\epsilon, 0) \cup (0, \infty)$, but not necessarily so in any neighbourhood of 0. We define an equivalence relation $\sim$ on the disjoint union $\bigsqcup_{\epsilon>0} \mathcal{F}_\epsilon$ as follows: two elements $f_1 \in \mathcal{F}_{\epsilon_1}$ and $f_2 \in \mathcal{F}_{\epsilon_2}$ are equivalent if there exists an $\epsilon > 0$, less than both ϵ_1 and ϵ_2, such that $f_1|_{(-\epsilon,\infty)} = f_2|_{(-\epsilon,\infty)}$. The space $\mathcal{F}$ is the set $\{\bigsqcup_{\epsilon>0} \mathcal{F}_\epsilon / \sim\}$ of all equivalence classes. We differentiate an element in $\mathcal{F}$ by differentiating any one of its representatives (in the distributional sense). The derivative is well defined and belongs to $\mathcal{F}$. Differentiation equips $\mathcal{F}$ with the structure of a module over the ring $A = \mathbb{C}[\frac{\mathrm{d}}{\mathrm{dt}}]$ of ordinary differential operators.

In this paper, we study systems defined as the kernel of an operator of the type

$$M\left(\frac{\mathrm{d}}{\mathrm{dt}}\right) : \mathcal{F}^k \to \mathcal{F}^\ell, \; \ell \leqslant k$$

where $M(\frac{\mathrm{d}}{\mathrm{dt}}) \in \mathcal{M}_{\ell,k}$, the set of all $\ell \times k$ matrices with entries from the ring A. Such operators are precisely the linear, shift invariant and support decreasing operators, which we abbreviate to *differential operators* henceforth. Singular descriptor state space systems are thus a special case of our study.

We next recall the definition of impulse controllability; it requires the notions of a state map, the set of initial conditions, and that of the consistency of an initial condition.

Given the differential operator $M(\frac{\mathrm{d}}{\mathrm{dt}}) \in \mathcal{M}_{\ell,k}$, we construct a differential operator $X_M(\frac{\mathrm{d}}{\mathrm{dt}})$, called its state map, by the shift-and-cut method of [8].

Define $\sigma : A \to A$ by $\sigma\left(c_0 + c_1\frac{\mathrm{d}}{\mathrm{dt}} + \cdots + c_{r+1}\frac{\mathrm{d}^{\mathrm{r}+1}}{\mathrm{dt}^{\mathrm{r}+1}}\right) = c_1 + \cdots + c_{r+1}\frac{\mathrm{d}^{\mathrm{r}}}{\mathrm{dt}^{\mathrm{r}}}$.

Higher-order actions σ^i of σ are defined recursively. The action $\sigma(M(\frac{\mathrm{d}}{\mathrm{dt}}))$ on a differential operator $M(\frac{\mathrm{d}}{\mathrm{dt}})$ is defined by the action of σ on each of its entries.

Suppose n is the highest degree amongst the entries in $M(\frac{\mathrm{d}}{\mathrm{dt}})$. Then, its state map, $X_M(\frac{\mathrm{d}}{\mathrm{dt}})$, is the operator obtained by stacking $\sigma(M(\frac{\mathrm{d}}{\mathrm{dt}})), \sigma^2(M(\frac{\mathrm{d}}{\mathrm{dt}})), \ldots, \sigma^n(M(\frac{\mathrm{d}}{\mathrm{dt}}))$ into a column and then retaining only those rows which form a basis, as a $\mathbb{C}$-vector space, for the row span of this tall matrix. Let the number of these rows of $X_M(\frac{\mathrm{d}}{\mathrm{dt}})$ be denoted by ℓ_X.

The map obtained by the above procedure (i.e. shift and cut, followed by retaining those rows which form a basis) that maps an operator $M(\frac{\mathrm{d}}{\mathrm{dt}}) \in \mathcal{M}_{\ell,k}$ to the operator $X_M(\frac{\mathrm{d}}{\mathrm{dt}}) \in \mathcal{M}_{\ell_X,k}$ is called the *canonical state map*.

Now define the set $S_X(M) \subseteq \mathbb{C}^{\ell_X}$ by

$$S_X(M) = \left\{ \left(X_M\left(\frac{\mathrm{d}}{\mathrm{dt}}\right) f \right)_{t=1} \mid f \in \mathcal{F}^k \right\}. \tag{4}$$

The set $S_X(M)$ is[1] the *space of all initial conditions*.

An element $\alpha \in S_X(M)$ is said to be *a consistent initial condition* if there exists an $\mathcal{L}^1_{loc}$ representative of $f \in \mathcal{F}^k$ such that $\alpha = \left(X_M(\frac{\mathsf{d}}{\mathsf{dt}})f\right)_{t=0^-}$.

Definition 2.1 The system defined by the kernel of $M(\frac{\mathsf{d}}{\mathsf{dt}}) : \mathcal{F}^k \to \mathcal{F}^\ell$ is said to be impulse controllable if every element in the space of initial conditions $S_X(M)$, constructed from the corresponding canonical state map, is a consistent initial condition.

We now point out an important difference between the systems defined above and those systems whose trajectories are functions or distributions on the entire real line (for instance Willems [13]). In the latter case, the object of importance is not the operator $M(\frac{\mathsf{d}}{\mathsf{dt}})$ itself, but the A-submodule M of A^k generated[2] by the ℓ rows of $M(\frac{\mathsf{d}}{\mathsf{dt}})$. Thus, two operators whose rows generate the same submodule of A^k define the same system and are hence equivalent. However, in the context of initial value problems that we study in this paper, operators whose rows generate the same submodule need not define the same system, due to differences in their impulsive responses to initial conditions. Thus, we need a different notion of equivalence, and for the purposes of this paper we use the following definition.

Definition 2.2 Differential operators $M(\frac{\mathsf{d}}{\mathsf{dt}})$ and $N(\frac{\mathsf{d}}{\mathsf{dt}})$ in $\mathcal{M}_{\ell,k}$ are equivalent over $\mathcal{F}$ if there exists an invertible matrix $C \in \mathbb{C}^{\ell\times\ell}$ such that $N(\frac{\mathsf{d}}{\mathsf{dt}}) = CM(\frac{\mathsf{d}}{\mathsf{dt}})$.

