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ORIGINAL ARTICLE

Quantized stabilization of strict-feedback nonlinear systems based on ISS cyclic-small-gain theorem

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Abstract This paper proposes a new tool for quantized nonlinear control design of dynamic systems transformable into the dynamically perturbed strict-feedback form. To address the technical challenges arising from measurement and actuator quantization, a new approach based on set-valued maps is developed to transform the closed-loop quantized system into a large-scale system composed of input-to-state stable (ISS) subsystems. For each ISS subsystem, the inputs consist of quantization errors and interacting states, and moreover, the ISS gains can be assigned arbitrarily. Then, the recently developed cyclic-small-gain theorem is employed to guarantee input-to-state stability with respect to quantization errors and to construct an ISS-Lyapunov function for the closed-loop quantized system. Interestingly, it is shown that, under some realistic assumptions, any n-dimensional dynamically perturbed strict-feedback nonlinear system can be globally practically stabilized by a quantized control law using $2n$ three-level dynamic quantizers.

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1 Introduction

The quantized stabilization problem is motivated by the practical requirement that the signals of the control devices, including controller, actuators and sensors, are quantized before transmission through digital communication channels to other devices in the closed-loop control system. Here, a quantizer can be mathematically modeled as a discontinuous function from a continuous region to a discrete set of numbers. A practical quantizer has a finite number of quantization levels, and the quantized signal is saturated if the original signal is outside the range of the quantizer. This motivates the dynamic quantization [1], by which the quantization levels are appropriately scaled in the process of quantized control to achieve global stabilization with finite-level quantizers. In this paper, we propose a new design tool for quantized feedback stabilization of dynamically perturbed strict-feedback systems, an important class of nonlinear systems widely studied in the nonlinear control literature.

Quantization errors cause major problems for quantized stabilization; see, for instance, [1–5] for linear systems and [6–12] for nonlinear systems. In particular, the book [10] discussed the relation between space discretization and perturbation. The recent papers [4,6,7,12] characterize the quantization error as a sector bounded perturbation. Reference [6] studied the conditions under which the quantization error does not cancel the stabilizing effect of a dissipativity-based control law. Reference [7] achieved semi-global quantized stabilization for nonlinear systems based on the semi-global stabilization approach in [13]. In [14], the problem caused by the finite-level quantizers was addressed for feedforward (upper-triangular) systems. Using the idea of scaling quantization levels, the authors of [1,9,11,15] studied dynamically quantized control with finite-level quantizers.

Quantized stabilization is closely related to robust stabilization by treating the state quantization errors as measurement errors. Despite its importance and relevance to many practical control problems, robust nonlinear stabilization with measurement errors has not been paid enough attention. In [16], it was discovered that if a system can be stabilized, then it can also be stabilized insensitively to small measurement errors. The authors of [17] introduced a robust nonlinear control design approach based on backstepping methodology (see e.g., [18]) and flattened Lyapunov functions to deal with bounded measurement errors. But with the method in [17], the influence of the measurement errors grows with the order of the system, which is daunting for high-order systems and for its further application to quantized stabilization.

The concept of input-to-state stability (ISS) invented by Sontag (see [19] for an excellent tutorial on ISS), appears to be fundamental to describe the robust stability of closed-loop quantized systems with respect to quantization errors in several results; see e.g., Liberzon and Nešić's dynamically quantized control results [9,11,20,21]. With the ISS small-gain theorem in [22] as a tool, reference [23] established a unified framework for control design of nonlinear systems with quantization and time scheduling via an emulation-like approach. Based on the small-gain theorem, the quantized controller should usually be designed to make the control error caused by quantization error smaller than the range of the quantizer, to ascertain global stabilization with finite-level quantization [11,23].

The objective of this paper is to propose a design tool for global stabilization of strict-feedback nonlinear systems with both state quantization and actuator quantization, by means of the recent developments in cyclic-small-gain theorems (see, e.g., [24,25]). Our recent contributions to quantized control [12,26] demonstrate the usefulness of the cyclic-small-gain theorem. For other related network small-gain theorems, the reader should consult [22,27–32]. In our novel controller design procedure, we will introduce a new set-valued map method to overcome the problems caused by the nonlinearity and discontinuity of the quantizers as well as the presence of dynamic uncertainty. Consequently, differential inclusions will be adopted to represent the closed-loop quantized system, and the motion of the closed-loop quantized system will be described with the extended Filippov solution introduced in [33,34]; see also [35] for the concept of Filippov solution. The gain assignment technique originally proposed in [22] will have to be further modified to adapt to the specifically designed set-valued maps such that the closed-loop quantized system can be transformed into a large-scale system composed of ISS subsystems. Moreover, the ISS gains of each subsystem will be assigned appropriately to satisfy the cyclic-small-gain condition and to guarantee the ISS of the closed-loop quantized system with respect to quantization errors. An ISS-Lyapunov function will also be constructed to evaluate the influence of the quantization error. Differently from the previously known results, the adjustment of the quantization levels will result in jumps of the state of the closed-loop quantized system. The dynamic quantization design method introduced in [11] will be modified for global quantized stabilization based on the ISS analysis of a specific discrete-time nonlinear system.

The rest of this paper is organized as follows. Section 2 presents some mathematical preliminaries. Section 3 describes the problem formulation and gives an expected quantized controller structure. In Sect. 4, we recursively design set-valued maps for the class of strict-feedback systems with quantized signals. Section 5 is devoted to the synthesis of dynamic quantization. Section 6 presents a remark on measurement feedback control as a by-product of the main result. Section 7 presents some conclusions and discusses some future directions. The proofs of some technical lemmas are in the Appendix.

2 Mathematical preliminaries

To make the paper self-contained, recall that a function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is positive definite if $\gamma(s) > 0$ for all $s > 0$ and $\gamma(0) = 0$. $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathscr{K}$ function (denoted by $\gamma \in \mathscr{K}$) if it is continuous, strictly increasing and $\gamma(0) = 0$; it is a $\mathscr{K}_\infty$ function (denoted by $\gamma \in \mathscr{K}_\infty$) if it is a $\mathscr{K}$ function and also satisfies $\gamma(s) \to \infty$ as $s \to \infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathscr{KL}$ (denoted by $\beta \in \mathscr{KL}$) if $\beta(\cdot, t)$ is a $\mathscr{K}$ function for each fixed t and $\beta(s, t)$ is decreasing to zero as $t \to \infty$ for each $s \in \mathbb{R}_+$. Id represents the identity function. For $\gamma_1, \gamma_2 \in \mathscr{K}$, inequality $\gamma_1 < \gamma_2$ means $\gamma_1(s) < \gamma_2(s)$ for all $s > 0$.

The ISS cyclic-small-gain theorem proposed in [24,25,28] is a tool to analyze the ISS property of large-scale nonlinear systems in the following form:

$$\dot{x}_i = f_i\left(x, u_i\right), \quad i = 1, \ldots, N \tag{1}$$

where $x_i \in \mathbb{R}^{n_i}$, $x = [x_1^{\mathrm{T}}, \ldots, x_N^{\mathrm{T}}]^{\mathrm{T}}$, $u_i \in \mathbb{R}^{m_i}$ and $f_i : \mathbb{R}^{n+m_i} \to \mathbb{R}^{n_i}$ with $n = \Sigma_{j=1}^{N} n_j$ is a locally Lipschitz map.

Although smooth ISS-Lyapunov functions have usually been used in the literature, for large-scale system (1) with locally Lipschitz dynamics, the smoothness requirement on the ISS-Lyapunov function can be reduced to locally Lipschitz continuity [34,31]. According to Rademacher's theorem (see, e.g., [36, p. 216]), locally Lipschitz continuity implies continuous differentiability almost everywhere.

It is assumed that each x_i-subsystem ($i = 1, \ldots, N$) in (1) admits a locally Lipschitz ISS-Lyapunov function $V_i : \mathbb{R}^{n_i} \to \mathbb{R}$ satisfying

1. There exist $\underline{\alpha}_i, \overline{\alpha}_i \in \mathscr{K}_\infty$ such that

$$\underline{\alpha}_i(|x_i|) \le V_i(x_i) \le \overline{\alpha}_i(|x_i|), \quad \forall x_i \in \mathbb{R}^{n_i}; \tag{2}$$

2. There exist $\gamma_{ij} \in \mathscr{K} \cup \{0\}$ ($j \neq i$) and $\gamma_i^u \in \mathscr{K} \cup \{0\}$ such that

$$\begin{aligned} V_i\left(x_i\right) \ge \max_{j=1,\ldots,N;\, j \neq i} &\left\{\gamma_{ij}\left(V_j\left(x_j\right)\right), \gamma_i^u\left(|u_i|\right)\right\} \\ &\Rightarrow \nabla V_i\left(x_i\right) f_i\left(x, u_i\right) \le -\alpha_i\left(V_i\left(x_i\right)\right) \end{aligned} \tag{3}$$

wherever ∇V_i exists, where α_i is continuous and positive definite.

The functions γ_{ij}, γ_i^u are known as the ISS gains of the subsystems. The following theorem presents a cyclic-small-gain condition to guarantee the ISS property of the large-scale system (1) with state x and input $u = [u_1^{\mathrm{T}}, \ldots, u_N^{\mathrm{T}}]^{\mathrm{T}}$.

Theorem 1 [24] *Consider the large-scale nonlinear system* (1). *Assume each x_i-subsystem admits an ISS-Lyapunov function V_i satisfying* (2) *and* (3). *Then, the large-scale nonlinear system* (1) *is ISS if for $r = 2, \ldots, N$,*

$$\gamma_{i_1 i_2} \circ \gamma_{i_2 i_3} \circ \cdots \circ \gamma_{i_r i_1} < \mathrm{Id} \tag{4}$$

where $1 \le i_k \le N$ and $i_k \neq i_{k'}$ if $k \neq k'$ for $1 \le k \le r$.

