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ORIGINAL ARTICLE

ISS with respect to boundary and in-domain disturbances for a coupled beam-string system

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Abstract

This paper addresses the robust stability of a boundary controlled system coupling two partial differential equations (PDEs), namely beam and string equations, in the presence of boundary and in-domain disturbances under the framework of input-to-state stability (ISS) theory. Well-posedness assessment is first carried out to determine the regularity of the disturbances required for guaranteeing the unique existence of the solution to the considered problem. Then, the method of Lyapunov functionals is applied in stability analysis, which results in the establishment of some ISS properties with respect to disturbances. As the analysis is based on the a priori estimates of the solution to the PDEs, it allows avoiding the invocation of unbounded operators while obtaining the ISS gains in their original expression without involving the derivatives of boundary disturbances.

Keywords Coupled partial differential equations · Boundary and in-domain disturbances · ISS

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1 Introduction

This paper addresses the robust stabilization problem of a boundary controlled system described by a pair of coupled partial differential equations (PDEs) in the presence of boundary and in-domain disturbances. The considered system is a model describing the dynamics in bending and twisting displacement, respectively, for a flexible aircraft wing [22]. This model is a linear version of the system presented in [3]. A very similar model of a flapping wing UAV is studied in [33] and [9]. The robust stability analysis presented in this work is carried out in the framework of input-to-state stability (ISS), which was first introduced by Sontag (see [35,36]) and has become one of the central concepts in the study of robust stability of control systems.

During the last two decades, a complete theory of ISS for nonlinear finite dimensional systems has been established and has been successfully applied to a very wide range of problems in nonlinear systems analysis and control (see, e.g., [13]). In recent years, a considerable effort has been devoted to extending the ISS theory to infinite dimensional systems governed by partial differential equations, including the characterization of ISS and iISS (integral input-to-state stability, which is a variant of ISS [37]) [6,11,12,25–32] and the establishment of ISS properties for different PDE systems [1,2,5,7,10,14–19,23,24,34,38–42].

In the formulation of PDEs, disturbances can be distributed over the domain and/or appear at isolated points in the domain or on the boundaries. Usually, pointwise disturbances will lead to a formulation involving unbounded operators [10,15,16,27], which is considered to be more challenging than the case of distributed disturbances [15]. To avoid dealing with unbounded operators, it is proposed in [1] to transform the boundary disturbance to a distributed one, which allows for the application of the tools established for the latter case, in particular the method of Lyapunov functionals. However, it is pointed out in [15,16] that such a method will end up establishing the ISS property with respect to boundary disturbance and some of its time derivatives, which is not strictly in the original form of ISS formulation. For this reason, the authors of [15,16] proposed a finite-difference scheme and eigenfunction expansion method with which the ISS in L^2-norm and in weighted L^∞-norm is derived directly from the estimates of the solution to the considered PDEs associated with a Sturm–Liouville operator. Although the aforementioned transformation of the disturbance from the boundary to the domain is still used, it is only for the purpose of well-posedness assessment, while the ISS property is expressed solely in terms of disturbances as expected. Nevertheless, the method employed in [15,16] may involve a very heavy computation when dealing with higher-order, coupled PDEs with complex boundary conditions including disturbances, as the one considered in the present work.

A monotonicity-based method has been introduced in [32] for studying the ISS of nonlinear parabolic equations with boundary disturbances. It has been shown that with the monotonicity the ISS of the original nonlinear parabolic PDE with constant boundary disturbances is equivalent to the ISS of a closely related nonlinear parabolic PDE with constant distributed disturbances and zero boundary conditions. As an application of this method, the ISS properties in L^p-norm ($\forall p > 2$) for some linear parabolic systems have been established.

It has been shown in [40] and [41] that the classical method of Lyapunov functionals is still effective in obtaining ISS properties w.r.t. boundary disturbances for certain semilinear parabolic PDEs with Dirichlet, and Neumann (or Robin) boundary conditions, respectively. In [40], the technique of De Giorgi iteration is used when Lyapunov method is involved in the establishment of ISS for PDEs with Dirichlet boundary disturbances. ISS in L^2-norm for Burgers' equations, and ISS in L^∞-norm for some linear PDEs, have been established in [40]. In [41], some technical inequalities have been developed, which allows dealing directly with the boundary disturbances in proceeding on ISS in L^2-norm for certain semilinear PDEs with Neumann (or Robin) boundary conditions via Lyapunov method. In [38], the ISS w.r.t. boundary disturbances in H^1-norm has also been established for linear hyperbolic PDEs using Lyapunov method.

It should be noticed that it is shown in [10] by means of admissibility that for a class of linear PDEs with boundary disturbances, ISS is equivalent to iISS if the corresponding semigroup is exponentially stable. Nevertheless, this is a quite strong condition and there may be difficulties to apply this assertion to systems for which the associated operators are not *a priori* dissipative, as dissipativity is a non-trivial property depending closely on, among other factors, the boundary conditions and the regularity of the disturbances.

The method adopted in the present work is also the application of Lyapunov theory in the establishment of the ISS and iISS properties of the considered system with respect to boundary and in-domain disturbances. However, greatly inspired by the methodology proposed in [15,16,41], stability analysis is based on the *a priori* estimates of the solution to the original PDEs, which allows avoiding the invocation of unbounded operators while obtaining the ISS and iISS properties expressed only in terms of the disturbances. The development of the solution consists in two steps. In the first step, we perform a well-posedness analysis to determine the regularity of the disturbances required for ensuring the existence of the solutions to the PDEs. Similar to [1,15,16], the technique of lifting is used in well-posedness analysis to avoid involving unbounded operators. In the second step, the ISS and iISS properties are established via the estimates of the solution to the original system. Instead of dealing with certain energy functional directly, the Lyapunov functional candidate for the system is actually derived from the regularity analysis of the solutions. In general, a Lyapunov functional candidate may be chosen according to the norms of the solution and their derivatives arising in the computation of *a priori* estimates of the solutions.

Note that the result presented in this work demonstrates that the appearance of the derivatives of boundary disturbances in ISS or iISS gains is not necessarily inherent to the Lyapunov method and may be avoided for certain settings. Therefore, we can expect that the well-established method of Lyapunov functionals can be applied to the establishment of ISS properties with respect to boundary disturbances for a wide range of PDEs. This constitutes the main contribution of the present work.

In the remainder of the paper, Sect. 2 introduces the dynamic model of the coupled beam-string system and presents the well-posedness assessment. Section 3 is devoted to the analysis of ISS and iISS properties of the considered system. Numerical simulation results for the considered system are presented in Sect. 4, followed by concluding remarks given in Sect. 5.

2 Problem formulation and well-posedness analysis

2.1 Notation

Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = (0, +\infty)$, and $\mathbb{R}_{\geq 0} = \{0\} \cup \mathbb{R}_+$. We define some function spaces for functions with one variable. For $a, b \in [-\infty, +\infty]$ and $p \in [1, +\infty)$, $L^p(a, b)$ is the space of all measurable functions f whose absolute value raised to the pth-power has a finite integral. The norm $\|\cdot\|$ on $L^p(a, b)$ is defined by $\|f\|_{L^p(a,b)} = \left(\int_a^b |f(x)|^p \mathrm{d}x\right)^{\frac{1}{p}}$. $L^\infty(a, b)$ is the space all measurable functions f whose absolute value is essential bounded. The norm $\|\cdot\|$ on $L^\infty(a, b)$ is defined by $\|f\|_{L^\infty(a,b)} = \operatorname{ess\,sup}_{a<x<b} |f(x)|$. For a positive integer m, $H^m(a, b) = H^m((a, b); \mathbb{R}) = \{f : (a, b) \to \mathbb{R} |\ f \in L^2(a, b)$ with each s-th order weak derivative $D^s f \in L^2(a, b),\ s = 1, 2, \ldots, m\}$. For a nonnegative integer m, $C^m(\mathbb{R}_{\geq 0}) = C^m(\mathbb{R}_{\geq 0}; \mathbb{R}) = \{f : \mathbb{R}_{\geq 0} \to \mathbb{R} |\ \frac{\mathrm{d}^s f}{\mathrm{d}x^s} (s = 0, 1, 2, \ldots, m)$ exist and are continuous on $\mathbb{R}_{\geq 0}\}$.

We define some function spaces for functions with two variables. For $t \in \mathbb{R}_{\geq 0}$, $l \in \mathbb{R}_{\geq 0}$ and $1 \leq p < +\infty$, the space $L^\infty(0, t; L^p(0, l))$ consists of all strongly measurable functions $f : [0, t] \to L^p(0, l)$ with the norm

$$\|f\|_{L^\infty(0,t;L^p(0,l))} = \operatorname*{ess\,sup}_{0<s<t} \|f(\cdot, s)\|_{L^p(0,l)} < +\infty.$$

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For a nonnegative integer m and a vector space H, $C^m(\mathbb{R}_{\geq 0}; H) = \{f : \mathbb{R}_{\geq 0} \to H |\ \frac{\partial^s f}{\partial t^s}(\cdot, t) \in H$, and $\frac{\partial^s f}{\partial t^s}(\cdot, t)$ is continuous on $\mathbb{R}_{\geq 0},\ s = 0, 1, 2, \ldots, m\}$.

Some well-known function classes commonly used in Lyapunov-based stability analysis are specified below:

$\mathcal{K} = \{\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} |\ \gamma(0) = 0, \gamma$ is continuous, strictly increasing$\}$;
$\mathcal{K}_\infty = \{\theta \in \mathcal{K} |\ \lim_{s\to\infty} \theta(s) = \infty\}$;
$\mathcal{L} = \{\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} |\ \gamma$ is continuous, strictly decreasing, $\lim_{s\to\infty} \gamma(s) = 0\}$;
$\mathcal{KL} = \{\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} |\ \beta(\cdot, t) \in \mathcal{K}, \forall t \in \mathbb{R}_{\geq 0}$, and $\beta(s, \cdot) \in \mathcal{L}, \forall s \in \mathbb{R}_+\}$.

