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ORIGINAL ARTICLE

A Kalman rank condition for the indirect controllability of coupled systems of linear operator groups

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Abstract In this article, we give a necessary and sufficient condition of Kalman type for the indirect controllability of systems of groups of linear operators, under some "regularity and locality" conditions on the control operator that will be made precise later and fit very well the case of distributed controls. Moreover, in the case of first order in time systems, when the Kalman rank condition is not satisfied, we characterize exactly the initial conditions that can be controlled. Some applications to the control of systems of Schrödinger or wave equations are provided. The main tool used here is the *fictitious control method* coupled with the proof of an *algebraic solvability* property for some related underdetermined system and some regularity results.

Keywords Controllability of abstract linear semi-groups · Indirect controllability of systems · Schrödinger and wave equations · Fictitious control method · Algebraic solvability

Mathematics Subject Classification 35F35 · 35G35 · 47D03 · 93B05 · 93B07

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1 Introduction

1.1 Presentation of the problem

The problem of controlling coupled systems of partial differential equations has drawn an increasing interest during the last decade, notably in the case of linear parabolic or hyperbolic linear systems, but also of more complex systems that naturally appear in numerous fields, including fluid mechanics, biology, population dynamics, medicine, etc. A very challenging issue for coupled system is the question of *indirect controllability*, which consists in understanding whether it is possible to act on them by a reduced number of controls (i.e., the number of controls is less that the number of equations), and on which equations it is necessary or not to act. This issue is interesting both from a theoretical and practical point of view.

From a theoretical point of view, this question is closely related to a more fundamental question on systems: *which* and *how* information propagates from one equation of the systems to another through the coupling terms, and notably how the coupling terms influence this propagation? Another related theoretical question comes from the field of inverse problems: using the well-known duality between controllability and observability, the question of controllability is equivalent to understanding if it is possible to recover the initial conditions of all components of an adjoint system thanks to partial observations on the system, giving notably some quantitative information about the unique continuation properties. To finish, let us mention that, as explained in details in [1, Section 1], indirect controllability is also closely related to insensitizing control and simultaneous control of coupled systems.

Concerning practical applications, indirect controllability is also crucial: for real-life models involving for example different kinds of physical quantities (velocity of a fluid, temperature, etc.), it might be impossible to act directly and all of them, and then it is important to understand if one can act on a complex system by just controlling for example one of the physical variables. Another point is that it is reasonable to try to limit the number of actuators or sensors for cost reasons. Hence, the questions raised here can also be of interest many other fields like automatic and engineering.

Our precise goal in the present work will be to understand how the algebraic structures of the coupling terms and of the control operator influence the properties of indirect controllability for conservative systems. More precisely, our initial motivation was to derive new controllability results in the spirit of [5] (which concerns systems of heat equation) in the case of conservative systems. Let us emphasize that for conservative systems, the strategy developed in [5] (for systems of heat equations) cannot be used at all because it was based on Carleman estimates. In this paper, we propose a possible strategy (that will be described briefly soon) that turns out to be valid in a much more general framework, so that the possible applications of this work cover a large class of problems, basically conservative partial differential equations with internal control.

For the sake of clarity, let us briefly explain on some examples the spirit of the present contribution on two examples, the Schrödinger or wave systems with internal control and constant coupling terms (the detailed results in these cases are given in Sect. 4). Let $T > 0$ and Ω be a smooth bounded open subset of $\mathbb{R}^N$. We denote by

$L^2(\Omega)$ the set of square-integrable functions defined on Ω with values in the complex plane $\mathbb{C}$. Let $n \in \mathbb{N}^*$ (where $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$) with $n \geqslant 2$. We consider the following control system of n Schrödinger equations with internal control

$$\begin{cases} \partial_t Y = i\Delta Y + AY + \mathbb{1}_\omega BV & \text{in} \quad (0,T) \times L^2(\Omega)^n, \\ Y(0) = Y^0 \in L^2(\Omega)^n, \end{cases} \tag{1.1}$$

with $(A, B) \in \mathcal{M}_n(\mathbb{C}) \times \mathcal{M}_{n,m}(\mathbb{C})$, $Y \in L^2(\Omega)^n$ the state and $V \in L^2(\Omega)^m$ the control assumed to be distributed on a non-empty open subset ω of Ω, that may not act on all the equations.

We also consider the following control system of n wave equations with internal control

$$\begin{cases} \partial_{tt} Y = \Delta Y + AY + \mathbb{1}_\omega BV & \text{in} \quad (0,T) \times L^2(\Omega)^n, \\ Y(0) = Y^0 \in H_0^1(\Omega)^n, \\ \partial_t Y(0) = Y^1 \in L^2(\Omega)^n. \end{cases} \tag{1.2}$$

In this context, a natural issue is the following: is it possible to find necessary and sufficient algebraic conditions on A and B of Kalman type that ensure the null controllability of systems (1.1) or (1.2), under some appropriate geometric conditions on ω (and in sufficiently large time for (1.2)). The general method that we will use in the article to answer this question is sometimes called *fictitious control method* and was first introduced in [18] in the context of affine control systems of ordinary differential equations without drift and was revisited in the context of partial differential equations in [27]. It has been then used in different contexts, notably in [2,17,22]. Let us explain the strategy on Eq. (1.1) (this is the same on Eq. 1.2). We first control the equations with n controls (one on each equation) and we try to eliminate the control on the last equation thanks to algebraic manipulations. More precisely, we decompose the problem into two different steps:

Analytic problem
Find a solution (Z, V) in an appropriate space to the control problem by n controls which are regular enough, i.e., solve

$$\begin{cases} \partial_t Z = i\Delta Z + AZ + \mathbb{1}_\omega V & \text{in } (0,T) \times \Omega, \\ Z = 0 & \text{on } (0,T) \times \partial\Omega, \\ Z(0,\cdot) = Y^0, \ Z(T,\cdot) = 0 & \text{in } \Omega, \end{cases} \tag{1.3}$$

where $V = (V_1, \ldots V_n)$. Solving Problem (1.3) is easier than solving the null controllability at time T of System (1.1), because we control System (1.3) with a control on each equation.

Algebraic problem
For $f := \mathbb{1}_\omega V$, find a pair (X, W) (where W has now only m components) in an appropriate space satisfying the following control problem:

$$\begin{cases} \partial_t X = i\Delta X + AX + BW + f & \text{in } (0,T) \times \Omega, \\ X = 0 & \text{on } (0,T) \times \partial\Omega, \\ X(0,\cdot) = X(T,\cdot) = 0 & \text{in } \Omega, \end{cases} \tag{1.4}$$

and such that the spatial support of X is strongly included in ω. We will solve this problem using the notion of *algebraic solvability* of differential systems, which is based on ideas coming from [28, Section 2.3.8]. The idea is to write System (1.4) as an *underdetermined* system in the variables X and W and to see f as a source term, so that we can write Problem (1.4) under the abstract form

$$\mathcal{L}(X, W) = f,$$

where

$$\mathcal{L}(X, W) := \partial_t X - i\Delta X - AX - BW.$$

The goal will be then to find a partial differential operator $\mathcal{M}$ satisfying

$$\mathcal{L} \circ \mathcal{M} = Id. \tag{1.5}$$

When (1.5) is satisfied, we say that System (1.4) is *algebraically solvable*. This exactly means that one can find a solution (X, W) to System (1.4) which can be written as a linear combination of some derivatives of the source term f.

Conclusion

If we can solve the analytic and algebraic problems, then it is easy to check that $(Y, V) = (Z - X, -W)$ will be a solution to System (1.1) in an appropriate space and will satisfy $Y(T) \equiv 0$ in Ω.

For more details concerning this method, we refer to [22, Section 2.3], [17, Section 3.1], [33, Section 1.3] and Sects. 2 and 3. Thanks to this method, we are able, for systems like (1.1) and (1.2) and under some additional conditions, to find some sufficient condition of controllability (the Kalman matrix $[B|AB|\ldots A^{n-1}B]$ has to be of maximal rank) that can be proved to be also necessary. In the case of Eq. 1.1, if the Kalman matrix is not satisfied, the same method also enables us to characterize the initial conditions that can be controlled. The examples of the wave and Schrödinger equations are treated in details in Sect. 4 of this work, and all the results presented in Sect. 4 are new. This approach may be to some extent generalized to abstract linear groups of operators under appropriate assumptions that are explained in details in Sect. 1.2.

Our paper is organized as follows. In Sect. 1.2, we explain in details the abstract framework chosen here and give the main results. In Sect. 1.3, we recall some previous results and explain precisely the scope of the present contribution. In Sect. 1.4 we present some related open problems. Sections 2 and 3 are, respectively, devoted to proving Theorems 1 and 2. In Sect. 4, we conclude with some applications, giving new results for the indirect controllability of Schrödinger equations and wave equations with internal control.

1.2 Abstract setting and main results

Let us introduce some notations. Let $T > 0$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{U}$, $\mathcal{H}$ two Hilbert spaces on $\mathbb{K}$ (that will be always identified with their dual in what follows) and a linear continuous application $\mathcal{C}:\mathcal{U}\to\mathcal{H}$, which will be our control operator (one may think as an example to consider $\mathcal{C}$ as a distributed control). We consider

- $\mathcal{L}:\mathcal{D}(\mathcal{L})\subset\mathcal{H}\to\mathcal{H}$ a closed unbounded operator with dense domain, which is supposed to be the generator of a strongly continuous **group** on $\mathcal{H}$,
- $\mathcal{Q}:\mathcal{D}(\mathcal{Q})\subset\mathcal{H}\to\mathcal{H}$ another closed unbounded operator with dense domain, which is supposed to be **self-adjoint and negative with compact resolvent**.

One can think as an example to consider $\mathcal{L}=i\Delta$ and $\mathcal{Q}=\Delta$, which are typical situations where our general result can be applied (see Sect. 4).

Let $n\in\mathbb{N}^*$ with $n\geqslant 2$ and $m\in\mathbb{N}^*$, and let $(A,B)\in\mathcal{M}_n(\mathbb{K})\times\mathcal{M}_{n,m}(\mathbb{K})$ two matrices. For every $k\in\mathbb{N}^*$, we introduce the operators $L_k:\mathcal{D}(\mathcal{L})^k\subset\mathcal{H}^k\to\mathcal{H}^k$, $Q_k:\mathcal{D}(\mathcal{Q})^k\subset\mathcal{H}^k\to\mathcal{H}^k$ and $C_k:\mathcal{U}^k\to\mathcal{H}^k$ such that, for every $\varphi\in\mathcal{D}(\mathcal{L})^k$ and $\psi\in\mathcal{U}^k$,

$$L_k(\varphi)=\begin{pmatrix}\mathcal{L}(\varphi_1)\\ \mathcal{L}(\varphi_2)\\ \vdots\\ \mathcal{L}(\varphi_k)\end{pmatrix},\ Q_k(\varphi)=\begin{pmatrix}\mathcal{Q}(\varphi_1)\\ \mathcal{Q}(\varphi_2)\\ \vdots\\ \mathcal{Q}(\varphi_k)\end{pmatrix}\ \text{ and } C_k(\psi)=\begin{pmatrix}\mathcal{C}(\psi_1)\\ \mathcal{C}(\psi_2)\\ \vdots\\ \mathcal{C}(\psi_k)\end{pmatrix}. \tag{1.6}$$

We consider the (first and second order in time) systems of n linear equations

$$\partial_t Y=L_n(Y)+AY+BC_mV \quad \text{in} \quad (0,T)\times\mathcal{H}^n, \tag{Ord1}$$

and

$$\partial_{tt} Y=Q_n(Y)+AY+BC_mV \quad \text{in} \quad (0,T)\times\mathcal{H}^n, \tag{Ord2}$$

where $V:=(v_1,\ldots,v_m)\in\mathcal{U}^m$ is called a control. One can think of Eq. (Ord1) as a generalization of the Schrödinger system (1.1), whereas one can think of Eq. (Ord2) as a generalization of the wave system (1.2).

(Ord1) (resp. (Ord2)) can be seen as a "system" version of the "scalar" controlled equation $\partial_t z=\mathcal{L}z+\mathcal{C}u$ (resp. $\partial_{tt}z=\mathcal{L}z+\mathcal{C}u$), where we add some coupling terms of zero order through the matrix A and where we impose a precise structure on the control through the matrix B. Note that one may have $m<n$, which means that the number of controls can be strictly less than the number of equations, and notably some equations might be uncontrolled. In this setting, the structure of the coupling terms is crucial in order to obtain some controllability results, and in some sense these coupling terms can be used to act indirectly on the equations that are not controlled.