By considering the subsystems (1) as vertices and the gains as the weights of directed arcs (i.e., connections between subsystems), the interconnection structure of the large-scale nonlinear system can be represented with a system digraph. Condition (4) is called cyclic-small-gain condition and means that the composition of the ISS gains along every simple cycle in the large-scale nonlinear system is less than the identity function Id.

For the ISS gains γ_{ij}'s ($1 \le i \le N$, $j \neq i$) satisfying condition (4), according to Lemma A.1 of [27], we can find $\mathscr{K}_\infty$ functions $\hat{\gamma}_{ij}$'s ($1 \le i \le N$, $j \neq i$) which are continuously differentiable on $(0, \infty)$ and slightly larger than the corresponding

γ_{ij}'s such that condition (4) still holds by replacing the γ_{ij}'s with the $\hat{\gamma}_{ij}$'s. Motivated by the ISS-Lyapunov function construction in [27], a locally Lipschitz ISS-Lyapunov function can be constructed for the large-scale system (1) as

$$V(x) = \max_{i=1,\ldots,n} \{\sigma_i(V_i(x_i))\} \tag{5}$$

where σ_i's are specific compositions of the $\hat{\gamma}_{(\cdot)}$'s.

The influence of the external input u can be represented as

$$\theta(u) = \max_{i=1,\ldots,n} \{\sigma_i \circ \gamma_i^u(|u_i|)\}. \tag{6}$$

Denote $f(x,u) = [f_1^{\mathrm{T}}(x,u_1),\ldots,f_N^{\mathrm{T}}(x,u_N)]^{\mathrm{T}}$. With the Lyapunov-based ISS cyclic-small-gain theorem presented in [25], we have

$$V(x) \geq \theta(u) \Rightarrow \nabla V(x) f(x,u) \leq -\alpha(V(x)) \tag{7}$$

wherever ∇V exists, with α being a continuous and positive definite function.

The type of ISS-Lyapunov functions (5) was used by [31] for the construction of an ISS-Lyapunov function for the matrix-small-gain theorem. Also notice that (5) was utilized in [25] in the construction of ISS-Lyapunov function for the ISS cyclic-small-gain theorem.

Recently, the authors of [33,34] extended the concepts of ISS and ISS-Lyapunov function to discontinuous systems and also proposed an extended Filippov solution for interconnected discontinuous systems described by differential inclusions. Based on the concept of extended Filippov solution, a discontinuous version of the ISS small-gain theorem was developed. Motivated by the results in [33,34], a discontinuous version of the cyclic-small-gain theorem can also be developed for large-scale discontinuous systems:

$$\dot{x}_i \in F_i(x,u_i), \quad i = 1,\ldots,N \tag{8}$$

where $F_i : \mathbb{R}^{n+m_i} \rightsquigarrow \mathbb{R}^{n_i}$ is a convex, compact and upper semi-continuous set-valued map and the variables are defined in the same way as for (1).

References [33,34] considered the case of multiple Lyapunov functions to carry out a general ISS theory for discontinuous nonlinear systems. Here we just consider a simplified case where each x_i-subsystem in (8) admits a common Lyapunov function V_i satisfying (2) and

$$\begin{aligned} V_i(x_i) \geq & \max_{j=1,\ldots,N;\, j\neq i} \{\gamma_{ij}(V_j(x_j)), \gamma_i^u(|u_i|)\} \\ & \Rightarrow \max_{f_i \in F_i(x,u_i)} \nabla V_i(x_i) f_i \leq -\alpha_i(V_i(x_i)) \end{aligned} \tag{9}$$

wherever ∇V_i exists, which is a modification of (3).

With the cyclic-small-gain condition (4) satisfied by the large-scale discontinuous system, an ISS-Lyapunov function V can be constructed in a quite similar way as in (5), and property (7) should be modified as

$$V(x) \geq \theta(u) \Rightarrow \max_{f \in F(x,u)} \nabla V(x) f \leq -\alpha(V(x)) \tag{10}$$

wherever ∇V exists, with $F(x, u) = [F_1^{\mathrm{T}}(x, u_1), \ldots, F_N^{\mathrm{T}}(x, u_N)]^{\mathrm{T}}$.

As we will show in this paper, the above discontinuous variant plays a central role in the recursive quantized control design for dynamic networks or high-dimensional nonlinear systems transformable into a dynamic network.

3 Problem formulation

3.1 System description

In this paper, we study quantized control of the following dynamically perturbed strict-feedback nonlinear system:

$$\dot{z} = g(z, x_1) \tag{11}$$

$$\dot{x}_i = x_{i+1} + \Delta_i(\bar{x}_i, z), \quad i = 1, \ldots, n \tag{12}$$

$$x_{n+1} \overset{\text{def}}{=} u \tag{13}$$

where $[x_1, \ldots, x_n]^{\mathrm{T}} := x \in \mathbb{R}^n$ is the measurable state, $z \in \mathbb{R}^{n_z}$ represents the state of the inverse dynamics and is not measurable, $u \in \mathbb{R}$ is the control input, $\bar{x}_i = [x_1, \ldots, x_i]^{\mathrm{T}}$, and Δ_i's ($i = 1, \ldots, n$) are unknown locally Lipschitz continuous functions. We study the general case in which both the measurement x and the control input u are quantized.

Throughout the paper, the following assumptions are made on system (11)–(13).

Assumption 1 System (11)–(13) with $u = 0$ is forward complete and small-time final-state norm-observable with x as the output, i.e., for $u = 0$,

$$\begin{aligned} &\forall t_d > 0 \ \exists \varphi \in \mathscr{K}_\infty \text{ such that} \\ &|X(t_d)| \leq \varphi(\|x\|_{[0,t_d]}), \quad \forall X(0) \in \mathbb{R}^{n+n_z} \end{aligned} \tag{14}$$

where $X := [z^T, x^T]^T$.

Remark 1 Reference [37] discussed the equivalent characterizations of initial-state norm-observability, final-state norm-observability and $\mathscr{K}\mathscr{L}$ norm-observability for forward complete (or unboundedness observable) nonlinear systems with external inputs. In Assumption 1, we just use the concept of small-time final-state norm-observability for nonlinear systems with no external inputs.

Remark 2 Assumption 1 is needed for global quantized stabilization; see [21]. However, it is important to mention that Assumption 1 is not needed if a bound is known on the initial state of system (11)–(13). In this case, only semi-global quantized stabilization can be achieved. Also, see [15,23].

Assumption 2 For each Δ_i with $i = 1, \ldots, n$, there exists a known $\lambda_{\Delta_i} \in \mathscr{K}_\infty$ such that for all $\bar{x}_i, z$,

$$|\Delta_i(\bar{x}_i, z)| \leq \lambda_{\Delta_i}(|(\bar{x}_i, z)|). \tag{15}$$

Assumption 3 The z-subsystem (11) with x_1 as the input admits a locally Lipschitz ISS-Lyapunov function $V_0 : \mathbb{R}^{n_z} \to \mathbb{R}_+$ satisfying

1. There exist $\underline{\alpha}_0, \overline{\alpha}_0 \in \mathscr{K}_\infty$ such that

$$\underline{\alpha}_0(|z|) \leq V_0(z) \leq \overline{\alpha}_0(|z|), \quad \forall z \in \mathbb{R}^{n_z}; \tag{16}$$

2. There exist a $\chi_z^{x_1} \in \mathscr{K}$ and a continuous and positive definite α_0 such that

$$V_0(z) \geq \chi_z^{x_1}(|x_1|) \Rightarrow \nabla V_0(z) g(z, x_1) \leq -\alpha_0(V_0(z)) \tag{17}$$

   wherever ∇V_0 exists.

Remark 3 Under Assumption 3, system (11)–(13) represents an important class of minimum-phase nonlinear systems, which have been studied extensively by many authors in the context of (non-quantized) robust and adaptive nonlinear control. The reader may consult [13,18,38,39] and references therein for the details.

3.2 Quantization

In this paper, a quantizer $q(r, \mu)$ with original signal $r \in \mathbb{R}$ and variable $\mu > 0$ is defined as $q(r, \mu) = \mu q^o(r/\mu)$, where $q^o : \mathbb{R} \to \mathbb{R}$ is a piecewise constant function for which there exists a constant $M > 0$ such that $|q^o(a) - M| \leq 1$ if $a > M$; $|q^o(a) - a| \leq 1$ if $|a| \leq M$; $|q^o(a) + M| \leq 1$ if $a < -M$; $q^o(0) = 0$.

Then, quantizer $q(r, \mu)$ satisfies:

$$|q(r, \mu) - M\mu| \leq \mu, \quad \text{if } r > M\mu; \tag{18}$$

$$|q(r, \mu) - r| \leq \mu, \quad \text{if } |r| \leq M\mu; \tag{19}$$

$$|q(r, \mu) + M\mu| \leq \mu, \quad \text{if } r < -M\mu; \tag{20}$$

$$q(0, \mu) = 0. \tag{21}$$

$M\mu$ is the quantization range of quantizer $q(r, \mu)$ and μ represents the quantization error bound (i.e., the maximum of $|q(r, \mu) - r|$ when $|r| \leq M\mu$). In dynamic quantization design, μ is known as the "zooming" variable. Given fixed M, the basic idea of dynamic quantization is to dynamically update μ (and thus $M\mu$) in discrete-time to improve the control performance in the existence of the quantization error. Increasing μ, referred to as "zooming-out", enlarges the quantization error bound μ and the quantization range $M\mu$. Decreasing μ, referred to as "zooming-in", reduces the quantization error bound μ and the quantization range $M\mu$.