2.2 System setting

Let $l \in \mathbb{R}_{\geq 0}$ be the length of the wing. Denote by $w(y, t) : [0, l] \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $\phi(y, t) : [0, l] \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ the bending and twisting displacements, respectively, at the location $y \in [0, l]$ along the wing span and at time $t \geq 0$. In the present work, we consider the dynamics of a flexible aircraft wing expressed by the following initial-boundary value problem (IBVP) representing a coupled beam-string system with boundary control [22]:

$$w_{tt} + (a_1 w_{yy} + b_1 w_{tyy})_{yy} = c_1\phi + p_1\phi_t + q_1 w_t + d_1, \tag{1a}$$
$$\phi_{tt} - (a_2\phi_y + b_2\phi_{ty})_y = c_2\phi + p_2\phi_t + q_2 w_t + d_2, \tag{1b}$$
$$w(0,t) = w_y(0,t) = \phi(0,t) = 0,$$
$$(a_1 w_{yy} + b_1 w_{tyy})_y(l,t) = d_3(t),\ (a_2\phi_y + b_2\phi_{ty})(l,t) = d_4(t), \tag{1c}$$
$$w(y,0) = w^0, w_t(y,0) = w_1^0, \phi(y,0) = \phi^0, \phi_t(y,0) = \phi_1^0, \tag{1d}$$

where (1a) and (1b) are defined in $(0,l)\times\mathbb{R}_{\geq 0}$, $a_i > 0, b_i > 0, c_i \geq 0\, (i = 1, 2)$, $p_1 \geq 0$, $p_2 \leq 0$, $q_1 \leq 0$ and $q_2 \geq 0$ are constants depending on structural and aerodynamic parameters, $w^0, w_1^0 \in H^2(0,l)$, $\phi^0, \phi_1^0 \in H^1(0,l)$, $d_1, d_2 \in C^1(\mathbb{R}_{\geq 0}; L^2(0,l))$, and $d_3, d_4 \in C^2(\mathbb{R}_{\geq 0}; \mathbb{R})$. Functions $d_1(y,t)$ and $d_2(y,t)$ represent disturbances distributed over the domain, while functions $d_3(t)$ and $d_4(t)$ represent disturbances at the boundary $y = l$. In general, d_1 and d_2 can represent modeling errors and aerodynamic load perturbations, and d_3 and d_4 can represent actuation and sensing errors.

Remark 1 (1) is a model of flexible aircraft wing with Kelvin–Voigt damping, in which the constants $\frac{b_1}{a_1}$ in (1a) and $\frac{b_2}{a_2}$ in (1b) represent the coefficients of bending Kelvin–Voigt damping and torsional Kelvin–Voigt damping, respectively (see [22] for instance).

2.3 Well-posedness analysis

In this section, we prove the well-posedness of System (1). To this end, consider the Hilbert space

$$\mathcal{H} := \Big\{(f,g,h,z) \in H^2(0,l)\times L^2(0,l)\times H^1(0,l)\times L^2(0,l) :$$
$$f(0) = f_y(0) = h(0) = 0,\ f, f_y, h \in \mathrm{AC}[0,l]\Big\},$$

endowed with the inner product

$$\langle (f_1,g_1,h_1,z_1),(f_2,g_2,h_2,z_2)\rangle_{\mathcal{H}} = \int_0^l (a_1 f_{1yy} f_{2yy} + g_1 g_2 + a_2 h_{1y} h_{2y} + z_1 z_2)\mathrm{d}y.$$

Introducing the state vector $X = (f,g,h,z)$, the norm $\|\cdot\|_{\mathcal{H}}$ on $\mathcal{H}$ induced by the inner product can be expressed as:

$$\|X\|_{\mathcal{H}}^2 = \|\sqrt{a_1} f_{yy}\|_{L^2(0,l)}^2 + \|g\|_{L^2(0,l)}^2 + \|\sqrt{a_2} h_y\|_{L^2(0,l)}^2 + \|z\|_{L^2(0,l)}^2.$$

In order to reformulate System (1) in an abstract form evolving in the space $\mathcal{H}$, we define the following operators. First, we introduce the unbounded operator $\mathcal{A}_{1,d}$:

$D(\mathcal{A}_{1,d}) \subset \mathcal{H} \to \mathcal{H}$ defined by

$$\mathcal{A}_{1,d}X := \big(g, -(a_1 f_{yy} + b_1 g_{yy})_{yy}, z, (a_2 h_y + b_2 z_y)_y\big) \tag{2}$$

on the following domain:

$$\begin{aligned} D(\mathcal{A}_{1,d}) := \Big\{ & (f, g, h, z) \in \mathcal{H} : g \in H^2(0,l),\ z \in H^1(0,l), \\ & (a_1 f_{yy} + b_1 g_{yy}) \in H^2(0,l), (a_2 h_y + b_2 z_y) \in H^1(0,l), \\ & f(0) = f_y(0) = 0,\ g(0) = g_y(0) = 0, \\ & h(0) = 0,\ z(0) = 0, (a_1 f_{yy} + b_1 g_{yy})(l) = 0, \\ & f, f_y, g, g_y, h, z, (a_1 f_{yy} + b_1 g_{yy}) \in \mathrm{AC}[0,l], \\ & (a_1 f_{yy} + b_1 g_{yy})_y, (a_2 h_y + b_2 z_y) \in \mathrm{AC}[0,l] \Big\}, \end{aligned}$$

where $\mathrm{AC}[0,l]$ denotes the set of all absolutely continuous functions on $[0,l]$. The contribution of other terms are embedded into the bounded operator $\mathcal{A}_2 \in \mathcal{L}(\mathcal{H})$ defined as

$$\mathcal{A}_2 X := (0, c_1 h + p_1 z + q_1 g, 0, c_2 h + p_2 z + q_2 g), \tag{3}$$

with domain $D(\mathcal{A}_2) = \mathcal{H}$ (the bounded property is a direct consequence of the Poincaré's inequality). Finally, we consider the boundary operator $\mathcal{B} : D(\mathcal{B}) = D(\mathcal{A}_{1,d}) \to \mathcal{H}$ defined as

$$\mathcal{B}X := ((a_1 f_{yy} + b_1 g_{yy})_y(l), (a_2 h_y + b_2 z_y)(l)). \tag{4}$$

Thus, System (1) can be represented in the following abstract system:

$$\begin{cases} \dot{X} = \big[\mathcal{A}_{1,d} + \mathcal{A}_2\big] X + (0, d_1, 0, d_2) \\ \mathcal{B}X = U \\ X_0 \in D(\mathcal{A}_{1,d}),\ \text{s.t. } \mathcal{B}X_0 = U(0) \end{cases} \tag{5}$$

where $U \triangleq (d_3, d_4)$.

In order to assess the well-posedness of (5), we introduce the unbounded disturbance-free operator $\mathcal{A}_1 = D(\mathcal{A}_1) \subset \mathcal{H} \to \mathcal{H}$ defined on the domain $D(\mathcal{A}_1) = D(\mathcal{A}_{1,d}) \cap \ker(\mathcal{B})$ by $\mathcal{A}_1 = \mathcal{A}_{1,d}\big|_{D(\mathcal{A}_1)}$. We also consider the lifting operator $\mathcal{T} \in \mathcal{L}(\mathbb{R}^2, \mathcal{H})$ defined by

$$\mathcal{T}(d_3, d_4) := \left(y \to -\frac{d_3}{6a_1} y^2 (3l - y), 0, y \to \frac{d_4}{a_2} y, 0 \right). \tag{6}$$

with $\|\mathcal{T}\| = \sqrt{l \times \max(1/a_2, l^2/(3a_1))}$ when $\mathbb{R}^2$ is endowed with the usual l^2-norm. A direct computation shows that $R(\mathcal{T}) \subset D(\mathcal{A}_{1,d})$, $\mathcal{A}_{1,d}\mathcal{T} = 0_{\mathcal{L}(\mathbb{R}^2,\mathcal{H})}$ and $\mathcal{B}\mathcal{T} = I_{\mathbb{R}^2}$, where $R(\mathcal{T})$ is the range of the operator $\mathcal{T}$. Thus, we can define a system in the following abstract form:

$$\begin{cases} \dot{V} = [\mathcal{A}_1 + \mathcal{A}_2]\, V + \mathcal{A}_2 \mathcal{T} U - \mathcal{T}\dot{U} + (0, d_1, 0, d_2) \\ V_0 \in D(\mathcal{A}_1) \end{cases} \tag{7}$$

By [4, Th 3.3.3], we have the following relationship between the solutions of abstract systems (5) and (7).

Lemma 1 *Let $X_0 \in D(\mathcal{A}_{1,d})$, $d_1, d_2 \in C^1(\mathbb{R}_{\geq 0}; L^2(0,l))$, and $d_3, d_4 \in C^2(\mathbb{R}_{\geq 0}; \mathbb{R})$ such that $\mathcal{B}X_0 = (d_3(0), d_4(0))$. Then $X \in C^0(\mathbb{R}_{\geq 0}; D(\mathcal{A}_{1,d})) \cap C^1(\mathbb{R}_{\geq 0}; \mathcal{H})$ with $X(0) = X_0$ is a solution of (5) if and only if $V = X - \mathcal{T}U \in C^0(\mathbb{R}_{\geq 0}; D(\mathcal{A}_1)) \cap C^1(\mathbb{R}_{\geq 0}; \mathcal{H})$ is a solution of (7) for the initial condition $V_0 = X_0 - \mathcal{T}U(0)$.*

We can now use Lemma 1 to assess the well-posedness of the original abstract problem (5).