It is usual to write the second-order system (Ord2) as a first-order system (see Sect. 3.1). However, we emphasize that (Ord2) *is not* a particular case of (Ord1), the reason being that if we transform (Ord2) into a first-order system, we will not be able to find any matrix A such that (Ord2) can be written as (Ord1). Finding such a matrix would require that the coupling terms involve simultaneously Y and Y_t (see notably (3.3)), which is not the case here.

It is well known that for the controllability of coupled systems like (Ord2), the natural state space $\mathcal{D}(\mathcal{Q}^{\frac{1}{2}})^n\times\mathcal{H}^n$ is not always possible. For instance taking zero as initial data, we cannot reach any target state in $\mathcal{D}(\mathcal{Q}^{\frac{1}{2}})^n \times \mathcal{H}^n$ because of the regularity of solutions of (Ord2) (one might for example think of a upper triangular matrix A with a control acting only on the last equation, see [23]). The same phenomena does not occur for system like (Ord1), however, in both cases, we will always assume that the initial conditions are regular enough, namely $Y(0) \in \mathcal{D}(\mathcal{L}^{n-1})^n$ (resp. $(Y(0), \partial_t Y(0)) \in \mathcal{D}(\mathcal{Q}^{n-\frac{1}{2}})^n \times \mathcal{D}(\mathcal{Q}^{n-1})^n$), which is enough to ensure that system (Ord1) (resp. (Ord2)) with initial condition in these spaces admits a unique solution in $C^0([0,T]; \mathcal{H}^n)$ (resp. in $C^0([0,T]; \mathcal{D}(\mathcal{Q}^{\frac{1}{2}})^n \times \mathcal{H}^n)$.

The main goal of this article is to analyze the null controllability of System (Ord1) and System (Ord2), which would (partially) generalize the results of [5] in the case of conservative systems. Let us recall the definition of these notions. It will be said that

- System (Ord1) (resp. System (Ord2)) is *null controllable* at time T if for every initial condition $Y^0 \in \mathcal{D}(\mathcal{L}^{n-1})^n$ (resp. $(Y^0, Y^1) \in \mathcal{D}(\mathcal{Q}^{n-\frac{1}{2}})^n \times \mathcal{D}(\mathcal{Q}^{n-1})^n$), there exists a control $V \in C^0([0,T]; \mathcal{U}^m)$ such that the solution Y to System (Ord1) with initial condition $Y(0) = Y^0$ (resp. to System (Ord2) with initial condition $(Y(0), \partial_t Y(0)) = (Y^0, Y^1)$) satisfies

$$Y(T) \equiv 0 \text{ in } \mathcal{H}^n \quad (\text{resp. } Y(T) \equiv 0 \textit{ and } \partial_t Y(T) \equiv 0 \text{ in } \mathcal{H}^n \times \mathcal{H}^n).$$

- System (Ord1) (resp. System (Ord2)) is *exactly controllable* at time T if for every initial condition $Y^0 \in \mathcal{D}(\mathcal{L}^{n-1})^n$ (resp. $(Y^0, Y^1) \in \mathcal{D}(\mathcal{Q}^{n-\frac{1}{2}})^n \times \mathcal{D}(\mathcal{Q}^{n-1})^n$) and every $Y_T \in \mathcal{D}(\mathcal{L}^{n-1})^n$ (resp. $(Y_T, Z_T) \in \mathcal{D}(\mathcal{Q}^{n-\frac{1}{2}})^n \times \mathcal{D}(\mathcal{Q}^{n-1})^n$), there exists a control $V \in C^0([0,T]; \mathcal{U}^m)$ such that the solution Y to System (Ord1) with initial condition $Y(0) = Y^0$ (resp. to System (Ord2) with initial condition $(Y(0), \partial_t Y(0)) = (Y^0, Y^1)$) satisfies

$$Y(T) \equiv Y_T \text{ in } \mathcal{D}(\mathcal{L}^{n-1})^n \text{ (resp. } Y(T) \equiv Y_T \text{ in } \mathcal{D}(\mathcal{Q}^{n-\frac{1}{2}})^n \text{ and } \partial_t Y(T) \equiv Z_T \text{ in } \mathcal{D}(\mathcal{Q}^{n-1})^n).$$

Let us remark that since L_n (*resp.* $\hat{Q}_n$ *defined in* (3.2) , see Sect. 3.1) is a generator of a group, then System (Ord1) (resp. System (Ord2)) is null controllable at time T if and only if it is exactly controllable at time T (see for example [19, p. 55]). Hence, from now on, we will only concentrate on the null controllability of Systems (Ord1) and (Ord2).

Our main assumptions (that will be commented afterward) will be the following.

Assumptions
There exists a linear continuous application $\overline{\mathcal{C}} : \mathcal{U} \to \mathcal{H}$ such that

Assumption 1.1 (Scalar null controllability)

Case (Ord1) The control system

$$\partial_t z = \mathcal{L} z + \overline{\mathcal{C}} u, \tag{1.7}$$

is exactly controllable at time T^*.

Case (Ord2) The control system

$$\partial_{tt} z = \mathcal{Q} z + \overline{\mathcal{C}} u, \tag{1.8}$$

is exactly controllable at time T^*.

Assumption 1.2 (Regularity and locality)

Case (Ord1) $\mathcal{L}^k(\overline{\mathcal{C}\mathcal{C}}^* \mathcal{D}(\mathcal{L}^{*k})) \subset \mathcal{C}(\mathcal{U})$ for all $k \in \{0, \ldots, n-1\}$,
Case (Ord2) $\mathcal{Q}^{\frac{k}{2}}(\overline{\mathcal{C}\mathcal{C}}^* \mathcal{D}(\mathcal{Q}^{\frac{k}{2}})) \subset \mathcal{C}(\mathcal{U})$ for all $k \in \{0, \ldots, 2n-2\}$.

Remark 1

1. It might happen that the operator $\mathcal{C}$ itself verify Assumptions 1.1 and 1.2, hence in some sense our assumptions are **more general** than just stating the same assumptions replacing $\overline{\mathcal{C}}$ by $\mathcal{C}$. However, the price to pay is that Assumption 1.1 is **stronger** under this form than stating the same assumptions replacing $\overline{\mathcal{C}}$ by $\mathcal{C}$ (see the next point).
2. Assumption 1.1 may seem quite artificial since it does not seem to be related to the controllability of systems

$$\partial_t z = \mathcal{L} z + \mathcal{C} u, \tag{1.9}$$

and

$$\partial_{tt} z = \mathcal{Q} z + \mathcal{C} u, \tag{1.10}$$

that would be the natural minimum conditions one might expect.
However, one can easily prove that Assumptions 1.1 and 1.2 imply the controllability at time T^* of (1.9) and (1.10). Let us explain it for (1.9) (this exactly the same reasoning for Eq. 1.10). Thanks to the Hilbert Uniqueness Method (HUM, see [32]), we know that the control function u in (1.7) with minimal L^2-norm is necessarily in $R(\overline{\mathcal{C}}^*)$ (where R denotes the range of an operator), which implies that $\overline{\mathcal{C}} u \in R(\overline{\mathcal{C}}\,\overline{\mathcal{C}}^*)$, hence thanks to Assumption 1.2 (with $k = 0$), we obtain $\overline{\mathcal{C}} u \in R(C)$ and this proves that (1.9) is indeed controllable.
3. One consequence of Assumption 1.2 is that we have $\overline{\mathcal{C}}\,\overline{\mathcal{C}}^* \mathcal{D}(\mathcal{L}^{*k}) \in \mathcal{D}(\mathcal{L}^k)$ for every $k \in \{0, \ldots, n-1\}$, hence in some sense $\overline{\mathcal{C}}$ has to "preserve the regularity", which is very natural in the context of conservative systems of second order like (Ord2) (see notably [21]). However, in many applications, this is in general false for the operator $\mathcal{C}$ itself (and it is the main reason why we introduce $\overline{\mathcal{C}}$). For example, consider some open subset Ω of $\mathbb{R}^N$ ($N \in \mathbb{N}^*$), and consider $\mathcal{H} = \mathcal{U} = L^2(\Omega)$. Assume that $\mathcal{L}$ is a differential operator defined on some open subset Ω and the application $\mathcal{C} : L^2(\Omega) \to L^2(\Omega)$ is defined by $\mathcal{C} u = \mathbb{1}_\omega u$, where ω is some open subset of Ω. Then it is clear that the property $\mathcal{C}\mathcal{C}^* \mathcal{D}(\mathcal{L}^{*k}) \subset \mathcal{D}(\mathcal{L}^k)$, which is equivalent to $\mathbb{1}_\omega \mathcal{D}(\mathcal{L}^{*k}) \subset \mathcal{D}(\mathcal{L}^k)$, is *always false* as soon as $k > 0$, $\mathcal{L}$ is of order more than 1 and ω is different from Ω (because $\mathbb{1}_\omega \in L^2(\Omega)$ but does not belong to any higher-order Sobolev space). Hence, roughly speaking, the linear application $\overline{\mathcal{C}}$ has to be thought as a "regularization" of the linear application $\mathcal{C}$. In the case of distributed control, a natural candidate for $\overline{\mathcal{C}}$ is $\overline{\mathcal{C}} u = \tilde{\mathbb{1}}_\omega u$, where $\tilde{\mathbb{1}}_\omega \in \mathcal{C}_c^\infty(\Omega)$

is some "regularization" of the indicator function $\mathbb{1}_\omega$, defined for example such that

$$\tilde{\mathbb{1}}_\omega := \begin{cases} 1 & \text{on} \quad \omega_0, \\ 0 & \text{on} \quad \Omega\backslash\omega, \end{cases}$$

where ω_0 is some well-chosen open set included in ω. This will be explained into more details in Sects. 4.1 and 4.2.

4. Adding the condition $\mathcal{L}^k(\overline{\mathcal{C}}\,\overline{\mathcal{C}}^*\mathcal{D}(\mathcal{L}^{*k})) \subset \mathcal{C}(\mathcal{U})$ for all $k \in \{0, \ldots, n-1\}$ is necessary in our method to prove that our control is in $(\mathcal{C}(\mathcal{U}))^m$. This notably ensures that the operator $\mathcal{M}$ and $\tilde{\mathcal{M}}$ defined in (2.18) and (3.17), respectively, are "local" in the sense that they send an element of the range of $\overline{\mathcal{C}}\,\overline{\mathcal{C}}^*$ into an element of the range of $\mathcal{C}$.

In the sequel, we will denote by $[A|B] \in \mathcal{M}_{n,nm}(\mathbb{K})$ the Kalman matrix, which is given by

$$[A|B] = (B|AB|A^2B|\cdots|A^{n-1}B). \tag{1.11}$$

Our result gives a necessary and sufficient condition for exact (or null) controllability of System (Ord1) and (Ord2).

Theorem 1 *Let us assume that $\mathcal{L}$ satisfies Assumptions 1.1 and 1.2. Let $Y^0 \in (\mathcal{D}(\mathcal{L})^{n-1})^n$. Then, for every $T > T^*$, there exists a control V in $C^0([0,T] \times \mathcal{U}^m)$ such that the solution of (Ord1) corresponding to the initial condition $Y(0) = Y^0$ in $\mathcal{H}^n$ satisfies*

$$Y(T) \equiv 0 \quad in \quad \mathcal{H}^n$$

if and only $Y^0 \in [A|B](\mathcal{H}^{nm})$.

Remark 2 Concerning Theorem 1, the reversibility of the equation allows us to obtain the same conclusion if we replace the final condition $Y(T) = 0$ by $Y(T) = Y^T$ for some $Y^T \in \mathcal{D}(\mathcal{L}^{n-1})^n \cap [A|B](\mathcal{H}^{nm})$.

When rank$([A|B]) = n$, Theorem 1 gives us a necessary and sufficient condition for the null controllability of System (Ord1).

Corollary 1.1 *Let us assume that $\mathcal{L}$ satisfies Assumptions 1.1 and 1.2. Then, for every $T > T^*$, the control system (Ord1) is exactly controllable at time T if and only if rank$([A|B]) = n$.*

Concerning system (Ord2), we have the following result.

Theorem 2 *Let us assume that $\mathcal{Q}$ satisfies Assumptions 1.1 and 1.2. Then, for every $T > T^*$, the control system (Ord2) is exactly controllable at time T if and only if rank$([A|B]) = n$.*

Remark 3 We were not able to derive the same kind of result as in Theorem 1 in the case of second-order in time systems, so that we do not know if it would be true in this context.

Using the transmutation technique (for instance the version given in [25]) and Sect. 3.2, one can deduce easily the following result in the parabolic case, assuming that the corresponding "scalar" hyperbolic system is controllable.