Remark 4 In several previously known quantized control results (see e.g., [15]), two positive parameters, say M', δ', are used to formulate a quantizer q' as: $|q'(r,\mu')-r| \leq \delta'\mu'$ if $|r| \leq M'\mu'$; $|q'(r,\mu')| > (M'-\delta')\mu'$ if $|r| > M'\mu'$. Indeed, if we define $M = M'/\delta'$, $\mu = \delta'\mu'$ and a new quantizer $q(r,\mu) = q'(r,\mu/\delta')$, then we can see properties (18)–(20) hold for the new quantizer q. Moreover, properties (18) and (20) explicitly represent the saturation property of the quantizer. It will be demonstrated that the explicit description of saturation in (18) and (20) is necessary in our recursive control design later, in which we try to take the saturation property into account.

Remark 5 It is shown in [40] that the simplest three-level quantizer satisfies the properties (18)–(20) with $M = 3$, and a larger number of levels means a larger M.

3.3 Quantized controller structure and control objective

In this paper, we introduce a new quantized control structure following an ISS small-gain design approach. Based on the ISS small-gain design method proposed in [41], we can recursively design a non-quantized controller for system (11)–(13) as

$$v_i = \breve{\kappa}_i(x_i - v_{i-1}), \quad i = 1, \ldots, n-1 \tag{22}$$

$$u = \breve{\kappa}_n(x_n - v_{n-1}) \tag{23}$$

where $v_0 = 0$ and $\breve{\kappa}_i$'s are appropriately designed continuous functions for $i = 1, \ldots, n$. The maps defined in (22) are usually known as virtual control laws.

Our solution to the quantized control problem of system (11)–(13) is to add quantizers before and after each (virtual) control law defined in (22) through a recursive design approach.

Following this idea, the quantized controller would be in the following form:

$$v_i = q_{i2}(\kappa_i(q_{i1}(x_i - v_{i-1}, \mu_{i1})), \mu_{i2}), \quad i = 1, \ldots, n-1 \tag{24}$$

$$u = q_{n2}(\kappa_n(q_{n1}(x_n - v_{n-1}, \mu_{n1})), \mu_{n2}) \tag{25}$$

where $v_0 = 0$, the q_{ij}'s are quantizers with zooming variables μ_{ij}'s for $i = 1, \ldots, n$, $j = 1, 2$, and the κ_i's for $i = 1, \ldots, n$ are nonlinear functions. The κ_i in (24)–(25) is not necessarily the same as the $\breve{\kappa}_i$ in (22)–(23) because of the implementation of the quantizers.

Remark 6 The new quantized controller structure (24)–(25) leads to no restrictiveness in practical applications. Indeed, the quantized control structure making use of adjustable centers of the quantizers can also be found in the very recent work [42].

We make the following assumption on all the employed quantizers in (24)–(25).

Assumption 4 Each quantizer q_{ij} for $i = 1, \ldots, n$, $j = 1, 2$ with zooming variable μ_{ij} satisfies

$$|q_{ij}(r, \mu_{ij}) - M_{ij}\mu_{ij}| \leq \mu_{ij}, \quad \text{if } r > M_{ij}\mu_{ij}; \tag{26}$$
$$|q_{ij}(r, \mu_{ij}) - r| \leq \mu_{ij}, \quad \text{if } |r| \leq M_{ij}\mu_{ij}; \tag{27}$$
$$|q_{ij}(r, \mu_{ij}) + M_{ij}\mu_{ij}| \leq \mu_{ij}, \quad \text{if } r < -M_{ij}\mu_{ij}; \tag{28}$$
$$q_{ij}(0, \mu_{ij}) = 0. \tag{29}$$

where $M_{i1} > 2$ and $M_{i2} > 1$.

In dynamic quantization, each zooming variable $\mu_{ij} > 0$ for $i = 1, \ldots, n$, $j = 1, 2$ is a piecewise constant signal and updated in discrete-time. Without loss of generality, we assume that the piecewise constant zooming variables are right-continuous on the time-line. Motivated by Liberzon [15], we will still design a dynamic quantization composed of two stages, zooming-out stage and zooming-in stage. In this paper, we consider the case where the update time sequences for all the zooming variables are the same and denoted by $\{t_0, t_1, t_2, \ldots\}$, in which $t_{k+1} - t_k = t_d$ with constant $t_d > 0$ for $k \in \mathbb{Z}_+$.

The update dynamics of each μ_{ij} is expected to be in the following form:

$$\mu_{ij}(t_{k+1}) = Q_{ij}(\mu_{ij}(t_k)), \quad k \in \mathbb{Z}_+. \tag{30}$$

In the zooming-out stage, $Q_{ij} = Q_{ij}^{\text{out}}$; in the zooming-in stage, $Q_{ij} = Q_{ij}^{\text{in}}$.

The objective of this paper is to design a quantized controller in the form of (24)–(25) with dynamic quantization (30) to globally stabilize system (11)–(13) such that all the signals including the state x in the closed-loop quantized system are bounded, and moreover, to steer x_1 to an arbitrarily small neighborhood of the origin.

The configuration of the total quantized control system is shown in Fig. 1.

4 Recursive control design with set-valued maps–static quantization

A technical obstacle for quantized feedback control design is that the quantized control system in question must be made robust with respect to the quantization errors. The nonlinearity and dimensionality of system (11)–(13) and the saturation and discontinuity of quantization cause the major difficulties.

The objective of this section is to develop a recursive design procedure for κ_i in (24)–(25) by taking into account the effects of static quantization, such that the closed-loop quantized system admits nested invariant sets for further dynamic quantization designs. In Sect. 4.1, we introduce set-valued maps S_i depending on κ_i ($1 \leq i \leq n$) to a recursive design to deal with the quantization errors and the closed-loop system will be transformed into a large-scale system composed of e_i-subsystems represented by differential inclusions. Section 4.2 presents a modified gain assignment technique to render the e_i-subsystems ISS with *any* specified ISS gains by appropriately choosing the set-valued maps. In Sect. 4.3, we show that we can find quantized control laws

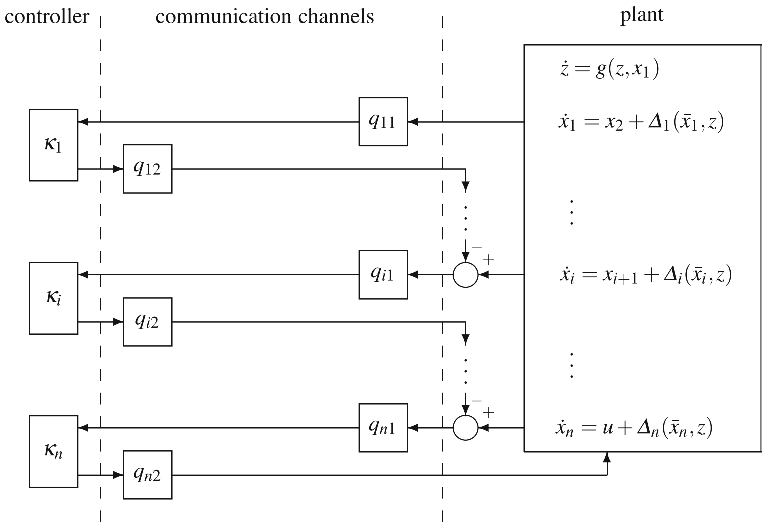


Fig. 1 The quantized control structure

in the form of (24)–(25) belonging to the chosen set-valued maps. The cyclic-small-gain theorem is employed to guarantee the ISS property and the existence of nested invariant sets for the closed-loop quantized system in Sect. 4.4.

In this section, we make the following assumption on the zooming variables.

Assumption 5 For $i = 1, \ldots, n$, $j = 1, 2$, the zooming variable μ_{ij} is constant with respect to time.

4.1 Main steps of the recursive design

4.1.1 Initial step: the e_1-subsystem

Let $e_1 = x_1$. The e_1-subsystem is in the following form:

$$\dot e_1 = x_2 + \Delta_1(\bar x_1, z). \tag{31}$$

Define a set-valued map S_1 as

$$S_1(\bar x_1, \mu_{11}, \mu_{12}) = \{\kappa_1(x_1 + b_{11}) + b_{12} : |b_{11}| \le \max\{c_{11}|e_1|, \mu_{11}\}, |b_{12}| \le \mu_{12}\} \tag{32}$$

where κ_1 is a continuously differentiable, odd, strictly decreasing and radially unbounded function and $0 < c_{11} < 1$ is a constant, both of which will be determined later. It should be noted that b_{11}, b_{12} defined in (32) are used as auxiliary variables to define set-valued map S_1.

Define

$$e_2 = \mathbf{d}(x_2, S_1(\bar{x}_1, \mu_{11}, \mu_{12})) \tag{33}$$

where $\mathbf{d}$ represents the directed distance from a point to an interval and is defined as

$$\mathbf{d}(\xi, \Omega) = \begin{cases} \xi - \max \Omega, & \text{if } \xi > \max \Omega; \\ \xi - \min \Omega, & \text{if } \xi < \min \Omega; \\ 0, & \text{otherwise.} \end{cases} \tag{34}$$

where $\xi \in \mathbb{R}$ and $\Omega \subset \mathbb{R}$ is an interval.

We rewrite the e_1-subsystem (31) as

$$\dot{e}_1 = x_2 - e_2 + \Delta_1(\bar{x}_1, z) + e_2 \tag{35}$$

where we have $x_2 - e_2 \in S_1(\bar{x}_1, \mu_{11}, \mu_{12})$ from (34).