Theorem 1 *For any $d_1, d_2 \in C^1(\mathbb{R}_{\geq 0}; L^2(0,l))$, and $d_3, d_4 \in C^2(\mathbb{R}_{\geq 0}; \mathbb{R})$, the abstract problem (5) admits a unique solution $X \in C^0(\mathbb{R}_{\geq 0}; D(\mathcal{A}_{1,d})) \cap C^1(\mathbb{R}_{\geq 0}; \mathcal{H})$ for any given $X_0 \in D(\mathcal{A}_{1,d})$ such that $\mathcal{B}X_0 = (d_3(0), d_4(0))$.*

Proof Let $X_0 \in D(\mathcal{A}_{1,d})$ such that $\mathcal{B}X_0 = U(0)$. It is known that $\mathcal{A}_1$ generates a C^0-semigroup on $\mathcal{H}$ [22]. As $\mathcal{A}_2 \in \mathcal{L}(\mathcal{H})$, $\mathcal{A}_1 + \mathcal{A}_2$ generates a C^0-semigroup on $\mathcal{H}$ (see [4, Th 3.2.1]). Furthermore, $\mathcal{A}_2\mathcal{T}U - \mathcal{T}\dot{U} + (0, d_1, 0, d_2) \in C^1(\mathbb{R}_{\geq 0}; \mathcal{H})$ due to $\mathcal{T} \in \mathcal{L}(\mathbb{R}^2, \mathcal{H})$ and $\mathcal{A}_2 \in \mathcal{L}(\mathcal{H})$. Then, from [4, Th 3.1.3], (7) admits a unique solution $V \in C^0(\mathbb{R}_{\geq 0}; D(\mathcal{A}_1)) \cap C^1(\mathbb{R}_{\geq 0}; \mathcal{H})$ for the initial condition $V(0) = V_0 = X_0 - \mathcal{T}U(0)$. We deduce then from Lemma 1 that there exists a unique solution $X \in C^0(\mathbb{R}_{\geq 0}; D(\mathcal{A}_{1,d})) \cap C^1(\mathbb{R}_{\geq 0}; \mathcal{H})$ to (5) associated to the initial condition $X(0) = X_0$. □

3 Stability assessment

In this section we establish the stability property of System (1). Let $D(\mathcal{A}_{1,d})$, $\mathcal{H}$ and the norm $\|\cdot\|_{\mathcal{H}}$ be defined as in Sect. 2.3. Let (w, ϕ) be the unique solution of System (1) satisfying $(w, w_t, \phi, \phi_t) \in C^0(\mathbb{R}_{\geq 0}; D(\mathcal{A}_{1,d})) \cap C^1(\mathbb{R}_{\geq 0}; \mathcal{H})$. For simplicity, throughout this section, we express the state variable and its initial value as $X = (w, w_t, \phi, \phi_t)$ and $X_0 = (w^0, w_1^0, \phi^0, \phi_1^0)$. Define the energy function

$$E(t) = \frac{1}{2}\int_0^l \left(|w_t|^2 + a_1|w_{yy}|^2 + |\phi_t|^2 + a_2|\phi_y|^2\right)\mathrm{d}y. \tag{8}$$

Then $\|X(\cdot,t)\|_{\mathcal{H}}^2 = 2E(t)$ for all $t \geq 0$.

Definition 1 System (1) is said to be input-to-state stable (ISS) with respect to disturbances $d_1, d_2 \in C^1(\mathbb{R}_{\geq 0}; L^2(0,l))$ and $d_3, d_4 \in C^2(\mathbb{R}_{\geq 0}) \cap L^\infty(\mathbb{R}_{\geq 0})$, if there exist functions $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ such that the solution of System (1) satisfies

$$\begin{aligned}\|X(\cdot,t)\|_{\mathcal{H}} \leq \beta(\|X_0\|_{\mathcal{H}},t) + \gamma_1(\|d_1\|_{L^\infty(0,t;L^2(0,l))}) + \gamma_2(\|d_2\|_{L^\infty(0,t;L^2(0,l))})\\ + \gamma_3(\|d_3\|_{L^\infty(0,t)}) + \gamma_4(\|d_4\|_{L^\infty(0,t)}),\ \forall t \geq 0.\end{aligned} \tag{9}$$

Moreover, System (1) is said to be exponential input-to-state stable (EISS) with respect to disturbances d_1, d_2, d_3, and d_4 if there exist $\beta' \in \mathcal{K}_\infty$ and a constant $\lambda > 0$ such that (9) holds with $\beta(\|X_0\|_{\mathcal{H}},t) = \beta'(\|X_0\|_{\mathcal{H}})\mathrm{e}^{-\lambda t}$.

Definition 2 System (1) is said to be integral input-to-state stable (iISS) with respect to disturbances $d_1, d_2 \in C^1(\mathbb{R}_{\geq 0}; L^2(0,l))$ and $d_3, d_4 \in C^2(\mathbb{R}_{\geq 0}) \cap L^\infty(\mathbb{R}_{\geq 0})$, if there exist functions $\beta \in \mathcal{KL}, \theta_1, \theta_2, \theta_3, \theta_4 \in \mathcal{K}_\infty$ and $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \mathcal{K}$, such that the solution of System (1) satisfies

$$\begin{aligned}\|X(\cdot,t)\|_{\mathcal{H}} \leq \beta(\|X_0\|_{\mathcal{H}},t) + \theta_1\left(\int_0^t \gamma_1(\|d_1(\cdot,s)\|_{L^2(0,l)})\mathrm{d}s\right)\\ + \theta_2\left(\int_0^t \gamma_2(\|d_2(\cdot,s)\|_{L^2(0,l)})\mathrm{d}s\right)\\ + \theta_3\left(\int_0^t \gamma_3(|d_3(s)|)\mathrm{d}s\right) + \theta_4\left(\int_0^t \gamma_4(|d_4(s)|)\mathrm{d}s\right),\ \forall t \geq 0.\end{aligned} \tag{10}$$

Moreover, System (1) is said to be exponential integral input-to-state stable (EiISS) with respect to disturbances d_1, d_2, d_3, and d_4 if there exist $\beta' \in \mathcal{K}_\infty$ and a constant $\lambda > 0$ such that (10) holds with $\beta(\|X_0\|_{\mathcal{H}},t) = \beta'(\|X_0\|_{\mathcal{H}})\mathrm{e}^{-\lambda t}$.

In order to obtain the stability of the solutions, we make the following assumptions:

$$l^2\sqrt{2l}\|d_3\|_{L^\infty(\mathbb{R}_{\geq 0})} < 2a_1, \tag{11a}$$

$$\sqrt{2l}(1+l\sqrt{l})(1+K_m)(1+c_1+c_2-p_2+q_2+\|d_4\|_{L^\infty(\mathbb{R}_{\geq 0})}) < a_2, \tag{11b}$$

$$l^2\sqrt{2l}(1+l^3)(c_1+p_1-q_1+q_2+\|d_3\|_{L^\infty(\mathbb{R}_{\geq 0})}) < 2b_1, \tag{11c}$$

$$\sqrt{2l}(1+l^3)(1+p_1+c_2-p_2+q_2+\|d_4\|_{L^\infty(\mathbb{R}_{\geq 0})}) < b_2, \tag{11d}$$

where $K_m = \max\left\{\frac{1}{\sqrt{a_1}}, \frac{1}{\sqrt{a_2}}, \frac{l^2}{2\sqrt{a_2}}, \frac{l^4}{4\sqrt{a_1}}\right\}$.

For notational simplicity, we denote hereafter $\|\cdot\|_{L^2(0,l)}$ by $\|\cdot\|$.

Theorem 2 *Assume that*

(i) $d_1, d_2 \in C^1(\mathbb{R}_{\geq 0}; L^2(0,l))$*;*
(ii) $d_3, d_4 \in C^2(\mathbb{R}_{\geq 0}) \cap L^\infty(\mathbb{R}_{\geq 0})$*;*
(iii) all conditions in (11) are satisfied.

Then System (1) is EISS and EiISS, having the following estimates:

$$\begin{aligned}\|X(\cdot,t)\|_{\mathcal{H}} \leq Ce^{-\frac{\mu_m}{4}t}\|X_0\|_{\mathcal{H}} + C\Bigg(\|d_1\|_{L^\infty(0,t;L^2(0,l))} + \|d_2\|_{L^\infty(0,t;L^2(0,l))}\\ + \|d_3\|^{\frac{1}{2}}_{L^\infty(0,t)} + \|d_4\|^{\frac{1}{2}}_{L^\infty(0,t)}\Bigg),\end{aligned} \tag{12}$$

and

$$\|X(\cdot,t)\|_{\mathcal{H}} \leq Ce^{-\frac{\mu_m}{4}t}\|X_0\|_{\mathcal{H}} + C\bigg(\int_0^t \|d_1(\cdot,s)\|^2 ds\bigg)^{\frac{1}{2}} + C\bigg(\int_0^t \|d_2(\cdot,s)\|^2 ds\bigg)^{\frac{1}{2}}$$
$$+\, C\left(\int_0^t \big(|d_3(s)|ds\right)^{\frac{1}{2}} + C\bigg(\int_0^t |d_4(s)|ds\bigg)^{\frac{1}{2}}. \tag{13}$$

where $C > 0$ and $\mu_m > 0$ are some constants independent of t.

Proof We introduce first the following notations:

$$f_1(\phi,\phi_t,w_t,d_1) = c_1\phi + p_1\phi_t + q_1 w_t + d_1,$$
$$f_2(\phi,\phi_t,w_t,d_2) = c_2\phi + p_2\phi_t + q_2 w_t + d_2.$$

In order to find an appropriate Lyapunov functional candidate, multiplying (1a) by w_t and considering the fact that $w \in C^1(\mathbb{R}_{\geq 0}; H^2(0,l)) \cap C^2(\mathbb{R}_{\geq 0}; L^2(0,l))$ with $(a_1 w_{yy} + b_1 w_{tyy})(\cdot,t) \in H^2(0,1)$, we get

$$\begin{aligned}\int_0^l f_1(\phi,\phi_t,w_t,d_1)w_t\mathrm{d}y &= \int_0^l (w_{tt} + (a_1 w_{yy} + b_1 w_{tyy})_{yy})w_t\mathrm{d}y\\ &= \int_0^l w_{tt}w_t\mathrm{d}y + a_1\int_0^l w_{yy}w_{tyy}\mathrm{d}y\\ &\quad + b_1\int_0^l w_{tyy}^2\mathrm{d}y + d_3(t)w_t(l,t)\\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|w_t\|^2 + a_1\|w_{yy}\|^2\big) + b_1\|w_{tyy}\|^2 + d_3(t)w_t(l,t),\end{aligned}$$

which gives

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|w_t\|^2 + a_1\|w_{yy}\|^2\big)\\ &\quad = -b_1\|w_{tyy}\|^2 - d_3(t)w_t(l,t) + \int_0^l f_1(\phi,\phi_t,w_t,d_1)w_t\mathrm{d}y.\end{aligned} \tag{14}$$

Multiplying (1a) by ϕ_t and since $\phi \in C^1(\mathbb{R}_{\geq 0}; H^1(0,l)) \cap C^2(\mathbb{R}_{\geq 0}; L^2(0,l))$ with $(a_2\phi_y + b_2\phi_{ty})(\cdot,t) \in H^1(0,l)$, we get

$$\begin{aligned}\int_0^l f_2(\phi,\phi_t,w_t,d_2)\phi_t\mathrm{d}y &= \int_0^l (\phi_{tt} - (a_2\phi_y + b_2\phi_{ty})_y)\phi_t\mathrm{d}y\\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\phi_t\|^2 + a_2\|\phi_y\|^2\big) + b_2\|\phi_{ty}\|^2 - d_4(t)\phi_t(l,t),\end{aligned}$$

which gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\phi_t\|^2 + a_2\|\phi_y\|^2\right) = -b_2\|\phi_{ty}\|^2 + d_4(t)\phi_t(l,t) + \int_0^l f_2(\phi,\phi_t,w_t,d_2)\phi_t\mathrm{d}y. \tag{15}$$