We denote the set of these equivalence classes by $\mathcal{S}_{\ell,k} := (\mathcal{M}_{\ell,k}/\sim)$, where, to emphasise, the equivalence $\sim$ is given by left multiplication by $\mathsf{GL}_\ell(\mathbb{C})$ as in the above definition and not by the action of a general unimodular matrix in $\mathsf{GL}_\ell(A)$ (for the reason that we have noted above).

Formally, we define a *system* to be an element of $\mathcal{S}_{\ell,k}$, and any operator in its equivalence class is a *representation* of it.

A more general notion of equivalence over the space $\mathcal{F}$ has been studied in [6], which involves the concept of zeros of a polynomial matrix at infinity (see below). This notion of equivalence is dependent on the specific degrees of the entries in the two matrices, but since in this paper we work with families of systems of arbitrary degree, we use instead the above stronger definition of equivalence. This definition is consistent with our study of initial value problems, because two differential operators whose rows span the same $\mathbb{C}$-subspace of $\mathcal{M}_{\ell,k}$ define the same system over the space $\mathcal{F}$. Thus, while the rows of the differential operator need not be A-independent, in view of the above definition we can, and do, assume that they are $\mathbb{C}$-independent. We also assume always that $\ell \leqslant k$, i.e. in this paper, we restrict our study to systems which are under-determined.

We now state a result for a class of systems to be impulse controllable. This notion is the ability to eliminate impulses in the system by using appropriate inputs, and it

[1] By shift invariance, the choice of $t = 1$ above is not important, any $t \neq 0$ suffices. We use the value $\left(X_M(\frac{\mathsf{d}}{\mathsf{dt}})f\right)_{t=1}$ as an initial value of a potential solution f to $M(\frac{\mathsf{d}}{\mathsf{dt}})$, i.e. for $\lim_{t\nearrow 0} f(t)$.

[2] Indeed, the kernel of the operator $M(\frac{\mathsf{d}}{\mathsf{dt}}) : (\mathcal{D}')^k \to (\mathcal{D}')^\ell$ is isomorphic to $\mathsf{Hom}_A(A^k/M,\ \mathcal{D}')$, the module of homomorphisms from A^k/M to the space of distributions $\mathcal{D}'$, for instance [9].

has been shown to be equivalent to the elimination of impulses by feedback control [2,4]. The result is formulated using the following notion of zeros at infinity.

Let $M(\frac{\mathsf{d}}{\mathsf{dt}}) \in \mathcal{M}_{\ell,k}$, and suppose that δ_i is the maximum degree of all the $i \times i$ minors of $M(\frac{\mathsf{d}}{\mathsf{dt}})$ (the degree of the 0 operator in A is defined to be $-\infty$). Then, $M(\frac{\mathsf{d}}{\mathsf{dt}})$ is said to have no zeros at infinity if each of the following $\ell - 1$ inequalities is satisfied:

$$\delta_1 \leqslant \delta_2 \leqslant \cdots \leqslant \delta_{\ell-1} \leqslant \delta_\ell. \tag{5}$$

Remark 2.1 Suppose that the above chain of inequalities is satisfied by $M(\frac{\mathsf{d}}{\mathsf{dt}})$ and that $\delta_\ell = -\infty$. Then, δ_1 is also equal to $-\infty$, and so $M(\frac{\mathsf{d}}{\mathsf{dt}})$ is the 0 operator. Hence, we may always assume that if $M(\frac{\mathsf{d}}{\mathsf{dt}})$ satisfies Eq. (5), then $\delta_\ell \geqslant 0$. By Proposition 3.1 of [9], this is equivalent to asserting that the rows of $M(\frac{\mathsf{d}}{\mathsf{dt}})$ are A-independent.

We next observe that if a nonzero $M(\frac{\mathsf{d}}{\mathsf{dt}})$ satisfies Eq. (5), then so does any differential operator $N(\frac{\mathsf{d}}{\mathsf{dt}})$ equivalent to it in the sense of Definition 2.2.

Clearly, δ_ℓ of one operator equals that of any other equivalent one, for the maximal minors of one are equal to the corresponding maximal minors of the other multiplied by the determinant of an element in $\mathsf{GL}_\ell(\mathbb{C})$. By the above remark, this $\delta_\ell \geqslant 0$. Furthermore, the rows of one operator are $\mathbb{C}$-linear combinations of the rows of the other. Thus, in order to establish our observation, it suffices to assume that if $r_1, r_2, \ldots, r_\ell$ are the rows of $M(\frac{\mathsf{d}}{\mathsf{dt}})$, then the first row of $N(\frac{\mathsf{d}}{\mathsf{dt}})$ equals $c_1 r_1 + c_2 r_2 + \cdots + c_\ell r_\ell$ (where the c_i are in $\mathbb{C}$), and that its other rows are again $r_2, \ldots, r_\ell$. It now follows that $c_1 \neq 0$, for otherwise, all the $\ell \times \ell$ minors of $N(\frac{\mathsf{d}}{\mathsf{dt}})$ would be 0, and its δ_ℓ would equal $-\infty$, a contradiction. By multi-linearity of the determinant, the $i \times i$ minors which involve the first row of $N(\frac{\mathsf{d}}{\mathsf{dt}})$ are equal to the corresponding minors of $M(\frac{\mathsf{d}}{\mathsf{dt}})$ multiplied by c_1. The other minors of $N(\frac{\mathsf{d}}{\mathsf{dt}})$, not involving the first row, are of course identical to the corresponding minors of $M(\frac{\mathsf{d}}{\mathsf{dt}})$. Hence, the corresponding δ_i of the two differential operators are equal for all i.

Thus, the chain of inequalities in Eq. (5) descends to equivalence classes of differential operators, i.e. to $\mathcal{S}_{\ell,k}$. On the other hand, elementary examples show that left multiplication by an element in $\mathsf{GL}_\ell(A)$ does not necessarily preserve the chain of inequalities in Eq. (5).

The following result from [4, Theorem 5.5] provides a necessary and sufficient condition for a subset of systems to be impulse controllable using the notion of zeros at infinity defined just before Eq. (5).