Remark 7 It is necessary to give a more detailed description of the set-valued map S_1. Consider the first-order nonlinear system $\dot{e}_1 = x_2 + \Delta_1(\bar{x}_1, z)$. Note that $e_1 = x_1$. With the gain assignment technique in [22,41,43], we can design a control law $x_2 = \kappa_1(x_1)$ to stabilize the e_1-system. In the existence of quantization errors, control law $x_2 = \kappa_1(x_1)$ should be modified as $x_2 = q_{12}(\kappa_1(q_{11}(x_1, \mu_{11})), \mu_{12})$. Clearly, the set-valued map S_1 takes into account the quantization errors of both quantizers q_{11} and q_{12}. In the control design procedure below, we will recursively design new set-valued maps and define new subsystems based on a similar idea.

Remark 8 In our recent paper [12], we employed set-valued maps to cover the sector-bounded uncertainties caused by logarithmic quantizers. In [12], the saturation property of the quantizers was not taken into account and the quantization levels are fixed. Moreover, in [12], the quantizers are directly connected to the state measurements, which is different from the quantized control structure (24)–(25).

4.1.2 Recursive step: the e_i-subsystems

Denote $\bar{\mu}_{i1} = [\mu_{11}, \ldots, \mu_{i1}]^{\mathrm{T}}$ and $\bar{\mu}_{i2} = [\mu_{12}, \ldots, \mu_{i2}]^{\mathrm{T}}$ for $i = 1, \ldots, n$. For each $i = 2, \ldots, n$, define a set-valued map S_i as

$$\begin{aligned} S_i(\bar{x}_i, \bar{\mu}_{i1}, \bar{\mu}_{i2}) = \{&\kappa_i(x_i - \varsigma_{i-1} + b_{i1}) + b_{i2} : \varsigma_{i-1} \in S_{i-1}(\bar{x}_{i-1}, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2}), \\ &|b_{i1}| \leq \max\{c_{i1}|e_i|, \mu_{i1}\}, |b_{i2}| \leq \mu_{i2}\} \end{aligned} \tag{36}$$

where κ_i is a continuously differentiable, odd, strictly decreasing and radially unbounded function and $0 < c_{i1} < 1$ is a constant, both of which will be determined later. The definition of S_i guarantees its convexity, compactness and upper semi-continuity of the set-valued map S_i. Here, b_{i1}, b_{i2} are auxiliary variables used to define the set-valued map S_i.

Remark 9 It can be observed that $S_1(\bar{x}_1, \bar{\mu}_{11}, \bar{\mu}_{12})$ defined in (32) is in the form of (36) with $S_0(\bar{x}_0, \bar{\mu}_{01}, \bar{\mu}_{02}) := \{0\}$.

For each $i = 2, \ldots, n$, define e_{i+1} as

$$e_{i+1} = \mathbf{d}(x_{i+1}, S_i(\bar{x}_i, \bar{\mu}_{i1}, \bar{\mu}_{i2})). \tag{37}$$

Lemma 1 shows that with the recursive construction of set-valued maps and new state variables in (36) and (37), we can represent the e_i-subsystems for $i = 1, \ldots, n$ with differential inclusions.

Lemma 1 *Consider the $(x_1, \ldots, x_n)$-system in (12)–(13). Under Assumptions 2 and 5, with the definitions in (32), (33), (36) and (37), each e_i-subsystem for $1 \leq i \leq n$ can be represented with the differential inclusion:*

$$\dot{e}_i \in S_i(\bar{x}_i, \bar{\mu}_{i1}, \bar{\mu}_{i2}) + \Phi_i^*(e_{i+1}, \bar{x}_i, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2}, z) \tag{38}$$

where Φ_i^* is a convex, compact and upper semi-continuous set-valued map, and there exists a $\lambda_{\Phi_i^*} \in \mathscr{K}_\infty$ such that for all $(e_{i+1}, \bar{x}_i, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2}, z)$, any $\phi_i^* \in \Phi_i^*(e_{i+1}, \bar{x}_i, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2}, z)$ satisfies

$$|\phi_i^*| \leq \lambda_{\Phi_i^*}(|(\bar{e}_{i+1}, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2}, z)|) \tag{39}$$

where $\bar{e}_i := [e_1, \ldots, e_i]^{\mathrm{T}}$.

The proof of Lemma 1 is in Appendix A.

With Lemma 1, through the recursive design approach, we have transformed the $(x_1, \ldots, x_n)$-system into the new $(e_1, \ldots, e_n)$-system with each e_i-subsystem ($i = 1, \ldots, n$) of form (38). The extended Filippov solution of each e_i-subsystem can be defined with differential inclusion (38) because both set-valued maps S_i and Φ_i^* are convex, compact and upper semi-continuous; see [33,34].

4.2 ISS of the subsystems

For each e_i-subsystem with $i = 1, \ldots, n$, we define the following ISS-Lyapunov function candidate

$$V_i(e_i) = \alpha_V(|e_i|) \tag{40}$$

where $\alpha_V(s) = \frac{1}{2}s^2$ for $s \in \mathbb{R}_+$. For convenience of notations, define $V_{n+1}(e_{n+1}) = \alpha_V(|e_{n+1}|)$. We simply use V_i instead of $V_i(e_i)$ in the following discussions.

Denote $e_0 = z$. Then, we have

$$\dot{e}_0 = g(e_0, e_1). \tag{41}$$

Define $\gamma_{e_0}^{e_1}(s) = \chi_z^{x_1} \circ \alpha_V^{-1}(s)$ for $s \in \mathbb{R}_+$. With Assumption 3 satisfied, we have

$$V_0 \geq \gamma_{e_0}^{e_1}(V_1) \Rightarrow \nabla V_0 g(e_0, e_1) \leq -\alpha_0(V_0). \tag{42}$$

Lemma 2 is a modification of the gain assignment technique employed in the previous ISS small-gain design results [22,41,43], and states that, for $i = 1, \dots, n$, by appropriately designing κ_i, each e_i-subsystem can be rendered to be ISS with V_i defined in (40) as ISS-Lyapunov function.

Lemma 2 *Consider the e_i-subsystem ($i = 1, \dots, n$) in the form of (38) with S_i defined in (32) and (36). Under Assumptions 2 and 5, for any specified constants $\varepsilon_i > 0, \iota_i > 0, 0 < c_{i1}, c_{i2} < 1, \gamma_{e_i}^{e_k} \in \mathscr{K}_\infty$ for $k = 0, \dots, i-1, i+1$, and $\gamma_{e_i}^{\mu_{k1}}, \gamma_{e_i}^{\mu_{k2}} \in \mathscr{K}_\infty$ for $k = 1, \dots, i-1$, one can find a continuously differentiable, odd, strictly decreasing and radially unbounded κ_i for the set-valued map S_i such that the e_i-subsystem is ISS with $V_i(e_i) = \alpha_V(|e_i|) = \frac{1}{2}e_i^2$ as an ISS-Lyapunov function satisfying*

$$\begin{aligned} V_i \geq \max_{k=1,\dots,i-1} & \left\{ \begin{array}{l} \gamma_{e_i}^{e_0}(V_0), \gamma_{e_i}^{e_k}(V_k), \gamma_{e_i}^{e_{i+1}}(V_{i+1}), \\ \gamma_{e_i}^{\mu_{k1}}(\mu_{k1}), \gamma_{e_i}^{\mu_{k2}}(\mu_{k2}), \gamma_{e_i}^{\mu_{i1}}(\mu_{i1}), \gamma_{e_i}^{\mu_{i2}}(\mu_{i2}), \varepsilon_i \end{array} \right\} \\ \Rightarrow \max_{\psi_i \in \Psi_i(e_{i+1}, \bar{x}_i, \bar{\mu}_{i1}, \bar{\mu}_{i2}, z)} & \nabla V_i \psi_i \leq -\iota_i V_i(e_i) \end{aligned} \tag{43}$$

where

$$\gamma_{e_i}^{\mu_{i1}}(s) = \alpha_V\left(\frac{1}{c_{i1}}s\right) \tag{44}$$

$$\gamma_{e_i}^{\mu_{i2}}(s) = \alpha_V\left(\frac{1}{1-c_{i1}}\bar{\kappa}_i^{-1}\left(\frac{1}{c_{i2}}s\right)\right) \tag{45}$$

$$\Psi_i(e_{i+1}, \bar{x}_i, \bar{\mu}_{i1}, \bar{\mu}_{i2}, z) := S_i(\bar{x}_i, \bar{\mu}_{i1}, \bar{\mu}_{i2}) + \Phi_i^*(e_{i+1}, \bar{x}_i, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2}, z) \tag{46}$$

with $\bar{\kappa}_i(s) = |\kappa_i(s)|$ for $s \in \mathbb{R}_+$.

The proof of Lemma 2 is in Appendix B.

Remark 10 The gain assignment lemmas in the earlier papers [22,41,43] deal with nonlinear systems represented with differential equations but not differential inclusions. Furthermore, in our control design for the e_i-subsystem, we deal with the effects of the set-valued map S_{i-1}, the control error $c_{i1}|e_i|$ and the quantization errors μ_{i1} and μ_{i2}. Also, it is worth noting that (43) yields a *common* Lyapunov function for each e_i-subsystem, while the general result on ISS discontinuous systems in [33,34] relies upon multiple Lyapunov functions.

4.3 Quantized controller

In the subsections above, we designed set-valued maps to derive a large-scale system composed of ISS subsystems described by differential inclusions. In this subsection, we show that the quantized control law u in the form of (24)–(25) with the κ_i's defined above belongs to the set-valued map S_n under realizable conditions. In this way, the closed-loop quantized system with the designed quantized control law u can be represented with a large-scale system composed of ISS subsystems.

Recall $\bar{\kappa}_i(s) = |\kappa_i(s)|$ for $s \in \mathbb{R}_+$. Lemma 3 provides conditions under which the quantized control law u in the form of (24)–(25) belongs to the set-valued map S_n.