In order to deal with the items containing $\|w_{yy}\|^2$ and $\|\phi_y\|^2$, multiplying (1a) and (1b) by w and ϕ, respectively, yields

$$\begin{aligned}\int_0^l w_{tt}w\mathrm{d}y =& -a_1\|w_{yy}\|^2 - d_3(t)w(l,t) - \int_0^l w_{yy}w_{tyy}\mathrm{d}y\\ &+\int_0^l f_1(\phi,\phi_t,w_t,d_1)w\mathrm{d}y,\\ \int_0^l \phi_{tt}\phi\mathrm{d}y =& -a_2\|\phi_y\|^2 + d_4(t)\phi(l,t) - \int_0^l \phi_y\phi_{ty}\mathrm{d}y + \int_0^l f_2(\phi,\phi_t,w_t,d_2)\phi\mathrm{d}y.\end{aligned}$$

Note that for any $\eta \in C^2(\mathbb{R}_{\geq 0}; L^2(0,l))$, there holds $\frac{\mathrm{d}}{\mathrm{d}t}\int_0^l \eta\eta_t\mathrm{d}y = \int_0^l \eta_t^2\mathrm{d}y + \int_0^l \eta\eta_{tt}\mathrm{d}y$. Then we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^l ww_t\mathrm{d}y =& -a_1\|w_{yy}\|^2 - d_3(t)w(l,t) - \int_0^l w_{yy}w_{tyy}\mathrm{d}y\\ &+\int_0^l w_t^2\mathrm{d}y + \int_0^l f_1(\phi,\phi_t,w_t,d_1)w\mathrm{d}y, \qquad (16)\\ \frac{\mathrm{d}}{\mathrm{d}t}\int_0^l \phi\phi_t\mathrm{d}y =& -a_2\|\phi_y\|^2 + d_4(t)\phi(l,t) - \int_0^l \phi_y\phi_{ty}\mathrm{d}y\\ &+\int_0^l \phi_t^2\mathrm{d}y + \int_0^l f_2(\phi,\phi_t,w_t,d_2)\phi\mathrm{d}y. \qquad (17)\end{aligned}$$

We define the augmented energy

$$\mathcal{E}(t) = E(t) + \varepsilon_1\int_0^l \phi\phi_t\mathrm{d}y + \varepsilon_2\int_0^l ww_t\mathrm{d}y, \tag{18}$$

where $0 < \varepsilon_1 < 1$ and $0 < \varepsilon_2 < 1$ are constants to be chosen later.

Note that (see [22])

$$\left|\int_0^l ww_t\mathrm{d}y\right| \leq \frac{\max\{1, l^4/2\}}{\sqrt{a_1}}E(t),$$

and

$$\left|\int_0^l \phi\phi_t\mathrm{d}y\right| \leq \frac{\max\{1, l^2/2\}}{\sqrt{a_2}}E(t).$$

Choosing $0 < \varepsilon_1, \varepsilon_2 < \frac{1}{K_m}$, we have

$$\frac{1}{1+K_m\varepsilon_m}\mathcal{E}(t) \le E(t) \le \frac{1}{1-K_m\varepsilon_m}\mathcal{E}(t), \tag{19}$$

where $\varepsilon_m = \max\{\varepsilon_1, \varepsilon_2\}$.

Based on (14) to (18) and "Appendix A", we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}E(t) + \varepsilon_1\frac{\mathrm{d}}{\mathrm{d}t}\int_0^l \phi\phi_t\mathrm{d}y + \varepsilon_2\frac{\mathrm{d}}{\mathrm{d}t}\int_0^l ww_t\mathrm{d}y \\
&= -\, b_1\|w_{tyy}\|^2 - d_3(t)w_t(l,t) + \int_0^l f_1(\phi,\phi_t,w_t,d_1)w_t\mathrm{d}y - b_2\|\phi_{ty}\|^2 \\
&\quad + d_4(t)\phi_t(l,t) + \int_0^l f_2(\phi,\phi_t,w_t,d_2)\phi_t\mathrm{d}y + \varepsilon_1\Big(-a_2\|\phi_y\|^2 + d_4(t)\phi(l,t) \\
&\quad - \int_0^l \phi_y\phi_{ty}\mathrm{d}y + \int_0^l \phi_t^2\mathrm{d}y + \int_0^l f_2(\phi,\phi_t,w_t,d_2)\phi\mathrm{d}y\Big) + \varepsilon_2\Big(-a_1\|w_{yy}\|^2 \\
&\quad - d_3(t)w(l,t) - \int_0^l w_{yy}w_{tyy}\mathrm{d}y + \int_0^l w_t^2\mathrm{d}y + \int_0^l f_1(\phi,\phi_t,w_t,d_1)w\mathrm{d}y\Big) \\
&= -b_1\|w_{tyy}\|^2 - \varepsilon_2 a_1\|w_{yy}\|^2 + \varepsilon_2\|w_t\|^2 - b_2\|\phi_{ty}\|^2 \\
&\quad - \varepsilon_1 a_2\|\phi_y\|^2 + \varepsilon_1\|\phi_t\|^2 - \varepsilon_1\int_0^l \phi_y\phi_{ty}\mathrm{d}y - \varepsilon_2\int_0^l w_{yy}w_{tyy}\mathrm{d}y \\
&\quad + \int_0^l f_1(\phi,\phi_t,w_t,d_1)(w_t+\varepsilon_2 w)\mathrm{d}y + \int_0^l f_2(\phi,\phi_t,w_t,d_2)(\phi_t+\varepsilon_1\phi)\mathrm{d}y \\
&\quad - \big(w_t(l,t)+\varepsilon_2 w(l,t)\big)d_3(t) + \big(\phi_t(l,t)+\varepsilon_1\phi(l,t)\big)d_4(t) \\
&\le (\varepsilon_2+\Lambda_1)\|w_t\|^2 + (\Lambda_2-\varepsilon_2 a_1)\|w_{yy}\|^2 + (\varepsilon_1+\Lambda_3)\|\phi_t\|^2 \\
&\quad + (\Lambda_4-\varepsilon_1 a_2)\|\phi_y\|^2 + (\Lambda_5-b_2)\|\phi_{ty}\|^2 + (\Lambda_6-b_1)\|w_{tyy}\|^2 + \Lambda_7 \\
&\le (\varepsilon_2+\Lambda_1)\|w_t\|^2 + (\Lambda_2-\varepsilon_2 a_1)\|w_{yy}\|^2 + (\varepsilon_1+\Lambda_3)\|\phi_t\|^2 \\
&\quad + (\Lambda_4-\varepsilon_1 a_2)\|\phi_y\|^2 + \frac{2}{l^2}(\Lambda_5-b_2)\|\phi_t\|^2 + \frac{4}{l^4}(\Lambda_6-b_1)\|w_t\|^2 + \Lambda_7 \\
&\le \Big(\varepsilon_2+\Lambda_1+\frac{4}{l^4}(\Lambda_6-b_1)\Big)\|w_t\|^2 + (\Lambda_2-\varepsilon_2 a_1)\|w_{yy}\|^2 \\
&\quad + \Big(\varepsilon_1+\Lambda_3+\frac{2}{l^2}(\Lambda_5-b_2)\Big)\|\phi_t\|^2 + (\Lambda_4-\varepsilon_1 a_2)\|\phi_y\|^2 + \Lambda_7,
\end{aligned} \tag{20}$$

with the coefficients satisfying

$$\Lambda_5 - b_2 < \Lambda_5' - b_2 < 0, \tag{21a}$$
$$\Lambda_6 - b_1 < \Lambda_6' - b_1 < 0, \tag{21b}$$
$$\varepsilon_2 + \Lambda_1 + \frac{4}{l^4}(\Lambda_6 - b_1) < \varepsilon_2 + \Lambda_1 + \frac{4}{l^4}(\Lambda_6' - b_1) < 0, \tag{21c}$$
$$\Lambda_2 - \varepsilon_2 a_1 < \Lambda_2' - \varepsilon_2 a_1 < 0, \tag{21d}$$

$$\varepsilon_1 + \Lambda_3 + \frac{2}{l^2}(\Lambda_5 - b_2) < \varepsilon_1 + \Lambda_3 + \frac{2}{l^2}(\Lambda_5' - b_2) < 0, \tag{21e}$$

$$\Lambda_4 - \varepsilon_1 a_2 < \Lambda_4' - \varepsilon_1 a_2 < 0, \tag{21f}$$

where $\Lambda_1, \Lambda_2, ..., \Lambda_7$ and $\Lambda_2', \Lambda_4', ..., \Lambda_7'$ are defined in (30) in "Appendix A". The proof of the above inequalities is given in "Appendix B".

Setting $\mu_m = \min\left\{-\varepsilon_2 - \Lambda_1 - \frac{4}{l^4}(\Lambda_6' - b_1), -\Lambda_2' + \varepsilon_2 a_1, -\varepsilon_1 - \Lambda_3 - \frac{2}{l^2}(\Lambda_5' - b_2), -\Lambda_4' + \varepsilon_1 a_2\right\} > 0$, which is independent of t, we obtain from (19) and (20):

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) \leq &- \mu_m E(t) + \Lambda_7 \\
\leq &- \frac{\mu_m}{1 + K_m\varepsilon_m}\mathcal{E}(t) + \Lambda_7 \\
\leq &- \frac{\mu_m}{2}\mathcal{E}(t) + \Lambda_7 \\
= &- \frac{\mu_m}{2}\mathcal{E}(t) + \frac{\|d_1(\cdot, t)\|^2}{2}\left(\frac{1}{r_7} + \frac{\varepsilon_2}{r_8}\right) + \frac{\|d_2(\cdot, t)\|^2}{2}\left(\frac{1}{r_9} + \frac{\varepsilon_1}{r_{10}}\right) \\
&+ 2\sqrt{2l}\big(|d_3(t)| + |d_4(t)\big) \\
\leq &- \frac{\mu_m}{2}\mathcal{E}(t) + C_1\Big(\|d_1(\cdot, t)\|^2 + \|d_2(\cdot, t)\|^2 + |d_3(t)| + |d_4(t)|\Big), \qquad (22) \\
\leq &- \frac{\mu_m}{2}\mathcal{E}(t) + C_1\Big(\|d_1\|^2_{L^\infty(0,t;L^2(0,l))} + \|d_2\|^2_{L^\infty(0,t;L^2(0,l))} \\
&+ \|d_3\|_{L^\infty(0,t)} + \|d_4\|_{L^\infty(0,t)}\Big), \qquad (23)
\end{aligned}$$

where $C_1 > 0$ is a constant independent of t.

