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ORIGINAL ARTICLE

iISS and ISS dissipation inequalities: preservation and interconnection by scaling

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Abstract In analysis and design of nonlinear dynamical systems, (nonlinear) scaling of Lyapunov functions has been a central idea. This paper proposes a set of tools to make use of such scalings and illustrates their benefits in constructing Lyapunov functions for interconnected nonlinear systems. First, the essence of some scaling techniques used extensively in the literature is reformulated in view of preservation of dissipation inequalities of integral input-to-state stability (iISS) and input-to-state stability (ISS). The iISS small-gain theorem is revisited from this viewpoint. Preservation of ISS dissipation inequalities is shown to not always be necessary, while preserving iISS which is weaker than ISS is convenient. By establishing relationships between the Legendre–Fenchel transform and the reformulated scaling techniques, this paper proposes a way to construct less complicated Lyapunov functions for interconnected systems.

Keywords Nonlinear dynamical systems · Integral input-to-state stability · Dissipation inequalities · Lyapunov functions

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1 Introduction

Input-to-state stability (ISS) is a notion that bounds the magnitude of state trajectories of a system in terms of the magnitude of the initial state and the maximum instantaneous magnitude of the input signal [34]. The existence of such an estimate implies boundedness of the state with respect to any bounded inputs. Integral input-to-state stability (iISS) provides a superset of ISS [35]. An iISS system may not exhibit bounded states even for inputs converging to zero and, in fact, iISS guarantees the boundedness of the state only with respect to inputs of finite energy. These two notions, ISS and iISS, have contributed greatly to the continuing development of nonlinear control theory (e.g., [1,4,6,22,25,27,33,38]). ISS and iISS can allow one to analyze or synthesize large-scale interconnected systems based on the knowledge of component subsystems. ISS and iISS can be characterized in terms of both trajectories, as described above, and Lyapunov functions [2,34,35,37]. Nonlinear scaling has sometimes played an important role in utilizing Lyapunov functions [6,7,9,10,28,30] to cope with nonlinearities in systems.

This paper focuses on preservation of ISS and iISS dissipation inequalities under scalings as studied in [23]. Let $x(t)$ and $w(t)$ denote the state and the input of a system, respectively. Consider an energy-like function $V(x) = x^2$ satisfying

$$\langle \nabla V(x),\, f(x, w)\rangle \le -\frac{2x^2}{1+x^2} + \left(\frac{w}{1+|w|}\right)^2. \tag{1}$$

In the literature, an inequality estimating an upper bound of $\langle \nabla V(x),\, f(x, w)\rangle$ as (1) is called a dissipation inequality associated with the system $\dot{x} = f(x, w)$ [37,39]. According to a Lyapunov characterization of ISS in [37], the function V is an ISS Lyapunov function,[1] which means that the system $\dot{x} = f(x, w)$ is ISS. In fact, the implication

$$x^2 \ge \frac{w^2}{1+2|w|} \;\Rightarrow\; \langle \nabla V(x),\, f(x, w)\rangle \le -\frac{x^2}{1+x^2} \tag{2}$$

holds true. Apply the scaling $\mu(s) = s^2$ to the function V to obtain $W = \mu(V)$. It is not obvious that the scaled function W admits an inequality of the form (1) which separates the input w from the state x completely. From (1) we have

$$\langle \nabla W(x),\, f(x, w)\rangle \le 2x^2 \left[-\frac{2x^2}{1+x^2} + \left(\frac{w}{1+|w|}\right)^2\right]. \tag{3}$$

The right-hand side of (3) cannot be bounded from above by any function of w which does not involve x. Nevertheless, the Lyapunov characterization of ISS in [37] demonstrates that the scaled function W is still an ISS Lyapunov function since $x^2 \ge$

[1] The decay rate $x^2/(1+x^2)$ in (2) is not of class $\mathcal{K}_\infty$ (i.e., the decay rate does not approach infinity as x approaches infinity), although the formal definition in [37, Definition 2.2] employs $\mathcal{K}_\infty$. However, as indicated in [37] and verified easily, a positive definite decay rate, e.g., (2), can imply ISS.

$w^2/(1+2|w|)$ implies $\langle \nabla W(x), f(x,w)\rangle \le -2x^4/(1+x^2)$, due to (3). Thus, for a function V, being an ISS Lyapunov function[2] is not equivalent to V satisfying a dissipation inequality with the complete separation of state and input.

A similar argument is valid for iISS by considering

$$\langle \nabla V(x), f(x,w)\rangle \le -\frac{2x^2}{1+x^2} + w^2. \tag{4}$$

This inequality guarantees that V is an iISS Lyapunov function [2], but the scaled function W with $\mu(s) = s^2$ cannot yield a dissipation inequality of the same type in spite of the fact that the system $\dot{x} = f(x,w)$ is iISS. Although these facts have appeared in a variety of work in the literature, the developed techniques have not been elaborated with the explicit goal of preserving ISS or iISS dissipation inequalities under scalings. A study in [36] focused on manipulation of ISS dissipation inequalities by scalings. However, [36] only considered unbounded decay rates for which the above violation of preservation never occurs.

One of the most useful tools in the ISS framework is the ISS small-gain theorem which is available in terms of both trajectories and Lyapunov functions [20,21]. There are fundamental obstacles to extending the ISS small-gain theorem to iISS systems [12,13]. In the trajectory-based formulation, the absence of instantaneous gain for large inputs and the incompatibility of signal spaces prevent us from applying contraction arguments globally. When it comes to Lyapunov functions, iISS systems which are not ISS do not admit the implication characterization of the form (2) that is effective for ISS systems [20]. Therefore, manipulating dissipation inequalities (1) of subsystems by scalings has been the central issue in establishing stability of interconnections involving iISS systems [12]. For example, as one sees in [8, Equation (119)] and [14, Equations (132), (133)], to construct a Lyapunov function of an interconnected system, an iISS/ISS dissipation inequality of each subsystem was transformed by a scaling into another iISS/ISS dissipation inequality. Scaling techniques used for solving such particular problems were not made available explicitly so that one could appreciate them in solving similar but different problems. Moreover, one may arrive at a better scaling technique if the essence of scalings is spotlighted as in [23].

This paper gives insights into scaling techniques and derives conditions under which an iISS/ISS dissipation inequality is transformed into another iISS/ISS dissipation inequality by a scaling. Explicit formulas of such scalings and resulting dissipation inequalities are shown based on an extended use of the classical division technique in [36], whereby the evaluation of a dissipation inequality is divided into two cases. In addition to limitations of scalings, useful flexibility of scalings in dealing with interconnected systems is demonstrated. This paper also clarifies relationships between the extended division technique proposed in Sect. 4.1 and the application of the Legendre–Fenchel transform proposed in [23]. The transform approach is modified in this paper to effectively cover iISS systems which are not ISS. These novel results on scalings allow us to propose a method to reduce the complexity of composite Lyapunov functions of interconnected systems whose subsystems are not necessarily ISS.

[2] in the sense of [37, Definition 2.2], i.e., an implication-form characterization

A preliminary result on the Legendre–Fenchel transform was reported in [16] which deals with strictly ISS systems. The conference paper [16] only sketched out possible complexity reduction of Lyapunov functions of interconnected systems consisting of ISS subsystems. The present paper not only effectively covers iISS systems, but also provides proofs and precise procedures which are not given in [16]. Furthermore, other results based on the extended division technique are elaborated and improved from [16].

2 iISS and ISS dissipation inequalities

Consider the system

$$\dot{x} = f(x, w) \tag{5}$$

whose state $x(t)$ evolves in Euclidean space $\mathbb{R}^N$. The input w is any measurable and locally essentially bounded function $w : \mathbb{R}_+ := [0, \infty) \to \mathbb{R}^M$. Suppose that Eq. (5) admits a unique maximally defined solution for each $x(0) \in \mathbb{R}^N$, i.e., assuming local Lipschitzness of $f : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N$ is sufficient. We make use of the standard symbols. The Euclidean norm of the space $\mathbb{R}^N$ is denoted by $|\cdot|$, and $\langle\cdot,\cdot\rangle$ denotes the inner product. A continuous function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{P}$ and one writes $\gamma \in \mathcal{P}$ if $\gamma(s) > 0$ for all $s \in \mathbb{R}_+\backslash\{0\}$, and $\gamma(0) = 0$. A class $\mathcal{P}$ function is said to be of class $\mathcal{K}$ if it is strictly increasing. It is of class $\mathcal{K}_\infty$ if, in addition, $\lim_{s\to\infty} \gamma(s) = \infty$ is satisfied. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if, for each fixed $t \in \mathbb{R}_+$, the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ and, for each fixed $s > 0$, $\beta(s, \cdot)$ is strictly decreasing and $\lim_{t\to\infty} \beta(s, t) = 0$. The symbol **Id** denotes the identity map on $\mathbb{R}_+$. For a constant $b \in \mathbb{R}^N$ and a map $\gamma : S \to \mathbb{R}^N$ with $S \subseteq \mathbb{R}$, $\gamma(s) \equiv b$ indicates $\gamma(s) = b$ for all $s \in S$, or it is simply denoted by $\gamma = b$. This paper also employs some useful notations for simplicity. For a continuous map $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$, the map $\gamma^{\ominus}$: $\overline{\mathbb{R}}_+ := [0, \infty] \to \overline{\mathbb{R}}_+$ is defined as $\gamma^{\ominus}(s) = \sup\{v \in \mathbb{R}_+ : s \geq \gamma(v)\}$. For example, given a function $\gamma \in \mathcal{K}$, by definition, $\gamma^{\ominus}(s) = \infty$ holds for all $s \geq \lim_{\tau\to\infty} \gamma(\tau)$, and $\gamma^{\ominus}(s) = \gamma^{-1}(s)$ elsewhere. For a continuous map $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\liminf_{l\to\infty} \gamma(l) = 0$, we have $\gamma^{\ominus}(s) = \infty$ for all $s \in \overline{\mathbb{R}}_+$. A non-decreasing map $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is extended to a map $\overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ by letting $\gamma(x) := \sup_{\{y\in\mathbb{R}_+: y\leq x\}} \gamma(y)$.

This paper focuses on the following properties.

Definition 1 ([35]) System (5) is said to be integral input-to-state stable (iISS) if there exist $\chi \in \mathcal{K}_\infty$, $\beta \in \mathcal{KL}$ and $\mu \in \mathcal{K}$ such that, for any initial condition $x(0) \in \mathbb{R}^N$ and any measurable, locally essentially bounded input $w : \mathbb{R}_+ \to \mathbb{R}^M$, the corresponding solution satisfies

$$\chi(|x(t)|) \leq \beta(|x(0)|, t) + \int_0^t \mu(|w(\tau)|)\mathrm{d}\tau, \quad \forall t \in \mathbb{R}_+. \tag{6}$$

Definition 2 ([34]) System (5) is said to be input-to-state stable (ISS) if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that, for any initial condition $x(0) \in \mathbb{R}^N$ and any measurable, locally essentially bounded input $w : \mathbb{R}_+ \to \mathbb{R}^M$, the corresponding solution satisfies

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma\left(\sup_{\tau\in[0,t)} |w(\tau)|\right), \quad \forall t \in \mathbb{R}_+. \tag{7}$$

Each of (6) and (7) implies global asymptotic stability (GAS) of $x = 0$ when $w(t) \equiv 0$. It is known that an ISS system is always iISS [2,35]. The following terminology is adopted in this paper.

Definition 3 A continuously differentiable function $V : \mathbb{R}^N \to \mathbb{R}_+$ is said to be an iISS Lyapunov function if there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$, $\alpha \in \mathcal{P}$ and $\sigma \in \mathcal{K}$ such that

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|), \quad \forall x \in \mathbb{R}^N \tag{8}$$

$$\langle \nabla V(x), f(x, w)\rangle \leq -\alpha(V(x)) + \sigma(|w|), \quad \forall x \in \mathbb{R}^N,\ w \in \mathbb{R}^M. \tag{9}$$

Definition 4 A continuously differentiable function $V : \mathbb{R}^N \to \mathbb{R}_+$ is said to be an ISS Lyapunov function if there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$, $\alpha \in \mathcal{K}$ and $\sigma \in \mathcal{K}$ satisfying (8), (9) and

$$\lim_{s\to\infty} \alpha(s) \geq \lim_{s\to\infty} \sigma(s). \tag{10}$$

It is known [2,37] that the existence of an iISS (resp. ISS) Lyapunov function is necessary as well as sufficient for system (5) to be iISS (resp. ISS). Equation (9) is often called a dissipation inequality. Since it establishes iISS, Eq. (9) is referred to as an iISS dissipation inequality. Property (10) is sufficient to determine if an iISS system is ISS and, hence, when (10) is satisfied (9), is referred to as an ISS dissipation inequality. A popular definition of an ISS Lyapunov function is based on a so-called implication-form characterization (e.g., [37, Definition 2.2]) instead of the dissipation-form (9). However, the existence of an ISS Lyapunov function of one form implies and is implied by the existence of the other form.

This paper uses the following terminology.

Definition 5 ([23]) A function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a scaling if it is of class $\mathcal{K}_\infty$, continuously differentiable, and satisfies $\mu'(s) > 0$ for all $s \in (0, \infty)$.

For clear presentation of ideas, throughout this paper, the derivative μ' is decomposed into a constant component $b \geq 0$ and the remaining component; that is, for $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ continuous,

$$\mu'(s) = b + \lambda(s), \quad \forall s \in \mathbb{R}_+, \text{ and} \tag{11}$$

$$\mu'(s) \not\equiv b \;\Rightarrow\; \left\{\mu'(s) > b,\ \forall s \in (0, \infty)\right\}. \tag{12}$$

Note that property (12) implies $\lambda(s) > 0$ for all $s \in (0, \infty)$ unless $\lambda(s) \equiv 0$. The decomposition fulfilling (11) and (12) is assumed throughout.