Lemma 3 *Under Assumption 4, if*

$$\frac{1}{M_{i1}} < c_{i1} \leq 0.5, \quad \frac{1}{M_{i2}} < c_{i2} < 1 \tag{47}$$

for all $i = 1, \ldots, n$ *and if*

$$|e_i| \leq M_{i1}\mu_{i1}, \tag{48}$$

$$\bar{\kappa}_i((1 - c_{i1})|e_i|) \leq M_{i2}\mu_{i2} \tag{49}$$

for all $i = 1, \ldots, n$, *then* v_i *for* $i = 1, \ldots, n-1$ *and* u *defined in* (24)–(25) *satisfy*

$$v_i \in S_i(\bar{x}_i, \bar{\mu}_{i1}, \bar{\mu}_{i2}), \quad i = 1, \ldots, n-1, \tag{50}$$

$$u \in S_n(\bar{x}_n, \bar{\mu}_{n1}, \bar{\mu}_{n2}). \tag{51}$$

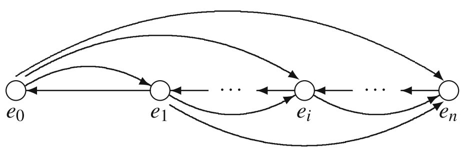
The proof of Lemma 3 is given in Appendix C by fully using the properties of the quantizers and the set-valued maps.

Remark 11 Conditions (48) and (49) imply that the signals $|e_i|$ and $|\bar{\kappa}_i((1 - c_i)|e_i|)|$ should be covered by the quantization ranges $M_{i1}\mu_{i1}$ and $M_{i2}\mu_{i2}$, respectively, such that the quantized control law u belongs to the set-valued map S_n. This is caused by the saturation property of the quantizers.

4.4 Small-gain based synthesis and nested invariant sets of the closed-loop quantized system

Recall $e = [e_0^T, e_1, \ldots, e_n]^{\mathrm{T}}$. By considering the e_i-subsystems as vertices and the gain interconnections between them as directed arcs, the interconnection structure of the $(e_0, e_1, \ldots, e_n)$-system can be represented with a digraph, as shown in Fig. 2. The purpose of this subsection is to fine tune the ISS gains to yield ISS property of the closed-loop quantized system with e as the state by using the cyclic-small-gain theorem.

Fig. 2 The interconnection digraph of the $(e_0, e_1, \ldots, e_n)$-system



Recall $\bar{e}_i = [e_1, \ldots, e_i]^{\mathrm{T}}$. For each $(e_0, \bar{e}_i)$-subsystem ($i = 1, \ldots, n$), given the $(e_0, \bar{e}_{i-1})$-subsystem, by designing the set-valued map S_i for the e_i-subsystem, we can assign the ISS gains from states $e_0, \ldots, e_{i-1}$ to state e_i. According to the recursive design, we assign the ISS gains $\gamma_{e_i}^{e_k}$ for $k = 0, \ldots, i-1$ such that

$$\begin{aligned} \gamma_{e_0}^{e_1} \circ \gamma_{e_1}^{e_2} \circ \gamma_{e_2}^{e_3} \circ \cdots \circ \gamma_{e_{i-1}}^{e_i} \circ \gamma_{e_i}^{e_0} &< \mathrm{Id} \\ \gamma_{e_1}^{e_2} \circ \gamma_{e_2}^{e_3} \circ \cdots \circ \gamma_{e_{i-1}}^{e_i} \circ \gamma_{e_i}^{e_1} &< \mathrm{Id} \\ &\vdots \\ \gamma_{e_{i-1}}^{e_i} \circ \gamma_{e_i}^{e_{i-1}} &< \mathrm{Id} \end{aligned} \tag{52}$$

where Id represents the identity function. Applying this reasoning repeatedly, we can guarantee (52) for all $i = 1, \ldots, n$. In this way, the e-system satisfies the cyclic-small-gain condition.

In the digraph of the e-system shown in Fig. 2, the e_1-subsystem is reachable from the subsystems of $e_0, e_2, \ldots, e_n$, i.e., there are sequences of directed arcs from the subsystems of $e_0, e_2, \ldots, e_n$ to the e_1-subsystem.

Motivated by the ISS-Lyapunov function construction in [27], we construct an ISS-Lyapunov function candidate for the e-system as

$$V(e) = \max_{i=0,\ldots,n} \{\sigma_i(V_i(e_i))\} \tag{53}$$

with $\sigma_1(s) = s$, $\sigma_i(s) = \hat{\gamma}_{e_1}^{e_2} \circ \cdots \circ \hat{\gamma}_{e_{i-1}}^{e_i}(s)$ ($i = 2, \ldots, n$) and $\sigma_0(s) = \max_{i=1,\ldots,n}\{\sigma_i \circ \hat{\gamma}_{e_i}^{e_0}(s)\}$ for $s \in \mathbb{R}_+$, where the $\hat{\gamma}_{(\cdot)}^{(\cdot)}$'s are $\mathscr{K}_\infty$ functions continuously differentiable on $(0, \infty)$ and slightly larger than the corresponding $\gamma_{(\cdot)}^{(\cdot)}$'s, and still satisfy the cyclic-small-gain condition.

The following lemma states that we can appropriately choose the κ_i's for the set-valued maps S_i's such that the cyclic-small-gain condition (52) is satisfied and the closed-loop quantized system with state e admits specific ISS properties.

Lemma 4 *Consider the e-system composed of the e_i-subsystems in the form of* (41) *and* (38) *satisfying* (42) *and* (43)*. If the ISS gains defined in* (42) *and* (43) *satisfy* (52) *for all $i = 1, \ldots, n$ and if $u \in S_n(\bar{x}_n, \bar{\mu}_{n1}, \bar{\mu}_{n2})$, then the ISS-Lyapunov function candidate V defined in* (53) *for the e-system satisfies*

$$V(e) \geq \theta(\bar{\mu}_{n1}, \bar{\mu}_{n2}, \bar{\varepsilon}_n) \Rightarrow \max_{\psi \in \Psi(e,x,\bar{\mu}_{n1},\bar{\mu}_{n2})} \nabla V(e)\psi \leq -\alpha(V(e)) \tag{54}$$

wherever ∇V exists, where α is a continuous and positive definite function, and

$$\theta(\bar{\mu}_{n1}, \bar{\mu}_{n2}, \bar{\varepsilon}_n) := \max_{i=1,\ldots,n} \left\{\sigma_i \left(\max_{k=1,\ldots,i} \{\gamma_{e_i}^{\mu_{k1}}(\mu_{k1}), \gamma_{e_i}^{\mu_{k2}}(\mu_{k2}), \varepsilon_i\}\right)\right\} \tag{55}$$

$$\Psi(e, x, \bar{\mu}_{n1}, \bar{\mu}_{n2}) := [\{g^{\mathrm{T}}(e_0, e_1)\}, \Psi_1(e_2, \bar{x}_1, \bar{\mu}_{11}, \bar{\mu}_{12}), \ldots, \Psi_n(0, \bar{x}_n, \bar{\mu}_{n1}, \bar{\mu}_{n2})]^{\mathrm{T}} \tag{56}$$

with $\bar{\varepsilon}_n := [\varepsilon_1, \ldots, \varepsilon_n]^{\mathrm{T}}$.

Proof In the case of $u \in S_n(\bar{x}_n, \bar{\mu}_{n1}, \bar{\mu}_{n2})$, we have $e_{n+1} = 0$ and $V_{n+1}(e_{n+1}) = 0$. With the cyclic-small-gain condition (52) satisfied for all $i = 1, \ldots, n$, by directly combining the methods in [33,34] and the cyclic-small-gain theorem in [25], as reviewed in Sect. 2, (54) can be proved. This ends the proof.

For specified σ_i for $i = 1, \ldots, n$, if we can design the $\gamma_{e_i}^{\mu_{k1}}$'s ($k = 1, \ldots, i-1$) and the $\gamma_{e_i}^{\mu_{k2}}$'s ($k = 1, \ldots, i-1$) small enough, we can achieve

$$\theta(\bar{\mu}_{n1}, \bar{\mu}_{n2}, \bar{\varepsilon}_n) = \max_{i=1,\ldots,n} \left\{\sigma_i \circ \gamma_{e_i}^{\mu_{i1}}(\mu_{i1}), \sigma_i \circ \gamma_{e_i}^{\mu_{i2}}(\mu_{i2}), \sigma_i(\varepsilon_i)\right\} \tag{57}$$

for all $\mu_{i1}, \mu_{i2}, \varepsilon_i > 0$ for $i = 1, \ldots, n$.

As pointed out in [15], nested invariant sets with sizes determined by the zooming variables $\bar{\mu}_{n1}$ and $\bar{\mu}_{n2}$ are vital for the implementation of dynamic quantization. Lemma 5 summarizes this section by showing the existence of the nested invariant sets for the closed-loop quantized system designed based on Lemmas 1–4.