We infer from Comparison Lemma (see, [20, Lemma 3.4]) and (23) that

$$\begin{aligned}
\mathcal{E}(t) \leq &\mathcal{E}(0)\mathrm{e}^{-\frac{\mu_m}{2}t} + \frac{2C_1}{\mu_m}\Big(\|d_1\|^2_{L^\infty(0,t;L^2(0,l))} + \|d_2\|^2_{L^\infty(0,t;L^2(0,l))} \\
&+ \|d_3\|_{L^\infty(0,t)} + \|d_4\|_{L^\infty(0,t)}\Big)(1 - \mathrm{e}^{-\frac{\mu_m}{2}t}) \\
\leq &\mathcal{E}(0)\mathrm{e}^{-\frac{\mu_m}{2}t} + \frac{2C_1}{\mu_m}\Big(\|d_1\|^2_{L^\infty(0,t;L^2(0,l))} + \|d_2\|^2_{L^\infty(0,t;L^2(0,l))} \\
&+ \|d_3\|_{L^\infty(0,t)} + \|d_4\|_{L^\infty(0,t)}\Big) \\
\leq &\mathcal{E}(0)\mathrm{e}^{-\frac{\mu_m}{2}t} + C_2\Big(\|d_1\|^2_{L^\infty(0,t;L^2(0,l))} + \|d_2\|^2_{L^\infty(0,t;L^2(0,l))} \\
&+ \|d_3\|_{L^\infty(0,t)} + \|d_4\|_{L^\infty(0,t)}\Big),
\end{aligned}$$

where $C_2 > 0$ is a constant independent of t. We conclude by (19) and $\varepsilon_m < \frac{1}{K_m}$ that

$$0 \leq E(t) \leq \frac{1}{1 - K_m\varepsilon_m}\mathcal{E}(t)$$

$$
\begin{aligned}
&\leq \frac{1}{1-K_m\varepsilon_m}\mathcal{E}(0)\mathrm{e}^{-\frac{\mu_m}{2}t} + \frac{C_2}{1-K_m\varepsilon_m}\Big(\|d_3\|_{L^\infty(0,t)} + \|d_4\|_{L^\infty(0,t)} \\
&\quad + \|d_1\|^2_{L^\infty(0,t;L^2(0,l))} + \|d_2\|^2_{L^\infty(0,t;L^2(0,l))}\Big) \\
&\leq \frac{1+K_m\varepsilon_m}{1-K_m\varepsilon_m}E(0)\mathrm{e}^{-\frac{\mu_m}{2}t} + \frac{C_2}{1-K_m\varepsilon_m}\Big(\|d_3\|_{L^\infty(0,t)} + \|d_4\|_{L^\infty(0,t)} \\
&\quad + \|d_1\|^2_{L^\infty(0,t;L^2(0,l))} + \|d_2\|^2_{L^\infty(0,t;L^2(0,l))}\Big) \\
&\leq C_3E(0)\mathrm{e}^{-\frac{\mu_m}{2}t} + C_3\Big(\|d_3\|_{L^\infty(0,t)} + \|d_4\|_{L^\infty(0,t)} \\
&\quad + \|d_1\|^2_{L^\infty(0,t;L^2(0,l))} + \|d_2\|^2_{L^\infty(0,t;L^2(0,l))}\Big),
\end{aligned}
$$

where $C_3 > 0$ is a constant independent of t. Noting that since $\|X(\cdot,t)\|^2_{\mathcal{H}} = 2E(t)$ for all $t \geq 0$, and $(a+b)^{\frac{1}{2}} \leq a^{\frac{1}{2}} + b^{\frac{1}{2}}$ for all $a \geq 0, b \geq 0$, the claimed result (12) follows immediately.

Similarly, we get by (22) and Comparison Lemma

$$
\begin{aligned}
\mathcal{E}(t) \leq \mathcal{E}(0)\mathrm{e}^{-\frac{\mu_m}{2}t} &+ C_4\int_0^t \big(|d_3(s)| + |d_4(s)|\big)\mathrm{d}s \\
&+ C_4\int_0^t \big(\|d_1(\cdot,s)\|^2 + \|d_2(\cdot,s)\|^2\big)\mathrm{d}s,
\end{aligned}
$$

where $C_4 > 0$ is a constant independent of t. Hence, it follows from (19) that

$$
\begin{aligned}
E(t) \leq C_5E(0)\mathrm{e}^{-\frac{\mu_m}{2}t} &+ C_5\int_0^t \big(|d_3(s)| + |d_4(s)|\big)\mathrm{d}s \\
&+ C_5\int_0^t \big(\|d_1(\cdot,s)\|^2 + \|d_2(\cdot,s)\|^2\big)\mathrm{d}s,
\end{aligned}
$$

where $C_5 > 0$ is a constant independent of t. Finally, we conclude (13) as above.

Note that

$$
\begin{aligned}
\|\phi(\cdot,t)\|^2_{L^\infty(0,l)} &\leq 2l\|\phi_y\|^2 \leq \frac{4l}{a_2}E(t), \\
\|w_y(\cdot,t)\|^2_{L^\infty(0,l)} &\leq \frac{l^2}{2}\|w_{yy}\|^2 \leq \frac{l^2}{a_1}E(t), \\
\|w(\cdot,t)\|^2_{L^\infty(0,l)} &\leq 2l\|w_y\|^2 \leq l^3\|w_{yy}\|^2 \leq \frac{2l^3}{a_1}E(t).
\end{aligned}
$$

We have the following boundedness estimates for the solution of System (1).

Corollary 1 *Under the same assumptions as in Theorem 2, the following estimates hold true:*

$$
\|w(\cdot,t)\|^2_{L^\infty(0,l)} + \|w_y(\cdot,t)\|^2_{L^\infty(0,l)} + \|\phi(\cdot,t)\|^2_{L^\infty(0,l)}
$$

$$\leq CE(0)e^{-\frac{\mu_m}{2}t} + C\Big(\|d_1\|^2_{L^\infty(0,t;L^2(0,l))} + \|d_2\|^2_{L^\infty(0,t;L^2(0,l))}\Big) + C\Big(\|d_3\|_{L^\infty(0,t)} + \|d_4\|_{L^\infty(0,t)}\Big),$$

and

$$\begin{aligned}&\|w(\cdot,t)\|^2_{L^\infty(0,l)} + \|w_y(\cdot,t)\|^2_{L^\infty(0,l)} + \|\phi(\cdot,t)\|^2_{L^\infty(0,l)}\\ &\leq CE(0)e^{-\frac{\mu_m}{2}t} + C\int_0^t \Big(\|d_1(\cdot,s)\|^2_{L^2(0,l)} + \|d_2(\cdot,s)\|^2_{L^2(0,l)}\Big)ds\\ &\quad + C\int_0^t \big(|d_3(s)| + |d_4(s)|\big)ds,\end{aligned}$$

where $C > 0$ and $\mu_m > 0$ are some constants independent of t.

Note that the boundedness assumption on d_3 and d_4 can be relaxed and the structural conditions in (11) can be simplified. Indeed, we estimate I_4 and I_5 in "Appendix A" as follows:

$$\begin{aligned}I_4 &:= -(w_t(l,t) + \varepsilon_2 w(l,t))d_3(t)\\ &\leq \frac{1}{2r_{13}}d_3^2(t) + \frac{r_{13}}{2}(w_t^2(l,t) + \varepsilon_2^2 w^2(l,t))\\ &\leq \frac{1}{2r_{13}}d_3^2(t) + lr_{13}(\|w_{ty}\|^2 + \varepsilon_2^2\|w_y\|^2)\\ &\leq \frac{1}{2r_{13}}d_3^2(t) + \frac{l^3 r_{13}}{2}(\|w_{tyy}\|^2 + \varepsilon_2^2\|w_{yy}\|^2),\ \forall r_{13} > 0,\end{aligned}$$

and

$$\begin{aligned}I_5 &:= (\phi_t(l,t) + \varepsilon_1\phi(l,t))d_4(t)\\ &\leq \frac{1}{2r_{14}}d_4^2(t) + l^3 r_{14}(\|\phi_{ty}\|^2 + \varepsilon_1^2\|\phi_y\|^2),\ \forall r_{14} > 0.\end{aligned}$$

Then the parameters $\Lambda_2, \Lambda_4, \Lambda_5, \Lambda_6, \Lambda_7$ in "Appendix A" become (other parameters retain unchanged)

$$\begin{aligned}\Lambda_2 &= \frac{\varepsilon_2}{2r_{12}} + \lambda_2 + \frac{\varepsilon_2^2 l^3 r_{13}}{2},\\ \Lambda_4 &= \frac{\varepsilon_1}{2r_{11}} + \lambda_4 + \lambda_8 + lr_{14}\varepsilon_1^2,\\ \Lambda_5 &= \frac{\varepsilon_1}{2}r_{11} + lr_{14},\\ \Lambda_6 &= \frac{\varepsilon_2}{2}r_{12} + \frac{l^3 r_{13}}{2},\\ \Lambda_7 &= \lambda_5 + \lambda_9 + \frac{1}{2r_{13}}d_3^2(t) + \frac{1}{2r_{14}}d_4^2(t).\end{aligned}$$

If we replace the conditions (11) by

$$\varepsilon_2 + \Lambda_1 + \frac{4}{l^4}(\Lambda_6 - b_1) < 0, \tag{24a}$$

$$\Lambda_2 - \varepsilon_2 a_1 < 0, \tag{24b}$$

$$\varepsilon_1 + \Lambda_3 + \frac{2}{l^2}(\Lambda_5 - b_2) < 0, \tag{24c}$$

$$\Lambda_4 - \varepsilon_1 a_2 < 0, \tag{24d}$$

for some $r_1, r_2, ..., r_{14}, \varepsilon_1, \varepsilon_2$, and relax the boundedness of d_3 and d_4, then we have:

Theorem 3 *Under the assumptions given in (24) and assuming that $d_1, d_2 \in C^1(\mathbb{R}_{\geq 0}; L^2(0, l))$ and $d_3, d_4 \in C^2(\mathbb{R}_{\geq 0})$, System (1) is EISS and EiISS, having the following estimates:*

$$\|X(\cdot, t)\|_{\mathcal{H}} \leq Ce^{-\frac{\mu_m}{4}t}\|X_0\|_{\mathcal{H}} + C\Big(\|d_1\|_{L^\infty(0,t;L^2(0,l))} + \|d_2\|_{L^\infty(0,t;L^2(0,l))}\Big) + C\Big(\|d_3\|_{L^\infty(0,t)} + \|d_4\|_{L^\infty(0,t)}\Big),$$

and

$$\|X(\cdot, t)\|_{\mathcal{H}} \leq Ce^{-\frac{\mu_m}{4}t}\|X_0\|_{\mathcal{H}} + C\bigg(\int_0^t \|d_1(\cdot, s)\|^2_{L^2(0,l)} ds\bigg)^{\frac{1}{2}} + C\bigg(\int_0^t \|d_2(\cdot, s)\|^2_{L^2(0,l)} ds\bigg)^{\frac{1}{2}} + C\bigg(\int_0^t d_3^2(s) ds\bigg)^{\frac{1}{2}} + C\bigg(\int_0^t d_4^2(s) ds\bigg)^{\frac{1}{2}},$$

where $C > 0$ and $\mu_m > 0$ are constants independent of t.