Consider the transformation of V in (9) by

$$W(x) = \mu(V(x)). \tag{13}$$

The scaling (13) applied to (9) gives

$$\begin{aligned}\langle \nabla W(x), f(x,w)\rangle &\le \mu'(V(x))\,[-\alpha(V(x))+\sigma(|w|)]\\ &= -\lambda(V(x))\alpha(V(x))+\lambda(V(x))\sigma(|w|)\\ &\quad - b\alpha(V(x))+b\sigma(|w|),\end{aligned} \tag{14}$$

for all $x \in \mathbb{R}^N,\ w \in \mathbb{R}^M$. The scaling also replaces (8) with $\underline{\alpha}_W(|x|) \le W(x) \le \overline{\alpha}_W(|x|)$ where $\underline{\alpha}_W := \mu \circ \underline{\alpha}$ and $\overline{\alpha}_W := \mu \circ \overline{\alpha}$. The effect of the bias b in (14) is trivial,[3] while the utilization and understanding of the effect of the function $\lambda(\cdot)$ is the key.

The iISS dissipation inequality (9) with a given pair $\alpha \in \mathcal{P}$ and $\sigma \in \mathcal{K}$ is said to be (qualitatively) preserved under a scaling μ if there exist $\hat{\alpha} \in \mathcal{P}$ and $\hat{\sigma} \in \mathcal{K}$ such that, for any iISS Lyapunov function V as in Definition 3, the scaled function W defined by (13) satisfies

$$\langle \nabla W(x), f(x,w)\rangle \le -\hat{\alpha}(W(x))+\hat{\sigma}(|w|),\ \forall x \in \mathbb{R}^N,\ w \in \mathbb{R}^M. \tag{15}$$

In a similar manner, the ISS dissipation inequality (9) with a given pair $\alpha, \sigma \in \mathcal{K}$ satisfying (10) is said to be (qualitatively) preserved under a scaling μ if there exist $\hat{\alpha}, \hat{\sigma} \in \mathcal{K}$ such that, for any ISS Lyapunov function V as in Definition 4, the scaled function W defined by (13) satisfies (15) and $\lim_{s\to\infty} \hat{\alpha}(s) \ge \lim_{s\to\infty} \hat{\sigma}(s)$. In the rest of this paper, the adverb "qualitatively" is omitted for the sake of brevity.

Remark 1 That property (10) implies ISS is a straightforward consequence of the results in [37], and it has often appeared in the literature such as [8,11,12] and also attempted in [32]. The fact has been proved implicitly in dealing with more involved problems such as [14,18,29]. The proof is again employed as a part of Theorem 3 in this paper. See Footnote 4 associated with Theorem 3.

Remark 2 Instead of (9), it is also popular to define iISS and ISS Lyapunov functions with

$$\langle \nabla V(x), f(x,w)\rangle \le -\alpha(|x|)+\sigma(|w|). \tag{16}$$

It is standard practice and easy to interchange (9) and (16) in results and tools around iISS and ISS Lyapunov functions by making use of (8). In the case of $\alpha \in \mathcal{K}$, one possible procedure is to interchange (9) and (16) by simply $-\alpha(V(x)) \le -\alpha(\underline{\alpha}(|x|))$ and $-\alpha(|x|) \le -\alpha(\overline{\alpha}^{-1}(V(x)))$. Property (10) remains the same under these interchanges.

[3] The use of $\mu'(s) = b$ is sufficient for verifying stability of interconnections of components admitting linear gains [5].

3 A necessary condition and a coarse estimate for scaling iISS dissipation inequalities

When system (5) is not ISS, for the scaled function $W(x)$ given by (13) to satisfy the iISS dissipation inequality, the scaling μ cannot grow too rapidly. More precisely, we prove the following in Appendix 1:

Theorem 1 *Let $V : \mathbb{R}^N \to \mathbb{R}_+$ be a continuous function satisfying (8). Suppose that $\alpha \in \mathcal{P}$ and $\sigma \in \mathcal{K}$ satisfy*

$$\liminf_{s\to\infty} \alpha(s) < \lim_{s\to\infty} \sigma(s). \tag{17}$$

If there exists a scaling $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\mu'(V(x))\,[-\alpha(V(x)) + \sigma(|w|)] \le -\hat{\alpha}(W(x)) + \hat{\sigma}(|w|),\ \forall x \in \mathbb{R}^N, w \in \mathbb{R}^M \tag{18}$$

holds with some $\hat{\alpha} \in \mathcal{P}$ and $\hat{\sigma} \in \mathcal{K}$, then

$$\liminf_{s\to\infty} \mu'(s) < \infty. \tag{19}$$

Furthermore, if $\lim\limits_{s\to\infty} \alpha(s)$ exists, then

$$\limsup_{s\to\infty} \mu'(s) < \infty. \tag{20}$$

Recall that (17) holds unless system (5) is ISS. Thus, in the case of scalings with unbounded derivative, the iISS dissipation inequality (9) with respect to V does not yield another iISS dissipation inequality with respect to the scaled function $\mu(V)$ unless system (5) is ISS. To allow unbounded μ' and preserve iISS dissipation inequalities, systems are required to be ISS. On the contrary, a scaling with bounded μ' always guarantees that any iISS dissipation inequality in V results in another iISS dissipation inequality in the scaled function $\mu(V)$. Indeed, the following holds obviously and is used or stated widely (e.g., [8,13,23]).

Proposition 1 *The iISS dissipation inequality (9) with a pair $\alpha \in \mathcal{P}$ and $\sigma \in \mathcal{K}$ is preserved under a scaling $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ if (20) holds.*

In fact, property (20) yields (15) with

$$\hat{\sigma}(s) = U\sigma(s), \quad U := \limsup_{s\to\infty} \mu'(s). \tag{21}$$

However, this estimation is very coarse and not satisfactory in many applications unless the scaling μ is linear. To make use of nonlinear scalings effectively, the rest of this paper focuses on ways to avoid such conservativeness.

4 Preservation of iISS and ISS dissipation inequalities

4.1 Extended use of changing supply technique [36]

The following theorem is proved in Appendix 2 on the basis of techniques used in [8,14].

Theorem 2 *Consider $\alpha \in \mathcal{P}$ and $\sigma \in \mathcal{K}$. Suppose that $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is a scaling with decomposition (11), (12) whose derivative μ' is non-decreasing and satisfies the implication*

$$\liminf_{s\to\infty} \alpha(s) < \infty \;\Rightarrow\; \lim_{s\to\infty} \mu'(s) < \infty. \tag{22}$$

Then the iISS dissipation inequality (9) with the pair $\alpha \in \mathcal{P}$ and $\sigma \in \mathcal{K}$ is preserved under the scaling μ. Furthermore, a pair $\hat{\alpha} \in \mathcal{P}$ and $\hat{\sigma} \in \mathcal{K}$ satisfying (15) is given by

$$\hat{\alpha} = [(\mathbf{Id} - (\mathbf{Id}+\omega)^{-1})\circ\tilde{\alpha}\circ\mu^{-1}][\lambda\circ\mu^{-1}] + b\alpha\circ\mu^{-1} \tag{23a}$$
$$\hat{\sigma} = [\mu'\circ\tilde{\alpha}^{\ominus}\circ(\mathbf{Id}+\omega)\circ\sigma]\sigma, \tag{23b}$$

where $\tilde{\alpha}, \omega : \mathbb{R}_+ \to \mathbb{R}_+$ are any functions satisfying

$$\liminf_{l\to\infty} \alpha(l) = 0 \;\Rightarrow\; \tilde{\alpha} = \alpha \tag{24}$$
$$\liminf_{l\to\infty} \alpha(l) > 0 \;\Rightarrow\; \{\tilde{\alpha} \in \mathcal{K},\quad \tilde{\alpha}(s) \le \alpha(s),\ \forall s \in \mathbb{R}_+\} \tag{25}$$
$$\mathbf{Id} + \omega \in \mathcal{K}_\infty,\quad \omega(\tilde{\alpha}(s)) > 0,\ s \in (0,\infty). \tag{26}$$

Moreover, $\hat{\alpha} \in \mathcal{K}$ (resp. $\hat{\alpha} \in \mathcal{K}_\infty$) holds if $\omega, \tilde{\alpha}, \alpha \in \mathcal{K}$ (resp. $\omega, \tilde{\alpha}, \alpha \in \mathcal{K}_\infty$). Finally, in the case of $\lim_{s\to\infty} \mu'(s) < \infty$, property (15) is also satisfied with $\hat{\alpha} \in \mathcal{P}$ and $\hat{\sigma} \in \mathcal{K}$ defined by taking the limit of (23) as $\Omega \to \infty$ for each $s \in \mathbb{R}_+$, where

$$\omega(s) = \Omega s, \qquad \forall s \in \mathbb{R}_+ \tag{27}$$

replaces (26).

The proof of Theorem 2 given in Appendix 2 naturally extends the classical division technique [36] which assumes $\alpha \in \mathcal{K}_\infty$. The proof of Theorem 2, meanwhile, does not rely on $\alpha \in \mathcal{K}_\infty$ in treating the two cases $\tilde{\alpha}(V) \ge (\mathbf{Id}+\omega)\circ\sigma(|w|)$ and $\tilde{\alpha}(V) < (\mathbf{Id}+\omega)\circ\sigma(|w|)$ separately to evaluate (14). The formula (23) covers not only $\alpha \in \mathcal{K}\backslash\mathcal{K}_\infty$, but also $\alpha \in \mathcal{P}\backslash\mathcal{K}$. The final claim using (27) in Theorem 2 demonstrates that Proposition 1 is obtained as a special case of (23) and can be interpreted in terms of the simple choice (27). The larger the Ω is, the larger both $\hat{\alpha}$ and $\hat{\sigma}$ become. As stated in Proposition 1, taking $\Omega \to \infty$ results in a conservative estimate of (15). Letting $\Omega \to \infty$ completely ignores any qualitative change induced on $\hat{\sigma}$ by nonlinear scalings.

Remark 3 Theorem 2 employs (22) to align with the developments in Sect. 4.2 below. However, it can be verified from the proof of Theorem 2 that (22) can be replaced by the milder condition

$$\lambda \circ \tilde{\alpha}^{\ominus} \circ (\mathbf{Id} + \omega) \circ \sigma(s) < \infty, \quad \forall s \in \mathbb{R}_+. \tag{28}$$

This milder condition is employed in [8,12,14] and Proposition 6 of this paper to deal with interconnected systems. Condition (28) is also used in Theorems 3 and 5.

Remark 4 If we fix $\omega(s) = (\tau - 1)s$ for all $s \in \mathbb{R}_+$ and some constant $\tau > 1$, then (23) simplifies to

$$\hat{\alpha} = \left[\left(1 - \frac{1}{\tau}\right)\tilde{\alpha} \circ \mu^{-1}\right][\lambda \circ \mu^{-1}] + b\alpha \circ \mu^{-1} \tag{29a}$$

$$\hat{\sigma} = [\mu' \circ \tilde{\alpha}^{\ominus} \circ \tau\sigma]\sigma. \tag{29b}$$

In the case of $\lim_{l\to\infty} \mu'(l) < \infty$, the use of (27) in (23) with $\Omega \to \infty$ is equivalent to taking the limit of $\hat{\alpha}(s)$ and $\hat{\sigma}(s)$ in (29) as $\tau \to \infty$ for each $s \in \mathbb{R}_+$. Employing the general function $\omega(s)$ instead of $(\tau - 1)s$ allows (10) to be satisfied with equality in addressing preservation of ISS dissipation inequality. For details, compare Theorem 3 with Corollary 1 stated below.

Similar to Theorem 2, preservation of ISS dissipation inequalities can be addressed as in the next theorem. Note that $\alpha \in \mathcal{K}$ can be assumed without any loss of generality since the right-hand side of (14) with $\liminf_{l\to\infty} \alpha(l) = 0$ never yields an ISS dissipation inequality.

Theorem 3 *Consider $\alpha, \sigma \in \mathcal{K}$ satisfying (10). Suppose that $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is a scaling with decomposition (11), (12) whose derivative μ' is non-decreasing and satisfies*

$$\liminf_{s\to\infty}\left(\frac{\alpha(s)}{(\mathbf{Id}+\omega)^{-1} \circ \alpha(s)} - 1\right)\lambda(s) \geq \lim_{s\to\infty} \lambda \circ \alpha^{\ominus} \circ (\mathbf{Id} + \omega) \circ \sigma(s) \tag{30}$$

for a continuous function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\mathbf{Id} + \omega \in \mathcal{K}_\infty, \quad \omega(\alpha(s)) > 0, \; s \in (0, \infty) \tag{31}$$

$$\lim_{s\to\infty} \alpha(s) \geq \lim_{s\to\infty} (\mathbf{Id} + \omega) \circ \sigma(s). \tag{32}$$

Then the ISS dissipation inequality (9) with the pair $\alpha, \sigma \in \mathcal{K}$ is preserved under the scaling μ. Furthermore, a pair $\hat{\alpha}, \hat{\sigma} \in \mathcal{K}$ satisfying (15) and

$$\lim_{s\to\infty} \hat{\alpha}(s) \geq \lim_{s\to\infty} \hat{\sigma}(s) \tag{33}$$

is given by

$$\hat{\alpha}(s) = k(s)(\eta(s) + b\alpha \circ \mu^{-1}(s)), \ \forall s \in \mathbb{R}_+ \tag{34a}$$
$$\hat{\sigma} = [\mu' \circ \alpha^{\ominus} \circ (\mathbf{Id}+\omega) \circ \sigma]\sigma, \tag{34b}$$

where $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ *is given by*

$$\eta = [(\mathbf{Id} - (\mathbf{Id} + \omega)^{-1}) \circ \alpha \circ \mu^{-1}][\lambda \circ \mu^{-1}] \in \mathcal{P}, \tag{35}$$

and $k : \mathbb{R}_+ \to \mathbb{R}_+$ *is any continuous function satisfying*

$$\limsup_{l\to\infty} k(l) = 1, \quad 0 < k(s) \le 1, \ \forall s \in (0, \infty) \tag{36}$$
$$\hat{\alpha} \text{ is strictly increasing.} \tag{37}$$

Finally, in the case of $\lim_{s\to\infty} \mu'(s) < \infty$*, property* (15) *is also satisfied with* $\hat{\alpha}$*,* $\hat{\sigma} \in \mathcal{K}$ *defined by taking the limit of* (34) *as* $\Omega \to \infty$ *for each* $s \in \mathbb{R}_+$*, where* (27) *replaces the pair of* (31) *and* (32)*.*

See Appendix 3 for the proof of Theorem 3. It is stressed that if we have (10), then there always exists a continuous function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (31) and (32).[4] It is also stressed that there always exists a continuous function $k : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (36) and (37) since $\eta \in \mathcal{P}$ holds and $\liminf_{l\to\infty} \eta(l) > 0$ is guaranteed by (30). Thus, when focusing strictly on ISS dissipation inequalities, property (30) is the only constraint on μ (through λ) in Theorem 3. Note that (30) holds whenever (27) is used with $\Omega \to \infty$.