Theorem 2.1 *Consider the system described by $M(\frac{\mathsf{d}}{\mathsf{dt}}) \in \mathcal{M}_{\ell,k}$. Suppose that its rows are A-independent. Then, the system is impulse controllable if and only if $M(\frac{\mathsf{d}}{\mathsf{dt}})$ has no zeros at infinity.*

We remarked above that we require the rows of $M(\frac{\mathsf{d}}{\mathsf{dt}})$ to be only $\mathbb{C}$-independent, and not necessarily A-independent as in the statement of the above theorem. However, with respect to the topology we impose on $\mathcal{M}_{\ell,k}$ in the next section, the rows of a generic matrix are in fact A-independent, and hence, the assumption in the above theorem is generically satisfied (Remark 3.2).

In the rest of the paper, we use the chain of inequalities in (5) to study impulse controllability in families of systems. Specifically, we ask: given a family of systems, how many of them are impulse controllable? Given an impulse controllable system, do perturbations of it move it to other impulse controllable systems? Given a system which is not impulse controllable, is it possible to perturb it to obtain one which is impulse controllable?

3 Perturbations of LTI systems

As explained in the introduction, we wish to study the persistence of impulse controllability under *perturbations*. Towards this, we need to first specify a notion of perturbation, i.e. we need to specify when one system is *close* to another. We accomplish this by defining a topology on the set of all systems.

We consider under-determined systems described by k attributes; these attributes are elements of the signal space $\mathcal{F}$ of distributions defined on $[0, \infty)$ whose singular support is contained in $\{0\}$ (as described in Sect. 2). Suppose that an LTI system is determined by ℓ laws, with $\ell \leqslant k$. It is then represented by the kernel of a differential operator in $\mathcal{M}_{\ell,k}$ whose ℓ rows are precisely these laws. Two equivalent differential operators (in the sense of Definition 2.1) define the same system, and thus, the set $\mathcal{S}_{\ell,k}$ of these systems is a quotient of $\mathcal{M}_{\ell,k}$. We first topologise $\mathcal{M}_{\ell,k}$ and then equip $\mathcal{S}_{\ell,k}$ with the quotient topology. Let $P : \mathcal{M}_{\ell,k} \to \mathcal{S}_{\ell,k}$ be the quotient map.

We now describe the topology on $\mathcal{M}_{\ell,k}$. Let $A(r)$ be the subset of A consisting of differential operators of order at most r. We identify $A(r)$ with the affine space $\mathbb{C}^{r+1}$ by identifying $p = c_0 + c_1 \frac{\mathrm{d}}{\mathrm{dt}} + \cdots + c_r \frac{\mathrm{d^r}}{\mathrm{dt^r}}$ with the point $(c_0, c_1, \ldots, c_r) \in \mathbb{C}^{r+1}$. By means of this identification, we carry over the Zariski topology[3] of the affine space $\mathbb{C}^{r+1}$ to $A(r)$. The topological space $A(r)$ injects continuously into $A(r+1)$ as a Zariski closed subspace by means of the map $(c_0 + \cdots + c_r \frac{\mathrm{d^r}}{\mathrm{dt^r}}) \mapsto (c_0 + \cdots + c_r \frac{\mathrm{d^r}}{\mathrm{dt^r}} + 0 \frac{\mathrm{d^{r+1}}}{\mathrm{dt^{r+1}}})$. Thus, $\{A(r), r \geqslant 0\}$ is a directed system, and its (strict) direct limit, $\varinjlim A(r)$, is the topological space of all differential operators, also denoted by A. In the notation of the previous section, this is the construction of the Zariski topology on $\mathcal{M}_{1,1}$.

The construction of the topology in the general case is similar: an element $M(\frac{\mathrm{d}}{\mathrm{dt}}) \in \mathcal{M}_{\ell,k}(r)$, the set of $\ell \times k$ matrices whose entries are all in $A(r)$, can be identified with a point in the affine space $\mathbb{C}^{\ell k(r+1)}$. Again, we carry over the Zariski topology on $\mathbb{C}^{\ell k(r+1)}$ to $\mathcal{M}_{\ell,k}(r)$ via this correspondence. The space $\mathcal{M}_{\ell,k}(r)$ injects continuously into $\mathcal{M}_{\ell,k}(r+1)$ as a Zariski closed subset, and the (strict) direct limit, $\varinjlim \mathcal{M}_{\ell,k}(r)$, of the directed system $\{\mathcal{M}_{\ell,k}(r), r \geqslant 0\}$, is the space $\mathcal{M}_{\ell,k}$ with the Zariski topology. The quotient $\mathcal{S}_{\ell,k}$ is equipped with the quotient topology, and we refer to it as the Zariski topology on the set of systems with k attributes that are governed by ℓ laws.

Remark 3.1 Suppose C is an invertible $\ell \times \ell$ matrix with coefficients in $\mathbb{C}$, then the endomorphism m_C of $\mathcal{M}_{\ell,k}(r)$ given by left multiplication by C is an isomorphism of affine space for each r and hence an isomorphism of $\mathcal{M}_{\ell,k}$. Thus, if an element

[3] We refer to any standard book on Algebra for definitions, for instance, D. Eisenbud: Commutative algebra with a view towards algebraic geometry.

$M(\frac{\mathsf{d}}{\mathsf{dt}})$ belongs to an open subset U, then $CM(\frac{\mathsf{d}}{\mathsf{dt}})$ belongs to the open subset $m_C(U)$. The notion of equivalence in $\mathcal{M}_{\ell,k}$ then implies that $P^{-1}(P(U))$ is also open and hence that $P : \mathcal{M}_{\ell,k} \to \mathcal{S}_{\ell,k}$ is an open map. For each r, P induces an open map $P(r) : \mathcal{M}_{\ell,k}(r) \to \mathcal{S}_{\ell,k}(r)$, whereby definition $\mathcal{S}_{\ell,k}(r)$ is the set of systems defined by operators of degree at most r.