Remark 2 If $d_3(t) = k_1(w_t(l, t) + \varepsilon_2 w(l, t))$ and $d_4(t) = -k_2(\phi_t(l, t) + \varepsilon_1\phi(l, t))$ appear as the feedback controls with constants $k_1 \geq 0$ and $k_2 \geq 0$, and (w, ϕ) is the solution of System (1), then the following estimates hold:

$$E(t) \leq CE(0)\mathrm{e}^{-\frac{\mu_m}{2}t} + C\Big(\|d_1\|^2_{L^\infty(0,t;L^2(0,l))} + \|d_2\|^2_{L^\infty(0,t;L^2(0,l))}\Big),$$

and

$$E(t) \leq CE(0)\mathrm{e}^{-\frac{\mu_m}{2}t} + C\int_0^t \Big(\|d_1(\cdot, s)\|^2_{L^2(0,l)} + \|d_2(\cdot, s)\|^2_{L^2(0,l)}\Big)\mathrm{d}s,$$

where $C > 0$ and $\mu_m > 0$ are some constants independent of t, d_1, and d_2.

Indeed, in this case, I_4 and I_5 given in (28) and (29) in "Appendix A" become $I_4 = -(w_t(l, t)+\varepsilon_2 w(l, t))(a_1 w_{yy}+b_1 w_{tyy})_y(l, t) = -k_1(w_t(l, t)+\varepsilon_2 w(l, t))^2 \leq 0$

and $I_5 = (\phi_t(l,t) + \varepsilon_1\phi(l,t))(a_2\phi_y + b_2\phi_{ty})(l,t) = -k_2(\phi_t(l,t) + \varepsilon_1\phi(l,t))^2 \leq 0$. Then taking in (30) $M_1 = M_2 = 0$ and proceeding as the proof of Theorem 2, one may get the desired results.

Note that, under the above assumptions, a disturbance-free setting (i.e., $d_1 = d_2 = 0$ in (1a) and (1b)) was considered in [22], and the exponential stability was obtained.

Remark 3 A more generic setting is to replace the boundary conditions given in (1c) by $(a_1 w_{yy} + b_1 w_{tyy})_y(l,t) = d_3(t) + k_1(w_t(l,t) + \varepsilon_2 w(l,t))$ and $(a_2\phi_y + b_2\phi_{ty})(l,t) = d_4(t) - k_2(\phi_t(l,t) + \varepsilon_1\phi(l,t))$, where $d_3(t)$, $d_4(t)$ are disturbances, $k_1 \geq 0$ and $k_2 \geq 0$. Under the same assumptions on d_1, d_2, d_3, and d_4 as in Theorem 2 or Theorem 3, if (w, ϕ) is the solution of System (1) with the above boundary conditions, then it can verify that the ISS and iISS properties given in Theorem 2 or Theorem 3 hold.

Remark 4 As pointed out in [16,41], the assumptions on the continuities of the disturbances are required for assessing the well-posedness of the considered system. However, they are only sufficient conditions and can be weakened if solutions in a weak sense are considered. Moreover, as shown in the proof of Theorem 2, the assumptions on the continuities of disturbances can eventually be relaxed for the establishment of ISS estimates.

4 Simulation results

The ISS properties of System (1) are illustrated in this section. Numerical simulations are performed based on the Galerkin method. The numerical values of the parameters are set to $a_1 = 3$, $b_1 = 0.3$, $c_1 = 0.06$, $p_1 = q_1 = 0.04$, $a_2 = 5$, $b_2 = 0.5$, $c_2 = 0.08$, $p_2 = q_2 = 0.06$, and $l = 1$. The four perturbation signals are selected as follows:

$$\begin{aligned}
d_1(y,t) &= 2(1 + \mathrm{e}^{-0.3t})(1 + \sin(0.5\pi t) + 3\sin(5\pi t))y,\\
d_2(y,t) &= -0.2(1 + \mathrm{e}^{-0.3t})(1 + \sin(0.5\pi t) + 3\sin(5\pi t))y,\\
d_3(t) &= (1 + 2\mathrm{e}^{-0.2t})\cos(0.2\pi t)\sin(3\pi t),\\
d_4(t) &= 0.5(1 + \mathrm{e}^{-0.2t})\sin(0.2\pi t)\cos(3\pi t),
\end{aligned}$$

while the initial conditions are set to $w_0 = 0.15y^2(y - 3l)/(6l^2)\,\mathrm{m}$ and $\phi_0(y) = 8y^2/l^2$ deg. The system response is depicted in Fig. 1 for the flexible displacements over the time and spatial domains. The behavior at the tip, exhibiting the displacements with maximal amplitude, is depicted in Fig. 2. It can be seen that the nonzero initial condition vanishes due to the exponential stability of the underlying C_0-semigroup. Furthermore, the amplitude of the flexible displacements under bounded in-domain and boundary perturbations remains bounded, which confirms the theoretical analysis.

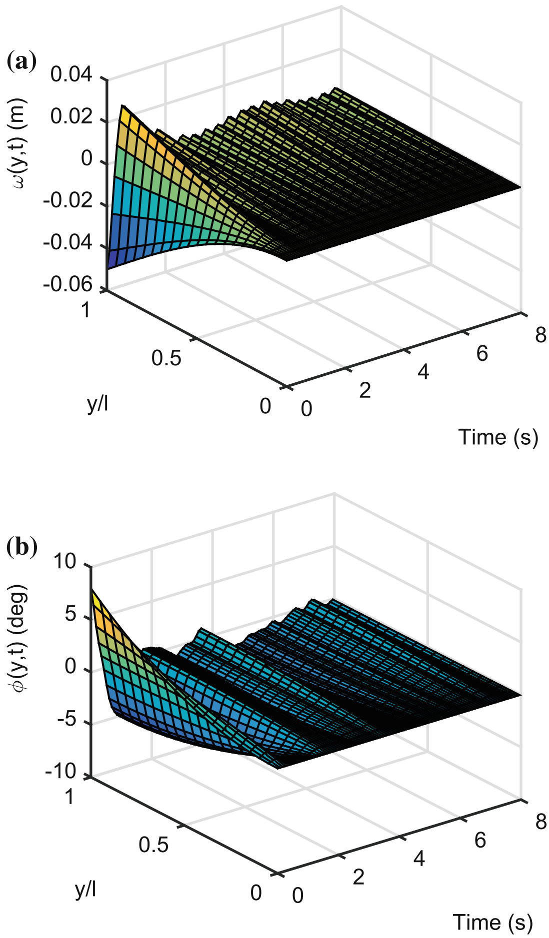

Fig. 1 Bending and twisting displacements simulation results. **a** Bending displacement $\omega(y, t)$, **b** twisting displacement $\phi(y, t)$

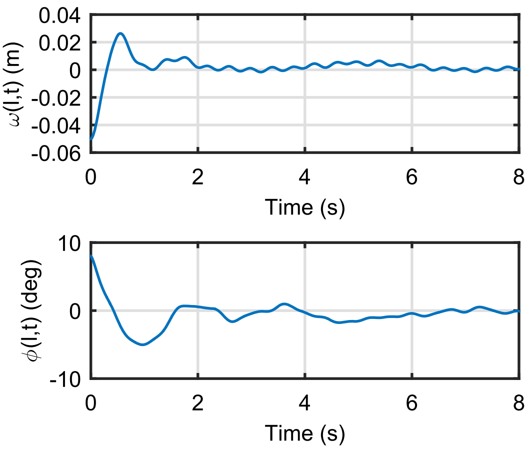

Fig. 2 Bending and twisting displacements at the tip

5 Concluding remarks

The present work established the exponential input-to-state stability (EISS) and exponential integral input-to-state stability (EiISS) of a system of boundary controlled partial differential equations (PDEs) with respect to boundary and in-domain disturbances. Compared to the ISS property with respect to in-domain disturbances, the case of boundary disturbances is more challenging due to essentially regularity issues. This difficulty has been overcome by using a priori estimates of the solution to the original PDEs, which leads to ISS gains in the expected form. It should be noted that the Lyapunov functional candidate used in this work is greatly inspired by the results reported [22]. As a further direction of research, it may be interesting to introduce and develop tools allowing the establishment of the ISS property for a wider range of problems in a more systematic manner, such as the attempt presented in [41].