If (10) is replaced by

$$\lim_{s\to\infty} \alpha(s) = \infty \ \text{ or } \ \lim_{s\to\infty} \alpha(s) > \lim_{s\to\infty} \sigma(s), \tag{38}$$

which removes the equality from (10), we can let $\omega(s) = (\tau - 1)s$ by picking $\tau > 1$ without any loss of generality, and (31) is satisfied. Condition (32) can be satisfied with a sufficiently small $\tau > 1$ and hence the only constraint on μ in Theorem 3, as given by (30), reduces to a simpler condition. This direct consequence of Theorem 3 is summarized as follows:

Corollary 1 *Consider* $\alpha, \sigma \in \mathcal{K}$ *such that there exists a constant* $\tau > 1$ *satisfying*

$$\lim_{s\to\infty} \alpha(s) \ge \tau \lim_{s\to\infty} \sigma(s). \tag{39}$$

Suppose that $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ *is a scaling with decomposition* (11)*,* (12) *whose derivative* μ' *is non-decreasing and satisfies*

$$\lim_{s\to\infty} (\tau - 1)\lambda(s) \ge \lim_{s\to\infty} \lambda \circ \alpha^{\ominus} \circ \tau\sigma(s). \tag{40}$$

[4] The condition $V(x) \ge \alpha^{\ominus} \circ (\mathbf{Id} + \omega) \circ \sigma(|w|)$ gives a hypothesizing clause of the implication-form characterization of ISS from (9).

Then the ISS dissipation inequality (9) *with the pair* $\alpha, \sigma \in \mathcal{K}$ *is preserved under the scaling* μ. *Furthermore, a pair* $\hat{\alpha}, \hat{\sigma} \in \mathcal{K}$ *satisfying* (15) *and* (33) *is given by*

$$\hat{\alpha} = \left[\left(1 - \frac{1}{\tau}\right)\alpha \circ \mu^{-1}\right][\lambda \circ \mu^{-1}] + b\alpha \circ \mu^{-1} \tag{41}$$

$$\hat{\sigma} = [\mu' \circ \alpha^{\ominus} \circ \tau\sigma]\sigma. \tag{42}$$

It is obvious that (40) is met whenever $\tau \geq 2$. Hence, we have the following.

Corollary 2 *Consider* $\alpha, \sigma \in \mathcal{K}$ *satisfying*

$$\lim_{s\to\infty} \alpha(s) \geq 2 \lim_{s\to\infty} \sigma(s). \tag{43}$$

Then the ISS dissipation inequality (9) *with the pair* $\alpha, \sigma \in \mathcal{K}$ *is preserved under the scaling* μ. *Furthermore, a pair* $\hat{\alpha}, \hat{\sigma} \in \mathcal{K}$ *satisfying* (15) *and* (33) *is given by* (41) *and* (42)*, respectively, with* $\tau = 2$.

As discussed above, Theorem 3 and Corollary 1 require only (30) and (40), respectively, to guarantee preservation of ISS dissipation inequalities. The next proposition, proved in Appendix 4, shows that (40) can always be satisfied if the scalings μ are restricted appropriately.

Proposition 2 *Suppose* $\alpha, \sigma \in \mathcal{K}$. *Let* $c > 1$ *be such that*

$$\lim_{s\to\infty} \alpha(s) \geq c \lim_{s\to\infty} \sigma(s). \tag{44}$$

Then, there exists $\varphi \geq 0$ *such that*

$$\exists \tau \in (1, c) \quad \text{s.t.} \quad \left(\frac{\tau}{c}\right)^{\varphi} \leq \tau - 1. \tag{45}$$

Furthermore, if a scaling μ *with decomposition* (11)*,* (12) *satisfies*

$$\lambda(s) = \alpha(s)^{\varphi}\beta(s), \quad \forall s \in \mathbb{R}_+ \tag{46}$$

for some non-decreasing continuous function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$, *then properties* (39) *and* (40) *hold.*

Interestingly, the form (46) subsumes scalings used in previous results [12,15] on interconnected systems. Indeed, such knowledge allows the authors to arrive at Proposition 2. For the choice (46) of scalings, the constraint (40) in Corollary 2 reduces to (38) which has already been assumed for the dissipation inequality (9) to be ISS. To address (30) instead of (40), we can reduce (38) into the precise ISS condition (10). Another class of $\mu \in \mathcal{K}_\infty$, i.e., another formula of λ extending (46), fulfilling (30) can be obtained for a general ω instead of $\omega(s) = (\tau - 1)s$ on the basis of the technique employed in [14], although it is more complicated than (46).

4.2 Application of the Legendre–Fenchel transform

This section briefly reviews and discusses a technique proposed in [23].[5] Then, a modification is introduced for reducing conservativeness when the dissipation inequality does not guarantee ISS. The Legendre–Fenchel transform is defined for a continuously differentiable function $\kappa \in \mathcal{K}_\infty$ satisfying $\kappa' \in \mathcal{K}_\infty$ as

$$\ell\kappa(s) := \int_0^s (\kappa')^{-1}(l)\mathrm{d}l, \quad \forall s \in \mathbb{R}_+. \tag{47}$$

This definition is equivalent to $\ell\kappa(s) = s(\kappa')^{-1}(s) - \kappa \circ (\kappa')^{-1}(s)$ (see [24, Lemma 15.i]). By definition, $\ell\kappa \in \mathcal{K}_\infty$. For arbitrary $s, t \in \mathbb{R}_+$, the following general version of Young's inequality is presented in [31]:

$$st \le \kappa(s) + \ell\kappa(t). \tag{48}$$

Applying (48) to $\lambda(V(x))\sigma(|w|)$ in (14) with $s = \lambda(V(x))$ and $t = \sigma(|w|)$ in the case $\mu' \neq b$ leads directly to the following.

Proposition 3 *Consider $\alpha \in \mathcal{P}$ and $\sigma \in \mathcal{K}$. Suppose that $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is a scaling with decomposition (11), (12) whose derivative μ' is non-decreasing. If there exists a continuously differentiable function $\kappa \in \mathcal{K}_\infty$ such that $\kappa' \in \mathcal{K}_\infty$ and*

$$\hat{\alpha} := [\mu' \circ \mu^{-1}][\alpha \circ \mu^{-1}] - \kappa \circ \lambda \circ \mu^{-1} \in \mathcal{P} \tag{49}$$

are satisfied, then the iISS dissipation inequality (9) with the pair $\alpha \in \mathcal{P}$ and $\sigma \in \mathcal{K}$ is preserved under the scaling μ. Furthermore, a pair $\hat{\alpha} \in \mathcal{P}$ and $\hat{\sigma} \in \mathcal{K}$ satisfying (15) is given by (49) and

$$\hat{\sigma} := \begin{cases} b\sigma, & \text{if } \ \mu'(s) \equiv b \\ \ell\kappa \circ \sigma + b\sigma, & \text{otherwise.} \end{cases} \tag{50}$$

The claim of Proposition 3 in the case $\mu'(s) \not\equiv b$ follows directly from

$$\lambda(V(x))\sigma(|w|) \le \kappa \circ \lambda(V(x)) + \ell\kappa \circ \sigma(|w|) \tag{51}$$

in (14). The claim of Proposition 3 in the case of $\mu'(s) \equiv b$, which does not rely on (47), is trivial. Although Proposition 3 does not state this explicitly, as shown in the following proposition and proved in Appendix 5, (49) requires (22) and (52).

Proposition 4 *Consider a scaling μ whose derivative μ' is non-decreasing. If (49) is satisfied for $\alpha \in \mathcal{P}$ and $\kappa, \kappa' \in \mathcal{K}_\infty$, then property (22) and the implication*

$$\liminf_{l\to\infty} \alpha(l) = 0 \ \Rightarrow \ \mu'(s) \equiv b \tag{52}$$

[5] It is assumed in [23] that μ' is strictly increasing. However, as demonstrated in this section, μ' can be taken to be non-decreasing.

hold true.

Property (52) implies that if (49) holds then $\alpha \in \mathcal{K}$ (or that α is bounded from below by a class $\mathcal{K}$ function) unless μ' is constant. Proposition 4 suggests that Proposition 3 is more demanding than Theorem 2 since it requires (52) and an extra condition of the existence of $\kappa \in \mathcal{K}_\infty$ satisfying $\kappa' \in \mathcal{K}_\infty$ and (49). Nevertheless, it will be demonstrated later in Sect. 4.3 that a desired function $\kappa \in \mathcal{K}_\infty$ exists under particular assumptions.

To appreciate Proposition 3, in particular the bound (50), note that Proposition 4 indicates that if (49) holds with $\alpha \in \mathcal{P}$ and $\kappa, \kappa' \in \mathcal{K}_\infty$, then

$$\lim_{s\to\infty} \alpha(s) < \infty \;\Rightarrow \left\{ \begin{array}{l} \exists m > 0 \quad \text{s.t.} \\ \lambda(V(x))\sigma(|w|) \le m\sigma(|w|),\ \forall x \in \mathbb{R}^N, w \in \mathbb{R}^M \end{array} \right\}. \tag{53}$$

Consequently, in bounding (14), the cross-term $\mu'(V(x))\sigma(|w|)$ can be directly bounded from above by $(b+m)\sigma(|w|)$ in the case of $\lim_{s\to\infty} \alpha(s) < \infty$.

However, when considering the bound (50), observe that it is not always possible to find an $\bar{m} > 0$ such that

$$\ell\kappa \circ \sigma(|w|) \le \overline{m}\sigma(|w|),\ \forall w \in \mathbb{R}^M. \tag{54}$$

Indeed, take any $\sigma \in \mathcal{K}_\infty$ and take $\kappa \in \mathcal{K}_\infty$ with $\kappa' \in \mathcal{K}_\infty$ as

$$\kappa(s) = c_p s^p + c_{p-1}s^{p-1} + \cdots + c_2 s^2 \tag{55}$$

where $p \ge 3$ and $c_i \ge 0$ for $i \in \{2, \ldots, p\}$ with at least one $c_i > 0$. Using [24, Lemma 15.i], we see that (54) implies

$$\ell\kappa \circ \sigma(s) = \sigma(s)(\kappa')^{-1}(\sigma(s)) - \kappa \circ (\kappa')^{-1}(\sigma(s)) \le \bar{m}\sigma(s)$$

which, in turn, implies $\kappa'(s)s - \kappa(s) \le \bar{m}\kappa'(s)$ for all $s \in \mathbb{R}_+$, due to $\sigma^{-1} \circ \kappa' \in \mathcal{K}_\infty$. However, with κ as in (55), this last inequality never holds true since $p \ge 3$. Hence, for any $\sigma \in \mathcal{K}_\infty$, the bound (54) cannot be satisfied with a polynomial $\kappa \in \mathcal{K}_\infty$ and, consequently, (50) is a conservative bound in the case of $\lim_{s\to\infty} \alpha(s) < \infty$, for instance, the non-ISS case.

The following corollary removes this conservativeness.

Corollary 3 *Consider $\alpha \in \mathcal{P}$ and $\sigma \in \mathcal{K}$. Suppose that $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is a scaling with decomposition (11), (12) whose derivative μ' is non-decreasing. If there exists a continuously differentiable function $\kappa \in \mathcal{K}_\infty$ such that $\kappa' \in \mathcal{K}_\infty$ and (49) is satisfied, then the iISS dissipation inequality (9) with the pair $\alpha \in \mathcal{P}$ and $\sigma \in \mathcal{K}$ is preserved under the scaling μ as in (15) with $\hat{\alpha} \in \mathcal{P}$ from (49) and*

$$\hat{\sigma}(s) := \begin{cases} b\sigma(s), & \text{if } \mu'(s) \equiv b \\ \min\left\{\ell\kappa \circ \sigma(s),\ \lim\limits_{l\to\infty} \lambda(l)\sigma(s)\right\} + b\sigma, & \text{otherwise.} \end{cases} \tag{56}$$

Obviously, it is possible to addresses preservation of ISS dissipation inequalities in the spirit of Corollary 3 as in the case of Proposition 3. The next proposition states this precisely.

Proposition 5 *Consider $\alpha, \sigma \in \mathcal{K}$ satisfying (10). Suppose that $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is a scaling whose derivative μ' is non-decreasing. If there exists a continuously differentiable function $\kappa \in \mathcal{K}_\infty$ such that $\kappa' \in \mathcal{K}_\infty$, $\hat{\alpha} \in \mathcal{K}$ and (33) are satisfied for the functions $\hat{\alpha} \in \mathcal{K}$ from (49) and $\hat{\sigma} \in \mathcal{K}$ from (56), then the ISS dissipation inequality (9) with the pair $\alpha, \sigma \in \mathcal{K}$ is preserved under the scaling μ.*

The idea of Proposition 5 is that, for a given scaling, a sufficient condition for the scaling to preserve the ISS dissipation inequality can be seen as a search for an appropriate $\kappa \in \mathcal{K}_\infty$. Unfortunately, Propositions 3 and 5 and Corollary 3 do not provide explicit guidance on how to achieve $\hat{\alpha} \in \mathcal{P}$, $\hat{\alpha} \in \mathcal{K}$, or (33). However, a notable advantage of Propositions 3, 5 and Corollary 3 is the straightforwardness of their proofs. In Sect. 5.3, this paper shows a way to make use of Corollary 3 and Proposition 5 in combination with the tools presented in Sect. 4.1 to find an appropriate $\kappa \in \mathcal{K}_\infty$.