Corollary 1.2 *Let us assume that $\mathcal{Q}$ satisfies Assumptions 1.1 and 1.2. Then, for every $T > 0$, the control system*

$$\partial_t Y = \mathcal{Q}_n(Y) + AY + BC_m V \quad in \quad (0, T) \times \mathcal{H}^n,$$

is null controllable at time T if and only if rank$([A|B]) = n$.

Remark 4 (a) $\text{rank}([A|B]) = n$ is called the Kalman rank condition by analogy with the finite-dimensional case.

(b) The assumption $T > T^*$ enables us to choose a regular control U in time for the analytic system (2.1) such that $U(0) = U(T) = 0$ (see [24]). This is necessary to ensure that during the resolution of the Algebraic Problem, we can construct a solution X of (2.9) or (3.11) such that $X(0) = X(T) = 0$. However, many of the controllability results known in the literature are either results in arbitrary small time or with an "open" condition on the minimal time of control, hence in practice controlling at any time $T > T^*$ rather that at time exactly T^* will not provide a weaker result than in the scalar case.

1.3 State of the art and precise scope of the paper

In all what follows, we will mainly concentrate on systems coupled with zero order terms, on distributed controls and on null or exact controllability results. The case of boundary controls (which are unbounded) is not covered by our abstract setting. However, there is also a huge literature on boundary control, approximate controllability and high-order coupling terms for coupled systems (see notably [7,15,22,26] for some recent contributions).

Concerning second-order parabolic equations, the case of coupled systems of heat equations with same diffusion coefficients and constant or time-dependent coupling terms is well-understood (see notably [5], where an algebraic Kalman rank condition similar to the one of the current article is given). In the case of different diffusion coefficients, a necessary and sufficient condition involving some differential operator related to the Kalman matrix was also given in [6]. This case was treated into more details in [9]. Another result concerning the case of two equations with different diffusion is given in [41], where the author also investigates the case of coupling different dynamics, e.g., a heat and a wave equation. However, as soon as the coupling coefficients depend on the space variable, the situation is far more intricate and in general we only have partial results, essentially with two equations, one control force and in the one-dimensional case (see [10] for example) or in simple geometries like cylinders (see [13]). Let us also mention that the nonlinear (and even semi-linear) has not been investigated too much up to now (see for example [4,16,27]). For further information on this topic, we refer to the recent survey [8].

The case of hyperbolic or dispersive systems seems to have been less studied and the results obtained are somehow quite different from the parabolic ones. Concerning the Schrödinger equation, the recent paper [31] considers the case of a cascade system of 2 equations with one control force under the condition that the coupling region and the control region intersect and verify some technical conditions ensuring that a Carleman estimate can be proved. Concerning systems of wave equation, let us mention [1], where a result of controllability in sufficiently large time for second order in time cascade or bidiagonal systems under coercivity conditions on the coupling terms is given. Another related result is also [3], where the case of two wave equations with one control and a coupling matrix A which is supposed to be symmetric and having some additional technical properties is investigated. Let us also mention a result in the one-dimensional and periodic case proved in [39]. In this last article, the authors also prove a result for the Schrödinger equation in arbitrary dimension on the torus; however, they only obtained a result in large time, which is rather counter-intuitive and should be only technical. The case of a cascade system of two wave equations with one control on a compact manifold without boundary was treated in [23], where the authors also give a necessary and sufficient condition of controllability depending on the geometry of the control domain and coupling region. Let us emphasize that in the four last references, the results obtained in the case of abstract systems of wave equations can be applied to get some interesting results in the case of abstract heat and Schrödinger equations thanks to the *transmutation method* (see [25] or [37,38]), leading however to strong (and in general artificial) geometric restrictions on the coupling region and control region. Let us also mention a recent result given in [2], which treats the case of some linear system of two periodic and one-dimensional non-conservative transport equations with same speed of propagation, space-time varying coupling matrix and one control and also a nonlinear case.

Regarding the previous presentation, let us precise the exact scope of this paper, which has a rather different spirit from most of the papers presented before concerning conservative systems.

- Our result is given in a very general setting, since we basically work on some group of operators (which are not necessarily differential operators) with a bounded control operator satisfying some technical conditions that appear to be verified in many cases in practice. Notably, our result fits very well (but is not restricted to) the case of conservative systems of PDEs with distributed control, where no general result was known in the case of constant coupling coefficients.
- Contrary to many results in the literature which concentrates on symmetric matrices, bidiagonal matrices or cascade matrices, our result does not require any structural conditions on the coupling matrix A, nor on the matrix B which is often assumed to be acting only on the last(s) equations. Moreover, we do not have any restriction on the number of equations n we treat. Hence, most of the techniques used in the literature will fail in our case. Another important point is that we are able here to give a necessary and sufficient condition of controllability and we also are able in the one-order in time case to characterize precisely the initial conditions that can be controlled, which—as far as we know—was only known for the finite-dimensional case and for linear second-order parabolic systems (see [6]).

- The main restriction is that we work with constant coupling coefficients (this implies that the coupling is made everywhere), which do not cover some interesting cases, notably the case of space-varying coefficients described before. Despite this, we believe that the important degree of generality of the present paper compensates this restriction and that our contribution is of interest in order to have a deeper understanding of the controllability properties of coupled systems.

1.4 Some related open problems

Let us address some related open questions and possible extensions of this work.

- An interesting question is the case of the local controllability of semi-linear equations, the main difficulty being that due to the difference of regularity between the initial condition and the control, standard inverse mapping theorems or fixed-point theorems cannot be used. A possible remedy would be to use a fixed-point strategy of Nash–Moser type as in [2].
- When systems of equations like (Ord1) and (Ord2) are concerned, a very natural question that might appear in many applications is what is called *partial controllability*, which means that we would like (for example) to bring only the first l ($l \in [|1, n-1|]$) components of the state variable to 0 without imposing any conditions the $n-l+1$ last components. It would be interesting to see if general conditions like the one found in [11] can be derived. Another interesting and close problem would be to see if the techniques employed here may be useful for the *synchronization* and *synchronization by groups* of solutions in the spirit of [34] and [36], by means of bounded controls.
- In the case of equation (Ord1), Assumption 1.2 and the regularity condition on the initial data do not seem to be necessary, and it would be natural to expect the same result by just assuming that Assumption 1.1 is true with $\overline{\mathcal{C}} = \mathcal{C}$, but the strategy used here prevented us to get this result. The case of admissible unbounded control operators $\mathcal{B} \in \mathcal{L}_c(U, \mathcal{D}((\mathcal{L}^*)')$ is also still open in this context (see [35], where the Kalman rank condition is proved to be necessary in order to obtain the approximate controllability).
- A natural extension of the Kalman rank condition is what is called the *Silverman-Meadows condition* (see [40]) in the case of matrix A and B depending on the time variable, that we did not manage to treat with the same strategy.
- One could also investigate more general coupled systems of the form

$$\partial_t Y = DL_n(Y) + AY + BC_m V \quad \text{in} \quad (0, T) \times \mathcal{H}^n$$

and

$$\partial_{tt} Y = DQ_n(Y) + AY + BC_m V \quad \text{in} \quad (0, T) \times \mathcal{H}^n,$$

where D is some constant matrix, for example a diagonal matrix with (possibly) distinct coefficients and try to derive results similar to [6].

Concerning the three last points, it is possible that one can use a different point of view based on suitable observability inequalities in the spirit of [41] (giving quite similar controllability results, in weaker spaces). This will be investigated in a future work.

2 Proof of Theorem 1

In the sequel, we focus our attention on the null controllability of the system (Ord1). Suppose that Assumptions 1.1 and 1.2 are satisfied and let $T > T^*$. We will always consider some initial condition Y^0 belonging to $\mathcal{D}(\mathcal{L}^{n-1})^n$. Let $k \in \mathbb{N}^*$. Using the same convention as in (1.6) we introduce the operator $\overline{C}_k : \mathcal{U}^k \to \mathcal{H}^k$ such that, for every $\psi \in \mathcal{U}^k$,

$$\overline{C}_k(\psi) = \begin{pmatrix} \overline{\mathcal{C}}(\psi_1) \\ \overline{\mathcal{C}}(\psi_2) \\ \vdots \\ \overline{\mathcal{C}}(\psi_k) \end{pmatrix}.$$

2.1 First part of the proof of Theorem 1

In all this section, we assume Y^0 is in $[A|B](\mathcal{H}^{nm})$ and we want to prove that the solution to (Ord1) with initial condition Y^0 can be brought to 0 at time T.

Analytic problem

We consider the control problem

$$\begin{cases} \partial_t Z = L_n(Z) + AZ + \overline{C}_n(U) \quad \text{in} \quad (0,T) \times \mathcal{H}^n, \\ Z(0) = Y^0. \end{cases} \tag{2.1}$$

Let us emphasize that in this step we act on *all equations* with n distinct controls, one on each equation. U is here called a *fictitious control* because it will disappear at the end of the reasoning. Let us first prove that (2.1) is controllable and give some regularity results on the control and the solution.

Proposition 2.1 *If Assumption 1.1 is satisfied, then the control system (2.1) is null controllable at time T^*. Moreover, for any $T > 0$, one can choose a control U such that*

$$U(t,.) \in [A|B](\mathcal{H}^{nm}) \quad \text{for every} \quad t \in (0,T), \tag{2.2}$$

$$U \in H_0^{n-1}(0,T;\mathcal{U}^n) \bigcap \cap_{k=0}^{n-1} C^k([0,T]; \overline{C}_n^* \mathcal{D}((L_n^*)^{n-1-k}), \tag{2.3}$$

$$Z \in \cap_{k=0}^{n-1} C^k([0,T]; \mathcal{D}(L_n^{n-1-k})). \tag{2.4}$$

Proof of Proposition 2.1. Using the change of variables $Z = e^{tA}\tilde{Z}$ and $U = e^{tA}\tilde{U}$, we obtain that the solution Z of system (2.1) is null controllable at time T^* if and only if the system

$$\left\{ \partial_t \tilde{Z} = L_n(\tilde{Z}) + \overline{C}_n(\tilde{U}) \quad \text{in} \quad (0, T^*) \times \mathcal{H}^n, \right. \tag{2.5}$$

is null controllable at time T^*. Remark that system (2.5) is totally uncoupled. Hence, since the control system (1.7) is controllable at time T^*, using the definitions of L_n and $\overline{C}_n$ given in (1.6), we easily obtain that the control system (2.5) is controllable at time T^*.

Moreover $\overline{C}_n \in \mathcal{L}(\mathcal{U}^n, \mathcal{H}^n)$ and the operator L_n is a generator of a group on $\mathcal{H}^n$. Then, if we consider now some $T > T^*$, applying [24, Corollary 1.5] (with $s = n-1$), we deduce that there exists $(\tilde{Z}, \tilde{U})$ a solution of (2.5) such that $\tilde{Z}(T) = 0$ and

$$\tilde{Z} \in \cap_{k=0}^{n-1} C^k([0, T]; \mathcal{Z}_{n-1-k}),$$

where $\mathcal{Z}_j$ is defined by induction by

$$\mathcal{Z}_0 = \mathcal{H}^n, \ \ \mathcal{Z}_j = L_n^{-1}(\mathcal{Z}_{j-1} + \overline{C}_n \, \overline{C}_n^*(\mathcal{D}((L_n^*)^j))).$$

The spaces $\mathcal{Z}_j$ are in general not known explicitly, however, in our case, using Assumption 1.2 (see also the second point of Remark 1), it is clear that notably

$$\overline{C_n} \, \overline{C_n}^* \mathcal{D}((L_n^*)^j) \subset \mathcal{D}(L_n^j),$$

from which we deduce easily by induction that for every $j \in [|1, n|]$, we have

$$\mathcal{Z}_j \subset \mathcal{D}(L_n^j),$$

which establishes (2.4).

Moreover, [24, Theorem 1.4] (with $s = n - 1$) notably implies that one can choose $\tilde{U}$ belonging to $H_0^{n-1}(0, T; \mathcal{U}^n)$. Finally, to prove (2.3), it is enough to prove that

$$\tilde{U} \in \cap_{k=0}^{n-1} C^k([0, T]; \overline{C}_n^* \mathcal{D}((L_n^*)^{n-1-k})),$$

which is an immediate consequence of the proof of Corollary 1.5 of [24, Page 1387] (and notably equality (3.19) in this reference).

It remains to prove (2.2). As in [24, Equality (1.3)], we fix $\delta > 0$ such that $T - 2\delta > T^*$ and we consider $\eta \in C^{n-1}(\mathbb{R})$ such that

$$\eta(t) = \begin{cases} 0 & \text{if } t \notin (0, T), \\ 1 & \text{if } t \in [\delta, T - \delta]. \end{cases}$$

Then, $\tilde{U}$ can also be chosen as the one of minimal $L^2(0, T; dt/\eta; U)$ among all possible controls for which the solution of (2.1) satisfies $\tilde{Z}(T) = 0$, properties (2.3) and (2.4) being still verified.