The above construction of the Zariski topology on $\mathcal{M}_{\ell,k}$, and hence on $\mathcal{S}_{\ell,k}$, assumes that a perturbation of a system governed by ℓ laws results again in a system governed by the same number ℓ of laws. We could replace this with the more general assumption that a perturbation of a system in $\mathcal{S}_{\ell,k}$ could result in a system in $\mathcal{S}_{\ell',k}$, where $\ell \leqslant \ell' \leqslant k$. We incorporate this by appending to the ℓ rows of any differential operator representing the system, the $(\ell' - \ell) \times k$ matrix $0_{\ell'-\ell,k}$, whose every entry is the 0 operator in A. The system defined by this enlarged differential operator is identical to the first, but the differential operator representing it now is in $\mathcal{M}_{\ell',k}$. Two equivalent operators in $\mathcal{M}_{\ell,k}$ remain equivalent in $\mathcal{M}_{\ell',k}$ after appending $0_{\ell'-\ell,k}$, and thus, this construction descends to the quotient. In other words, the above injection of $\mathcal{M}_{\ell,k}$ in $\mathcal{M}_{\ell',k}$ induces an injection $\mathcal{S}_{\ell,k} \hookrightarrow \mathcal{S}_{\ell',k}$, and we can thereby study perturbations of a system where the number of its defining laws has increased.

Thus, we can consider the chain of inclusions

$$\mathcal{M}_{1,k} \hookrightarrow \cdots\ \mathcal{M}_{\ell,k} \hookrightarrow \mathcal{M}_{\ell+1,k}\ \cdots \hookrightarrow \mathcal{M}_{k,k}$$

and the corresponding chain

$$\mathcal{S}_{1,k} \hookrightarrow \cdots\ \mathcal{S}_{\ell,k} \hookrightarrow \mathcal{S}_{\ell+1,k}\ \cdots \hookrightarrow \mathcal{S}_{k,k}.$$

We observe that $\mathcal{M}_{\ell,k}$ embeds in $\mathcal{M}_{\ell+1,k}$ as a Zariski closed subset and thus also $\mathcal{S}_{\ell,k}$ in $\mathcal{S}_{\ell+1,k}$. Hence, the topology we have defined implies that generically the number of laws governing a system cannot decrease but can only increase. (This is analogous to the statement that the number 0 can be easily perturbed to become nonzero, but it is unlikely that a nonzero number, when perturbed, becomes 0.) We can therefore either study perturbations of a system defined by ℓ laws within the space of systems all defined by ℓ laws, i.e. within $\mathcal{M}_{\ell,k}$, or study perturbations which result in ℓ' laws, $\ell' > \ell$, by studying the inclusion $\mathcal{M}_{\ell,k} \hookrightarrow \mathcal{M}_{\ell',k}$.

In this paper, we are concerned only with genericity questions or, in other words, concerned about systems belonging to an open dense set. Hence, we confine ourselves to the case where the number of laws defining a system remains constant under perturbations, as the image of $\mathcal{S}_{\ell,k} \hookrightarrow \mathcal{S}_{\ell+1,k}$ is proper closed. In other words, in the space of systems that are described by $(\ell + 1)$ laws, those systems that could be described by a fewer number of laws are a Zariski closed subset, and those that need to be described by $(\ell + 1)$ laws are open dense.

We first collect a few elementary properties of these spaces.

Lemma 3.1 *The map $\alpha : \mathcal{M}_{\ell_1,k} \times \mathcal{M}_{\ell_2,k} \to \mathcal{M}_{\ell_1+\ell_2,k}$, given by appending the rows of an $\ell_2 \times k$ matrix to the rows of an $\ell_1 \times k$ matrix (entries in A), is an isomorphism.*

Proof The above map induces, for every r, a map $\alpha(r) : \mathcal{M}_{\ell_1,k}(r) \times \mathcal{M}_{\ell_2,k}(r) \to \mathcal{M}_{\ell_1+\ell_2,k}(r)$. This map is the isomorphism $\mathbb{C}^{\ell_1 k(r+1)} \times \mathbb{C}^{\ell_2 k(r+1)} \simeq \mathbb{C}^{(\ell_1+\ell_2)k(r+1)}$, and the lemma now follows. □

Lemma 3.2 *Let $I = \{i_1, \ldots, i_{\ell_1}\}$ be a set of ℓ_1 indices between 1 and ℓ, and $J = \{j_1, \ldots, j_{k_1}\}$ a set of k_1 indices between 1 and k. Let $s : \mathcal{M}_{\ell,k} \to \mathcal{M}_{\ell_1,k_1}$ map an $\ell \times k$ matrix with entries in A to its $\ell_1 \times k_1$ submatrix determined by the indices I and J. Then, s is continuous and open.*

Proof The map $s(r) : \mathcal{M}_{\ell,k}(r) \to \mathcal{M}_{\ell_1,k_1}(r)$ induced by s, is continuous and open for every r as it is a projection $\mathbb{C}^{\ell k(r+1)} \to \mathbb{C}^{\ell_1 k_1(r+1)}$ of an affine space onto the affine space given by a subset of its coordinates. Hence it follows that s is continuous and open. □

Lemma 3.3 *Let* det *: $\mathcal{M}_{i,i} \to A$ map a square matrix of size i with entries in A to its determinant. This map is continuous. Hence, the set of differential operators satisfying $\delta_i = 0$ is a Zariski closed subset of $\mathcal{M}_{i,i}$.*

Proof The map det induces a map $\mathcal{M}_{i,i}(r) \to A(ir)$ of affine spaces, which is continuous with respect to the Zariski topology as it is given by algebraic operations. □

Corollary 3.1 *The subset $\mathcal{M}^*_{\ell,k}$ of differential operators in $\mathcal{M}_{\ell,k}$ which satisfy $\delta_\ell \geqslant 0$ is open dense.*

Proof To say that $\delta_\ell \geqslant 0$ is to say that at least one maximal minor is nonzero. By the above two lemmas, this is an open subset of $\mathcal{M}_{\ell,k}$. □

Remark 3.2 Theorem 2.1 requires the rows of the defining matrix to be A-independent, whereas systems over the signal space $\mathcal{F}$ are defined by matrix differential operators whose rows can be assumed to be only $\mathbb{C}$-independent. In other words, Theorem 2.1 is not applicable for every system in $\mathcal{S}_{\ell,k}$. However, as A-independence of the rows of a differential operator is equivalent to the condition that at least one of its maximal minors is nonzero (Remark 2.1), this subset of differential operators, namely $\mathcal{M}^*_{\ell,k}$, is by the above corollary, a Zariski open subset of $\mathcal{M}_{\ell,k}$. Thus, Theorem 2.1 is valid for this large class of systems. In what follows, our strategy of proof is to identify an open dense subset in $\mathcal{M}^*_{\ell,k}$ using Theorem 2.1 and then to observe that this subset is also open dense in $\mathcal{M}_{\ell,k}$.