Remark 5 Corollary 3 can be stated with another pair of simpler, but more conservative $\hat{\alpha}, \hat{\sigma} \in \mathcal{K}$ that are not directly defined using the Legendre–Fenchel transform. The assumption $\kappa, \kappa' \in \mathcal{K}_\infty$ directly yields that $\kappa(s) \leq s\kappa'(s)$ and $\kappa \circ (\kappa')^{-1}(s) \geq 0$ hold for all $s \in \mathbb{R}_+$. Thus, using the property $\ell\kappa(s) = s(\kappa')^{-1}(s) - \kappa \circ (\kappa')^{-1}(s)$ of the Legendre–Fenchel transform, in the case of $\mu'(s) \not\equiv b$, an upper bound for (51) is given by

$$\kappa \circ \lambda(V(x)) + \ell\kappa \circ \sigma(|w|) \leq \lambda(V(x))\kappa'(\lambda(V(x))) + \sigma(|w|)(\kappa')^{-1}(\sigma(|w|)) \quad (57)$$

for all $x \in \mathbb{R}^N$ and $w \in \mathbb{R}^M$. Therefore, the functions

$$\hat{\alpha}_Y := [\mu' \circ \mu^{-1}][\alpha \circ \mu^{-1}] - [\lambda \circ \mu^{-1}][\kappa' \circ \lambda \circ \mu^{-1}] \in \mathcal{P} \quad (58)$$

$$\hat{\sigma}_Y(s) := \begin{cases} b\sigma(s), & \text{if } \mu'(s) \equiv b \\ \min\left\{\sigma(s)[(\kappa')^{-1} \circ \sigma(s)],\ \lim_{l\to\infty} \lambda(l)\sigma(s)\right\} + b\sigma(s), & \text{otherwise.} \end{cases} \quad (59)$$

can replace $\hat{\alpha}$ in (49) and $\hat{\sigma}$ in (56), respectively, in the statement of Corollary 3. Note that these functions do not depend on κ or $\ell\kappa$, but only on κ'. Furthermore, note that these functions satisfy

$$\hat{\alpha}(s) \text{ in } (49) \geq \hat{\alpha}_Y(s), \quad \hat{\sigma}(s) \text{ in } (56) \leq \hat{\sigma}_Y(s), \quad \forall s \in \mathbb{R}_+. \quad (60)$$

4.3 Relationship between the two approaches

This subsection demonstrates that under appropriate assumptions, the Legendre–Fenchel transform can be used to prove the preservation of an iISS/ISS dissipation inequality. For this purpose, we introduce the following notation:

$$\hat{\alpha}_{D,\tau} = \{\hat{\alpha} \text{ given by (41)}\}, \quad \hat{\sigma}_{D,\tau} = \{\hat{\sigma} \text{ given by (42)}\}$$
$$\hat{\alpha}_{L} = \{\hat{\alpha} \text{ given by (49)}\}, \quad \hat{\sigma}_{L} = \{\hat{\sigma} \text{ given by (56)}\}.$$

Recall that the pair (29a), (29b) can be represented by the pair (41), (42) when $\alpha \in \mathcal{K}$ is assumed. The following, proved in Appendix 6, is the main result in this subsection.

Theorem 4 *Consider $\alpha, \sigma \in \mathcal{K}$. Suppose that $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is a scaling whose derivative μ' is non-decreasing and satisfies $\lambda \in \mathcal{K}$ and the implication (22). Then for each $\tau > 1$ and for each $R > 0$, there exists a continuously differentiable function $\kappa \in \mathcal{K}_\infty$ such that $\kappa' \in \mathcal{K}_\infty$, $\hat{\alpha}_L$, $\hat{\sigma}_L \in \mathcal{K}$ and*

$$\hat{\alpha}_L(s) \geq \hat{\alpha}_{D,\tau}(s), \quad \forall s \in \mathbb{R}_+ \tag{61}$$
$$\hat{\sigma}_L(s) \leq \hat{\sigma}_{D,\tau}(s), \quad \forall s \in [0, R) \tag{62}$$
$$\hat{\sigma}_L(s) \leq \lim_{l\to\infty} \hat{\sigma}_{D,\tau}(l), \quad \forall s \in \mathbb{R}_+. \tag{63}$$

Furthermore, if

$$\lim_{s\to\infty} \mu'(s) < \infty \;\Rightarrow\; \lim_{s\to\infty} \alpha(s) < \infty \tag{64}$$

is satisfied, then (62) can be satisfied with $R = \infty$ simultaneously.

With the help of Theorem 2 and Remark 4, it is verified from (61)–(63) established by Theorem 4 that when (22) holds, the modified bound using the Legendre–Fenchel transform proposed in Corollary 3 can establish the preservation of an iISS dissipation inequality under a scaling μ. Furthermore, according to Corollary 1, when (39) and (40) are satisfied additionally, the modified bound using the Legendre–Fenchel transform can achieve the preservation of an ISS dissipation inequality under a scaling μ. As demonstrated in Sect. 4.2, the minimum in (56) is crucial for covering iISS systems which are not ISS. Theorem 4 indicates that the modified bound using the Legendre–Fenchel transform has the potential to give a tighter iISS/ISS dissipation inequality, and to get rid of the condition (40) in preserving ISS. Section 5.3 proposes a way to exploit this point.

The following presents an explicit formula of κ that can give $\ell\kappa$ analytically, which is obtained directly from the proof of Theorem 4 given in Appendix 6.

Corollary 4 *Consider $\alpha, \sigma \in \mathcal{K}$. Suppose that $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is a scaling whose derivative μ' is non-decreasing and satisfies $\lambda \in \mathcal{K}$ and (22). For $\tau > 1$ and $R > 0$, let*

$$\tilde{\lambda}(s) = \lambda(s) + q \max\{s - \alpha^{\ominus}(\tau\sigma(R)), 0\}, \quad \forall s \in \mathbb{R}_+, \tag{65}$$

where $q = 0$ *if* (64) *holds, otherwise* $q > 0$. *Let* $\tilde{L} := \lim_{l\to\infty} \tilde{\lambda}(l) \in \overline{\mathbb{R}}_+$, *If the continuous function* $\kappa : [0, \tilde{L}) \to \infty$ *defined by*

$$\kappa(s) = \frac{1}{\tau} s \left[\alpha \circ \tilde{\lambda}^{\ominus}(s)\right] \tag{66}$$

is continuously differentiable and satisfies

$$\lim_{s\to 0^+} \kappa'(s) = 0 \tag{67}$$

$$\kappa'(s) : \textit{strictly increasing}, \ \forall s \in [0, \tilde{L}), \tag{68}$$

$$\tilde{L} < \infty \ \Rightarrow \ \lim_{l\to \tilde{L}^-} \kappa'(l) < \infty, \tag{69}$$

then $\hat{\alpha}_L$ *and* $\hat{\sigma}_L$ *are of class* $\mathcal{K}$ *and satisfy* (61), (62), *and* (63). *Furthermore, if* (64) *is satisfied, then* (62) *is satisfied with* $R = \infty$ *simultaneously.*

The proof of Theorem 4 also gives the following corollary which provides an explicit formula for κ' instead of κ.

Corollary 5 *Consider* $\alpha, \sigma \in \mathcal{K}$. *Suppose that* $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ *is a scaling whose derivative* μ' *is non-decreasing and satisfies* $\lambda \in \mathcal{K}$ *and* (22). *For* $\tau > 1$ *and* $R > 0$, *let* $\tilde{\lambda}$ *be given by* (65), *where* $q = 0$ *if* (64) *holds, otherwise* $q > 0$. *Define*

$$\kappa'(s) = \frac{1}{\tau} \alpha \circ \tilde{\lambda}^{\ominus}(s), \quad \forall s \in [0, \tilde{L}), \tag{70}$$

where $\tilde{L} := \lim_{l\to\infty} \tilde{\lambda}(l) \in \overline{\mathbb{R}}_+$, *and let* $\kappa : [0, \tilde{L}) \to \mathbb{R}_+$ *be the antiderivative of* κ' *satisfying* $\kappa(0) = 0$. *Then* $\hat{\alpha}_L$ *and* $\hat{\sigma}_L$ *are of class* $\mathcal{K}$ *and satisfy* (61)–(63). *Furthermore, if* (64) *is satisfied, then* (62) *is satisfied with* $R = \infty$ *simultaneously.*

The formulas (66) and (70) specify $\kappa(s)$ and $\kappa'(s)$, respectively, for only $s \in [0, \tilde{L})$ instead of $s \in \mathbb{R}_+$ since (49) and (56) giving α_L and σ_L, respectively, do not require $\kappa(s)$ in the interval $[\tilde{L}, \infty)$. See Appendix 6 for details. To obtain α_L in (49) and σ_L in (56) based on Corollary 5, integration to obtain κ from κ' is needed. Computing an integral analytically is generally harder than computing a derivative analytically. At the price of (67), (68), and (69), Corollary 4 allows one to avoid computing κ from κ'. Notice that in contrast to α_L and σ_L, the conservative estimates α_Y and α_Y given in (58) and (59) do not require the integral κ.

Remark 6 Property (62) indicates that $\hat{\sigma}_L(s) > \hat{\sigma}_{D,\tau}(s)$ may hold for some $s \in [R, \infty)$ unless $R = \infty$. It is, however, important to note that $R > 0$ can be made arbitrarily large and (63) is always ensured. In addition, according to the proof of Theorem 4, specifically the components that yield Corollaries 4 and 5, it is guaranteed that the smaller the $q > 0$ is, the smaller $\hat{\sigma}_L(s)$ becomes in the interval $[R, \infty)$. Therefore, the function $\hat{\sigma}_L(s)$ can be made small to be arbitrarily close to $\hat{\sigma}_{D,\tau}(s)$ in the interval $s \in [R, \infty)$ by choosing sufficiently small $q > 0$.

Remark 7 When (39) and (64) are satisfied, the two functions κ presented in Corollaries 4 and 5 satisfy

$$\ell\kappa \circ \sigma(s) \le \lim_{l\to\infty} \lambda(l)\sigma(s), \quad \forall s \in \mathbb{R}_+. \tag{71}$$

In other words, in the case of (64), if the dissipation inequality (9) is ISS strictly in the sense of (38), modifying the Legendre–Fenchel transform is not necessary for the iISS and ISS preservations, and the two functions $\hat{\sigma}$ given in (50) and (56) become identical.

Remark 8 The assumption $\lambda \in \mathcal{K}$ implies $\mu'(s) \not\equiv b$. As the Legendre–Fenchel transform is neither utilized nor necessary in Proposition 3 when $\mu'(s) \equiv b$, this case is not considered in Theorem 4 and Corollaries 4 and 5.

Remark 9 As explained after Proposition 4, the relation investigated in this subsection requires $\alpha \in \mathcal{K}$. If a system admits (8) and (9) with $\alpha \in \mathcal{K}$, it is not only iISS, but also ISS with respect to small inputs, which is the notion introduced in [3].

5 Utilization for interconnections

5.1 iISS small-gain theorem via preservation of iISS dissipation inequalities

This subsection briefly illustrates how scalings have been utilized for addressing stability properties of interconnected systems by revisiting the Lyapunov approach to iISS interconnections originating from [8]. This revisit leads us to an interesting observation and a new approach in the subsequent subsections. Consider the interconnected system Σ described by

$$\Sigma : \begin{cases} \Sigma_1 : \dot{x}_1 = f_1(x_1, x_2, w_1) \\ \Sigma_2 : \dot{x}_2 = f_2(x_1, x_2, w_2), \end{cases} \tag{72}$$

where $x_i \in \mathbb{R}^{N_i}$ and $w_i \in \mathbb{R}^{M_i}$. The functions $f_i : \mathbb{R}^{N_1}\times\mathbb{R}^{N_2}\times\mathbb{R}^{M_i} \to \mathbb{R}^{N_i}$, $i = 1, 2$, are assumed to be locally Lipschitz. System (5) is obtained with

$$\begin{aligned} &x = [x_1^T, x_2^T]^T \in \mathbb{R}^N, \quad w = [w_1^T, w_2^T]^T \in \mathbb{R}^M \\ &f(x, w) = [f_1(x_1, x_2, w_1)^T, f_2(x_1, x_2, w_2)^T]^T \in \mathbb{R}^N, \end{aligned}$$

where $N = N_1 + N_2$ and $M = M_1 + M_2$. Instead of the differential equations (72), we prescribe the two subsystems in terms of iISS dissipation inequalities as follows:

Assumption 1 Subsystems Σ_i, $i = 1, 2$, in (72) admit the existence of continuously differentiable functions $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$ and $\underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty$, $\alpha_i \in \mathcal{K}$, $\sigma_i \in \mathcal{K}$, $\theta_i \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}_i(|x_i|) \le V_i(x_i) \le \overline{\alpha}_i(|x_i|),\ \forall x_i \in \mathbb{R}^{N_i},\ i = 1, 2 \tag{73}$$

$$\begin{aligned}&\langle \nabla V_i(x_i),\ f_i(x_1, x_2, w_i)\rangle \le -\alpha_i(V_i(x_i)) + \sigma_i(V_{3-i}(x_{3-i})) + \theta_i(|w_i|),\\ &\forall x_i \in \mathbb{R}^{N_i},\ x_{3-i} \in \mathbb{R}^{N_{3-i}},\ w_i \in \mathbb{R}^{M_i},\ i = 1, 2.\end{aligned} \tag{74}$$

Property (74) implies that each subsystem Σ_i is iISS with respect to input (x_{3-i}, w_i). Thus, $x_i = 0$ of each Σ_i is GAS for $x_{3-i}(t) \equiv 0$ and $w_i(t) \equiv 0$. Consider

$$W(x) = \mu_1(V_1(x_1)) + \mu_2(V_2(x_2)). \tag{75}$$

Then scalings μ_i that preserve iISS dissipation inequalities of individual subsystem Σ_i result in

$$\begin{aligned}\langle \nabla W(x),\ f(x, w)\rangle &\le \sum_{i=1}^{2} \mu_i'(V_i)\left[-(1-\delta)\alpha_i(V_i) + \sigma_i(V_{3-i})\right]\\ &\quad + \sum_{i=1}^{2} \mu_i'(V_i)\left[-\delta\alpha_i(V_i) + \theta_i(|w_i|)\right]\\ &\le \sum_{i=1}^{2}[-\hat{\alpha}_i(W_i) + \hat{\sigma}_{3-i}(W_i) + \hat{\theta}_i(|w_i|)]\\ &\le -\hat{\alpha}(W(x)) + \hat{\theta}(|w|)\end{aligned} \tag{76}$$

for $\delta \in (0, 1)$. One can take $\delta = 0$ in the case of $w = 0$. Here, the achievement of the last inequality is not obvious, and it is heavily dependent on the choice of estimates $\hat{\alpha}_i$ and $\hat{\sigma}_{3-i}$ of the scaled dissipation inequality of each Σ_i. By combining Theorem 2 and Remarks 3 and 4 judiciously under condition (77) stated below, it can be verified that the last inequality in (76) can be achieved with the estimates (29a), (29b). To confirm this assertion precisely for $w = 0$, a proof of the next proposition is given in Appendix 7.