Hence, using [24, Proposition 1.3], $\tilde{U}$ can be written as

$$\tilde{U} = \eta(t) \overline{C}_n^* e^{(T-t)L_n^*} G_n^{-1} e^{T L_n} Y^0,'$$

where

$$G_n = \int_0^T e^{(T-t)L_n} \overline{C}_n \, \overline{C}_n^* e^{(T-t)L_n^*} \, \mathrm{d}t.$$

Note that since (2.5) is null controllable, G_n is indeed an invertible linear application. Using that $Y^0 = [A|B]\hat{Y}^0$ with $\hat{Y}^0 \in \mathcal{H}^{nm}$, and the formulas

$$\begin{aligned} L_n[A|B] &= [A|B]L_{nm}, \\ \overline{C}_n^*[A|B] &= [A|B]\overline{C}_{nm}^*, \\ \overline{C}_n[A|B] &= [A|B]\overline{C}_{nm}, \end{aligned}$$

we deduce

$$\tilde{U} = [A|B]\eta(t)\overline{C}_{nm}^* e^{(T-t)L_{nm}^*} G_{nm}^{-1} e^{TL_{nm}} \hat{Y}^0,$$

where

$$G_{nm} = \int_0^T e^{(T-t)L_{nm}} \overline{C}_{nm} \, \overline{C}_{nm}^* e^{(T-t)L_{nm}^*} \, \mathrm{d}t,$$

which is also invertible. Using that $\tilde{U} = e^{-tA}U$, we obtain

$$U = e^{tA}[A|B]\eta(t)\overline{C}_{nm}^* e^{(T-t)L_{nm}^*} G_{nm}^{-1} e^{TL_{nm}} \hat{Y}^0. \tag{2.6}$$

By the Cayley–Hamilton theorem, there exists $\beta = \begin{pmatrix} \beta_0 \\ \vdots \\ \beta_{n-1} \end{pmatrix} \in \mathbb{K}^n$ such that $A^n = \sum_{i=0}^{n-1} \beta_i A^i$. Let $\psi = \begin{pmatrix} \psi_0 \\ \vdots \\ \psi_{n-1} \end{pmatrix} \in \mathcal{H}^{nm}$, we have

$$\begin{aligned} A[A|B]\psi &= (AB, A^2B, \ldots, A^nB)\psi \\ &= (AB, A^2B, \ldots, \textstyle\sum_{i=0}^{n-1} \beta_i A^i B)\psi \\ &= [A|B]\hat{\psi} \end{aligned} \tag{2.7}$$

with

$$\hat{\psi} = \begin{pmatrix} \beta_0 \psi_{n-1} \\ \psi_0 + \beta_1 \psi_{n-1} \\ \vdots \\ \psi_{n-2} + \beta_{n-1} \psi_{n-1} \end{pmatrix} \in \mathcal{H}^{nm}.$$

Combining (2.6), (2.7) and using the fact that for every $t \in (0, T)$, there exists $\alpha(t) \in \mathbb{K}^n$ such that $e^{-tA} = \sum_{i=0}^{n-1} \alpha_i(t) A^i$, we obtain (2.2) and the proof of Proposition 2.1 is complete. □

Algebraic problem

Now, we would like to come back to the original system (Ord1) by algebraic manipulations.

Using Proposition 2.1, there exists (Z, U) solution of (2.1) verifying moreover (2.2), (2.3) and (2.4). Notably, there exists $\hat{U} \in \mathcal{U}^{nm}$ such that $\overline{C}_n(U) = [A|B]\overline{C}_{nm}(\hat{U})$. From now on, we will call $f := \overline{C}_{nm}(\hat{U})$, that will be considered as a source term. Our goal will be to find a pair $(X, \tilde{W}) \in C^0([0, T]; \mathcal{D}(L_n)) \cap C^1([0, T]; \mathcal{H}^n) \times C^0([0, T]; \mathcal{U}^m)$ satisfying the following problem:

$$\begin{cases} \partial_t X = L_n(X) + AX + BC_m\tilde{W} + [A|B]f \quad \text{in} \quad (0, T) \times \mathcal{H}^n, \\ X(0) = X(T) = 0. \end{cases} \tag{2.8}$$

Calling $C_m\tilde{W} = W$, we will rather solve (the unknowns being the variables X and W) the following problem

$$\begin{cases} \partial_t X = L_n(X) + AX + BW + [A|B]f \quad \text{in} \quad (0, T) \times \mathcal{H}^n, \\ X(0) = X(T) = 0, \end{cases} \tag{2.9}$$

(the fact that $W \in C_m\mathcal{U}^m$ will be a consequence of our construction and our assumptions). Let us remark that system (2.9) is *underdetermined* in the sense that we have more unknowns than equations. Hence, one can hope to find a trajectory (X, W) verifying $X(0) = X(T) = 0$ (which is a *crucial point here*), which would not be necessarily possible if the system were well-posed. We will use the notion of *algebraic solvability*, which is based on ideas coming from [28, Section 2.3.8] for differential systems and was already widely used in [2,17,22]. The idea is to write System (2.9) as an undetermined system in the variables X and W and to see f as a source term, so that we can write Problem (2.9) under the abstract form

$$\mathcal{P}(X, W) = [A|B]f, \tag{2.10}$$

where

$$\begin{aligned} \mathcal{P} : \mathcal{D}(\mathcal{P}) \subset L^2(0, T; \mathcal{H}^{n+m}) &\to L^2(0, T; \mathcal{H}^n) \\ (X\,,\ W) &\mapsto \partial_t X - L_n X - AX - BW. \end{aligned} \tag{2.11}$$

The goal will be then to find an operator $\mathcal{M}$ (involving time derivatives and powers of L_n) satisfying

$$\mathcal{P} \circ \mathcal{M} = [A|B]. \tag{2.12}$$

When (2.12) is satisfied, we say that System (2.9) is *algebraically solvable*. In this case, one can choose as a particular solution of (2.9) $(X, W) = \mathcal{M}(\overline{C}_{nm}(\hat{U}))$. This exactly means that one can find a solution (X, W) of System (2.9) which can be written as a linear combination of f, its derivatives in time, and some $L^k_{nm} f$ with $k \in \mathbb{N}^*$. Let us prove the following Proposition:

Proposition 2.2 *Let* $(A, B) \in \mathcal{M}_n(\mathbb{K}) \times \mathcal{M}_{n,m}(\mathbb{K})$. *There exists an operator* $\mathcal{M}$ *such that the equality* (2.12) *is satisfied. Moreover, the operator* $\mathcal{M}$ *is an operator of order at most*

$$\begin{cases} n-2 \text{ for the } n \text{ first components} \\ n-1 \text{ for the } m \text{ last components} \end{cases} \tag{2.13}$$

in time and in term of powers of L_n.

Proof of Proposition 2.2. We can remark that equality (2.12) is equivalent to

$$\mathcal{M}^* \circ P^* = [A|B]^*. \tag{2.14}$$

The adjoint operator $\mathcal{P}^* : \mathcal{D}(P^*) \subset L^2(0, T; \mathcal{H}^n) \to L^2(0, T; \mathcal{H}^n) \times L^2(0, T; \mathcal{H}^m)$ of the operator $\mathcal{P}$ is given for all $\varphi \in \mathcal{D}(P^*)$ by

$$\mathcal{P}^*\varphi := \begin{pmatrix} (P^*\varphi)_1 \\ \vdots \\ (\mathcal{P}^*\varphi)_n \\ (\mathcal{P}^*\varphi)_{n+1} \\ \vdots \\ (P^*\varphi)_{n+m} \end{pmatrix} = \begin{pmatrix} -\partial_t\varphi - L_n^*\varphi - A^*\varphi \\ -B^*\varphi \end{pmatrix}. \tag{2.15}$$

Since $(A, B) \in \mathcal{M}_n(\mathbb{K}) \times \mathcal{M}_{n,m}(\mathbb{K})$ are constant matrices, we have the following commutation properties:

$$B^*(A^*)^i L_n^* = L_m^* B^*(A^*)^i$$

and

$$B^*(A^*)^i \partial_t = \partial_t B^*(A^*)^i.$$

By definition, we have

$$B^*\varphi = -\begin{pmatrix} (P^*\varphi)_{n+1} \\ \vdots \\ (P^*\varphi)_{n+m} \end{pmatrix}.$$

Now, for $i = \{1, \ldots, n-1\}$, applying $B^*(A^*)^{i-1}$ to $-\partial_t\varphi - L_n^*\varphi - A^*\varphi$, we have

$$B^*(A^*)^{i-1} \begin{pmatrix} (P^*\varphi)_1 \\ \vdots \\ (P^*\varphi)_n \end{pmatrix} = -(\partial_t + L_m^*)(B^*(A^*)^{i-1}\varphi) - B^*(A^*)^i\varphi, \text{ i.e.,}$$

$$B^*(A^*)^i\varphi = -B^*(A^*)^{i-1} \begin{pmatrix} (P^*\varphi)_1 \\ \vdots \\ (P^*\varphi)_n \end{pmatrix} - (\partial_t + L_m^*)(B^*(A^*)^{i-1}\varphi).$$

By induction, we find, for every $i \in \{1, \ldots, n-1\}$,

$$B^*(A^*)^i \varphi = \sum_{j=0}^{i-1} (-1)^{j+1} \left((\partial_t + L_m^*)^j B^*(A^*)^{i-1-j} \begin{pmatrix} (P^*\varphi)_1 \\ \vdots \\ (P^*\varphi)_n \end{pmatrix} \right) + (-1)^{i+1} (\partial_t + L_m^*)^i \begin{pmatrix} (P^*\varphi)_{n+1} \\ \vdots \\ (P^*\varphi)_{n+m} \end{pmatrix}. \tag{2.16}$$

We introduce the operator $\mathcal{M}^* : \mathcal{D}(\mathcal{M}^*) \subset L^2\left(0, T; \mathcal{H}^{n+m}\right) \to L^2\left(0, T; \mathcal{H}^{nm}\right)$ defined by

$$\mathcal{M}^* \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{n+m} \end{pmatrix} := \begin{pmatrix} -\psi_{n+1} \\ \vdots \\ -\psi_{n+m} \\ -B^* \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix} + (\partial_t + L_m^*) \begin{pmatrix} \psi_{n+1} \\ \vdots \\ \psi_{n+m} \end{pmatrix} \\ \vdots \\ \sum_{j=0}^{n-2} (-1)^{j+1} \left((\partial_t + L_m^*)^j B^*(A^*)^{n-2-j} \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix} \right) + (-1)^n (\partial_t + L_m^*)^{n-1} \begin{pmatrix} \psi_{n+1} \\ \vdots \\ \psi_{n+m} \end{pmatrix} \end{pmatrix}. \tag{2.17}$$

Thanks to (2.16) and (2.17), $\mathcal{M}^*$ verifies equality (2.14). Using the definition of $\mathcal{M}^*$ given in (2.17), we deduce that

$$\mathcal{M} : \mathcal{D}(\mathcal{M}) \subset L^2(0, T; \mathcal{H}^{nm}) \to L^2(0, T; \mathcal{H}^{n+m})$$
$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \\ f_{m+1} \\ \vdots \\ f_{2m} \\ \vdots \\ f_{nm} \end{pmatrix} \mapsto \mathcal{M} f,$$

defined by

$$\mathcal{M}f = \begin{pmatrix} \sum_{i=1}^{n-1} \sum_{j=0}^{i-1} (-1)^{j+1} (-\partial_t + L_n)^j A^{i-1-j} B \begin{pmatrix} f_{jm+1} \\ \vdots \\ f_{(j+1)m} \end{pmatrix} \\ \sum_{i=0}^{n-1} (-1)^{i+1} (-\partial_t + L_m)^i \begin{pmatrix} f_{im+1} \\ \vdots \\ f_{(i+1)m} \end{pmatrix} \end{pmatrix}, \tag{2.18}$$

satisfies (2.12). Thus, in the n first components, the higher-order term is $(-\partial_t + \mathcal{L})^{n-2}$ and in the m last components, the higher-order term is $(-\partial_t + \mathcal{L})^{n-1}$, which concludes the proof. □