The subset of differential operators in $\mathcal{M}^*_{\ell,k}$ whose entries are all of order at most r is denoted $\mathcal{M}^*_{\ell,k}(r)$. These subsets are stable under our notion of equivalence (Definition 2.1), and so define subsets of systems, denoted $\mathcal{S}^*_{\ell,k}$ and $\mathcal{S}^*_{\ell,k}(r)$, respectively. They are Zariski open dense subsets of $\mathcal{S}_{\ell,k}$ and $\mathcal{S}_{\ell,k}(r)$.

Lemma 3.4 *The spaces $\mathcal{M}^*_{\ell,k}$ and $\mathcal{S}^*_{\ell,k}$ are irreducible (i.e. they cannot be written as the union of two proper closed subsets). Thus, nonempty open subsets in them are also dense.*

Proof Suppose $\mathcal{M}_{\ell,k} = C_1 \cup C_2$ is a decomposition where C_1 and C_2 are proper closed subsets. Then, there is an r such that $C_i \cap \mathcal{M}_{\ell,k}(r)$, $i = 1, 2$ are both proper, but as their union is all of $\mathcal{M}_{\ell,k}(r)$, it follows that $\mathcal{M}_{\ell,k}(r)$ is not irreducible. This is a contradiction as $\mathcal{M}_{\ell,k}(r)$ is isomorphic to the affine space $\mathbb{C}^{\ell k(r+1)}$. As $\mathcal{S}_{\ell,k}$ is a quotient of an irreducible space, it is also irreducible. Since $\mathcal{M}^*_{\ell,k}$ is open in $\mathcal{M}_{\ell,k}$, the lemma now follows. □

4 Genericity

We now prove a key result of the paper. We recall our standing assumption that our discussion is about under-determined systems, i.e. $\ell \leqslant k$ always.

Theorem 4.1 *The set $\mathcal{C}_{\ell,k} \subset \mathcal{S}_{\ell,k}$ of impulse controllable systems contains an open dense set, i.e. impulse controllability is a generic property.*

Proof We first show that the set $\mathcal{B}_{\ell,k}$ of differential operators in $\mathcal{M}_{\ell,k}$ that define impulse controllable systems contains an open dense set, and to show it, we establish this statement for $\mathcal{M}_{\ell,k}(r)$, for every $r \geqslant 0$ (by definition of the direct limit topology on $\mathcal{M}_{\ell,k}$). By Theorem 2.1, a system defined by $M(\frac{\mathsf{d}}{\mathsf{dt}}) \in \mathcal{M}^*_{\ell,k}(r)$ is impulse controllable if and only if the chain of inequalities (5) holds. We now show that there is an open dense set in $\mathcal{M}_{\ell,k}(r)$ for which $\delta_i = ir$, $1 \leqslant i \leqslant \ell$ for every $r \geqslant 0$.

Define $\mathsf{det}_i : \mathcal{M}_{\ell,k}(r) \to A(ir)$ by mapping $M(\frac{\mathsf{d}}{\mathsf{dt}})$ to the determinant of the $i \times i$ submatrix given by the first i rows and columns. This map is continuous by Lemmas 3.2 and 3.3. The set U_i of differential operators in $A(ir)$ whose order is *equal* to ir is open, as it is the complement of the closed subspace $A(ir-1)$ of $A(ir)$. Clearly the image of det_i intersects U_i, hence $\mathsf{det}_i^{-1}(U_i)$ is nonempty and open, and so open dense (as $\mathcal{M}_{\ell,k}(r)$ is irreducible). Every differential operator in $\mathsf{det}_i^{-1}(U_i)$ satisfies $\delta_i = ir$.

Consider the open dense subset $V = \bigcap_{i=1}^{\ell} \mathsf{det}_i^{-1}(U_i)$ in $\mathcal{M}_{\ell,k}(r)$; every differential operator in it satisfies $\delta_i = ir$ for $1 \leqslant i \leqslant \ell$ and hence also satisfies the chain of inequalities (5). Thus, the set of operators in $V \cap \mathcal{M}^*_{\ell,k}(r)$, all of which define impulse controllable systems by Theorem 2.1, is open dense in $\mathcal{M}^*_{\ell,k}(r)$, for all $r \geqslant 0$. Hence, $\mathcal{B}_{\ell,k}$ contains an open dense subset of $\mathcal{M}^*_{\ell,k}$ and so also an open dense subset of $\mathcal{M}_{\ell,k}$.

The image $P(\mathcal{B}_{\ell,k}) = \mathcal{C}_{\ell,k}$ in $\mathcal{S}_{\ell,k}$ is clearly dense, and by Remark 3.1, this image also contains an open subset of $\mathcal{S}_{\ell,k}$. This completes the proof of the theorem. □

In Willems' behavioural approach to dynamical systems [13], the notion of feedback control is studied as a special case of the *interconnection* of two systems. We next ask if the interconnection of two impulse controllable systems results in a system which is also impulse controllable.

A system $S \in \mathcal{S}_{\ell,k}$ is said to be the interconnection of systems $S_1 \in \mathcal{S}_{\ell_1,k}$ and $S_2 \in \mathcal{S}_{\ell_2,k}$ for $\ell = \ell_1 + \ell_2$, if a differential operator defining S is obtained by appending the rows of an operator representing S_1 to the rows of an operator representing S_2. This construction is easily seen to be independent of the choice of the representing operators.