Proposition 6 *Suppose that Assumption 1 is satisfied with the restriction $w_i = 0$, $i = 1, 2$. If there exists $c > 1$ such that*

$$\alpha_1^{\ominus} \circ c\sigma_1 \circ \alpha_2^{\ominus} \circ c\sigma_2(s) \le s, \quad \forall s \in \mathbb{R}_+ \tag{77}$$

holds, then the function $W : \mathbb{R}^N \to \mathbb{R}_+$ given by (75) with the scalings $\mu_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2$, satisfying

$$\mu_i'(s) = \alpha_i(s)^{\varphi}\sigma_{3-i}(s)^{\varphi+1},\ i = 1, 2 \tag{78}$$

is a Lyapunov function establishing global asymptotic stability of $x = 0$ of the interconnected system (72) with $w(t) \equiv 0$, where $\varphi \ge 0$ is any non-negative real number satisfying

$$\exists \tau \in (1, c] \quad \text{s.t.} \quad \left(\frac{\tau}{c}\right)^{\varphi+1} < \tau - 1. \tag{79}$$

It is stressed that there always exists $\varphi \geq 0$ satisfying (79). Condition (77) is called the iISS small-gain condition. This type of small-gain theorem with different μ_i's is proved in earlier results (e.g., [8,12,14,15]) without explicitly relying on the preservation of iISS dissipation inequalities. For a more general choice of a function other than a constant c in (77), see [14]. In the presence of the disturbance, i.e., $w(t) \not\equiv 0$, we can make use of (22) instead of (28) to establish iISS of (72). We prove the following in Appendix 8.

Proposition 7 *Suppose that Assumption 1 and*

$$\left\{\lim_{s\to\infty} \alpha_i(s) = \infty \quad \text{or} \quad \lim_{s\to\infty} \sigma_{3-i}(s) < \infty\right\}, \ i = 1, 2 \tag{80}$$

are satisfied. If there exists $c > 1$ such that (77) holds, then the function $W : \mathbb{R}^N \to \mathbb{R}_+$ given by (75) with the scalings $\mu_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2$, satisfying (78) is an iISS Lyapunov function of the interconnected system (72), where $\varphi \geq 0$ is any non-negative real number satisfying (79). Moreover, if $\alpha_i \in \mathcal{K}_\infty$ also holds for $i = 1, 2$, the function W is an ISS Lyapunov function of (72).

5.2 ISS small-gain theorem with/without preservation of ISS dissipation inequalities

If both subsystems are ISS with respect to their feedback inputs, i.e., Σ_i is ISS with respect to x_{3-i} for $i = 1, 2$, then Proposition 6 reduces[6] to the ISS small-gain condition [21]. In fact, if

$$\lim_{s\to\infty} \alpha_i(s) = \infty \ \text{ or } \ \lim_{s\to\infty} \alpha_i(s) > \lim_{s\to\infty} \sigma_i(s) \tag{81}$$

is fulfilled for $i = 1, 2$, the left-hand side of (77) is the composite of ISS gain functions of two subsystems. Recall that (81) implies ISS of subsystem Σ_i with respect to the input x_{3-i} from subsystem Σ_{3-i}.

Now, notice that the scaling (78) is in the form of (46) of Proposition 2 for the preservation of ISS dissipation inequalities under scalings with the additional assumption (44). The following theorem, proved in Appendix 9, demonstrates that the scalings that preserve ISS dissipation inequalities via Corollary 1 and Proposition 2 can always provide a Lyapunov function of the interconnected system (72) if both subsystems are ISS with respect to the feedback inputs (i.e., (81) holds) and the small-gain condition (77) is satisfied.

Theorem 5 *Suppose that Assumption 1 holds and there exists $c > 1$ such that*

(i) (77) *is satisfied;*
(ii) $\lim_{s\to\infty} \alpha_i(s) \geq c \lim_{s\to\infty} \sigma_i(s)$ *holds for $i = 1, 2$.*

[6] For the no-gap case where only (10) instead of (38) is fulfilled by both subsystems, see [14].

If the scaling $\mu_i : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies* $\mu' = \alpha_i^{\varphi} \sigma_{3-i}^{\varphi+1}$ *for each* $i = 1, 2$, *with* $\varphi \geq 0$ *fulfilling* (45), *then the function* $W : \mathbb{R}^N \to \mathbb{R}_+$ *given by* (75) *is a Lyapunov function establishing GAS of* $x = 0$ *of the interconnected system* (72) *with* $w(t) \equiv 0$. *Furthermore, if* (80) *is satisfied, the function* W *is an iISS Lyapunov function of* (72). *Moreover, if* $\alpha_i \in \mathcal{K}_\infty$ *also holds for* $i = 1, 2$, *the function* W *is an ISS Lyapunov function.*

Note that (81) for $i = 1, 2$ is equivalent to the existence of $c > 1$ satisfying (ii). Theorem 5 demonstrates that the use of scalings preserving ISS can establish stability of interconnections of ISS subsystems. However, preserving ISS dissipation inequalities is not the only approach, and it can cause unnecessary complexity in Lyapunov functions. In fact, ISS properties of subsystems are not necessary for establishing stability properties of interconnected systems, which is indeed the key idea of the iISS small-gain theorem [8]. It has been proved in [14] that only one of the two subsystems Σ_i is necessarily ISS with respect to the feedback input x_{3-i}. Forcing ISS in scaling on both subsystems can deform the Lyapunov function of the interconnection excessively. In fact, the smaller the c is, the larger the exponent φ becomes to fulfill (45). This results in a higher order nonlinearity in the scaling (46). This point is illustrated by the next example.

Example 1 Consider two dissipation inequalities (74) given with

$$\alpha_1(s) = \frac{s}{s+1}, \ \sigma_1(s) = \frac{4s}{5(s+1)}, \ \theta_1 = 0 \tag{82}$$

$$\alpha_2(s) = s, \quad \sigma_2(s) = \frac{s}{4}, \quad \theta_2 = 0. \tag{83}$$

Since (81) holds, both subsystems Σ_i, $i = 1, 2$, are ISS with respect to the feedback input x_{3-i}. The small-gain condition (77) becomes $c^2/5 \leq 1$. Pick $c = 2.2$ and $\tau = 2$ for which (79) is satisfied with $\varphi = 0$. Then Proposition 6 establishes global asymptotic stability of $x = 0$ for (72) using the Lyapunov function (75) with

$$\mu_1'(s) = \frac{s}{4}, \quad \mu_2'(s) = \frac{4s}{5(s+1)}. \tag{84}$$

However, due to $1 = \lim_{s\to\infty} \alpha_1(s) < c \lim_{s\to\infty} \sigma_1(s) = 8.8/5$, property *(ii)* of Theorem 5 is not met, and the scaling μ_1 with $\varphi = 0$ does not preserve the ISS dissipation inequality of Σ_1. A choice of c to preserve ISS of both subsystems is $c = 5/4$. It can be verified as described in [19] that the existence of τ satisfying (45) is guaranteed if and only if $\varphi > 16.06\ldots$. Then the scalings defined by (78) are

$$\mu_1'(s) = \left[\frac{s}{s+1}\right]^{17} \left[\frac{s}{4}\right]^{18}, \ \mu_2'(s) = s^{17} \left[\frac{4s}{5(s+1)}\right]^{18} \tag{85}$$

and the resulting Lyapunov function (75) is unnecessarily complex when compared to that using the scalings in (84). This illustrates that requiring dissipation inequalities of both subsystems to retain ISS in constructing a Lyapunov function of an interconnection can lead to an unnecessary complexity.

5.3 Simpler Lyapunov functions using the Legendre–Fenchel transform

For constructing less complicated Lyapunov functions of interconnected systems, this subsection proposes an idea of utilizing the relationship established by Theorem 4 between the Legendre–Fenchel transform and the extended use of the classical division technique of changing supply rates. Recall that Propositions 6 and 7 are proved based on the extended division technique. For example, using κ proposed in Corollary 4, the Legendre–Fenchel transform provides us with an iISS dissipation inequality that is tighter than or the same as the one provided by the extended division technique. Applying this fact to the scaling (78) of each subsystem in an interconnection, one can expect some reduction of conservativeness in establishing stability properties of the interconnected system. Here, it is essential to remember that the small-gain criterion that can be given by the extended division technique has no conservativeness [14], provided that the information of subsystems is given only in terms of such dissipation inequalities as in Assumption 1. This fact, however, does not exclude the possibility of a composite Lyapunov function to be somehow conservative. One may be able to make use of the Legendre–Fenchel transform to greatly decrease the exponent φ in (78) of the scalings, which results in reduction of the order of nonlinearities in the Lyapunov function constructed for the interconnected system. We prove the following result in Appendix 10.

Theorem 6 *Suppose that Assumption 1 is satisfied with α_i and $\sigma_i \in \mathcal{K}$, $i = 1, 2$, which are continuously differentiable, and satisfy (80) and*

$$\left\{ \frac{\alpha_i'(s)\sigma_{3-i}(s)}{\sigma_{3-i}'(s)} : \textit{non-decreasing}, \ \forall s \in \mathbb{R}_+ \right\}, \ i = 1, 2. \tag{86}$$

Assume that there exists $c > 1$ such that (77) holds. Let $\tau, \varphi > 0$ be such that (79) holds. For $\psi > 0$, let $\lambda_{i,\psi} : \mathbb{R}_+ \to \mathbb{R}_+$ and $\kappa_{i,\psi} : [0, L_{i,\psi}) \to \mathbb{R}_+$ be

$$\lambda_{i,\psi}(s) = \alpha_i(s)^{\psi}\sigma_{3-i}(s)^{\psi+1}, \forall s \in \mathbb{R}_+ \tag{87}$$

$$\kappa_{i,\psi}(s) = \frac{1}{\tau} s \left(\alpha_i \circ \lambda_{i,\psi}^{\ominus}(s)\right), \forall s \in [0, L_{i,\psi}) \tag{88}$$

for $i = 1, 2$, where $L_{i,\psi} := \lim_{l\to\infty} \lambda_{i,\psi}(l) \in \overline{\mathbb{R}}_+$. Furthermore, define $\tilde{\alpha}_{i,\psi}, \tilde{\sigma}_{i,\psi} \in \mathcal{K}$ by

$$\tilde{\alpha}_{i,\psi}(s) := \left(1 - \frac{1}{\tau}\right) \lambda_{i,\psi}(s)\alpha_i(s), \forall s \in \mathbb{R}_+ \tag{89a}$$

$$\tilde{\sigma}_{i,\psi}(s) := \min\left\{\ell\kappa_{i,\psi} \circ \sigma_i(s), L_{i,\psi}\sigma_i(s)\right\}, \forall s \in \mathbb{R}_+ \tag{89b}$$

for $i = 1, 2$, and $\psi > 0$. Then

(i) *There exists* $\epsilon > 0$ *such that*

$$\tilde{\alpha}_{i,\varphi}(s) - \tilde{\sigma}_{3-i,\varphi}(s) \geq \epsilon\lambda_{i,\varphi}(s)\alpha_i(s), \quad \forall s \in \mathbb{R}_+, i = 1, 2. \tag{90}$$

(ii) *If there exists* $\psi \in (0, \varphi]$ *such that*

$$\tilde{\alpha}_{i,\psi}(s) - \tilde{\sigma}_{3-i,\psi}(s) \geq \epsilon\lambda_{i,\psi}(s)\alpha_i(s), \quad \forall s \in \mathbb{R}_+\,, i = 1, 2, \tag{91}$$

then the function $W : \mathbb{R}^N \to \mathbb{R}_+$ *given by* (75) *with* $\mu_i \in \mathcal{K}_\infty$ *satisfying* $\mu_i' = \lambda_{i,\psi}$ *for* $i = 1, 2$*, is an iISS Lyapunov function of* (72)*. Moreover, if* $\alpha_i \in \mathcal{K}_\infty$ *holds for* $i = 1, 2$*, additionally, the function* W *is an ISS Lyapunov function.*

Note that if the inequality in (91) is replaced by the milder condition $>$ 0 for $s \in (0, \infty)$, then the origin $x = 0$ of the interconnected system (72) is globally asymptotically stable for $w(t) \equiv 0$. Although the parameter ψ we compute such that (91) holds is allowed to be a real number, it is practically satisfactory for obtaining a Lyapunov function to restrict ψ to integers. The next examples illustrate Theorem 6 by showing two cases where ψ achieving (91) can be much smaller than φ given by (79).

Example 2 For the interconnected system (72), consider two dissipation inequalities (74) with

$$\alpha_1(s) = \frac{s}{s+1}, \ \sigma_1(s) = \frac{4s}{5(s+1)}, \ \theta_1 = s^2 \tag{92}$$

$$\alpha_2(s) = s, \quad \sigma_2(s) = \frac{8s}{5(s+2)}, \ \theta_2 = s. \tag{93}$$

Since (81) holds for $i = 1, 2$, subsystem Σ_i is ISS with respect to the input x_{3-i} for each $i = 1, 2$. The small-gain condition (77) becomes $16c^2/25 \leq 1$. Pick $c = 1.25$ and $\tau = 1.2$. Then property (79) is satisfied if and only if $\varphi > 38.43$. Although Proposition 6 establishes global asymptotic stability of $x = 0$ for (72), it produces a Lyapunov function of the form (75) with scalings given by

$$\mu_1'(s) = \left[\frac{s}{s+1}\right]^{\psi}\left[\frac{8s}{5(s+2)}\right]^{\psi+1}, \ \mu_2'(s) = s^{\psi}\left[\frac{4s}{5(s+1)}\right]^{\psi+1} \tag{94}$$

with a large ψ such as $\psi = 39$.