Conclusion: combination of the Analytic and Algebraic Problems.
Thanks to Proposition 2.1, there exists (Z, U) solution of (2.1) verifying moreover (2.2), (2.3) and (2.4). One can notably write

$$\overline{C}_n(U) = [A|B]\overline{C}_{nm}(\hat{U}) \tag{2.19}$$

for some $\hat{U} \in L^2((0, T); \mathcal{U}^{nm})$. Using Proposition 2.2, we define (X, W) by

$$\begin{pmatrix} X \\ W \end{pmatrix} := \mathcal{M}\left(\overline{C}_{nm}\hat{U}\right), \tag{2.20}$$

where $\hat{U}$ is defined in (2.19). Using (2.3), we know that

$$\overline{C}_n U \in H_0^{n-1}(0, T; \overline{C}_n(\mathcal{U}^n)) \bigcap \cap_{k=0}^{n-1} C^k([0, T]; \overline{C}_n\, \overline{C}_n^* \mathcal{D}((L_n^*)^{n-1-k})),$$

which implies using Assumption 1.2 that

$$\overline{C}_n U \in H_0^{n-1}(0, T; \overline{C}_n(\mathcal{U}^n)) \bigcap \cap_{k=0}^{n-1} C^k([0, T]; (\mathcal{D}(\mathcal{L}^{n-1-k}))^n).$$

Using now (2.19), we obtain

$$\overline{C}_{nm}\hat{U} \in H_0^{n-1}(0, T; \overline{C}_{nm}(\mathcal{U}^{nm})) \bigcap \cap_{k=0}^{n-1} C^k([0, T]; (\mathcal{D}(\mathcal{L}^{n-1-k}))^{nm}). \tag{2.21}$$

Using (2.20) together with (2.13) and (2.21), we obtain that

$$(X, W) \in \left(H_0^1(0, T; \overline{C}_n(\mathcal{U}^n)) \bigcap \cap_{k=0}^{1} C^k([0, T]; \mathcal{D}(L_n^{1-k}))\right) \times \left(L^2(0, T; \overline{C}_m(\mathcal{U}^m)) \bigcap C^0([0, T]; \mathcal{H}^m)\right),$$

i.e.,

$$(X, W) \in \left(H_0^1(0, T; \overline{C}_n(\mathcal{U}^n)) \bigcap \cap_{k=0}^1 C^k([0, T]; \mathcal{D}(L_n^{1-k})) \right) \times C^0([0, T]; \overline{C}_m(\mathcal{U}^m)). \tag{2.22}$$

Notably, there exists $\tilde{W} \in C^0([0, T]; \mathcal{U}^m)$ such that $W = C_m \tilde{W}$. Thus, coming back to (2.8), we infer that $(X, \tilde{W})$ is a solution to the problem

$$\begin{cases} \partial_t X = L_n(X) + AX + BC_m\tilde{W} + [A|B]f \quad \text{in} \quad (0, T) \times \mathcal{H}^n, \\ X(0) = X(T) = 0. \end{cases} \tag{2.23}$$

Hence, combining (2.1), (2.23) and the regularity results given in (2.4) and (2.22), the fictitious control f disappears and the pair $(Y, V) := (Z - X, -\tilde{W})$ is a solution to System (Ord1) in the space

$$C^0([0, T]; \mathcal{D}(L_n)) \cap C^1([0, T]; \mathcal{H}^n) \times C^0([0, T]; \mathcal{U}^m)$$

satisfying

$$\begin{aligned} Y(0) &= Y^0 \quad \text{in } \mathcal{D}(\mathcal{L}^{n-1})^n, \\ Y(T) &\equiv 0 \quad \text{in } \mathcal{H}^n, \end{aligned}$$

which concludes the first part of the proof of Theorem 1. □

2.2 Second part of the proof of Theorem 1

In this section, we assume Y^0 is NOT in $[A|B](\mathcal{H}^{nm})$ and we want to prove that we cannot bring the solution of (Ord1) with initial Y^0 to 0. We argue by contradiction.

For the sake of completeness, we mimic the proof of [5, Theorem 1.5]. Without loss of generality, we can only consider the case where we have one control force $m = 1$, that is to say $B \in \mathcal{M}_{n,1}(\mathbb{K})$ $(m = 1)$, the general case being quite similar (see notably [5, Lemma 3.1 and Page 14]).

Let $l \in \mathbb{N}$ such that $\text{rank}[A|B] = l < n$. It is clear that $\{B, AB, \ldots, A^{l-1}B\}$ is linearly independent. We introduce

$$X = \text{span}\{B, AB, \ldots, A^{l-1}B\}.$$

Since $A^l B \in X$, we know that there exists $\alpha \in \mathbb{K}^l$ such that

$$A^l B = \alpha_1 B + \alpha_2 AB + \cdots + \alpha_l A^{l-1} B. \tag{2.24}$$

Let $p_{l+1}, \ldots, p_n$ be $n - l$ vectors in $\mathbb{K}^n$ such that the set

$$\{B, AB, \ldots, A^{l-1}B, p_{l+1}, \ldots, p_n\}$$

is a basis of $\mathbb{K}^n$. Introducing

$$P = (B|AB|\dots|A^{l-1}B|p_{l+1}|\dots, p_n),$$

we have $Pe_1 = B$ with $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ and

$$P^{-1}AP = \begin{pmatrix} D_{11} & D_{12} \\ 0 & D_{22} \end{pmatrix}$$

for some $D_{12} \in \mathcal{M}_{l,n-l}(\mathbb{K})$, $D_{22} \in \mathcal{M}_{n-l}(\mathbb{K})$ and $D_{11} \in \mathcal{M}_l$ which is given by

$$D_{11} = \begin{pmatrix} 0 & 0 & 0 & \cdots & \alpha_1 \\ 1 & 0 & 0 & \cdots & \alpha_2 \\ 0 & 1 & 0 & \cdots & \alpha_3 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \alpha_l \end{pmatrix}$$

By the change of variables $W = P^{-1}Y$ and using $L_n P^{-1} = P^{-1}L_n$ and $\partial_t P^{-1} = P^{-1}\partial_t$, we observe that there exists a control V in $L^2((0,T)\times\mathcal{H})$ such that the solution of (Ord1) corresponding to the initial condition $Y(0) = Y^0$ in $\mathcal{H}^n$ satisfies $Y(T) \equiv 0$ in $\mathcal{H}^n$ if and only if the solution $W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}$ to

$$\begin{cases} \partial_t W = L_n W + \begin{pmatrix} D_{11} & D_{12} \\ 0 & D_{22} \end{pmatrix} W + C_1 V e_1, \\ W(0) = W^0 := P^{-1}Y^0 \end{cases} \tag{2.25}$$

verifies $W(T) \equiv 0$ in $\mathcal{H}^n$. Besides, it is easy to see that $Y^0 \in [A|B](\mathcal{H}^{nm})$ if and only if there exists $W_1^0 \in \mathcal{H}^l$ such that $Y^0 = P\begin{pmatrix} W_1^0 \\ 0 \end{pmatrix}$. If $Y^0 \notin [A|B](\mathcal{H}^{nm})$ then $Y^0 = P\begin{pmatrix} W_1^0 \\ W_2^0 \end{pmatrix}$ with $W_1^0 \in \mathcal{H}^l$, $W_2^0 \in \mathcal{H}^{n-l}$ and $W_2^0 \neq 0$. Thus, by uniqueness we conclude that $W_2(T) \neq 0$. Hence the solution W of (2.25) cannot be driven to zero at time T and (Ord1) cannot driven from Y^0 to 0 at time T, which concludes the proof. □

3 Proof of Theorem 2

Let us recall that we consider here an operator $\mathcal{Q}$ which is assumed to be self-adjoint, negative with compact resolvent. Suppose that Assumptions 1.1 and 1.2

are satisfied and let $T > T^*$. We will always consider some initial condition $(Y^0, Y^1) \in \mathcal{D}(\mathcal{Q}^{n-\frac{1}{2}})^n \times \mathcal{D}(\mathcal{Q}^{n-1})^n$.

During this section, we will assume that $\mathbb{K} = \mathbb{C}$ for the sake of simplicity. The case $\mathbb{K} = \mathbb{R}$ can be easily deduced for example by complexifying in the usual way the spaces $\mathcal{H}$ and $\mathcal{U}$ into, respectively, $\mathcal{H}_{\mathbb{C}}$ and $\mathcal{U}_{\mathbb{C}}$: one can then extend easily $\mathcal{Q}$ on $\mathcal{H}_{\mathbb{C}}$ and $\mathcal{C}$ on $\mathcal{U}_{\mathbb{C}}$. Going back to the initial version (i.e., System Ord2) is then possible by looking at the real part. Let us point out that we do not require that A is trigonalizable in $\mathbb{R}$ in what follows.

3.1 First part of the proof of Theorem 2

In this section, we assume that the Kalman rank condition is satisfied and we want to prove the controllability of (Ord2). We proceed as in the proof of Theorem 1. Let us emphasize that the main difference with the previous case is that the changing of variables exhibited during the proof of Proposition 2.1 does not work anymore; hence, we have to change totally the proof of the analytic part, which will now rely on a classical compactness-uniqueness argument similar to the one given for example in [20, Section 3]. Concerning the algebraic part, the computations are essentially the same.

Analytic problem

We consider the controlled system

$$\begin{cases} \partial_{tt} Z = \mathcal{Q}_n(Z) + AZ + \overline{\mathcal{C}}_n(U) \quad \text{in} \quad (0,T) \times \mathcal{H}^n, \\ Z(0) = Z^0, \\ \partial_t Z(0) = Z^1. \end{cases} \tag{3.1}$$

Let us first introduce some notations and first-order framework. Let H_α be the Hilbert space defined by $\mathcal{H}_\alpha = \mathcal{D}(\mathcal{Q}^\alpha)$ for any $\alpha \geq 0$ and $H_{-\alpha}$ is the dual space of H_α with respect to the pivot space $\mathcal{H}$. We denote by $X = \left(\mathcal{H}_{\frac{1}{2}} \times \mathcal{H}\right)^n$ our state space. We introduce the operator $\hat{\mathcal{Q}} : \mathcal{D}(\hat{\mathcal{Q}}) = \left(\mathcal{H}_1 \times \mathcal{H}_{\frac{1}{2}}\right)^n \subset X \to X$ such that

$$\hat{\mathcal{Q}} = \begin{pmatrix} A_\mathcal{Q} & 0 & \cdots & 0 \\ 0 & A_\mathcal{Q} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & A_\mathcal{Q} \end{pmatrix} \text{ with } A_\mathcal{Q} = \begin{pmatrix} 0 & I_d \\ \mathcal{Q} & 0 \end{pmatrix}. \tag{3.2}$$

The system (3.1) can be written as a first order system

$$\hat{Z}_t = (\hat{\mathcal{Q}} + \hat{A})\hat{Z} + \hat{C}U, \tag{3.3}$$

with $\hat{Z} = \begin{pmatrix} Z_1 \\ (Z_1)_t \\ \vdots \\ Z_n \\ (Z_n)_t \end{pmatrix}$, $\hat{C}\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} 0 \\ \overline{\mathcal{C}}u_1 \\ \vdots \\ 0 \\ \overline{\mathcal{C}}u_n \end{pmatrix}$ and $\hat{A} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_{11} & 0 & a_{12} & \cdots & 0 & a_{1n} & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_{n1} & 0 & a_{n2} & \cdots & 0 & a_{nn} & 0 \end{pmatrix}$.

Since we identify $\mathcal{H}$ with its dual, we shall define $\left(\mathcal{H}_{\frac{1}{2}} \times \mathcal{H}\right)' = \mathcal{H} \times \mathcal{H}_{-\frac{1}{2}}$ and the duality product defined for $(y_0, y_1) \in \mathcal{H} \times \mathcal{H}_{-\frac{1}{2}}$, $(z_0, z_1) \in \mathcal{H}_{\frac{1}{2}} \times \mathcal{H}$ by

$$\left\langle \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}, \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \right\rangle_{\left(\mathcal{H}\times\mathcal{H}_{-\frac{1}{2}}\right)\times\left(\mathcal{H}_{\frac{1}{2}}\times\mathcal{H}\right)} = \langle y_0, z_1\rangle_H + \langle y_1, z_0\rangle_{\mathcal{H}_{-\frac{1}{2}}\times\mathcal{H}_{\frac{1}{2}}}.$$

With this scalar product, we have

$$\hat{Q}^* = -\begin{pmatrix} A_{Q^*} & 0 & \cdots & 0 \\ 0 & A_{Q^*} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & A_{Q^*} \end{pmatrix} \text{ with } \begin{cases} X' = (\mathcal{H} \times \mathcal{H}_{-\frac{1}{2}})^n, \\ \mathcal{D}(\hat{Q}^*) = \left(\mathcal{H}_{\frac{1}{2}} \times \mathcal{H}\right)^n, \end{cases}$$

and $\hat{C}^* : X^* \to \mathcal{U}^n$ is given by

$$\hat{C}^*\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2n} \end{pmatrix} = \begin{pmatrix} \overline{\mathcal{C}}^* x_1 \\ \overline{\mathcal{C}}^* x_3 \\ \vdots \\ \overline{\mathcal{C}}^* x_{2n-1} \end{pmatrix}. \tag{3.4}$$

Thus, $\hat{C}\hat{C}^* : X^* \to X$ is exactly

$$\hat{C}\hat{C}^* = \begin{pmatrix} B_{\overline{\mathcal{C}}} & 0 & \cdots & 0 \\ 0 & B_{\overline{\mathcal{C}}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & B_{\overline{\mathcal{C}}} \end{pmatrix} \text{ with } B_{\overline{\mathcal{C}}} = \begin{pmatrix} 0 & 0 \\ \overline{\mathcal{C}}\,\overline{\mathcal{C}}^* & 0 \end{pmatrix}.$$

and for $i \in \mathbb{N}^*$, $\mathcal{D}(\hat{Q}^i) = (\mathcal{H}_{\frac{i+1}{2}} \times \mathcal{H}_{\frac{i}{2}})^n$ and $\mathcal{D}((\hat{Q}^*)^i) = \mathcal{D}(\hat{Q}^{i-1})$.