Corollary 4.1 *Let* $\ell_1 + \ell_2 = \ell \leqslant k$. *Then, there are open dense sets* $U_i \subset \mathcal{S}_{\ell_i,k}$, $i = 1, 2$, *such that the interconnection of any* $S_1 \in U_1$ *and any* $S_2 \in U_2$ *is impulse controllable. Hence, the interconnection of two generic impulse controllable systems is also impulse controllable.*

Proof Let $U(r)$ be the open dense subset of $\mathcal{M}_{\ell,k}(r)$ consisting of differential operators that define impulse controllable systems (guaranteed by the above theorem). By Lemma 3.1, $\alpha^{-1}(U)$ is open in $\mathcal{M}_{\ell_1,k}(r) \times \mathcal{M}_{\ell_2,k}(r)$. Its projection to each factor is open, as the projection of a product of affine spaces to a factor is an open map. The images of these two open sets in $\mathcal{S}_{\ell_i,k}(r)$, $i = 1, 2$, are open sets $U_i(r)$ with the property that the interconnection of any two from each of them is impulse controllable. As this is true for every r, the corollary follows. □

We now study persistence of the property of impulse controllability under *structured* perturbations. By this, we mean the following: suppose $\mathcal{T}_{\ell,k} \subset \mathcal{S}_{\ell,k}$, and suppose the allowed perturbations of a system in $\mathcal{T}_{\ell,k}$ leaves it in $\mathcal{T}_{\ell,k}$. Such perturbations are said to be structured. We ask if impulse controllability continues to remain generic; in other words, does $\mathcal{T}_{\ell,k}$ with the subspace topology contain an open dense set of impulse controllable systems? This is equivalent to asking whether $P^{-1}(\mathcal{T}_{\ell,k}) \subset \mathcal{M}_{\ell,k}$ contains an open dense set (in the subspace topology).

For instance, suppose we allow only those perturbations of a system of degree r which do not increase the degree; in other words, suppose $\mathcal{T}_{\ell,k}$ equals $\mathcal{S}_{\ell,k}(r)$. The proof of Theorem 4.1 shows that impulse controllability is generic for this subset of systems; indeed, the proof of genericity for $\mathcal{S}_{\ell,k}$ required us to prove the claim first for $\mathcal{S}_{\ell,k}(r)$, for each $r \geqslant 0$.

The following proposition is elementary.

Proposition 4.1 *Any open* $\mathcal{T}_{\ell,k} \subset \mathcal{S}_{\ell,k}$ *is generically impulse controllable.*

Proof The set $\mathcal{C}_{\ell,k} \cap \mathcal{T}_{\ell,k}$ of impulse controllable systems in $\mathcal{T}_{\ell,k}$ contains an open dense set since $\mathcal{S}_{\ell,k}$ is irreducible. □

From this proposition follows the next corollary.

Corollary 4.2 (Systems bounded below in degree) *Let* $\mathcal{N}_{\ell,k}(r_{i,j}) \subset \mathcal{M}_{\ell,k}$ *be the subset of those differential operators whose* (i, j)*-th entry has degree at least* $r_{i,j}$, *where* $r_{i,j} \in \{-\infty, 0, 1, 2 \ldots\}$. *Then, the set of systems* $P(\mathcal{N}_{\ell,k}(r_{i,j}))$ *in* $\mathcal{S}_{\ell,k}$ *is generically impulse controllable.*

Proof In the ring A, the set of operators of degree at least r is open, as its complement, the set of operators of degree at most $r - 1$ is closed. Thus, $\mathcal{N}_{\ell,k}(r_{i,j})$ is open in $\mathcal{M}_{\ell,k}$, and so is the set of systems determined by it. □

We now study impulse controllability for Zariski closed subsets of systems other than $\mathcal{S}_{\ell,k}(r)$, which we have already discussed above. We consider the important case of systems bounded above in degree, but not uniformly. These include state space and descriptor systems for which the notion of impulse controllability was first introduced. The entries of the system matrix are of different degrees, and structured perturbations might preserve these differences. In other words, when a system is perturbed, it can be that different parameters are subject to different changes, independent of each other. We use the following notation.

Let $I = \{1, \ldots, \ell\}$, $J = \{1, \ldots, k\}$, and let $\pi = \{(1, j_1), \ldots, (\ell, j_\ell)\}$ be a choice of ℓ elements from $I \times J$, where $j_1, \ldots, j_\ell$ are all distinct. Let Π be the set of all such choices; its cardinality is $k(k-1)\cdots(k-\ell+1)$. Let $\rho = (r_{i,j})$ be an $\ell \times k$ matrix whose entries are from $\{-\infty, 0, 1, 2, \ldots\}$; ρ is a *degree* matrix. Define $r_\pi^-(\rho) = \min\{r_{1,j_1}, \ldots, r_{\ell,j_\ell}\}$, $r_\pi(\rho) = r_{1,j_1} + \cdots + r_{\ell,j_\ell}$, and let $r_\pi^*(\rho)$ be the sum of those terms in $r_\pi(\rho)$ which are not equal to $-\infty$. Let $R(\rho) = \max\{r_\pi(\rho)\}$ and $R^*(\rho) = \max\{r_\pi^*(\rho)\}$, where the maximum is over all $\pi \in \Pi$; clearly $R(\rho) \leqslant R^*(\rho)$.

The purpose of this notation is to carve out closed subsets of $\mathcal{M}_{\ell,k}$.

Given $\rho = (r_{i,j})$, a degree matrix as above, let $\mathcal{M}_{\ell,k}(\rho) \subset \mathcal{M}_{\ell,k}$ be the subset of differential operators whose (i, j)th entry has degree at most $r_{i,j}$; it is the $\mathbb{C}$-affine space of dimension $\sum(r_{i,j}+1)$, where the sum is over those (i, j) for which $r_{i,j} \geqslant 0$. (If every $r_{i,j} = r$, then we use the earlier notation of Sect. 3, namely $\mathcal{M}_{\ell,k}(r)$, for this subset.) Let $\mathcal{M}^*_{\ell,k}(\rho)$ denote the subset $\mathcal{M}_{\ell,k}(\rho) \cap \mathcal{M}^*_{\ell,k}$ of those operators such that at least one maximal minor is nonzero. It is open dense in $\mathcal{M}_{\ell,k}(\rho)$ if $R(\rho) > -\infty$. Systems they define are denoted by $\mathcal{S}_{\ell,k}(\rho)$ and $\mathcal{S}^*_{\ell,k}(\rho)$, respectively.

We elucidate the above construction by an example.