Now, we shall make use of Theorem 6. Obviously, (80) is satisfied. To check if (86) is fulfilled, define

$$D_i = \frac{\alpha_i'\sigma_{3-i}}{\sigma_{3-i}'}, \quad Z_i = \alpha_i''\sigma_{3-i}\sigma_{3-i}' + \alpha_i'\sigma_{3-i}'\sigma_{3-i}' - \alpha_i'\sigma_{3-i}\sigma_{3-i}''.$$

Then we have

$$D_i' = \frac{Z_i}{(\sigma_{3-i}')^2}, \quad i = 1, 2. \tag{95}$$

The functions in (92) and (93) yield $Z_1(s) = 256/(25(s+1)^3(s+2)^4) > 0$, $Z_2(s) = 16(2s+1)/(25(s+1)^4) > 0$, $\sigma_1'(s) > 0$ and $\sigma_2'(s) > 0$ for all $s > 0$. Thus, due to (95), (86) is fulfilled. It is numerically verified that $\psi = 1$ achieves (91) with $\epsilon = 0.05$ for $i = 1, 2$. Therefore, the function W given by (75) and (94) with $\psi = 1$ is an iISS Lyapunov function of the interconnected system (72).

Example 3 Consider the interconnected system (72) satisfying (74) in Assumption 1 with

$$\alpha_1(s) = \frac{s}{s+1}, \ \sigma_1(s) = \frac{5s}{4s+6}, \ \theta_1 = s^2 \tag{96}$$

$$\alpha_2(s) = s, \quad \sigma_2(s) = \frac{5s}{7s+6}, \ \theta_2 = s. \tag{97}$$

For these functions, the x_1-subsystem is not ISS but iISS, while the x_2-subsystem is ISS. For (96), (97), the small-gain condition (77) is calculated as $25c^2/36 \leq 1$, which is satisfied by $c = 6/5$. Pick $\tau = 1.1 < 1.2 = c$. Then (79) holds if and only if $\varphi > 25.46...$ holds.[7] Thus, from (78) we obtain

$$\mu_1'(s) = \left[\frac{s}{s+1}\right]^{\psi}\left[\frac{5s}{7s+6}\right]^{\psi+1}, \quad \mu_2'(s) = s^{\psi}\left[\frac{5s}{4s+6}\right]^{\psi+1} \tag{98}$$

with $\psi = 26$, which gives an iISS Lyapunov function of (72) in the form of (75). Next, one can verify (80) and (86) from (96) and (97). Thus, Theorem 6 can be applied to (96) and (97). Numerical computation confirms that (91) is achieved with $\psi = 2$ and $\epsilon = 0.05$. Therefore, the use of $\psi = 2$ in (98) also gives an iISS Lyapunov function of (72). Finally, it is worth noting that if the Legendre–Fenchel transformation is applied to only the x_2-subsystem which is ISS, the numerical computation yields only $\psi = 5$ which is larger than $\psi = 2$. This illustrates that application of the Legendre–Fenchel transformation is beneficial in terms of order reduction of the Lyapunov function when applied to both ISS and iISS subsystems.

Note that the functions of (82), (83) do not satisfy (80) for $i = 1$, and one cannot invoke Theorem 6. Since (91) poses a differential inequality involving composite and inverse mappings for each i, computing ψ analytically can be complicated. However, condition (91) which one wants to solve for a constant ψ has only a single variable s for each i. Point-wise numerical evaluation of (91) along the one-dimensional space is not extremely demanding. In fact, constructing a Lyapunov function W from V_1 and V_2 is a problem on the two-dimensional space of combinations (V_1, V_2). If Q denotes the resolution of discretizing a compact domain of each one-dimensional space, the

[7] It is not necessary to use $\tau = 1.1$. There exists τ satisfying (79) if and only if $\varphi > 21.419...$

criterion (91) reduces the size of the problem from Q^2 to $2Q$. Applying the same idea to networks consisting of n subsystems on the basis of the development in [15], the number of grid points can be reduced from Q^n to nQ.

6 Concluding remarks

Following the recent work [23] on preserving iISS/ISS dissipation inequalities with respect to scalings, this paper has refined and exploited scaling techniques from previous studies on iISS systems and their interconnections. Given a set of systems specified by a dissipation inequality, this paper has clarified conditions under which a scaling of an iISS (resp. ISS) Lyapunov function is guaranteed to admit an iISS (resp. ISS) dissipation inequality. Presenting such conditions is reasonably straightforward, while finding appropriate scalings is more challenging. This paper has shown a useful pair of a condition and a set of scalings on the basis of an extended use of the classical division technique of changing supply rates proposed in [36]. This paper has also investigated sufficient conditions based on the Legendre–Fenchel transform for preserving (i)ISS dissipation inequalities under scalings as proposed in [23]. While [23] proposed some sufficient conditions, the problem of finding appropriate scalings remained. This paper has given a way to construct such scalings by relating the Legendre–Fenchel transform approach to the extended division technique. Furthermore, the Legendre–Fenchel transform approach to scalings has been modified to encompass non-ISS systems effectively.

This paper has also given insights into the preservation of iISS/ISS dissipation inequalities under scalings in view of stability of interconnected systems. The iISS small-gain theorem can be revisited by explicitly referring to both iISS and ISS preserving scalings. This paper has shown that preservation of iISS under scaling is always useful to establish stability of interconnected systems based on information of iISS/ISS dissipation inequalities of subsystems. Interestingly, both preserving and non-preserving scalings for ISS dissipation inequalities lead us to the same small-gain condition. However, the Lyapunov functions constructed by the two scalings for the same interconnected system are different. Imposing the preservation on scaling of ISS dissipation inequalities may unnecessarily cause high-order nonlinearities in the Lyapunov function. Indeed, the dissipation inequality of one subsystem does not have to retain ISS if the other in the loop is ISS with a strong decay rate. In the case where two subsystems interconnected with each other satisfy the small-gain condition only marginally, the dissipation inequality of each subsystem is allowed to be degraded only very slightly by scaling. When the small-gain condition can be satisfied only marginally, the extended division technique inevitably generates high-order nonlinearities in the composite Lyapunov function as observed in [8,12,14,15,19]. In such a situation, this paper has illustrated that for both iISS and ISS subsystems, the Legendre–Fenchel approach has a remarkable potential to greatly reduce the complexity of the composite Lyapunov function, although it does not improve the small-gain criterion that has already been proved necessary. The reduction is beneficial in control design such as Lyapunov redesign, $L_g V$ type control, and inverse optimal control and so on [4,26,33,34]. Moreover, the reduction can significantly simplify stability analysis of stochastic systems [17].

Appendix

Appendix 1: Proof of Theorem 1

Property (17) implies the existence of $w_L \in (0, \infty)$ and a sequence $\{s_i\}$ of real numbers such that $\lim_{i\to\infty} s_i = \infty$ and $\lim_{i\to\infty} \alpha(s_i) < \sigma(w_L)$. By virtue of (8), if $\liminf_{s\to\infty} \mu'(s) = \infty$ holds, then

$$\limsup_{|x|\to\infty} \mu'(V(x))\{-\alpha(V(x)) + \sigma(w_L)\} = \infty. \tag{99}$$

On the other hand, the assumptions $\hat{\alpha} \in \mathcal{P}$ and $\hat{\sigma} \in \mathcal{K}$ imply

$$\limsup_{|x|\to\infty} -\hat{\alpha}(W(x)) + \hat{\sigma}(|w|) < \infty, \quad \forall |w| \in \mathbb{R}_+. \tag{100}$$

The contradiction between (99) and (100) arising from (18) indicates that (19) must hold. If $\lim_{s\to\infty} \alpha(s)$ exists, property $\lim_{i\to\infty} \alpha(s_i) < \sigma(w_L)$ holds for any sequence $\{s_i\}$ of real numbers such that $\lim_{i\to\infty} s_i = \infty$. Hence, the claim (20) follows from (100).

Appendix 2: Proof of Theorem 2

The decomposition (11) yields

$$\begin{aligned}
&\mu'(V(x))\,[-\alpha(V(x)) + \sigma(|w|)] \\
&\quad = -b\alpha(V(x)) + b\sigma(|w|) + \lambda(V(x))\,[-\alpha(V(x)) + \sigma(|w|)]\,.
\end{aligned} \tag{101}$$

Obviously, in the case of $\lambda(s) \equiv 0$, inequality (15) holds with $\hat{\alpha} = b\alpha \circ \mu^{-1} \in \mathcal{P}$ and $\hat{\sigma} = b\sigma \in \mathcal{K}$ which are identical with the pair (23a), (23b). The assertions about $\hat{\alpha} \in \mathcal{K}$ and $\hat{\alpha} \in \mathcal{K}_\infty$ are straightforward. Note that ω is irrelevant in this case. Hence, the rest of the proof assumes $\lambda(s) \not\equiv 0$. Property (12) implies $\lambda(s) > 0$ for all $s \in (0, \infty)$. Since μ' is non-decreasing, so is λ.

First, suppose that $\liminf_{l\to\infty} \alpha(l) > 0$. This clearly guarantees the existence of $\tilde{\alpha} \in \mathcal{K}$ satisfying (25). It is also straightforward that there exists a continuous function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (26). Following the idea in [36], we evaluate $\lambda(V(x))\,[-\alpha(V(x)) + \sigma(|w|)]$ in (101) in the two cases $\tilde{\alpha}(V(x)) \geq (\mathbf{Id}+\omega)\circ\sigma(|w|)$ and $\tilde{\alpha}(V(x)) \leq (\mathbf{Id}+\omega)\circ\sigma(|w|)$ separately. Due to the non-decreasing property of λ and $\mathbf{Id}+\omega \in \mathcal{K}_\infty$, the combination of the evaluation in the two cases yields (15) with (23). Notice that (22) implying $\lim_{s\to\infty} \lambda(s) < \infty$ for $\lim_{s\to\infty} \tilde{\alpha}(s) < \infty$ ensures $\lambda \circ \tilde{\alpha}^{\ominus} \circ (\mathbf{Id}+\omega)\circ\sigma(s)$ is well-defined for all $s \in \mathbb{R}_+$. The non-decreasing property of λ and $\mathbf{Id}+\omega \in \mathcal{K}_\infty$ yields $\hat{\sigma} \in \mathcal{K}$. It is verified that

$$(\mathbf{Id}-(\mathbf{Id}+\omega)^{-1})\circ(\mathbf{Id}+\omega)=(\mathbf{Id}+\omega)-\mathbf{Id}=\omega.$$

Due to $\mathbf{Id}+\omega \in \mathcal{K}_\infty$, we have

$$(\mathbf{Id}-(\mathbf{Id}+\omega)^{-1})=\omega\circ(\mathbf{Id}+\omega)^{-1} \tag{102}$$

which gives

$$\hat{\alpha}=[\omega\circ(\mathbf{Id}+\omega)^{-1}\circ\tilde{\alpha}\circ\mu^{-1}][\lambda\circ\mu^{-1}]+b\alpha\circ\mu^{-1}. \tag{103}$$

From (26) it follows that $\omega\circ(\mathbf{Id}+\omega)^{-1}\circ\tilde{\alpha}(s)>0$ holds for all $s\in(0,\infty)$, and $\omega\circ(\mathbf{Id}+\omega)^{-1}\circ\tilde{\alpha}\in\mathcal{P}$. Thus, we have $\hat{\alpha}\in\mathcal{P}$. Finally, Eq. (103) also implies $\hat{\alpha}\in\mathcal{K}$ (resp. $\hat{\alpha}\in\mathcal{K}_\infty$) if $\omega,\tilde{\alpha},\alpha\in\mathcal{K}$ (resp. $\omega,\tilde{\alpha},\alpha\in\mathcal{K}_\infty$).

Next, suppose that $\liminf_{l\to\infty}\alpha(l)=0$. Then property (24) implies that $\tilde{\alpha}^{\ominus}(s)=\infty$ for all $s\in\overline{\mathbb{R}}_+$ by virtue of the definition of $\ominus$. Since $L:=\lim_{l\to\infty}\lambda(l)<\infty$ is ensured by (22), the formula (23b) gives $\hat{\sigma}=(b+L)\sigma\in\mathcal{K}$ which is independent of ω. The choice (24) also implies $\hat{\alpha}\in\mathcal{P}$ for (23a) for each given ω. On the other hand,

$$b\sigma(|w|)+\lambda(V(x))\sigma(|w|)\le(b+L)\sigma(|w|),\quad \forall x\in\mathbb{R}^N,\ \forall w\in\mathbb{R}^M \tag{104}$$

holds. From (24) we also obtain

$$\begin{aligned}
&b\alpha(V(x))+\lambda(V(x))\alpha(V(x))\\
&\quad\ge b\alpha\circ\mu^{-1}(W(x))+[\lambda\circ\mu^{-1}(W(x))][\tilde{\alpha}\circ\mu^{-1}(W(x))] &(105)\\
&\quad\ge b\alpha\circ\mu^{-1}(W(x))\\
&\qquad+[\lambda\circ\mu^{-1}(W(x))][(\mathbf{Id}-(\mathbf{Id}+\omega)^{-1})\circ\tilde{\alpha}\circ\mu^{-1}(W(x))] &(106)
\end{aligned}$$

for all $x\in\mathbb{R}^N$ and $w\in\mathbb{R}^M$ Applying these inequalities to (101), we arrive at not only (23), but also (23) with (27) with $\Omega\to\infty$.

Finally, suppose that $\lim_{s\to\infty}\mu'(s)<\infty$. Defining $L:=\lim_{l\to\infty}\mu'(l)<\infty$ again yields (104). Independently, (105) follows from (25). Thus, (101) is bounded from above by $-\hat{\alpha}(W(x))+\hat{\sigma}(|w|)$ defined by

$$\hat{\alpha}(s)=[\tilde{\alpha}\circ\mu^{-1}(s)][\lambda\circ\mu^{-1}(s)]+b\alpha\circ\mu^{-1}(s),\quad \hat{\sigma}=(b+L)\sigma(s).$$

These functions are identical with taking $\Omega\to\infty$ in (23) with (27) for each $s\in\mathbb{R}_+$.