We can now go back to the resolution of the analytic problem.

For any unbounded operator $\mathcal{R}$ defined on $\mathcal{H}^k_{\frac{1}{2}} \times \mathcal{H}^k$ for some $k \in \mathbb{N}^*$, we introduce the set

$$\mathcal{N}_{\mathcal{R}}(T) := \left\{ \begin{bmatrix} Z^0 \\ Z^1 \end{bmatrix} \in \mathcal{H}^k_{\frac{1}{2}} \times \mathcal{H}^k \,|\, \mathcal{C}_n^* Y_t = 0 \ \forall t \in [0,T], \ \ Y \text{ solution of } \begin{cases} \partial_{tt} Y = \mathcal{R} Y \\ Y(0) = Z^0, \ \partial_t Y Z(0) = Z^1 \end{cases} \right\},$$
$$= \left\{ \begin{bmatrix} Z^0 \\ Z^1 \end{bmatrix} \in \mathcal{H}^k_{\frac{1}{2}} \times \mathcal{H}^k \,|\, (0, \mathcal{C}_n^*) Z = 0 \ \forall t \in [0,T], \ \ Z \text{ solution of } \begin{cases} \partial_t Z = \begin{pmatrix} 0 & I_d \\ \mathcal{R} & 0 \end{pmatrix} Z \\ Z(0) = \begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix} \end{cases} \right\}. \tag{3.5}$$

The main proposition is the following.

Proposition 3.1 *If Assumption 1.1 is satisfied, for every $T > T^*$, the control system (3.1) is null controllable at time T. Moreover, one can choose U such that*

$$U \in H_0^{2n-2}(0,T; \mathcal{U}^n) \bigcap \cap_{k=0}^{2n-2} C^k([0,T], \overline{C}^*(\mathcal{H}_{n-1-\frac{k}{2}})^n). \tag{3.6}$$

Proof of Proposition 3.1. We will need the two following lemmas. □

Lemma 3.1 *If $\mathcal{N}_{Q_n^*+A^*}(T) = \{0\}$ and if the system*

$$\partial_{tt} Z = Q_n(Z) + \overline{C}_n(U) \quad in \quad (0,T) \times \mathcal{H}^n \tag{3.7}$$

is exactly controllable at time T then the system (3.1) is exactly controllable at time T.

Lemma 3.1 can be found in [20, Theorem 4] (with $\epsilon = 1$), its proof will then be omitted.

Lemma 3.2 *If $\mathcal{N}_{\mathcal{Q}^*}(T) = \{0\}$ then $\mathcal{N}_{Q_n^*+A^*}(T) = \{0\}$.*

Let us temporarily admit this lemma and explain how we can deduce Proposition 3.1. By Assumption 1.1, we have a unique continuation property for the adjoint system of (3.7), from which we obtain that $\mathcal{N}_{\mathcal{Q}^*}(T) = \{0\}$. Now, using Lemma 3.1 and Lemma 3.2, the system (3.1) is null controllable at time T. Since $\hat{\mathcal{C}} \in \mathcal{L}(\mathcal{U}^n, X)$, the control system (3.3) is null controllable at time T and the operator $\hat{Q} + \hat{A}$ is a generator of a strongly continuous group on X. Let $T > T^*$, combining one more time Theorem 1.4, Corollary 1.5 and the equality (3.19) in [24] (with $s = 2n-2$), if $Y_0 \in \mathcal{D}(\hat{Q}^{2n-2})$ one can choose U such that

$$U \in H_0^{2n-2}(0,T; \mathcal{U}^n) \bigcap \cap_{k=0}^{2n-2} C^k([0,T]; \hat{C}^* \mathcal{D}((\hat{Q}^*)^{2n-2-k})).$$

Since for $i \in \mathbb{N}^*$, $\mathcal{D}(\hat{Q}^i) = (\mathcal{H}_{\frac{i+1}{2}} \times \mathcal{H}_{\frac{i}{2}})^n$, $\mathcal{D}((\hat{Q}^*)^i) = \mathcal{D}(\hat{Q}^{i-1})$, and going back to the definition of $\hat{C}^*$ given in (3.4), we obtain (3.6). □

It remains to prove Lemma 3.2. The proof is based on the following property.

Lemma 3.3 *Let $a \in \mathbb{K}$. If $\mathcal{N}_{\mathcal{Q}^*}(T) = \{0\}$ then $\mathcal{N}_{\mathcal{Q}^*+aI_{\mathcal{H}}}(T) = \{0\}$.*

Proof of Lemma 3.3. Let us decompose the proof into three steps.

- If $C^*\varphi \neq 0$ for every eigenvector φ of $\mathcal{Q}^* + aI_{\mathcal{H}}$ then $\mathcal{N}_{\mathcal{Q}^*+aI_{\mathcal{H}}} = \{0\}$.
  To prove this property we refer to the proof of [20, Theorem 5] which relies on an easy compactness-uniqueness argument.
- If φ an eigenvector of $\mathcal{Q}^*$ associated to the eigenvalue $-\lambda < 0$ and $\mathcal{N}_{\mathcal{Q}^*}(T) = 0$ then $C^*\varphi \neq 0$.
  Indeed, in this case $\begin{pmatrix} \frac{\varphi}{i\sqrt{\lambda}} \\ \varphi \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} 0 & I_d \\ \mathcal{Q}^* & 0 \end{pmatrix}$ associated to the eigenvalue $i\sqrt{\lambda}$. Let us assume that $(0, C^*)\begin{pmatrix} \frac{\varphi}{i\sqrt{\lambda}} \\ \varphi \end{pmatrix} = C^*\varphi = 0$. Let Z_φ be the solution of
  $$\begin{cases} \partial_t Z_\varphi = \begin{pmatrix} 0 & I_d \\ \mathcal{Q}^* & 0 \end{pmatrix} Z_\varphi, \\ Z_\varphi(0) = \begin{pmatrix} \frac{\varphi}{i\sqrt{\lambda}} \\ \varphi \end{pmatrix}. \end{cases}$$
  Thus for all $t \in (0, T)$, we have
  $$Z(t) = e^{i\sqrt{\lambda}t} \begin{pmatrix} \frac{\varphi}{i\sqrt{\lambda}} \\ \varphi \end{pmatrix}.$$
  By assumption $(0, C^*)Z(t) = e^{i\sqrt{\lambda}t}C^*\varphi = 0$. From the definition of $\mathcal{N}_{\mathcal{Q}^*}(T)$, $(\phi/(i\sqrt{\lambda}), \varphi) \in \mathcal{N}_{\mathcal{Q}^*}(T) = \{0\}$, whence the contradiction.
- For every φ eigenvector of $\mathcal{Q}^* + aI_{\mathcal{H}}$, $C^*\varphi \neq 0$.
  Indeed, the couple (φ, λ) is a vector-eigenfunction of $\mathcal{Q}^*$ if and only if the couple $(\varphi, \lambda + a)$ is a vector-eigenfunction of $\mathcal{Q}^* + aI_{\mathcal{H}}$, and we can use the previous point.

Combining these 3 arguments provides Lemma 3.3. □

Proof of Lemma 3.2. Since $\mathbb{K} = \mathbb{C}$, A^* is trigonalizable. Hence, there exists an invertible matrix P such that $A^* = P\mathcal{T}P^{-1}$ with $\mathcal{T} = (t_{ij})$ some lower triangular matrix. Using the change of variables $V = P^{-1}Z$, we deduce that $\mathcal{N}_{\mathcal{Q}_n^*+A^*}(T) = \{0\}$ if and only if $\mathcal{N}_{\mathcal{Q}_n^*+\mathcal{T}^*}(T) = \{0\}$. The system

$$\begin{cases} Z_{tt} = \mathcal{Q}_n^* Z + \mathcal{T}Z, \\ Z(0) = Z_0, \end{cases} \tag{3.8}$$

can be written as

$$\begin{cases} Z_{tt}^1 = \mathcal{Q}_n^* Z^1 + t_{11}Z^1, \\ Z_{tt}^2 = \mathcal{Q}_n^* Z^2 + t_{21}Z^1 + t_{22}Z^2, \\ \vdots \\ Z_{tt}^n = \mathcal{Q}_n^* Z^n + t_{n1}Z^1 + t_{n2}Z^2 + \cdots + t_{nn}Z^n, \\ Z(0) = Z_0. \end{cases} \tag{3.9}$$

Let $Z_0 = \begin{pmatrix} Z_0^1 \\ \vdots \\ Z_0^n \end{pmatrix} \in \mathcal{N}_{\mathcal{Q}_n^*+\mathcal{T}^*}(T)$. By definition $\overline{C}_n^* Z = \begin{pmatrix} \overline{C}^* Z^1 \\ \vdots \\ \overline{C}^* Z^n \end{pmatrix} = 0$ with Z solution of (3.8), hence $Z_0^1 \in \mathcal{N}_{\mathcal{Q}^*+t_{11}I_{\mathcal{H}}}$. Since $\mathcal{N}_{\mathcal{Q}^*}(T) = 0$ and using Lemma 3.3, we have $Z_0^1 = 0$. Thus the system can be written as

$$\begin{cases} Z_{tt}^2 = \mathcal{Q}_n^* Z^2 + t_{22} Z^2, \\ \vdots \\ Z_{tt}^n = \mathcal{Q}_n^* Z^n + t_{n2} Z^2 + \cdots + t_{nn} Z^n, \\ Z(0) = Z_0. \end{cases}$$

Using the same reasoning as before by replacing Z^1 by $Z^2, Z^3, \ldots, Z^n$ successively, we get by induction that $Z_0^2 = \cdots = Z_0^n = 0$, thus we have $Z_0 = 0$ and we infer that $\mathcal{N}_{\mathcal{Q}_n^*+\mathcal{T}^*}(T) = 0$, which concludes the proof thanks to Lemma 3.1. □

Algebraic problem
For $f := \overline{C}_n(U)$, we want to find a pair $(X, \tilde{W}) \in C^0([0,T]; \mathcal{D}(\mathcal{Q}_n)) \cap C^1([0,T]; \mathcal{D}(\mathcal{Q}_n^{\frac{1}{2}})) \cap C^2([0,T]; \mathcal{H}^n) \times C^0([0,T]; \mathcal{U}^m)$ satisfying the following control problem:

$$\begin{cases} \partial_{tt} X = \mathcal{Q}_n(X) + AX + BC_m \tilde{W} + f & \text{in} \quad (0,T) \times \mathcal{H}^n, \\ X(0) = X(T) = \partial_t X(0) = \partial_t X(T) = 0. \end{cases} \tag{3.10}$$

As in the proof of Theorem 1, we will solve instead

$$\begin{cases} \partial_{tt} X = \mathcal{Q}_n(X) + AX + BW + f & \text{in} \quad (0,T) \times \mathcal{H}^n, \\ X(0) = X(T) = \partial_t X(0) = \partial_t X(T) = 0, \end{cases} \tag{3.11}$$

with $W \in \mathcal{C}_m(\mathcal{U}^m)$. We will mimic the proof of Proposition 2.2. In the same way we can write Problem (2.23) under the abstract form

$$\tilde{\mathcal{P}}(X, W) = f, \tag{3.12}$$

where

$$\begin{aligned} \tilde{\mathcal{P}} : \mathcal{D}(\tilde{\mathcal{P}}) \subset L^2(0,T; \mathcal{H}^{n+m}) &\to L^2(0,T; \mathcal{H}^n) \\ (X\,,\ W) &\mapsto \partial_{tt} X - \mathcal{Q}_n X - AX - BW. \end{aligned} \tag{3.13}$$

The goal will be then to find a partial differential operator $\mathcal{M}$ satisfying

$$\tilde{\mathcal{P}} \circ \mathcal{M} = I_n, \tag{3.14}$$

where $I_n \in \mathbb{K}$ is the identity matrix.