Define

$$B_1(\bar{\mu}_{n1}, \bar{\mu}_{n2}) = \max_{i=1,\ldots,n} \left\{ \begin{array}{l} \sigma_i \circ \alpha_V(M_{i1}\mu_{i1}), \\ \sigma_i \circ \alpha_V\left(\frac{1}{1-c_{i1}}\bar{\kappa}_i^{-1}(M_{i2}\mu_{i2})\right) \end{array} \right\}, \tag{58}$$

$$B_2(\bar{\mu}_{n1}, \bar{\mu}_{n2}) = \max_{i=1,\ldots,n} \left\{ \begin{array}{l} \sigma_i \circ \alpha_V\left(\frac{1}{c_{i1}}\mu_{i1}\right), \\ \sigma_i \circ \alpha_V\left(\frac{1}{1-c_{i1}}\bar{\kappa}_i^{-1}\left(\frac{1}{c_{i2}}\mu_{i2}\right)\right) \end{array} \right\}. \tag{59}$$

Lemma 5 *Consider the quantized control system consisting of the plant* (11)–(13) *and the quantized control law* (24)–(25)*. Under Assumptions* 2, 3, 4 *and* 5*, the closed-loop quantized system can be transformed into a large-scale system composed of e_i-subsystems in the form of* (41) *and* (38)*, and for specified constants c_{i1}, c_{i2} satisfying* (47) *for $i = 1, \ldots, n$, specified ISS gains $\gamma_{e_k}^{e_{k'}}$ ($k \neq k'$) satisfying the cyclic-small-gain condition* (52) *for all $i = 1, \ldots, n$, specified ISS gains $\gamma_{e_i}^{\mu_{k1}}, \gamma_{e_i}^{\mu_{k2}}$ for $i = 1, \ldots, n, k = 1, \ldots, i-1$ satisfying* (57) *and specified arbitrarily small constants ε_i for $i = 1, \ldots, n$, we can find continuously differentiable, odd, strictly decreasing and radially unbounded functions κ_i for $i = 1, \ldots, n$ such that* (43) *holds for $i = 1, \ldots, n$. Moreover, if*

$$\begin{aligned} \sigma_i \circ \alpha_V(M_{i1}\mu_{i1}) &= \sigma_i \circ \alpha_V\left(\frac{1}{1-c_{i1}}\bar{\kappa}_i^{-1}(M_{i2}\mu_{i2})\right) \\ &= \sigma_j \circ \alpha_V(M_{j1}\mu_{j1}) = \sigma_j \circ \alpha_V\left(\frac{1}{1-c_{j1}}\bar{\kappa}_j^{-1}(M_{j2}\mu_{j2})\right) \end{aligned} \tag{60}$$

for all $i, j = 1, \ldots, n$ and if

$$B_1(\bar{\mu}_{n1}, \bar{\mu}_{n2}) \geq \theta_0 \tag{61}$$

with $\theta_0 = \max_{i=1,\ldots,n}\{\sigma_i(\varepsilon_i)\}$, *then the ISS-Lyapunov function candidate* V *defined in* (53) *satisfies*

$$B_1(\bar{\mu}_{n1}, \bar{\mu}_{n2}) \geq V(e) \geq \max\{B_2(\bar{\mu}_{n1}, \bar{\mu}_{n2}), \theta_0\} \tag{62}$$

$$\Rightarrow \max_{\psi \in \Psi(e,x,\bar{\mu}_{n1},\bar{\mu}_{n2})} \nabla V(e)\psi \leq -\alpha(V(e)) \tag{63}$$

where Ψ *is defined in* (56).

Proof Under Assumptions 2 and 5, with Lemma 1, we can transform the closed-loop quantized system into a large-scale system with state e composed of e_i-subsystems in the form of (41) and (38).

Under Assumptions 2, 3, 4 and 5, by directly using Lemma 2, for any specified constants c_{i1}, c_{i2} satisfying (47) for $i = 1, \ldots, n$, any ISS gains $\gamma_{e_k}^{e_{k'}}$ $(k \neq k')$ satisfying the cyclic-small-gain condition (52) for all $i = 1, \ldots, n$, any specified ISS gains $\gamma_{e_i}^{\mu_{k1}}, \gamma_{e_i}^{\mu_{k2}}$ for $i = 1, \ldots, n$, $k = 1, \ldots, i-1$ satisfying (57) and specified arbitrarily small constants ε_i for $i = 1, \ldots, n$, we can find continuously differentiable, odd, strictly decreasing and radially unbounded functions κ_i for $i = 1, \ldots, n$ such that (43) holds for $i = 1, \ldots, n$.

The satisfaction of (47) by appropriately choosing the κ_i for $i = 1, \ldots, n$ guarantees $B_1(\bar{\mu}_{n1}, \bar{\mu}_{n2}) > B_2(\bar{\mu}_{n1}, \bar{\mu}_{n2})$ for all positive zooming variables μ_{i1}, μ_{i2}. By using (61), we have $B_1(\bar{\mu}_{n1}, \bar{\mu}_{n2}) \geq \max\{B_2(\bar{\mu}_{n1}, \bar{\mu}_{n2}), \theta_0\}$. Recall the definitions of $V_i(e_i)$ in (40) and $V(e)$ in (53). The equalities in (60) and the left inequality in (62) guarantee (48)–(49). Under Assumption 4, with (47) satisfied, using Lemma 3, we have $u \in S_n(\bar{x}_n, \bar{\mu}_{n1}, \bar{\mu}_{n2})$.

Note that (57) is satisfied by appropriately choosing κ_i for $i = 1, \ldots, n$. In virtue of (44), (45), (57) and (59), $\theta(\bar{\mu}_{n1}, \bar{\mu}_{n2}, \bar{\varepsilon}_n) = \max\{B_2(\bar{\mu}_{n1}, \bar{\mu}_{n2}), \theta_0\}$. With the cyclic-small-gain condition (52) satisfied by appropriately choosing κ_i for $i = 1, \ldots, n$ and $u \in S_n(\bar{x}_n, \bar{\mu}_{n1}, \bar{\mu}_{n2})$, Lemma 4 guarantees the implication in (62) and (63). This ends the proof.

Remark 12 Based on Lemma 5, we can appropriately adjust the zooming variables to make conditions (60) and (61) always satisfied such that the sets represented by functions B_1 and B_2 are nested invariant sets. In the next section, we will use the invariant sets to design dynamic quantization logic.

4.5 A guideline for quantized control law design

In this subsection, we provide a guideline to choosing the functions κ_i for the quantized control law (24)–(25) such that the closed-loop quantized system satisfies property (62)–(63). The guideline includes two major steps:

1. *Choose the ISS parameters of the* e_i*-subsystems.*
   (a) Choose constants c_{i1}, c_{i2} to satisfy (47) for $i = 1, \ldots, n$;
   (b) Choose ISS gains $\gamma_{e_i}^{e_j} \in \mathcal{K}_\infty$ $(j \neq i)$ and the corresponding functions $\hat{\gamma}_{e_i}^{e_j} > \gamma_{e_i}^{e_j}$ to satisfy the cyclic-small-gain condition (52) for all $i = 1, \ldots, n$, and calculate the σ_i for $i = 1, \ldots, n$ in (53);

(c) Choose ISS gains $\gamma_{e_i}^{\mu_{k1}}, \gamma_{e_i}^{\mu_{k2}}$ for $i = 1, \ldots, n, k = 1, \ldots, i-1$ such that (57) holds for all $\mu_{i1}, \mu_{i2}, \varepsilon_i > 0$ for $i = 1, \ldots, n$;
(d) Choose specified $\varepsilon_i, \iota_i > 0$ for $i = 1, \ldots, n$.

2. *Choose the κ_i for $i = 1, \ldots, n$ based on Lemma 2 with the ISS parameters chosen in Step 1.*

Remark 13 In Step 1, it is only required that Step (c) is after Step (b), because condition (60) in Step (c) depends on the σ_i calculated in Step (b). Under Assumptions 2–5, if the ISS parameters and the κ_i are chosen according to the guideline and if conditions (60) and (61) are satisfied, then from Lemma 5, the nested invariant sets exist.

5 Dynamic quantization

Because of the saturation property of the quantizers, the quantized control law designed in Sect. 4 can only guarantee local stabilization; see (62)–(63). In this section, based on the invariant sets given in Lemma 5, we design a dynamic quantization logic in the form of (30), composed of zooming-in stage and zooming-out stage, to dynamically adjust the zooming variables μ_{ij} ($i = 1, \ldots, n$, $j = 1, 2$) such that the closed-loop quantized system is globally stabilized. Recall that in dynamic quantization, the zooming variables $\mu_{ij}(t)$ are piecewise constant signals, and are adjusted on a discrete time sequence $\{t_0, t_1, t_2, \ldots\}$ where $t_{k+1} - t_k = t_d$ with constant $t_d > 0$.

To satisfy condition (60) in Lemma 5, we design dynamic quantization such that for all $t \in \mathbb{R}_+$

$$\sigma_i \circ \alpha_V(M_{i1}\mu_{i1}(t)) = \sigma_i \circ \alpha_V\left(\frac{1}{1-c_{i1}}\bar{\kappa}_i^{-1}(M_{i2}\mu_{i2}(t))\right) := \Theta(t) \tag{64}$$

for $i = 1, \ldots, n$. Equivalently, we have

$$\mu_{i1}(t) = \frac{1}{M_{i1}}\alpha_V^{-1} \circ \sigma_i^{-1}(\Theta(t)) := \Upsilon_{i1}(\Theta(t)), \tag{65}$$

$$\mu_{i2}(t) = \frac{1}{M_{i2}}\bar{\kappa}_i\left((1-c_{i1})\alpha_V^{-1} \circ \sigma_i^{-1}(\Theta(t))\right) := \Upsilon_{i2}(\Theta(t)) \tag{66}$$

for $i = 1, \ldots, n$. Note that Υ_{i1} and Υ_{i2} are invertible for $i = 1, \ldots, n$. Thus, the dynamic quantization logic (30) can be designed by choosing an appropriate update law for Θ, which can largely reduce the design complexity for all the zooming variables μ_{ij} ($i = 1, \ldots, n$, $j = 1, 2$). The update law for Θ is expected to be in the following form:

$$\Theta(t_{k+1}) = Q(\Theta(t_k)), \quad k \in \mathbb{Z}_+ \tag{67}$$

where $t_{k+1} - t_k = t_d$ with constant $t_d > 0$. In the zooming-out stage, $Q = Q^{\text{out}}$; in the zooming-in stage, $Q = Q^{\text{in}}$. With Q^{out} and Q^{in} designed, we can design the dynamic quantization logic (30) for μ_{ij} by choosing

$$Q_{ij}^{\text{out}} = \Upsilon_{ij} \circ Q^{\text{out}} \circ \Upsilon_{ij}^{-1}, \quad Q_{ij}^{\text{in}} = \Upsilon_{ij} \circ Q^{\text{in}} \circ \Upsilon_{ij}^{-1}. \tag{68}$$