Appendix 3: Proof of Theorem 3

In the case of $\lambda(s)\equiv 0$, the claim holds true obviously from $\alpha,\sigma\in\mathcal{K}$ and (10) since (34) gives $\hat{\alpha}=b\alpha\circ\mu^{-1}\in\mathcal{K}$ and $\hat{\sigma}=b\sigma\in\mathcal{K}$. Therefore, the rest considers the case of $\lambda(s)\not\equiv 0$. First, suppose that (30)–(32) are satisfied with a continuous function

$\omega : \mathbb{R}_+ \to \mathbb{R}_+$. Following the proof of Theorem 2 with $\tilde{\alpha} = \alpha$, we obtain $\hat{\sigma} \in \mathcal{K}$ in (34b). Note that property (28) is guaranteed by (32). The function $\hat{\alpha} \in \mathcal{K}$ which is obtained as in (23a) with $\tilde{\alpha} = \alpha$ and satisfies (15) is only of class $\mathcal{P}$. Hence, write (23a) as $\eta + b\alpha \circ \mu^{-1}$ by defining η as in (35). Rewrite $\eta \in \mathcal{P}$ as

$$\eta(s) = \left[\left(\frac{\alpha \circ \mu^{-1}(s)}{(\mathbf{Id}+\omega)^{-1} \circ \alpha \circ \mu^{-1}(s)} - 1\right)\right] \cdot \left[(\mathbf{Id}+\omega)^{-1} \circ \alpha \circ \mu^{-1}(s)\right] \cdot \left[\lambda \circ \mu^{-1}(s)\right]. \tag{107}$$

Applying (30) and (32) to (107) with the help of $\mu \in \mathcal{K}_\infty$, one arrives at

$$\liminf_{s\to\infty} \eta(s) \geq \lim_{s\to\infty} [\lambda \circ \alpha^{\ominus} \circ (\mathbf{Id}+\omega) \circ \sigma]\sigma.$$

Since this inequality implies $\liminf_{s\to\infty} \eta(s) > 0$ and we have $\alpha \circ \mu^{-1} \in \mathcal{K}$ in addition, there always exists a continuous function $k : \mathbb{R}_+ \to \mathbb{R}_+$ such that (36) and (37) are fulfilled. Defining $\hat{\alpha}$ as in (34a) with (36) and (37) ensures $\hat{\alpha} \in \mathcal{K}$ and $\hat{\alpha}(s) \leq \eta(s) + b\alpha \circ \mu^{-1}(s)$ for all $s \in \mathbb{R}_+$. Thus, the preservation of the iISS dissipation inequality (9) under the scaling μ is established by $\hat{\alpha}, \hat{\sigma} \in \mathcal{K}$ given in (34). Furthermore, by virtue of $\limsup_{l\to\infty} k(l) = 1$ and $\hat{\alpha} \in \mathcal{K}$, property (33) follows from (10), (30) and (107). This proves the preservation of the ISS dissipation inequality.

Finally, replace the pair of (31) and (32) by (27) with $\Omega \to \infty$ in the case of $\lim_{s\to\infty} \mu'(s) < \infty$. Define $L := \lim_{l\to\infty} \lambda(l) < \infty$. Then (34) becomes

$$\hat{\alpha} = k[\mu' \circ \mu^{-1}][\alpha \circ \mu^{-1}], \quad \hat{\sigma} = (b + L)\sigma, \tag{108}$$

and clearly satisfies $\hat{\sigma}, \hat{\sigma} \in \mathcal{K}$. Using (36) and (37) to modify (23) of Theorem 2 verifies that the functions in (108) achieve the preservation the iISS dissipation inequality (9) under the scaling μ. For (108), by virtue of $\limsup_{l\to\infty} k(l) = 1$ and $\hat{\alpha} \in \mathcal{K}$, property (33) is implied by (10). This establishes the preservation the ISS dissipation inequality.

Appendix 4: Proof of Proposition 2

The existence of $\tau, \varphi \geq 0$ fulfilling (45) is straightforward from $c > 1$. Property (44) with $\tau < c$ implies (39). Due to the property $\alpha_i \circ \alpha^{\ominus}(s) \leq s$ for all $s \in \mathbb{R}_+$, property (40) follows if

$$\lim_{s\to\infty} (\tau - 1)\, [\alpha(s)]^{\varphi}\, \beta(s) \geq \lim_{s\to\infty} [\tau\sigma(s)]^{\varphi} [\beta \circ \alpha^{\ominus} \circ \tau\sigma(s)]. \tag{109}$$

The non-decreasing property of β and (39) guarantee

$$\lim_{s\to\infty} \beta(s) \geq \lim_{s\to\infty} [\beta \circ \alpha^{\ominus} \circ \tau\sigma(s)].$$

Thus, if

$$\lim_{s\to\infty}(\tau-1)\,[\alpha(s)]^{\varphi}\geq\lim_{s\to\infty}[\tau\sigma(s)]^{\varphi} \tag{110}$$

is met, property (109) is satisfied. In the case where (45) holds for $\varphi=0$, we can easily verify (110). Therefore, we next assume that (45) holds for some $\varphi>0$. Property (110) is satisfied if we have

$$\lim_{s\to\infty}(\tau-1)^{\frac{1}{\varphi}}\alpha(s)\geq\lim_{s\to\infty}\tau\sigma(s).$$

This property is achieved if

$$(\tau-1)^{-\frac{1}{\varphi}}\tau\leq c \tag{111}$$

since we have (44). Property (111) is secured by (45).

Appendix 5: Proof of Proposition 4

In the case of $\mu'(s)\equiv b$, the implications (52) and (22) do not require anything, which proves the claim. Suppose that $\mu'(s)\not\equiv b$. Since $\mu\in\mathcal{K}_\infty$, property (49) is equivalent to

$$(\lambda+b)\alpha-\kappa\circ\lambda\in\mathcal{P}. \tag{112}$$

Clearly, this property implies

$$(\lambda(s)+b)\alpha(s)-\kappa\circ\lambda(s)\geq 0,\quad\forall s\in\mathbb{R}_+. \tag{113}$$

Recalling $\mu'(s)\not\equiv b$, properties $\lim_{s\to\infty}\lambda(s)>0$ and $\kappa\in\mathcal{K}_\infty$ imply $\lim_{s\to\infty}\kappa\circ\lambda(s)>0$. Since κ' is of $\mathcal{K}_\infty$, property (113) requires $\liminf_{s\to\infty}\alpha(s)>0$. Hence, property (52) must hold. Next, suppose $\lim_{s\to\infty}\mu'(s)=\infty$ which means $\lim_{s\to\infty}\lambda(s)=\infty$. Property $\kappa'\in\mathcal{K}_\infty$ in (113) again implies $\lim_{s\to\infty}\alpha(s)=\infty$. Therefore, (22) must hold.

Appendix 6: Proof of Theorem 4

First, we assume that (64) holds. Let $L:=\lim_{l\to\infty}\lambda(l)\leq\infty$. Suppose that $\alpha\circ\lambda^{\ominus}$ is piecewise differentiable on the interval $[0,L)$. Let κ' be any class $\mathcal{K}_\infty$ function satisfying

$$\frac{1}{\tau}\alpha\circ\lambda^{\ominus}(s)\leq\kappa'(s)\leq\frac{1}{\tau}\alpha\circ\lambda^{\ominus}(s)+\frac{1}{\tau}s\,[(\alpha\circ\lambda^{\ominus})'(s)],\quad\forall s\in[0,L), \tag{114}$$

where the second inequality is evaluated at all differentiable points. Note that $(\alpha\circ\lambda^{\ominus})'(s)\geq 0$ holds for almost all $s\in[0,L)$ due to $\alpha,\lambda\in\mathcal{K}$. Therefore, in the case of $\lim_{s\to\infty}\alpha(s) < \infty$, the existence of a function $\kappa' \in \mathcal{K}_\infty$ satisfying (114) follows from assumption (22) since $\lim_{s\to\infty}\lambda(s) < \infty$ and $\alpha, \lambda \in \mathcal{K}$. In the case of $\lim_{s\to\infty}\alpha(s) = \infty$, the existence is guaranteed by $\alpha\circ\lambda^{\ominus} \in \mathcal{K}_\infty$ under the assumption of (64).

Let κ denote the antiderivative of κ' satisfying $\kappa(0) = 0$. Then $\kappa \in \mathcal{K}_\infty$ follows from $\kappa' \in \mathcal{K}_\infty$. Define the map $\overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ as

$$\overline{\kappa}(s) = \frac{1}{\tau}s\,[\alpha\circ\lambda^{\ominus}(s)], \quad s \in \overline{\mathbb{R}}_+, \tag{115}$$

which satisfies

$$\overline{\kappa}'(s) = \frac{1}{\tau}\alpha\circ\lambda^{\ominus}(s) + \frac{1}{\tau}s\,[(\alpha\circ\lambda^{\ominus})'(s)] \tag{116}$$

for almost all $s \in [0, L)$. Thus, from (114) we obtain

$$\kappa(s) \le \overline{\kappa}(s), \quad \forall s \in [0, L). \tag{117}$$

This property together with (115) yields

$$\kappa\circ\lambda(s) \le \overline{\kappa}\circ\lambda(s) = \frac{1}{\tau}\lambda(s)\alpha(s), \quad \forall s \in \mathbb{R}_+. \tag{118}$$

Hence, we have

$$\hat{\alpha}_L(s) = [\mu'\circ\mu^{-1}(s)][\alpha\circ\mu^{-1}(s)] - \kappa\circ\lambda\circ\mu^{-1}(s) \ge \hat{\alpha}_{D,\tau}(s), \ \forall s \in \mathbb{R}_+ \tag{119}$$

and $\hat{\alpha}_{D,\tau} \in \mathcal{K}$ by virtue of $\mu \in \mathcal{K}_\infty$ and $\tau > 1$. Therefore, we arrive at (61) with $\hat{\alpha}_L \in \mathcal{K}$.

Next, applying $\lambda \in \mathcal{K}$ to both sides of the first inequality in (114) from the right, one obtains

$$\frac{1}{\tau}\alpha(s) \le \kappa'\circ\lambda(s), \quad \forall s \in \mathbb{R}_+.$$

Applying the non-decreasing function $\alpha^{\ominus}(\tau s)$ defined for $s \in [0, \lim_{l\to\infty}\alpha(l)/\tau)$ again yields

$$s \le \kappa'\circ\lambda\circ\alpha^{\ominus}(\tau s), \quad \forall s \in [0, \lim_{l\to\infty}\alpha(l)/\tau).$$

Applying $(\kappa')^{-1} \in \mathcal{K}_\infty$ to the above from the left, one obtains

$$(\kappa')^{-1}(s) \le \lambda\circ\alpha^{\ominus}(\tau s), \quad \forall s \in [0, \lim_{l\to\infty}\alpha(l)/\tau). \tag{120}$$

Here, recalling $\ell\kappa(s) = s(\kappa')^{-1}(s) - \kappa \circ (\kappa')^{-1}(s)$ and $\kappa, \kappa' \in \mathcal{K}_\infty$, we have $\ell\kappa(s) \le s(\kappa')^{-1}(s)$ for all $s \in \mathbb{R}_+$. Thus,

$$\ell\kappa(s) \le s\,[\lambda \circ \alpha^{\ominus}(\tau s)], \quad \forall s \in [0, \lim_{l\to\infty} \alpha(l)/\tau).$$

Hence, we have

$$\ell\kappa \circ \sigma(s) \le [\lambda \circ \alpha^{\ominus} \circ \tau\sigma(s)]\sigma(s), \quad \forall s \in [0, \lim_{l\to\infty} \sigma^{\ominus} \circ \tau^{-1}\alpha(l)).$$

Since we have the implication

$$\lim_{l\to\infty} \alpha(l) < \lim_{l\to\infty} \tau\sigma(l) \;\Rightarrow\; \lambda \circ \alpha^{\ominus} \circ \tau\sigma(s) = L, \quad \forall s \in [\lim_{l\to\infty} \sigma^{\ominus} \circ \tau^{-1}\alpha(l), \infty),$$

by virtue of (22), we arrive at

$$\min\{\ell\kappa \circ \sigma(s), L\sigma(s)\} \le [\lambda \circ \alpha^{\ominus} \circ \tau\sigma(s)]\sigma(s), \; \forall s \in \mathbb{R}_+.$$

Comparing this with (42) yields

$$\hat{\sigma}_L(s) = \min\{\ell\kappa \circ \sigma(s), L\sigma(s)\} + b\sigma \le \hat{\sigma}_{D,\tau}(s), \quad \forall s \in \mathbb{R}_+ \tag{121}$$

which implies (62) with $R = \infty$ and (63).

If $\alpha \circ \lambda^{\ominus} : [0, L) \to \mathbb{R}_+$ is not piecewise differentiable, the above arguments hold true by replacing (114) with

$$\kappa'(s) = \frac{1}{\tau}\alpha \circ \lambda^{\ominus}(s), \quad \forall s \in [0, L). \tag{122}$$

Note that (117) is guaranteed again since $\kappa' \in \mathcal{K}_\infty$ is chosen, due to (22).