Proposition 3.2 *Let $(A, B) \in \mathcal{M}_n(\mathbb{K}) \times \mathcal{M}_{n,m}(\mathbb{K})$. If rank$([A|B]) = n$, $\tilde{\mathcal{P}}$ has a right inverse denoted by $\mathcal{M}$. Moreover, the operator $\mathcal{M}$ is an operator of order*

$$\begin{cases} 2n-4 \text{ for } n \text{ first components} \\ 2n-2 \text{ for the } m \text{ last components} \end{cases} \tag{3.15}$$

in time and

$$\begin{cases} n-2 \text{ for } n \text{ first components} \\ n-1 \text{ for the } m \text{ last components} \end{cases} \tag{3.16}$$

in terms of powers of $\mathcal{Q}$.

Proof of Proposition 3.2. Changing ∂_t to ∂_{tt} in the proof of Proposition 2.2 we have

$$\tilde{\mathcal{P}} \circ \tilde{\mathcal{M}} = [A|B]$$

with

$$\tilde{\mathcal{M}} : \mathcal{D}(\tilde{\mathcal{M}}) \subset L^2(0, T; \mathcal{H}^{nm}) \to L^2(0, T; \mathcal{H}^{n+m})$$
$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \\ f_{m+1} \\ \vdots \\ f_{2m} \\ \vdots \\ f_{nm} \end{pmatrix} \mapsto \tilde{\mathcal{M}} f,$$

defined by

$$\tilde{\mathcal{M}} f = \begin{pmatrix} \sum_{i=1}^{n-1} \sum_{j=0}^{i-1} (-1)^{j+1} (-\partial_{tt} + Q_n)^j A^{i-1-j} B \begin{pmatrix} f_{jm+1} \\ \vdots \\ f_{(j+1)m} \end{pmatrix} \\ \\ \sum_{i=0}^{n-1} (-1)^{i+1} (-\partial_{tt} + Q_m)^i \begin{pmatrix} f_{im+1} \\ \vdots \\ f_{(i+1)m} \end{pmatrix} \end{pmatrix}. \tag{3.17}$$

Since rank$([A|B]) = n$, there exists $D \in \mathcal{M}_{nm,n}$ such that $[A|B]D = I_n$, where $I_n \in \mathbb{K}$ is the identity matrix. Introducing the operator $\mathcal{M} := \tilde{M} D$ we obtain (3.14). Moreover, in the n first components the higher order term is $(-\partial_{tt} + \mathcal{Q})^{n-2}$ and in the m last components the higher-order term is $(-\partial_{tt} + \mathcal{Q})^{n-1}$, which concludes the proof. □

Conclusion: combination of the Analytic and Algebraic Problems
The proof is similar to the one-order case, so that we just give the main arguments here. Let (X, W) be defined by

$$\begin{pmatrix} X \\ W \end{pmatrix} := \mathcal{M}\left(\overline{C}_n U\right), \tag{3.18}$$

with $\overline{C}_n U \in H_0^{2n-2}(0, T; \overline{C}_n(\mathcal{U}^n)) \bigcap \cap_{k=0}^{2n-2} C^k([0, T]; \overline{C}_n\, \overline{C}_n^*(\mathcal{H}^n_{n-1-\frac{k}{2}}))$ constructed in Proposition 3.1. Using Proposition 3.2 and Assumption 1.2 , we obtain that

$$(X, W) \in \left(H_0^2(0, T; (\overline{C}_n(\mathcal{U}^n))) \bigcap \cap_{k=0}^{2} C^k([0, T]; \mathcal{H}^n_{1-\frac{k}{2}})\right) \times C^0([0, T], \overline{C}_m(\mathcal{U}^m)).$$

Notably, there exists $\tilde{W} \in C^0([0, T], \mathcal{U}^m)$ such that $W = C_m \tilde{W}$. Moreover, using Proposition 3.1 we have $X(0) = X(T) = 0$ in $\mathcal{H}^n$ and we remark that $(X, \tilde{W})$ is a solution to the problem

$$\begin{cases} \partial_{tt} X = Q_n(X) + AX + BC_m \tilde{W} + f & \text{in} \quad (0, T) \times \mathcal{H}^n, \\ X(0) = X(T) = \partial_t X(0) = \partial_t X(T) = 0. \end{cases}$$

Thus the pair $(Y, V) := (Z - X, -\tilde{W})$ is a solution to System (Ord2) in $C^0([0, T]; \mathcal{D}(Q_n)) \cap C^1([0, T]; \mathcal{D}(Q_n^{\frac{1}{2}})) \cap C^2([0, T]; \mathcal{H}^n) \times C^0([0, T]; \mathcal{U}^m)$ and satisfies

$$\begin{aligned} &Y(0) = Y^0 \ \text{in} \ \mathcal{D}(\mathcal{Q}^{n-\frac{1}{2}})^n, \quad \partial_t Y(0) = Y^1 \ \text{in} \ \mathcal{D}(\mathcal{Q}^{n-1})^n \\ &Y(T) \equiv 0 \ \text{in} \ \mathcal{H}^n, \quad \partial_t Y(T) \equiv 0 \ \text{in} \ \mathcal{H}^n. \end{aligned}$$

□

3.2 Second part of the proof of Theorem 2

In this section, we assume that the Kalman condition is NOT satisfied and we want to prove that the null controllability of (Ord2) fails. We will use an argument based on the *transmutation method* in order to go back to a parabolic one-order system. The ideas are then essentially the same as in Sect. 2.2.

Using the transmutation technique (as in [25] for instance), we have the following lemma:

Lemma 3.4 *If the system*

$$\left\{ \partial_{tt} Y = \mathcal{Q}_n Y + AY + BC_m U \quad \textit{in} \quad (0, T) \times \mathcal{H}^n, \right. \tag{3.19}$$

is null controllable in some time T *then at any time* $\tilde{T} > 0$*, the system*

$$\left\{ \partial_t Z = \mathcal{Q}_n Z + AZ + BC_m V \quad \textit{in} \quad (0, \tilde{T}) \times \mathcal{H}^n, \right. \tag{3.20}$$

is null controllable.

We assume that $\text{rank}([A|B]) \neq n$. Thus, there exists $X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \in \mathcal{H}^n \backslash \{0\}$ such that $B^*X = B^*A^*X = \cdots B^*(A^*)^{n-1} X = 0$. We consider the following adjoint system of (3.20)

$$\begin{cases} -\partial_t \varphi = \mathcal{Q}_n^*(\varphi) + A^*\varphi & \text{in} \quad (0,T) \times \mathcal{H}^n, \\ \varphi(T) = \varphi^T. \end{cases} \tag{3.21}$$

It is well known that System (3.20) is null controllable at time T if and only if there exists a positive constant C_1 such that for every $\varphi^T \in \mathcal{H}^n$, the solution φ of (3.21) verifies

$$\|\varphi(0)\|_{\mathcal{H}^n} \leq C_1 \int_0^T \|C_m^* B^* \varphi\|_{\mathcal{U}^m} \, \mathrm{d}t. \tag{3.22}$$

Let φ_X the solution of (3.21) with $\varphi_X(T) = X$. Let $S = (S_t)_{t \in \mathbb{R}}$ be a strongly continuous group on $\mathcal{H}$, with generator $\mathcal{Q} : \mathcal{D}(\mathcal{Q}) \subset \mathcal{H} \to \mathcal{H}$. By the definition of $\mathcal{Q}_n$ given in (1.6), we obtain $C_m^* B^* \varphi_X = C_m^* B^* e^{tA^*} \begin{pmatrix} S^*(T-t)X_1 \\ \vdots \\ S^*(T-t)X_n \end{pmatrix}$. Since $(A, B) \in \mathcal{M}_n(\mathbb{K}) \times \mathcal{M}_{n,m}(\mathbb{K})$, for all $i \in \{0, \ldots, n-1\}$, $B^*(A^*)^i \begin{pmatrix} S^*(T-t)X_1 \\ \vdots \\ S^*(T-t)X_n \end{pmatrix}$ is solution of

$$\begin{cases} -\partial_t \varphi = \mathcal{Q}_m^*(\varphi) & \text{in} \quad (0,T) \times \mathcal{H}^m, \\ \varphi(T) = B^*(A^*)^i X = 0. \end{cases}$$

Thus

$$B^* \left(A^*\right)^i \begin{pmatrix} S^*(T-t)X_1 \\ \vdots \\ S^*(T-t)X_n \end{pmatrix} = 0.$$

Using the Cayley-Hamilton theorem, we obtain that $C_m^* B^* \varphi_X = 0$, from which we deduce that (3.22) is not satisfied. Thus, the System (3.20) is not controllable at time T. Using Lemma 3.4, we deduce that the system

$$\partial_{tt} Y = \mathcal{Q}_n Y + AY + BC_m U \quad \text{in} \quad (0,T) \times \mathcal{H}^n,$$

is not controllable, which concludes the proof. ◻

4 Applications

In this section, we will give some examples of applications in the case of systems of Schrödinger and wave equations with internal control. Let Ω be a smooth bounded open subset of $\mathbb{R}^N$. We will denote by $L^2(\Omega)$ the set of square-integrable functions defined on Ω with values in the complex plane $\mathbb{C}$.

We recall the following definition that will be widely used in what follows.

Definition 4.1 (GCC) We say that ω verifies the Geometric Optics Condition (GCC in short) if there exists $T^* > 0$ such that any rays of Geometric Optics in Ω enter the open set ω in time smaller than T^* (see [14]).

4.1 System of Schrödinger equations with internal control

Let us introduce the state space $\mathcal{H} = L^2(\Omega)$ and the control space $\mathcal{U} = L^2(\Omega) = \mathcal{H}$. We consider the linear continuous operator $\mathcal{C} : L^2(\Omega) \to L^2(\Omega)$ defined by $\mathcal{C}u = \mathbb{1}_\omega u$ with

$$\mathbb{1}_\omega = \begin{cases} 1 & \text{on} \quad \omega, \\ 0 & \text{on} \quad \Omega\backslash\omega. \end{cases}$$

We consider the following Schrödinger equation

$$\begin{cases} \partial_t z = i\Delta z + W(x)z + \mathbb{1}_\omega u, \\ z(0) = z_0, \end{cases} \tag{4.1}$$

where the potential W is in $C^\infty(\bar{\Omega}; \mathbb{C})$. It is well known that the operator $i\Delta + W(x) : \mathcal{H}^2(\Omega) \cap \mathcal{H}^1_0(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$ is a generator of a group on $L^2(\Omega)$. Let $n \in \mathbb{N}^*$, we have $\mathcal{D}((i\Delta + W(x))^n) = \mathcal{D}(((i\Delta + W(x))^*)^n) = H^{2n}_{(0)}(\Omega)$ where $H^{2n}_{(0)}(\Omega)$ is defined by

$$H^{2n}_{(0)}(\Omega) := \{v \in H^{2n}(\Omega) \text{ such that } v = \Delta v = \cdots = \Delta^{n-1} v = 0 \text{ on } \partial\Omega\}. \tag{4.2}$$

We consider the control system

$$\begin{cases} \partial_t Y = i\Delta Y + W(x)Y + AY + \mathbb{1}_\omega BV & \text{in} \quad (0, T) \times L^2(\Omega)^n, \\ Y(0) = Y^0, \end{cases} \tag{4.3}$$

with $(A, B) \in \mathcal{M}_n(\mathbb{C}) \times \mathcal{M}_{n,m}(\mathbb{C})$.

Our first result concerns the case where ω is strongly included in Ω.

Theorem 3 *Let $T > 0$ and let us assume that the open set ω of Ω satisfies GCC and $\overline{\omega} \subset \Omega$, then there exists a control V in $C^0((0, T) \times (L^2(\Omega))^m)$ such that the solution of (4.3) with initial condition $Y(0) = Y^0$ in $(H^{2n-2}_{(0)}(\Omega))^n$ satisfies*

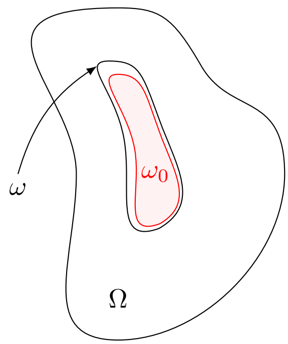

Fig. 1 An example when $\overline{\omega} \subset \Omega$

$$Y(T) \equiv 0 \quad in \quad (L^2(\Omega))^n$$

if and only $Y^0 \in [A|B]((L^2(\Omega))^{nm})$.