Example 4.1 (i) Let a 3×4 degree matrix be given by

$$\rho = \begin{bmatrix} 1 & 0 & 0 & -\infty \\ 0 & 1 & -\infty & -\infty \\ 0 & -\infty & -\infty & 0 \end{bmatrix}.$$

Let $\pi = \{(1, 4), (2, 2), (3, 1)\}$. Then, $r_\pi^-(\rho) = -\infty$, $r_\pi(\rho) = -\infty$, and $r_\pi^*(\rho) = 1$. Both $R(\rho)$ and $R^*(\rho)$ equal 2 (attained for $\pi' = \{(1, 1), (2, 2), (3, 4)\}$).

This ρ defines $\mathcal{M}_{3,4}(\rho)$, it is the affine space $\mathbb{C}^9$. The operator of Eq. (1) belongs to it; in fact it belongs to $\mathcal{M}^*_{3,4}(\rho)$.
(ii) Consider the degree matrix

$$\rho' = \begin{bmatrix} 10^{10} & 10^{10} & 0 & 1 \\ 0 & 10^{10} & -\infty & 0 \\ 10^{10} & 10^{10} & -\infty & 10^{10} \end{bmatrix}$$

For $\pi = \{(1, 4), (2, 2), (3, 1)\}$, $r_\pi^-(\rho') = 1$, $r_\pi(\rho') = r_\pi^*(\rho) = 2 \times 10^{10} + 1$. Both $R(\rho)$ and $R^*(\rho)$ equal 3×10^{10}, again attained for $\pi' = \{(1, 1), (2, 2), (3, 4)\}$. □

Definition 4.1 A degree matrix ρ is a *descriptor* matrix if for all i,

$$r_{i,j} \leqslant 1,\ 1 \leqslant j \leqslant \ell,\ \text{and}\ r_{i,j} \leqslant 0,\ \ell+1 \leqslant j \leqslant k.$$

The systems in $\mathcal{S}_{\ell,k}(\rho)$ are called *descriptor* systems defined by ρ. It is a closed subset of $\mathcal{S}_{\ell,k}(1)$, the space of first-order systems.

The degree matrix in (i) of the above example is a descriptor matrix.

Proposition 4.2 *Let ρ be a descriptor matrix such that $R(\rho) = R^*(\rho)$. Then, $\mathcal{S}_{\ell,k}(\rho)$ is generically impulse controllable.*

Proof Let $R(\rho) = R^*(\rho) = \ell_1$ say, where $0 \leqslant \ell_1 \leqslant \ell$. Let $\pi = \{(1, j_1), \dots, (\ell, j_\ell)\} \in \Pi$ be such that $r_\pi(\rho) = \ell_1$, this is the number of elements $(i, j_i) \in \pi$ such that $r_{i,j_i} = 1$. Let $\mathsf{det} : \mathcal{M}_{\ell,k}(\rho) \to A(\ell_1)$ map a differential operator to the determinant of the operator in $\mathcal{M}_{\ell,\ell}$ determined by the columns $j_1, \dots, j_\ell$. Let U be the open dense subset of $A(\ell_1)$ consisting of operators whose degree is equal to ℓ_1. Then, $\mathsf{det}^{-1}(U)$ is nonempty, hence open dense in $\mathcal{M}_{\ell,k}(\rho)$. Every differential operator in it satisfies the chain condition (5); indeed, $\delta_i = i$ for $i \leqslant \ell_1$, and $\delta_i = \ell_1$ for $i \geqslant \ell_1$. These differential operators define an open dense subset of impulse controllable systems in $\mathcal{S}_{\ell,k}(\rho)$. Hence, $\mathcal{S}_{\ell,k}(\rho)$ is generically impulse controllable. □

Proposition 4.3 *Suppose $\rho = (r_{i,j})$ is a descriptor matrix such that $R(\rho) < R^*(\rho)$. Then, an open dense set of systems in $\mathcal{S}_{\ell,k}(\rho)$ is not impulse controllable.*

Proof Let $R^*(\rho) = \ell_1$, and let $\pi = \{(1, j_1), \dots, (\ell, j_\ell)\} \in \Pi$ be such that $r^*_\pi(\rho) = \ell_1$. Let the number of elements $(i, j_i) \in \pi$ such that $r_{i,j_i} > -\infty$ be ℓ_2; necessarily $\ell_2 < \ell$, for otherwise $R(\rho) = R^*(\rho)$. Let $\mathsf{det} : \mathcal{M}_{\ell,k}(\rho) \to A(\ell_1)$ map a differential operator $M(\frac{\mathsf{d}}{\mathsf{dt}})$ to the determinant of the operator determined by its $\ell_2 \times \ell_2$ submatrix whose rows and columns are the above indices i and j_i for which $r_{i,j_i} > -\infty$. Let U be the open dense subset of $A(\ell_1)$ consisting of operators whose degree is equal to ℓ_1. Then, $\mathsf{det}^{-1}(U)$ is nonempty, hence open dense in $\mathcal{M}_{\ell,k}(\rho)$. Every operator in it satisfies $\delta_{\ell_2} = \ell_1$ and $\delta_{\ell_2+1} < \ell_1$. Thus, every operator in this open dense subset fails to satisfy the chain condition (5), and so the system defined by it is not impulse controllable. □

We combine the above propositions to obtain the following theorem.

Theorem 4.2 (Singular descriptor systems) *Let ρ be a descriptor matrix. The set of descriptor systems $\mathcal{S}_{\ell,k}(\rho)$ is generically impulse controllable if and only if $R(\rho) = R^*(\rho)$. Otherwise, $\mathcal{S}_{\ell,k}(\rho)$ is generically not impulse controllable.*

It follows as a corollary to Proposition 4.2 (for $R(\rho) = \ell$) that every regular state space system, i.e. $\dot{x} = Ax + Bu$, is impulse controllable.

We next provide a partial generalisation of Theorem 4.2 in which we consider closed subsets of higher-order systems. Given a degree matrix ρ, define for every $\pi \in \Pi$, the subset

$$\mathcal{M}^-_\pi(\rho) = \left\{ M\left(\frac{\mathsf{d}}{\mathsf{dt}}\right) = (m_{i,j}) | \mathsf{deg}(m_{i,j}) \leqslant r_{i,j}, (i, j) \in \pi,\ \mathsf{deg}(m_{i,j}) \right.$$
$$\left. \leqslant r_\pi(\rho)^-, (i, j) \notin \pi \right\} \subset \mathcal{M}_{\ell,k}(\rho).$$

It is a closed subset of $\mathcal{M}_{\ell,k}(\rho)$. In this notation, we have the following result.