Finally, suppose that (64) does not hold, i.e., assume that $\lim_{s\to\infty} \lambda(s) < \infty$ and $\lim_{s\to\infty} \alpha(s) = \infty$ are satisfied. Let $q > 0$ be arbitrary. Consider $p \in (0, \infty)$ which has yet to be determined. Let $\tilde{\lambda} \in \mathcal{K}_\infty$ be defined as

$$\tilde{\lambda}(s) = \lambda(s) + q\max\{s - p, 0\}, \quad \forall s \in \mathbb{R}_+. \tag{123}$$

Obviously, $\tilde{L} := \lim_{l\to\infty} \tilde{\lambda}(l) = \infty$, $\tilde{L} > L := \lim_{l\to\infty} \lambda(l)$ and in addition,

$$\lambda(s) \le \tilde{\lambda}(s), \quad \forall s \in \mathbb{R}_+ \tag{124}$$

$$\lambda(s) = \tilde{\lambda}(s), \quad \forall s \in [0, p]. \tag{125}$$

Assume that $\alpha \circ \tilde{\lambda}^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$ is piecewise differentiable. Let κ' be any class $\mathcal{K}_\infty$ function satisfying

$$\frac{1}{\tau}\alpha \circ \tilde{\lambda}^{-1}(s) \le \kappa'(s) \le \frac{1}{\tau}\alpha \circ \tilde{\lambda}^{-1}(s) + \frac{1}{\tau}s\,[(\alpha \circ \tilde{\lambda}^{-1})'(s)], \quad \forall s \in \mathbb{R}_+, \tag{126}$$

where the second inequality is evaluated at all differentiable points. Due to $\alpha, \tilde{\lambda} \in \mathcal{K}_\infty$, we have $(\alpha \circ \tilde{\lambda}^{-1})'(s) \geq 0$ for almost all $s \in \mathbb{R}_+$. Thus, the existence of a function $\kappa' \in \mathcal{K}_\infty$ satisfying (126) is guaranteed by virtue of $\alpha \circ \tilde{\lambda}^{-1} \in \mathcal{K}_\infty$. Let κ denote the antiderivative of κ' satisfying $\kappa(0) = 0$. Then $\kappa \in \mathcal{K}_\infty$ follows from $\kappa' \in \mathcal{K}_\infty$. Define $\overline{\kappa} \in \mathcal{K}_\infty$ by

$$\overline{\kappa}(s) = \frac{1}{\tau} s\,[\alpha \circ \tilde{\lambda}^{-1}(s)], \quad s \in \mathbb{R}_+. \tag{127}$$

Since $\overline{\kappa}'(s) = \frac{1}{\tau}\alpha \circ \tilde{\lambda}^{-1}(s) + \frac{1}{\tau} s\,[(\alpha \circ \tilde{\lambda}^{-1})'(s)]$ holds for almost all $s \in \mathbb{R}_+$, from (126) we obtain

$$\kappa(s) \leq \overline{\kappa}(s), \quad \forall s \in \mathbb{R}_+. \tag{128}$$

This property, (127) and (124) give

$$\kappa \circ \lambda(s) \leq \overline{\kappa} \circ \lambda(s) = \frac{1}{\tau}\lambda(s)\,[\alpha \circ \tilde{\lambda}^{-1} \circ \lambda(s)] \leq \frac{1}{\tau}\lambda(s)\alpha(s), \quad \forall s \in \mathbb{R}_+.$$

Hence, by virtue of $\mu \in \mathcal{K}_\infty$ and $\tau > 1$, we arrive at (61) with $\hat{\alpha}_L \in \mathcal{K}$. On the other hand, taking inverse of both sides of the first inequality in (126) yields

$$\tilde{\lambda} \circ \alpha^{-1}(\tau s) \geq (\kappa')^{-1}(s), \quad \forall s \in \mathbb{R}_+. \tag{129}$$

From $\ell\kappa(s) \leq s(\kappa')^{-1}(s)$ for all $s \in \mathbb{R}_+$ it follows that

$$\ell\kappa \circ \sigma(s) \leq [\tilde{\lambda} \circ \alpha^{-1} \circ \tau\sigma(s)]\sigma(s), \quad \forall s \in \mathbb{R}_+.$$

Due to (125), we have

$$\ell\kappa \circ \sigma(s) \leq [\lambda \circ \alpha^{-1} \circ \tau\sigma(s)]\sigma(s), \quad \forall s \in [0, \sigma^{\ominus} \circ \tau^{-1}\alpha(p)).$$

Since α is of class $\mathcal{K}_\infty$, for any given $R \in (0, \infty)$, there exists $p \in (0, \infty)$ such that $R = \sigma^{\ominus} \circ \tau^{-1}\alpha(p)$ holds. Therefore,

$$\min\{\ell\kappa \circ \sigma(s), L\sigma(s)\} \leq [\lambda \circ \alpha^{\ominus} \circ \tau\sigma(s)]\sigma(s), \ \forall s \in [0, R).$$

Using $L = \lim_{l\to\infty} \lambda(l)$ and (42), we obtain (62) and (63). If $\alpha \circ \tilde{\lambda}^{\ominus} : \mathbb{R}_+ \to \mathbb{R}_+$ is not piecewise differentiable, the above arguments hold true by replacing (126) with

$$\kappa'(s) = \frac{1}{\tau}\alpha \circ \tilde{\lambda}^{-1}(s), \quad \forall s \in \mathbb{R}_+. \tag{130}$$

This completes the proof.

Appendix 7: Proof of Proposition 6

Apply Theorem 2 and Remark 3 to each subsystem Σ_i with $\omega_i(s) = (\tau - 1)s$ for $\tau > 1$, $\tilde{\alpha}_i = \alpha_i$ and $\lambda_i = \mu_i'$. It can be verified that (52), (25) and (26) are satisfied. Property (77) also guarantees (28). The formulas in (29) with $\tilde{\alpha}_i = \alpha_i \in \mathcal{K}$ yield

$$\hat{\alpha}_i = \left(1 - \frac{1}{\tau}\right) \left[\alpha_i \circ \mu_i^{-1}(s)\right]^{\varphi+1} \left[\sigma_{3-i} \circ \mu_i^{-1}(s)\right]^{\varphi+1}, \tag{131}$$

$$\hat{\sigma}_i = \left[\alpha_i \circ \alpha_i^{\ominus} \circ \tau\sigma_i(s)\right]^{\varphi} \left[\sigma_{3-i} \circ \alpha_i^{\ominus} \circ \tau\sigma_i(s)\right]^{\varphi+1} \sigma_i(s) \tag{132}$$

Property $\alpha_i \circ \alpha^{\ominus}(s) \leq s$ for all $s \in \mathbb{R}_+$ and property (77) with $1 < \tau \leq c$ in (79) imply

$$\begin{aligned} \hat{\sigma}_i(s) &\leq \tau^{\varphi} \left[\sigma_{3-i} \circ \alpha_i^{\ominus} \circ \tau\sigma_i(s)\right]^{\varphi+1} [\sigma_i(s)]^{\varphi+1} \\ &\leq \tau^{\varphi} \left[\frac{1}{c}\alpha_{3-i}(s)\right]^{\varphi+1} [\sigma_i(s)]^{\varphi+1}, \quad s \in \mathbb{R}_+. \end{aligned} \tag{133}$$

Thus, we have

$$\sum_{i=1}^{2} \mu_i'(V_i)\{-\alpha_i(V_i) + \sigma_i(V_{3-i})\} \leq -\sum_{i=1}^{2} [\sigma_{3-i} \circ \mu_i^{-1}(W_i)]^{\varphi+1} q_i(W_i), \tag{134}$$

where $W_i = \mu_i(V_i)$ and

$$q_i(s) = \left(1 - \frac{1}{\tau}\right) [\alpha_i \circ \mu_i^{-1}(s)]^{\psi+1} - \tau^{\psi} \left[\frac{1}{c}\alpha_i \circ \mu_i^{-1}(s)\right]^{\psi+1}.$$

The existence of $\epsilon > 0$ such that

$$q_i(s) \geq \epsilon[\alpha_i \circ \mu_i^{-1}(s)]^{\psi+1}, \quad s \in \mathbb{R}_+ \tag{135}$$

is guaranteed by (79). From (134), (135), (73) and (74) it follows that W is a Lyapunov function proving GAS of $x = 0$ of (72).

Appendix 8: Proof of Proposition 7

Property (80) together with (78) implies

$$\left\{ \lim_{s\to\infty} \alpha_i(s) = \infty \quad \text{or} \quad \lim_{s\to\infty} \mu_i'(s) < \infty \right\}, \; i = 1, 2. \tag{136}$$

In the case of $\lim_{s\to\infty} \mu_i'(s) < \infty$, $i = 1, 2$, iISS and ISS of system (72) are easily verified by incorporating

$$\mu_i'(V_i)\theta_i(|w_i|) \le \lim_{l\to\infty} \mu_i'(l)\theta_i(|w_i|), \quad i = 1, 2 \tag{137}$$

into the proof of Proposition 6. In the remaining case, property (136) allows one to invoke a technique proposed in [8,14] as indicated by [12, Proposition 12].

Appendix 9: Proof of Theorem 5

Suppose that $\tau, \varphi \ge 0$ satisfy (45). Then $\tau < c$ implies $(\tau/c)^{\varphi} > (\tau/c)^{\varphi+1}$. Hence, property (79) is met. By virtue of (77), Propositions 6 and 7 prove all the claims.

Appendix 10: Proof of Theorem 6

First, the function $\lambda_{i,\psi}$ defined by (87) for each $i = 1, 2$ is of class $\mathcal{K}$ for all $\psi > 0$ since $\alpha_i, \sigma_{3-i} \in \mathcal{K}$. With the help of (80), property (87) with $\psi > 0$ also ensures

$$\left\{\lim_{s\to\infty} \alpha_i(s) < \infty \ \Leftrightarrow \ \lim_{s\to\infty} \lambda_{i,\psi}(s) < \infty\right\}, \ i = 1, 2 \tag{138}$$

for all $\psi > 0$, which corresponds to (64) as well as (22). Thus, for arbitrary given $\psi > 0$, Corollary 4 is applicable to the two pairs (α_i, σ_i), $i = 1, 2$, and the formula (66) with (65) and $k = 0$, which is exactly (88), guarantees (61) and (62) with $R = \infty$, provided that $\kappa_{i,\psi} : [0, L_{i,\psi}) \to \mathbb{R}_+$ is continuously differentiable, and that (67), (68) and (69) hold in terms of $\kappa_{i,\psi}$ for each $i \in \{1, 2\}$. Recall that the arguments to derive (119) and (121) allow $\kappa_{i,\psi}$ to be defined on only the interval $[0, L_i, \psi)$ in (88) instead of the entire $\mathbb{R}_+$. To confirm the continuous differentiability of $\kappa_{i,\psi}$ and (67), (68) and (69), using (87) and (88) and continuous differentiability of α_i and σ_{3-i}, we first obtain

$$\lambda_{i,\psi}'(s) = (\alpha_i(s)\sigma_{3-i}(s))^{\psi}\left\{\frac{\psi\alpha_i'(s)\sigma_{3-i}(s)}{\alpha_i(s)} + (\psi+1)\sigma_{3-i}'(s)\right\}, \ \forall s \in (0, \infty) \tag{139}$$

$$\kappa_{i,\psi} \circ \lambda_{i,\psi}(s) = \frac{1}{\tau}(\alpha_i(s)\sigma_{3-i}(s))^{\psi+1}, \ \forall s \in \mathbb{R}_+ \tag{140}$$

$$[\kappa_{i,\psi} \circ \lambda_{i,\psi}]'(s) = \frac{\psi+1}{\tau}(\alpha_i(s)\sigma_{3-i}(s))^{\psi}\left\{\alpha_i'(s)\sigma_{3-i}(s) + \alpha_i(s)\sigma_{3-i}'(s)\right\}, \quad \forall s \in \mathbb{R}_+. \tag{141}$$

The chain rule $[\kappa_{i,\psi} \circ \lambda_{i,\psi}]'(s) = \left(\kappa'_{i,\psi} \circ \lambda_{i,\psi}(s)\right) \lambda'_{i,\psi}(s)$ yields

$$\begin{aligned} \kappa'_{i,\psi} \circ \lambda_{i,\psi}(s) &= [\kappa_{i,\psi} \circ \lambda_{i,\psi}]'(s) \frac{1}{\lambda'_{i,\psi}(s)} \\ &= \frac{1}{\tau} \alpha_i(s) \left(1 + G_{i,\psi}(s)\right), \end{aligned} \tag{142}$$

where

$$\begin{aligned} G_{i,\psi}(s) &= \frac{\alpha'_i(s)\sigma_{3-i}(s)}{\psi\alpha'_i(s)\sigma_{3-i}(s) + (\psi+1)\alpha_i(s)\sigma'_{3-i}(s)}, \\ &= \frac{1}{\psi + (\psi+1)\dfrac{\alpha_i(s)\sigma'_{3-i}(s)}{\alpha'_i(s)\sigma_{3-i}(s)}}. \end{aligned}$$

Property $\alpha_i, \sigma_{3-i} \in \mathcal{K}$ implies $\alpha_i(s) \geq 0$ and $\sigma_{3-i}(s) \geq 0$ for $s \in \mathbb{R}_+$. Hence, for each $\psi > 0$, assumption (86) and $\alpha_i \in \mathcal{K}$ guarantee that $\kappa'_{i,\psi} \circ \lambda_{i,\psi}(s)$ given by (142) exits and is strictly increasing for all $s \in \mathbb{R}_+$. By definition, it also holds that

$$\lim_{s\to\infty} G_{i,\psi}(s) < \infty \tag{143}$$

for any $\psi > 0$. Therefore, properties $\kappa'_{i,\psi} \circ \lambda_{i,\psi}(0) = 0$ and $\lambda_{i,\psi} \in \mathcal{K}$ ensure for all $\psi > 0$ that $\kappa'_{i,\psi}(s)$ exists for $s \in [0, L_{i,\psi})$, and

$$\kappa'_{i,\psi}(0) = 0 \tag{144}$$

$$\kappa'_{i,\psi}(s) \text{ is strictly increasing } \forall s \in [0, L_{i,\psi}). \tag{145}$$

Recalling that $L_{i,\psi} < \infty$ implies $\lim_{s\to\infty} \alpha_i(s) < \infty$ due to (138), from (142) and (143) it follows that

$$L_{i,\psi} < \infty \ \Rightarrow \lim_{s\to L^-_{i,\psi}} \kappa'_{i,\psi}(s) < \infty. \tag{146}$$

Therefore, it is proved that $\kappa_{i,\psi}$ is continuously differentiable and fulfills and (67), (68) and (69) for each $i = 1, 2$. We can now invoke Corollary 4. By virtue of $\tilde{\alpha}_{i,\psi} = \lambda_{i,\psi}\alpha - \kappa_{i,\psi} \circ \lambda_{i,\psi}$ that can be verified from (118) with $\kappa = \overline{\kappa}$, substituting (119) and (121) into (134) and (135) in the proof of Proposition 6, there exists $\epsilon > 0$ satisfying (90). Hence, (i) is proved. Finally, following the arguments used to prove Propositions 6 and 7, the proof of (ii) is completed.

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