Proof of Theorem 3. Since $\overline{\omega} \subset \Omega$, one can construct a function $\tilde{\mathbb{1}}_\omega \in \mathcal{C}^\infty(\Omega)$ defined by

$$\tilde{\mathbb{1}}_\omega := \begin{cases} 1 & \text{on} \quad \omega_0, \\ 0 & \text{on} \quad \Omega \backslash \omega, \end{cases} \tag{4.4}$$

where ω_0 is some well-chosen open set strongly included in ω still verifying GCC, so that

$$\begin{cases} \partial_t z = (i\Delta + W(x))z + \tilde{\mathbb{1}}_\omega u, \\ z(0) = z_0, \end{cases} \tag{4.5}$$

is exactly controllable thanks to the result of [30] (see Fig. 1). Thus, we deduce that Assumption 1.1 is verified with $\overline{\mathcal{C}} : L^2(\Omega) \to L^2(\Omega)$ defined by $\overline{\mathcal{C}}u = \tilde{\mathbb{1}}_\omega u$.

To apply Theorem 1 we need to verify Assumption 1.2 where $\overline{\mathcal{C}}$ is described above. Let $\varphi \in \mathcal{D}(((i\Delta + W(x))^*)^k) = H^{2k}_{(0)}(\Omega)$, for all $k \in \{0, \ldots, n-1\}$, by definition of $\tilde{\mathbb{1}}_\omega \in \mathcal{C}^\infty_c(\Omega)$ we obtain $\overline{\mathcal{C}}\overline{\mathcal{C}}^* \varphi = \tilde{\mathbb{1}}^2_\omega \varphi \in H^{2k}_{(0)}(\Omega)$ and $(i\Delta + W(x))^k(\overline{\mathcal{C}}\overline{\mathcal{C}}^* \mathcal{D}(((i\Delta + W(x))^*)^k) \in C(L^2(\Omega))$. Thus, we can apply Theorem 1 whence the conclusion of the proof of Theorem 3. □

Remark 5 One can obtain the same result as in Theorem 3 with exactly the same proof by replacing the open set Ω with some regular compact connected Riemannian manifold without boundary M, with ω any open subset of M verifying GCC.

From now on we consider ω an open subset of $\overline{\Omega}$ such that $\omega \cap \partial\Omega \neq \emptyset$ (see Fig. 2). In general it is impossible to construct a function $\tilde{\mathbb{1}}_\omega \in \mathcal{C}^\infty(\Omega)$ as (4.4) such that the system (4.5) is controllable and $\tilde{\mathbb{1}}_\omega$ maps $H^{2k}_{(0)}(\Omega)$ into itself for all $k \in [3, \infty]$. More precisely, satisfying Assumption 1.2 would require that, for every $\varphi \in H^{2(n-1)}_{(0)}(\Omega)$, for every $k \leq n-2$, $\Delta^k(\tilde{\mathbb{1}}_\omega \varphi) = 0$ on $\partial\Omega$ which is not verified without additional conditions on $\tilde{\mathbb{1}}_\omega$. If $n = 2$ it is clear that $\tilde{\mathbb{1}}_\omega$ maps $L^2(\Omega, \mathbb{C})$ into itself. If $n = 3$, we need to assume moreover that $\nabla \tilde{\mathbb{1}}_\omega . \vec{n} = 0$ with $\vec{n}$ the unit normal vector. If $n > 3$, $\tilde{\mathbb{1}}_\omega$ has to satisfy strong global geometric conditions, for instance $\tilde{\mathbb{1}}_\omega$ can be chosen constant near any connected component of the boundary (for more information we

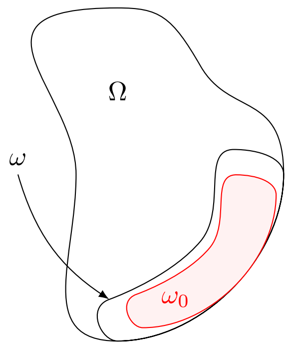

Fig. 2 An example when $\omega \subset \overline{\Omega}$ is an open set such that $\omega \cap \partial\Omega \neq \emptyset$

refer to [21, Section 4.2]). However, we cannot affirm anymore that the system (4.5) will still be controllable without loss of regularity or locality on the function $\tilde{\mathbb{1}}_\omega$, so that Assumption 1.2 will not be verified. Hence, we will focus our attention on some particular cases. We will first consider the case where the number of equations is less than or equal to three without additional conditions on Ω and then we will consider the case where Ω is the product of N open intervals in $\mathbb{R}^N$ and the case where Ω is a unit disk.

Theorem 4 *Let $T > 0$. Let ω be an open subset of $\overline{\Omega}$ such that $\omega \cap \partial\Omega \neq \emptyset$ and ω satisfies GCC. Let $n \leq 3$, then there exists a control V in $C^0([0,T];(L^2(\Omega))^m)$ such that the solution of (4.3) corresponding to the initial condition $Y(0) = Y^0$ in $(H^{2n-2}_{(0)}(\Omega))^n$ satisfies*

$$Y(T) \equiv 0 \quad in \quad (L^2(\Omega))^n,$$

if and only if $Y^0 \in [A|B]((L^2(\Omega))^{nm})$.

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Proof of Theorem 4. We can keep constructing a function $\tilde{\mathbb{1}}_\omega \in \mathcal{C}^\infty(\Omega)$ defined by

$$\tilde{\mathbb{1}}_\omega := \begin{cases} 1 & \text{on} \quad \omega_0, \\ 0 & \text{on} \quad \Omega\backslash\omega, \end{cases} \tag{4.6}$$

where ω_0 is some well-chosen open set included in ω still verifying GCC such that $\nabla\tilde{\mathbb{1}}_\omega.\vec{n} = 0$ and the system

$$\begin{cases} \partial_t z = i\Delta z + W(x)z + \tilde{\mathbb{1}}_\omega u, \\ z(0) = z_0, \end{cases} \tag{4.7}$$

is exactly controllable (see Fig. 2). Moreover, using [21, Section 4.2] and $\nabla\tilde{\mathbb{1}}_\omega.\vec{n} = 0$, we infer that $\mathbb{1}^2_\omega$ maps $H^{2k}_{(0)}(\Omega)$ into itself for $k \leq 2$ and by definition of $\tilde{\mathbb{1}}_\omega$, we have $(i\Delta + W(x))^k(\overline{\mathcal{C}\mathcal{C}}^*\mathcal{D}(((i\Delta + W(x))^*)^k)) \in C(L^2(\Omega))$ for $k \leq 2$. Thus, Assumption 1.2 is verified and we can conclude as in the proof of Theorem 3. □

It is well known that GCC is only a sufficient condition to ensure the exact controllability of the Schrödinger equation, but is not always necessary in some particular geometries. Let us give two examples. If Ω is the product of N open intervals, we do not need to impose restrictions on ω and if Ω is a unit disk, ω has to touch the boundary of Ω.

Theorem 5 *Let $T > 0$ and let us assume that the domain $\Omega \subset \mathbb{R}^N$ is the product of N open intervals, then there exists a control V in $C^0((0,T)\times(L^2(\Omega))^m)$ such that the solution of (4.3) corresponding to the initial condition $Y(0) = Y^0$ in $(H^{2n-2}_{(0)}(\Omega))^n$ satisfies*

$$Y(T) \equiv 0 \quad in \quad (L^2(\Omega))^n,$$

if and only $Y^0 \in [A|B]((L^2(\Omega))^{nm})$.

Proof of Theorem 5. From [29, Proposition 8.8], one can easily deduce that the system

$$\begin{cases} \partial_t z = i\Delta z + W(x)z + \mathbb{1}_{\tilde{\omega}} u, \\ z(0) = z_0, \end{cases}$$

is exactly controllable for any non-empty open $\tilde{\omega}$ subset of Ω. Consequently, without loss of generality, we can assume that $\overline{\omega} \subset \Omega$. Thus, we just have to copy the proof of Theorem 3 and we deduce the expected result. □

Theorem 6 *Let $T > 0$ and let us assume that the domain $\Omega \subset \mathbb{R}^2$ is the unit disk and let ω be an open subset of $\overline{\Omega}$ such that $\omega \cap \partial\Omega \neq \emptyset$. Let $n \leq 3$, then there exists a control V in $C^0([0,T]; (L^2(\Omega))^m)$ such that the solution of (4.3) corresponding to the initial condition $Y(0) = Y^0$ in $(H^{2n-2}_{(0)}(\Omega))^n$ satisfies*

$$Y(T) \equiv 0 \quad in \quad (L^2(\Omega))^n,$$

if and only $Y^0 \in [A|B]((L^2(\Omega))^{nm})$.

Proof of Theorem 6. Since the domain $\Omega \subset \mathbb{R}^2$ is the unit disk, from [12, Theorem 1.2], we deduce that the equation

$$\begin{cases} \partial_t z = i\Delta z + W(x)z + \mathbb{1}_{\tilde{\omega}} u, \\ z(0) = z_0, \end{cases}$$

is exactly controllable for any open $\tilde{\omega}$ subset of $\overline{\Omega}$ such that $\tilde{\omega} \cap \partial\Omega \neq \emptyset$. Thus, we just mimic the proof of Theorem 4 and we have directly the expected result. □

Remark 6 The same results can be obtained by replacing the Schrödinger equation by the plate equation (see for example [30, Section 5]).

4.2 System of wave equations with internal control

We consider the state space and control space as in the previous section, i.e., $\mathcal{H} = \mathcal{U} = L^2(\Omega)$. We introduce the operator $\mathcal{C} : L^2(\Omega) \to L^2(\Omega)$ such that $\mathcal{C}u = \mathbb{1}_\omega u$. We consider the following wave equation

$$\begin{cases} \partial_{tt} z = \Delta z + W(x)z + \mathbb{1}_\omega u, \\ z(0) = z_0, \\ \partial_t z(0) = z_1, \end{cases} \tag{4.8}$$

where the potential W is in $C^\infty(\bar{\Omega}; \mathbb{C})$. It is well known that the operator $\Delta + W(x) : \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$ is self-adjoint with compact resolvent, but it is not a negative operator in general. This is not a problem here because we know that for $\mu > 0$ large enough the operator $\Delta + W(x) - \mu$ becomes negative; hence, we can adapt the results of Theorem 2 in this case. Moreover, we know that if ω verifies GCC, then (4.8) is controllable at any time $T > T^*$, where T^* is the minimal time needed to ensure that all the rays of Geometric Optics in Ω enter the open set ω. Let $n \in \mathbb{N}^*$, we have $\mathcal{D}((\Delta + W(x))^n) = H^{2n}_{(0)}(\Omega)$ where $H^{2n}_{(0)}(\Omega)$ is defined in (4.2). We consider the control system

$$\begin{cases} \partial_{tt} Y = \Delta Y + W(x)Y + AY + \mathbb{1}_\omega BV \quad \text{in} \quad (0,T) \times L^2(\Omega)^n, \\ Y(0) = Y^0, \\ \partial_t Y(0) = Y^1, \end{cases} \tag{4.9}$$

with $(A, B) \in \mathcal{M}_n(\mathbb{C}) \times \mathcal{M}_{n,m}(\mathbb{C})$. Mimicking the proof of Theorems 3 and 4, we immediately obtain the following results:

Theorem 7 *Let us assume that ω satisfies GCC and $\overline{\omega} \subset \Omega$. Let $(Y^0, Y^1) \in H^{2n-1}_{(0)}(\Omega) \times H^{2n-2}_{(0)}(\Omega)$. Then, for every $T > T^*$, the control system (4.9) is exactly controllable at time T if and only if rank$([A|B]) = n$.*

Remark 7 As in the case of the Schrödinger equation, one can obtain the same result as in Theorem 7 with exactly the same proof by replacing the open set Ω with some regular compact connected Riemannian manifold without boundary M, with ω any open subset of M verifying GCC.

Theorem 8 *Let ω be an open subset of $\overline{\Omega}$ such that $\omega \cap \partial\Omega \neq \emptyset$ and ω satisfies GCC. Let $n \leq 3$. Let $(Y^0, Y^1) \in H^{2n-1}_{(0)}(\Omega) \times H^{2n-2}_{(0)}(\Omega)$, then, for every $T > T^*$, the control system (4.9) is exactly controllable at time T if and only if rank$([A|B]) = n$.*

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Compliance with ethical standards

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Conflict of interest The authors declare that they have no conflict of interest.

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