Using the definition of B_1 in (58), we also have

$$\Theta(t) = B_1(\bar{\mu}_{n1}(t), \bar{\mu}_{n2}(t)). \tag{69}$$

Before designing dynamic quantization, the relation between zooming variables $\bar{\mu}_{n1}, \bar{\mu}_{n2}$ and control error e should be clarified. For $i = 1, \ldots, n$, using the definitions of S_i in (36), the strictly decreasing property of κ_i implies

$$\begin{aligned} \max S_i(\bar{x}_i, \bar{\mu}_{i1}, \bar{\mu}_{i2}) = \kappa_i(x_i &- \max S_{i-1}(\bar{x}_{i-1}, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2}) \\ &- \max\{c_{i1}|e_i|, \mu_{i1}\}) + \mu_{i2}, \end{aligned} \tag{70}$$

and

$$\begin{aligned} \min S_i(\bar{x}_i, \bar{\mu}_{i1}, \bar{\mu}_{i2}) = \kappa_i(x_i &- \min S_{i-1}(\bar{x}_{i-1}, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2}) \\ &+ \max\{c_{i1}|e_i|, \mu_{i1}\}) - \mu_{i2}. \end{aligned} \tag{71}$$

Recall $e = [e_0^{\mathrm{T}}, e_1, \ldots, e_n]^{\mathrm{T}}$. Given the definitions of e_i for $i = 2, \ldots, n$ in (37), we can denote

$$e = e(X, \bar{\mu}_{n1}, \bar{\mu}_{n2}) \tag{72}$$

with $X = [z^{\mathrm{T}}, x^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{n+n_z}$. It can be observed that e is a continuous function of $X, \bar{\mu}_{n1}, \bar{\mu}_{n2}$. Clearly, the piecewise constant adjustment of $\bar{\mu}_{n1}, \bar{\mu}_{n2}$ causes jumps of e on the time-line. This makes a difference from the previously known results.

5.1 Zooming-out stage

The design of the zooming-out stage is motivated by [21]. The purpose of the zooming-out stage in this subsection is to increase the zooming variables μ_{ij} such that at some finite time t_{k^*}, the state of the closed-loop quantized system is in the larger invariant set corresponding to B_1 in (62). In this stage, the components κ_i for $i = 1, \ldots, n$ of the controller are set to be zero. Thus, $u = 0$.

The small-time norm-observability assumed in Assumption 1 guarantees that for $t_d > 0$, there exists a $\varphi \in \mathscr{K}_\infty$ such that

$$|X(t_k + t_d)| \le \varphi(\|x\|_{[t_k, t_k+t_d]}) \tag{73}$$

for any $k \in \mathbb{Z}_+$. Considering the definitions of V and e in (53) and (72), for $t_d > 0$, property (73) can be represented with the Lyapunov function V as

$$|V(e(X(t_k + t_d), 0, 0))| \le \bar{\varphi}(\|x\|_{[t_k, t_k+t_d]}) \tag{74}$$

for any $k \in \mathbb{Z}_+$, where $\bar{\varphi} \in \mathcal{K}_\infty$.

With the forward completeness property assumed in Assumption 1, we design a zooming-out logic $Q^{\text{out}} : \mathbb{R}_+ \to \mathbb{R}_+$ to increase Θ fast enough to dominate the growth rate of $\bar{\varphi}(|x|)$ such that at some finite time $t_{k^*} > 0$ with $k^* \in \mathbb{Z}_+$, it holds:

$$M_{i1}\mu_{i1}(t_{k^*}) \geq |x_i(t_{k^*})|, \quad i = 1, \ldots, n, \tag{75}$$

$$\Theta(t_{k^*}) \geq \bar{\varphi}(\|x\|_{[t_{k^*}-t_d, t_{k^*}]}). \tag{76}$$

Remark 14 Due to the saturation of the quantizer, if the input signal of a quantizer is outside the range of the quantizer, then we cannot estimate the bound of the signal. In the zooming-out stage, the κ_i's are set to be zero, and the input of the quantizer q_{i1} is x_i; see control law (24)–(25). Inequality (75) means that at some finite time t_{k^*}, x_i is in the quantization range of q_{i1}. Then, we can estimate the bound of $|x_i(t_{k^*})|$.

Using (74) and (76), we have

$$\Theta(t_{k^*}) \geq \max\{V(e(X(t_{k^*}), 0, 0)), \theta_0\}. \tag{77}$$

From the definitions of $\max S_i(\bar{x}_i, \bar{\mu}_{i1}, \bar{\mu}_{i2})$, $\min S_i(\bar{x}_i, \bar{\mu}_{i1}, \bar{\mu}_{i2})$ and e_{i+1}, one observes that increase of $\bar{\mu}_{n1}, \bar{\mu}_{n2}$ leads to increase of $\max S_i(\bar{x}_i, \bar{\mu}_{i1}, \bar{\mu}_{i2})$, decrease of $\min S_i(\bar{x}_i, \bar{\mu}_{i1}, \bar{\mu}_{i2})$ and thus decrease or hold of $|e_{i+1}|$ for $i = 1, \ldots, n-1$. Thus, with the zooming-out logic Q^{out}, we achieve that, at time $t_{k^*} > 0$ with $k^* \in \mathbb{Z}_+$, it holds that

$$\Theta(t_{k^*}) \geq \max\{V(e(X(t_{k^*}), \bar{\mu}_{n1}(t_{k^*}), \bar{\mu}_{n2}(t_{k^*}))), \theta_0\}. \tag{78}$$

With Q^{out} designed, we can design the zooming-out logic Q^{out}_{ij} for $i = 1, \ldots, n$, $j = 1, 2$ according to (68).

Remark 15 As mentioned in Remark 2, if a bound of the initial state $X(0)$ is known, we can directly set $\Theta(0)$ to satisfy (77) with $t_{k^*} = 0$. In this case, the zooming-out stage is not necessary and Assumption 1 is not required.

5.2 Zooming-in stage

The zooming-out stage achieves (78) at time t_{k^*} with $k^* \in \mathbb{Z}_+$. Suppose that at some $t_k > 0$ with $k \geq k^*$,

$$\Theta(t_k) \geq \max\{V(e(X(t_k), \bar{\mu}_{n1}(t_k), \bar{\mu}_{n2}(t_k))), \theta_0\}. \tag{79}$$

In this subsection, first we design a $Q^{\text{in}} : \mathbb{R}_+ \to \mathbb{R}_+$ for the zooming-in stage such that

$$\Theta(t_{k+1}) = Q^{\text{in}}(\Theta(t_k)) \geq \max\{V(e(X(t_{k+1}), \bar{\mu}_{n1}(t_{k+1}), \bar{\mu}_{n2}(t_{k+1}))), \theta_0\}. \tag{80}$$

This objective will be achieved by Lemmas 6 and 7. Then we show the convergence property of the update law (67) for Θ in the zooming-in stage by Lemma 8.

Remark 16 If (80) is achieved based on (79), we can recursively guarantee that the state e of the closed-loop quantized system is always in the larger invariant set represented by B_1 in spite of the update of Θ; see (62) and (69).

The following lemma describes the decreasing property of V during the time interval $[t_k, t_{k+1})$, based on which we will design the zooming-in update law Q^{in} for Θ.

Lemma 6 *Consider the closed-loop quantized system with V satisfying property* (62)–(63)*. If* (79) *holds at time t_k with $k \in \mathbb{Z}_+$, then there exists a continuous and positive definite function $\bar{\rho}$ such that*

$$(\text{Id} - \bar{\rho}) \in \mathscr{K}_\infty, \tag{81}$$

$$V(e(X(t_{k+1}), \bar{\mu}_{n1}(t_k), \bar{\mu}_{n2}(t_k))) \le \max\{(\text{Id} - \bar{\rho})(\Theta(t_k)), \theta_0\}. \tag{82}$$

The proof of Lemma 6 is in Appendix D.

From the definition in (72), the piece-wise constant update of the zooming variables $\bar{\mu}_{n1}, \bar{\mu}_{n2}$ causes jumps of e and thus jumps of V. Based on (82), we design the zooming-in logic Q^{in} to achieve (80) in spite of the jumps.

To clearly represent the relation between Lyapunov function V and X, Θ, define

$$W(\xi, s) = V(e(\xi, \bar{\Upsilon}_{n1}(s), \bar{\Upsilon}_{n2}(s))) \tag{83}$$

for $\xi \in \mathbb{R}^{n+n_z}$ and $s \in \mathbb{R}_+$, where

$$\bar{\Upsilon}_{n1}(s) = [\Upsilon_{11}(s), \ldots, \Upsilon_{n1}(s)]^{\text{T}}, \tag{84}$$

$$\bar{\Upsilon}_{n1}(s) = [\Upsilon_{12}(s), \ldots, \Upsilon_{n2}(s)]^{\text{T}}. \tag{85}$$

Then, $W(\xi, s)$ is a continuous function of (ξ, s).

Consider (ξ, s) satisfying

$$0 \le s \le \Theta(t_{k^*}) \tag{86}$$

$$W(\xi, s) \le \Theta(t_{k^*}). \tag{87}$$

From the definitions of V and W in (53) and (83), we can find a compact set $\Omega^o \subset \mathbb{R}^{n+n_z} \times \mathbb{R}_+$ such that all the (ξ, s) satisfying (86)–(87) belong to Ω^o. By using the property of continuous functions, we can find a continuous and positive definite function $\rho^o < \text{Id}$ such that for all $(\xi, s) \in \Omega^o$ and all $h \ge 0$, it holds that

$$|W(\xi, s - \rho^o(h)) - W(\xi, s)| \le h. \tag{88}$$

We propose the following update law for Θ in the zooming-in stage:

$$Q^{\text{in}}(\Theta) = \Theta - \rho^o\left(\frac{\Theta - \max\{\Xi(\Theta), \theta_0\}}{2}\right). \tag{89}$$

where $\Xi = (\mathrm{Id} - \bar{\rho})$. In the following procedure, we employ Lemma 7 to guarantee objective (80) and employ Lemma 8 to show the convergence property of Θ with update law defined in (89).