Theorem 4.3 *Let ρ be a degree matrix. Define $\mathcal{M}^-_+(\rho) \subset \mathcal{M}_{\ell,k}(\rho)$ by $\mathcal{M}^-_+(\rho) := \bigcup \mathcal{M}^-_\pi(\rho)$ with the union being over all $\pi \in \Pi$ for which $r_\pi(\rho) \geqslant 0$. Then, the set $P(\mathcal{M}^-_+(\rho))$ of systems in $\mathcal{S}_{\ell,k}$ determined by it is generically impulse controllable.*

Proof It suffices to prove that $\mathcal{M}_{\pi}^{-}(\rho)$ is generically impulse controllable for every π such that $r_{\pi}(\rho) \geqslant 0$. Fix such a π.

Let $M(\frac{\mathrm{d}}{\mathrm{dt}})$ be the following differential operator: for every $(i, j) \in \pi$, choose a $p(\frac{\mathrm{d}}{\mathrm{dt}}) \in A$ of degree $r_{i,j}$, and set all other entries equal to 0. This operator defines an impulse controllable system as it satisfies the chain of inequalities (5), and thus, the set of operators which define impulse controllable systems has nonempty intersection with $\mathcal{M}_{\pi}^{-}(\rho)$. Moreover, this intersection has nonempty interior as $M(\frac{\mathrm{d}}{\mathrm{dt}})$ is in the interior of $\mathcal{M}_{\pi}^{-}(\rho)$. (A neighbourhood of $M(\frac{\mathrm{d}}{\mathrm{dt}})$ is the set of differential operators with the entry 0 in the $(i, j) \notin \pi$ replaced by operators of degree $r_{\pi}^{-}(\rho)$.) But $\mathcal{M}_{\pi}^{-}(\rho)$ is a $\mathbb{C}$-affine space; hence, the intersection is dense as well, and the theorem follows. □

Example 4.2 (Example 1.1 revisited) In this example, we apply the results of this section to the three circuits shown in Fig. 1 and infer results about their generic impulse controllability/uncontrollability. The three cases are:

(a) Figure 1a with the voltage source present: described in Eq. (1),
(b) Figure 1b with the voltage source set equal to 0: described in Eq. (2), and
(c) Figure 1c with a capacitor across the port: described in Eq. (3).

The three degree matrices ρ_a, ρ_b and ρ_c are defined below:

$$\rho_a := \begin{bmatrix} 1 & 0 & 0 & -\infty \\ 0 & 1 & -\infty & -\infty \\ 0 & -\infty & -\infty & 0 \end{bmatrix}, \quad \rho_b := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\infty \\ 0 & -\infty & -\infty \end{bmatrix}$$
$$\text{and } \rho_c := \begin{bmatrix} 1 & 0 & 0 & -\infty \\ 0 & 1 & -\infty & -\infty \\ 0 & -\infty & -\infty & 0 \\ -\infty & -\infty & 0 & 1 \end{bmatrix}.$$

Note that ρ_a is also the degree matrix considered in Example 4.1(i), and the systems corresponding to ρ_a belong to $\mathcal{M}_{3,4}(\rho_a)$. Similarly, the systems described by circuits in Fig. 1b, c belong to $\mathcal{M}_{3,3}(\rho_b)$ and to $\mathcal{M}_{4,4}(\rho_c)$, respectively.
We next evaluate $R(\rho)$ and $R^*(\rho)$ as in Example 4.1.

(a) Consider the circuit in Fig. 1a. We verify that for ρ_a, we get $R(\rho_a) = 2$, achieved for $\pi = \{(1, 1), (2, 2), (3, 4)\}$. Further, $R^*(\rho_a)$ also equals 2, which is also achieved for the same $\pi = \{(1, 1), (2, 2), (3, 4)\}$. So the system described by it is impulse controllable, and in fact, we conclude using Proposition 4.2 that this system belongs to the Zariski open dense subset of impulse controllable systems in $\mathcal{S}_{3,4}(\rho_a)$ described there in Proposition 4.2.
(b) Next consider the autonomous system corresponding to the circuit shown in Fig. 1b and described in Eq. (2), for which the degree matrix ρ_b is above. For this case, one can verify that $R(\rho_b) = 1$ (achieved for $\pi = \{(1, 3), (2, 2), (3, 1)\}$), while $R^*(\rho_b) = 2$ (achieved for $\pi = \{(1, 1), (2, 2)\}$). Since $R(\rho_c) < R^*(\rho_c)$, the system is not impulse controllable. Indeed, using Proposition 4.3, this system belongs to the Zariski open neighbourhood in $\mathcal{S}_{3,3}(\rho_b)$ consisting of systems none of which are impulse controllable.

(c) In the context of the circuit in Fig. 1c, with equations described in Eq. (3), the degree matrix ρ_c is above. Obtain $R(\rho_c) = 2$ (achieved for $\pi = \{(1, 1), (2, 2), (3, 4), (4, 3)\}$), and $R^*(\rho_c) = 3$ (achieved for $\pi = \{(1, 1), (2, 2), (4, 4)\}$). Since $R(\rho_c) < R^*(\rho_c)$ again, using Proposition 4.3, this system also belongs to a Zariski open neighbourhood of systems within $\mathcal{S}_{4,4}(\rho_c)$ which are not impulse controllable.

Finally, consider the degree matrix ρ' of Example 4.1(ii). The system described by Eq. (1) is also in the Zariski open dense subset of impulse controllable systems in $\mathcal{S}_{3,4}(\rho')$ described by Theorem 4.3. This open subset strictly contains the set described in (a) above. □

In summary, we have equipped the set of all systems with a topology that is very coarse, as *every* open set in it is of full measure, everywhere dense, and of the second category. Indeed the closed sets are affine varieties and are therefore not only of zero measure, but are also nowhere dense. The genericity results established in the paper with respect to this topology thus provide very strong answers to the questions raised at the end of Sect. 2, in the sense that we allow perturbations corresponding to open subsets of the Zariski topology.

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