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A Proof of (20)

By Young's inequality (see, e.g., [8, Appendix B.2.d]) and Poincaré inequality (see, e.g., [21, Chap. 2, Remark 2.2]), we have

$$\begin{aligned}
\int_0^l \phi^2 \mathrm{d}y \leq & \frac{4l^2}{\pi}\int_0^l \phi_y^2 \mathrm{d}y \leq \frac{l^2}{2}\int_0^l \phi_y^2 \mathrm{d}y,\\
\int_0^l \phi w_t \mathrm{d}y \leq & \frac{1}{2r_1}\int_0^l \phi^2 \mathrm{d}y + \frac{r_1}{2}\int_0^l w_t^2 \mathrm{d}y \leq \frac{l^2}{4r_1}\int_0^l \phi_y^2 \mathrm{d}y + \frac{r_1}{2}\int_0^l w_t^2 \mathrm{d}y,\\
\int_0^l \phi_t w_t \mathrm{d}y \leq & \frac{1}{2r_2}\int_0^l \phi_t^2 \mathrm{d}y + \frac{r_2}{2}\int_0^l w_t^2 \mathrm{d}y,\\
\int_0^l \phi w \mathrm{d}y \leq & \frac{1}{2r_3}\int_0^l \phi^2 \mathrm{d}y + \frac{r_3}{2}\int_0^l w^2 \mathrm{d}y \leq \frac{l^2}{4r_3}\int_0^l \phi_y^2 \mathrm{d}y + \frac{r_3 l^4}{8}\int_0^l w_{yy}^2 \mathrm{d}y,\\
\int_0^l \phi_t w \mathrm{d}y \leq & \frac{1}{2r_4}\int_0^l \phi_t^2 \mathrm{d}y + \frac{r_4}{2}\int_0^l w^2 \mathrm{d}y \leq \frac{1}{2r_4}\int_0^l \phi_t^2 \mathrm{d}y + \frac{r_4 l^4}{8}\int_0^l w_{yy}^2 \mathrm{d}y,\\
\int_0^l w w_t \mathrm{d}y \leq & \frac{1}{2r_5}\int_0^l w^2 \mathrm{d}y + \frac{r_5}{2}\int_0^l w_t^2 \mathrm{d}y \leq \frac{l^4}{8r_5}\int_0^l w_{yy}^2 \mathrm{d}y + \frac{r_5}{2}\int_0^l w_t^2 \mathrm{d}y,\\
\int_0^l \phi\phi_t \mathrm{d}y \leq & \frac{1}{2r_6}\int_0^l \phi^2 \mathrm{d}y + \frac{r_6}{2}\int_0^l \phi_t^2 \mathrm{d}y \leq \frac{l^2}{4r_6}\int_0^l \phi_y^2 \mathrm{d}y + \frac{r_6}{2}\int_0^l \phi_t^2 \mathrm{d}y,\\
\int_0^l d_1 w_t \mathrm{d}y \leq & \frac{\|d_1(\cdot,t)\|^2}{2r_7} + \frac{r_7}{2}\int_0^l w_t^2 \mathrm{d}y,\\
\int_0^l d_1 w \mathrm{d}y \leq & \frac{\|d_1(\cdot,t)\|^2}{2r_8} + \frac{r_8}{2}\int_0^l w^2 \mathrm{d}y \leq \frac{\|d_1(\cdot,t)\|^2}{2r_8} + \frac{r_8 l^4}{8}\int_0^l w_{yy}^2 \mathrm{d}y,\\
\int_0^l d_2 \phi_t \mathrm{d}y \leq & \frac{\|d_2(\cdot,t)\|^2}{2r_9} + \frac{r_9}{2}\int_0^l \phi_t^2 \mathrm{d}y,
\end{aligned}$$

$$\begin{aligned}
\int_0^l d_2\phi \mathrm{d}y \le& \frac{\|d_2(\cdot,t)\|^2}{2r_{10}} + \frac{r_{10}}{2}\int_0^l \phi^2\mathrm{d}y \le \frac{\|d_2(\cdot,t)\|^2}{2r_{10}} + \frac{r_{10}l^2}{4}\int_0^l \phi_y^2\mathrm{d}y,\\
\int_0^l \phi_y\phi_{ty}\mathrm{d}y \le& \frac{1}{2r_{11}}\int_0^l \phi_y^2\mathrm{d}y + \frac{r_{11}}{2}\int_0^l \phi_{ty}^2\mathrm{d}y,\\
\int_0^l w_{yy}w_{tyy}\mathrm{d}y \le& \frac{1}{2r_{12}}\int_0^l w_{yy}^2\mathrm{d}y + \frac{r_{12}}{2}\int_0^l w_{tyy}^2\mathrm{d}y.
\end{aligned}$$

Then we get

$$\begin{aligned}
I_1 :=& \int_0^l f_1(\phi,\phi_t,w_t,d_1)(w_t+\varepsilon_2 w)\mathrm{d}y\\
=& \int_0^l (c_1\phi + p_1\phi_t + q_1 w_t + d_1)(w_t+\varepsilon_2 w)\mathrm{d}y\\
\le& \frac{1}{2}\left(c_1r_1 + p_1r_2 + 2 - q_1 + r_7 + \varepsilon_2 - q_1r_5\right)\|w_t\|^2\\
&+ \frac{\varepsilon_2 l^4}{8}\left(c_1r_3 + p_1r_4 - \frac{q_1}{r_5} + r_8\right)\|w_{yy}\|^2\\
&+ \frac{p_1}{2}\left(\frac{1}{r_2}+\frac{\varepsilon_2}{r_4}\right)\|\phi_t\|^2 + \frac{c_1l^2}{4}\left(\frac{1}{r_1}+\frac{\varepsilon_2}{r_3}\right)\|\phi_y\|^2 + \frac{\|d_1(\cdot,t)\|^2}{2}\left(\frac{1}{r_7}+\frac{\varepsilon_2}{r_8}\right)\\
:=& \lambda_1\|w_t\|^2 + \lambda_2\|w_{yy}\|^2 + \lambda_3\|\phi_t\|^2 + \lambda_4\|\phi_y\|^2 + \lambda_5, \qquad (25)\\
I_2 :=& \int_0^l f_2(\phi,\phi_t,w_t,d_2)(\phi_t+\varepsilon_1\phi)\mathrm{d}y\\
=& \int_0^l (c_2\phi + p_2\phi_t + q_2 w_t + d_2)(\phi_t+\varepsilon_1\phi)\mathrm{d}y\\
\le& \frac{q_2}{2}\left(r_2+\varepsilon_1 r_1\right)\|w_t\|^2 + \frac{1}{2}\left(c_2r_6 - 2p_2 + \frac{q_2}{r_2} + r_9 - \varepsilon_1 p_2 r_6\right)\|\phi_t\|^2\\
&+ \frac{l^2}{4}\left(\frac{c_2}{r_6} + 2\varepsilon_1 c_2 - \frac{\varepsilon_1 p_2}{r_6} + \frac{\varepsilon_1 q_2}{r_1} + \varepsilon_1 r_{10}\right)\|\phi_y\|^2 + \frac{\|d_2(\cdot,t)\|^2}{2}\left(\frac{1}{r_9}+\frac{\varepsilon_1}{r_{10}}\right)\\
:=& \lambda_6\|w_t\|^2 + \lambda_7\|\phi_t\|^2 + \lambda_8\|\phi_y\|^2 + \lambda_9, \qquad (26)\\
I_3 :=& \varepsilon_1\int_0^l \phi_y\phi_{ty}\mathrm{d}y + \varepsilon_2\int_0^l w_{yy}w_{tyy}\mathrm{d}y\\
\le& \frac{\varepsilon_1}{2}\left(\frac{1}{r_{11}}\|\phi_y\|^2 + r_{11}\|\phi_{ty}\|^2\right) + \frac{\varepsilon_2}{2}\left(\frac{1}{r_{12}}\|w_{yy}\|^2 + r_{12}\|w_{tyy}\|^2\right). \qquad (27)
\end{aligned}$$

We shall estimate $(w_t(l,t)+\varepsilon_2 w(l,t))d_3(t)$ and $(\phi_t(l,t)+\varepsilon_1\phi(l,t))d_4(t)$. Note that for any $f \in H^1([0,l])$ with $f(0)=0$ there holds $f^2(l) \le 2l\|f_y\|^2$. We compute

$$\begin{aligned}
I_4 :=& -(w_t(l,t)+\varepsilon_2 w(l,t))d_3(t)\\
\le& |d_3(t)||w_t(l,t)+\varepsilon_2 w(l,t)|\\
\le& |d_3(t)|(\sqrt{2l}\|w_{ty}\| + \varepsilon_2\sqrt{2l}\|w_y\|)
\end{aligned}$$

$$\leq \sqrt{2l}|d_3(t)|(2+\|w_{ty}\|^2+\varepsilon_2\|w_y\|^2)$$
$$\leq \sqrt{2l}|d_3(t)|\left(2+\frac{l^2}{2}\|w_{tyy}\|^2+\frac{\varepsilon_2 l^2}{2}\|w_{yy}\|^2\right). \tag{28}$$

Similarly, we get

$$I_5 := (\phi_t(l,t)+\varepsilon_1\phi(l,t))d_4(t) \leq \sqrt{2l}|d_4(t)|\big(2+\|\phi_{ty}\|^2+\varepsilon_1\|\phi_y\|^2\big). \tag{29}$$

Finally, we have

$$I_1+I_2+I_3+I_4+I_5$$
$$\leq \Lambda_1\|w_t\|^2+\Lambda_2\|w_{yy}\|^2+\Lambda_3\|\phi_t\|^2+\Lambda_4\|\phi_y\|^2+\Lambda_5\|\phi_{ty}\|^2+\Lambda_6\|w_{tyy}\|^2+\Lambda_7,$$

where

$$\Lambda_1 = \lambda_1+\lambda_6, \tag{30a}$$
$$\Lambda_2 = \frac{\varepsilon_2}{2r_{12}}+\lambda_2+\frac{\varepsilon_2 l^2}{2}\sqrt{2l}|d_3(t)| \leq \frac{\varepsilon_2}{2r_{12}}+\lambda_2+\frac{\varepsilon_2 l^2}{2}\sqrt{2l}M_1 := \Lambda_2', \tag{30b}$$
$$\Lambda_3 = \lambda_3+\lambda_7, \tag{30c}$$
$$\Lambda_4 = \frac{\varepsilon_1}{2r_{11}}+\lambda_4+\lambda_8+\varepsilon_1\sqrt{2l}|d_4(t)| \leq \frac{\varepsilon_1}{2r_{11}}+\lambda_4+\lambda_8+\varepsilon_1\sqrt{2l}M_2 := \Lambda_4', \tag{30d}$$
$$\Lambda_5 = \frac{\varepsilon_1}{2}r_{11}+\sqrt{2l}|d_4(t)| \leq \frac{\varepsilon_1}{2}r_{11}+\sqrt{2l}M_2 := \Lambda_5', \tag{30e}$$
$$\Lambda_6 = \frac{\varepsilon_2}{2}r_{12}+\frac{l^2}{2}\sqrt{2l}|d_3(t)| \leq \frac{\varepsilon_2}{2}r_{12}+\frac{l^2}{2}\sqrt{2l}M_1 := \Lambda_6', \tag{30f}$$
$$\Lambda_7 = \lambda_5+\lambda_9+2\sqrt{2l}(|d_3(t)|+|d_4(t)|) \leq \lambda_5+\lambda_9+2\sqrt{2l}(M_1+M_2) := \Lambda_7', \tag{30g}$$
$$M_1 = \|d_3\|_{L^\infty\mathbb{R}_{\geq 0}}, \tag{30h}$$
$$M_2 = \|d_4\|_{L^\infty\mathbb{R}_{\geq 0}}. \tag{30i}$$

B Proof of (21)

First, note that

$$\varepsilon_2+\Lambda_1+\frac{4}{l^4}(\Lambda_6'-b_1)<0$$
$$\Leftrightarrow \varepsilon_2+\frac{1}{2}(c_1r_1+p_1r_2-2q_1+r_7-\varepsilon_2 q_1 r_5)$$
$$+\frac{q_2}{2}(r_2+\varepsilon_1 r_1)+\frac{4}{l^4}\left(\frac{\varepsilon_2}{2}r_{12}+\frac{l^2\sqrt{2l}}{2}M_1-b_1\right)<0, \tag{31a}$$

$$
\begin{aligned}
&\Lambda_2' - \varepsilon_2 a_1 < 0 \\
&\quad \Leftrightarrow \frac{1}{2r_{12}} + \frac{l^4}{8}\left(c_1 r_3 + p_1 r_4 - \frac{q_1}{r_5} + r_8\right) + \frac{l^2\sqrt{2l}}{2} M_1 - a_1 < 0, \qquad (31b)\\
&\varepsilon_1 + \Lambda_3 + \frac{2}{l^2}(\Lambda_5 - b_2) < 0 \\
&\quad \Leftrightarrow \varepsilon_1 + \frac{1}{2}\left(c_2 r_6 - 2p_2 + \frac{q_2}{r_2} + r_9 - \varepsilon_1 p_2 r_6\right) \\
&\qquad + \frac{p_1}{2}\left(\frac{1}{r_2} + \frac{\varepsilon_2}{r_4}\right) + \frac{2}{l^2}\left(\frac{\varepsilon_1}{2} r_{11} + \sqrt{2l} M_2 - b_2\right) < 0, \qquad (31c)\\
&\Lambda_4' - \varepsilon_1 a_2 < 0 \\
&\quad \Leftrightarrow \frac{\varepsilon_1}{2r_{11}} + \frac{c_1 l^2}{4}\left(\frac{1}{r_1} + \frac{\varepsilon_2}{r_3}\right) + \varepsilon_1 \sqrt{2l} M_2 - \varepsilon_1 a_2 \\
&\qquad + \frac{l^2}{4}\left(\frac{c_2}{r_6} + 2\varepsilon_1 c_2 - \frac{\varepsilon_1 p_2}{r_6} + \frac{\varepsilon_1 q_2}{r_1} + \varepsilon_1 r_{10}\right) < 0, \qquad (31d)\\
&(21c) \Rightarrow (21b) \ \text{ and } \ (21e) \Rightarrow (21a). \qquad (31e)
\end{aligned}
$$

It suffices to prove the right-hand side of (31a)–(31d).
Indeed, we get from (11b)

$$
\begin{aligned}
&\frac{c_1 + c_2}{2}(l^2 K_m + \sqrt{l}) + \sqrt{2l} M_2 + \frac{l}{2} + \frac{l^2}{2}(c_2 - p_2 + q_2) \\
&\leq \frac{c_1 + c_2}{2}(K_m + 1)(l^2 + \sqrt{l}) + \sqrt{2l} M_2 + \frac{l}{2} + \frac{l^2}{2}(c_2 - p_2 + q_2) \\
&\leq (K_m + 1)(l^2 + \sqrt{l})\left(\frac{c_1 + c_2}{2} + \sqrt{2} M_2 + \frac{1}{2} + \frac{c_2 - p_2 + q_2}{2}\right) \\
&\leq \sqrt{l}(1 + l\sqrt{l})(K_m + 1)(1 + c_1 + q_2 + c_2 - p_2 + \sqrt{2} M_2) \\
&\leq \sqrt{2l}(1 + l\sqrt{l})(K_m + 1)(1 + c_1 + q_2 + c_2 - p_2 + M_2) \\
&< a_2,
\end{aligned}
$$

which implies

$$
\frac{c_1 + c_2}{2}\sqrt{l} + \sqrt{2l} M_2 + \frac{l}{2} + \frac{l^2}{2}(c_2 - p_2 + q_2) < a_2, \tag{32}
$$

and

$$
\frac{c_1 + c_2}{2} l^2 K_m + \sqrt{2l} M_2 + \frac{l}{2} + \frac{l^2}{2}(c_2 - p_2 + q_2) < a_2. \tag{33}
$$

Let

$$\varepsilon_0 = \frac{\frac{(c_1+c_2)l^2}{2}}{a_2 - \sqrt{2l}M_2 - \frac{l}{2} - \frac{l^2}{2}(c_2 - p_2 + q_2)}.$$

By (32) and (33), we have $\frac{(c_1+c_2)l^2}{2a_2} < \varepsilon_0 < \min\left\{\frac{1}{K_m}, l\sqrt{l}\right\}$.

We get from (11c)

$$(c_1 + p_1 - 4q_1 + q_2)l^4 + q_2 l^4 \varepsilon_0 + 8l^2\sqrt{2l}M_1 < 16b_1. \tag{34}$$

Indeed, we can compute

$$\begin{aligned}
&(c_1 + p_1 - 4q_1 + q_2)l^4 + q_2 l^4 \varepsilon_0 + 8l^2\sqrt{2l}M_1 \\
&\quad \le (c_1 + p_1 - 4q_1 + q_2)l^4 + q_2 l^4 l\sqrt{l} + 8l^2\sqrt{2l}M_1 \\
&\quad = l^2\sqrt{l}\left(l\sqrt{l}(c_1 + p_1 - 4q_1 + q_2) + l^3 q_2 + 8\sqrt{2}M_1\right) \\
&\quad \le l^2\sqrt{l}(1 + l^3)(c_1 + p_1 - 4q_1 + q_2 + 8\sqrt{2}M_1) \\
&\quad \le 8l^2\sqrt{l}(1 + l^3)(c_1 + p_1 - q_1 + q_2 + \sqrt{2}M_1) \\
&\quad \le 8l^2\sqrt{2l}(1 + l^3)(c_1 + p_1 - q_1 + q_2 + M_1) \\
&\quad < 16b_1.
\end{aligned}$$

We get from (11d)

$$l^2\left(p_1 + \frac{c_2}{4} - p_2 + q_2\right) + l^2\left(1 - \frac{p_2}{4} + \frac{1}{l^3}\right)\varepsilon_0 + 2\sqrt{2l}M_2 < 2b_2. \tag{35}$$

Indeed, we can compute

$$\begin{aligned}
&l^2\left(p_1 + \frac{c_2}{4} - p_2 + q_2\right) + l^2\left(1 - \frac{p_2}{4} + \frac{1}{l^3}\right)\varepsilon_0 + 2\sqrt{2l}M_2 \\
&\quad \le l^2\left(p_1 + \frac{c_2}{4} - p_2 + q_2\right) + l^2\left(1 - \frac{p_2}{4} + \frac{1}{l^3}\right)l\sqrt{l} + 2\sqrt{2l}M_2 \\
&\quad \le l^2\left(p_1 + \frac{c_2}{4} - p_2 + q_2\right) + \sqrt{l}\left(\left(1 - \frac{p_2}{4}\right)l^3 + 1\right) + 2\sqrt{2l}M_2 \\
&\quad \le l^2\left(p_1 + \frac{c_2}{4} - p_2 + q_2\right) + \sqrt{l}(l^3 + 1)\left(1 - \frac{p_2}{4}\right) + 2\sqrt{2l}M_2 \\
&\quad \le \sqrt{l}(l^3 + 1)\left(p_1 + \frac{c_2}{4} - p_2 + q_2 + 1 - \frac{p_2}{4} + 2\sqrt{2}M_2\right) \\
&\quad \le 2\sqrt{2l}(l^3 + 1)\left(1 + p_1 + c_2 - p_2 + q_2 + M_2\right)
\end{aligned}$$

$$< 2b_2.$$

Setting $r_{11} = \frac{1}{l}$, we have

$$\frac{(c_1 + c_2)l^2}{2} + \left((c_2 - p_2 + q_2)\frac{l^2}{2} + \frac{1}{2r_{11}} + \sqrt{2l}M_2 - a_2\right)\varepsilon_0 = 0, \tag{36}$$

with $(c_2 - p_2 + q_2)\frac{l^2}{2} + \frac{1}{2r_{11}} + \sqrt{2l}M_2 - a_2 < 0$ due to (32) or (33).

Regarding (34), (35) and (36), we may choose $\varepsilon_1 \in \left(\varepsilon_0, \min\{\frac{1}{K_m}, l\sqrt{l}\}\right)$ such that

$$\frac{1}{4}(c_1 + p_1 - 4q_1 + q_2)l^4 + \frac{1}{4}q_2 l^4 \varepsilon_1 + 2l^2\sqrt{2l}M_1 < 4b_1, \tag{37}$$

$$l^2(p_1 + \frac{c_2}{4} - p_2 + q_2) + l^2\left(1 - \frac{p_2}{4} + \frac{1}{l^3}\right)\varepsilon_1 + 2\sqrt{2l}M_2 < 2b_2, \tag{38}$$

$$\frac{(c_1 + c_2)l^2}{2} + \left((c_2 - p_2 + q_2)\frac{l^2}{2} + \frac{1}{2r_{11}} + \sqrt{2l}M_2 - a_2\right)\varepsilon_1 < 0. \tag{39}$$

Note that by (11a), (31b) holds with r_3, r_4, r_8 small enough and r_5, r_{12} large enough. Setting $r_1 = r_6 = \frac{1}{2}$, we get by (39),

$$\frac{\varepsilon_1}{2r_{11}} + \frac{c_1 l^2}{4}\frac{1}{r_1} + \frac{l^2}{4}\left(\frac{c_2}{r_6} + 2\varepsilon_1 c_2 - \frac{\varepsilon_1 p_2}{r_6} + \frac{\varepsilon_1 q_2}{r_1}\right) + \varepsilon_1\sqrt{l}M_2 - \varepsilon_1 a_2 < 0. \tag{40}$$

For the above r_3, one may choose $\varepsilon_2 < \frac{1}{K_m}$ small enough such that $\frac{\varepsilon_2}{r_3}$ small enough. Then by (40), (31d) holds with small r_{10} and $\frac{\varepsilon_2}{r_3}$.

Similarly, by (38), (31c) holds with $r_2 = \frac{1}{2}$ and $r_9, \frac{\varepsilon_2}{r_4}$ small enough. By (37), (31a) holds with $r_2 = \frac{1}{2}$ and $\varepsilon_2 r_{12}, \varepsilon_2 r_5, r_7$ small enough.

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