Lemma 7 shows that property (80) can be achieved with the zooming-in update law (89) for Θ, given (78) and (79) satisfied.

Lemma 7 *Consider the closed-loop quantized system with V satisfying property* (62)–(63)*. Suppose that condition* (78) *holds at some finite time t_{k^*} and condition* (79) *holds at some time t_k with $k \geq k^*$. Then, property* (80) *is satisfied at time t_{k+1} with the update law $\Theta(t_{k+1}) = Q^{\mathrm{in}}(\Theta(t_k))$ with Q^{in} defined in* (89)*.*

Proof With $0 < \rho^o < \mathrm{Id}$, it can be guaranteed that

$$\Theta(t_{k+1}) \leq \Theta(t_k) \tag{90}$$

and

$$\begin{aligned} \Theta(t_{k+1}) &= \Theta(t_k) - \rho^o \left(\frac{\Theta(t_k) - \max\{\Xi(\Theta(t_k)), \theta_0\}}{2} \right) \\ &\geq \frac{\Theta(t_k) + \max\{\Xi(\Theta(t_k)), \theta_0\}}{2} \end{aligned} \tag{91}$$

for $k \geq k^*$. Thus, $0 < \Theta(t_k) \leq \Theta(t_{k^*})$ for $k \geq k^*$.

From Lemma 6, (82) holds. Using (78), (82) and (83), we have

$$W(X(t_{k+1}), \Theta(t_k)) \leq \max\{\Xi(\Theta(t_k)), \theta_0\} \leq \Theta(t_{k^*}) \tag{92}$$

for $k \geq k^*$. Hence, $(X(t_{k+1}), \Theta(t_k)) \in \Omega^o$ for $k \geq k^*$. Given $(X(t_{k+1}), \Theta(t_k)) \in \Omega^o$, from (89) and (92), we obtain

$$\begin{aligned} &W(X(t_{k+1}), \Theta(t_{k+1})) \\ &\quad \leq W(X(t_{k+1}), \Theta(t_k)) + |W(X(t_{k+1}), \Theta(t_{k+1})) - W(X(t_{k+1}), \Theta(t_k))| \\ &\quad \leq W(X(t_{k+1}), \Theta(t_k)) + |W(X(t_{k+1}), Q^{\mathrm{in}}(\Theta(t_k)) - W(X(t_{k+1}), \Theta(t_k))| \\ &\quad \leq \max\{\Xi(\Theta(t_k)), \theta_0\} + \frac{\Theta(t_k) - \max\{\Xi(\Theta(t_k)), \theta_0\}}{2} \\ &\quad = \frac{\Theta(t_k) + \max\{\Xi(\Theta(t_k)), \theta_0\}}{2}. \end{aligned} \tag{93}$$

From (79), we have $\theta_0 \leq \Theta(t_k)$, which implies

$$\theta_0 \leq \frac{\Theta(t_k) + \max\{\Xi(\Theta(t_k)), \theta_0\}}{2}. \tag{94}$$

Properties (91), (93) and (94) together with the definition of W in (83) guarantee (80). This ends the proof of Lemma 7.

Lemma 8 shows the convergence property of the update law (67) for Θ with $Q = Q^{\text{in}}$ defined in (89). The proof of Lemma 8 can be found in [44, Section IV.C].

Lemma 8 *Suppose that at some $t_{k^*} > 0$ with $k^* \in \mathbb{Z}_+$, $\Theta(t_{k^*}) \geq \theta_0$. Then with Q^{in} defined in (89), update law $\Theta(t_{k+1}) = Q^{\text{in}}(\Theta(t_k))$ achieves*

$$\lim_{k\to\infty} \Theta(t_k) = \theta_0. \tag{95}$$

Remark 17 The zooming-in update law for Θ can be designed by finding the $\bar{\rho}$ with Lemma 6 and ρ^o by using the continuity of W. Lemmas 7 and 8 are used to prove the effectiveness of the the zooming-in update law Q^{in} defined in (89). With Q^{in} designed, we can design the zooming-in logic Q_{ij}^{in} for $i = 1, \ldots, n$, $j = 1, 2$ according to (68).

5.3 Main result

The main result of quantized control is summarized in Theorem 2.

Theorem 2 *For the system (11)–(13), under Assumptions 1–4, by choosing constants c_{i1}, c_{i2} satisfying (47) for $i = 1, \ldots, n$, ISS gains $\gamma_{e_k}^{e_{k'}}$ $(k \neq k')$ satisfying the cyclic-small-gain condition (52) for all $i = 1, \ldots, n$, ISS gains $\gamma_{e_i}^{\mu_{k1}}, \gamma_{e_i}^{\mu_{k2}}$ for $i = 1, \ldots, n, k = 1, \ldots, i-1$ satisfying (57) and constants $\varepsilon_i > 0$ for $i = 1, \ldots, n$, we can design the functions κ_i for $i = 1, \ldots, n$ in (24)–(25) and the dynamic quantization logic Q_{ij} for $i = 1, \ldots, n$, $j = 1, 2$ in (30) such that the closed-loop solutions z and x are globally bounded. Moreover, by choosing the constants $\varepsilon_i > 0$ for $i = 1, \ldots, n$ arbitrarily small, the output $x_1(t)$ can be steered to within an arbitrarily small neighborhood of origin.*

Proof With Assumption 1 satisfied, at some time $t_{k^*} > 0$ with $k^* \in \mathbb{Z}_+$, (78) can be achieved by the zooming-out logic Q_{ij}^{out} designed in Sect. 5.1.

With Assumptions 2–4 satisfied, using Lemma 5, by appropriately designing the functions κ_i for $i = 1, \ldots, n$ such that the ISS parameters satisfy the conditions (47), (52) and (57), the closed-loop quantized system has the nested invariant sets defined in (63).

Using Lemmas 6 and 7, (80) can be guaranteed with the zooming-in logic designed in Sect. 5.2, and it holds that $V(e(X(t_k), \bar{\mu}_{n1}(t_k), \bar{\mu}_{n2}(t_k))) \leq \Theta(t_k)$ for $k \geq k^*$. Moreover, Lemma 8 implies that $\overline{\lim}_{k\to\infty} V(e(X(t_k), \bar{\mu}_{n1}(t_k), \bar{\mu}_{n2}(t_k))) \leq \theta_0$. Recall the definition of V in (53). The closed-loop signal x_1 is driven to within the region $|x_1| \leq \alpha_V^{-1}(\theta_0)$. Recall the definition of θ_0 in Lemma 5. By designing ε_i $(i = 1, \ldots, n)$ arbitrarily small, the state x_1 can be steered to within an arbitrarily small neighborhood of the origin. This ends the proof.

Remark 18 The main result of the paper is still new when there is no inverse dynamics (i.e., the z-subsystem) in system (11)–(13). In this case, the assumption on small-time norm-observability in Assumption 1 is not needed.

6 A remark on robust measurement feedback control

Consider system (11)–(13) under additive state measurement disturbances without quantization. For $i = 1, \ldots, n$, denote x_i^m as the measurement of x_i and w_i as the corresponding measurement disturbance, that is, $x_i^m = x_i + w_i$. We show that the above designed procedure can be adopted to solve the robust measurement feedback control problem.

Assumption 6 For each $i = 1, \ldots, n$, there exists a constant $\mu_{i1} \geq 0$ such that

$$\|w_i\| \leq \mu_{i1}. \tag{96}$$

Under Assumptions 2, 3 and 6, robust control with disturbed measurements can be addressed in a similar way to the quantized control case. In the constructive control design, let $\mu_{i2} = 0$ for $i = 1, \ldots, n$. The measurement feedback control law is designed recursively as

$$v_i = \kappa_i(x_i^m - v_{i-1}) = \kappa_i(x_i - v_{i-1} + w_i), \quad i = 1, \ldots, n \tag{97}$$

$$u = v_n \tag{98}$$

with $v_0 = 0$. It can be directly checked that

$$v_1 \in S_1(\bar{x}_1, \bar{\mu}_{11}, \bar{\mu}_{12}) \Rightarrow \cdots \Rightarrow v_i \in S_i(\bar{x}_i, \bar{\mu}_{i1}, \bar{\mu}_{i2}) \Rightarrow \cdots \Rightarrow v_n \in S_n(\bar{x}_n, \bar{\mu}_{n1}, \bar{\mu}_{n2}). \tag{99}$$

An ISS-Lyapunov function $V(e)$ constructed as in (53) with θ defined as in (57) satisfies

$$V(e) \geq \theta(\bar{\mu}_{n1}, \bar{\mu}_{n2}, \bar{\varepsilon}_n) \Rightarrow \max_{\psi \in \Psi(e, x, \bar{\mu}_{n1}, \bar{\mu}_{n2})} \nabla V(e)\psi \leq -\alpha(V(e)) \tag{100}$$

wherever ∇V exists, where α is continuous and positive definite, and Ψ is defined in (56). Note that $\sigma_1 = \mathrm{Id}$. We can then achieve $\theta(\bar{\mu}_{n1}, \bar{\mu}_{n2}, \bar{\varepsilon}_n) = \gamma_{e_1}^{\mu_{11}}(\mu_{11}) = \alpha_V(\mu_{11}/c_{11})$ by designing σ_i $(i = 2, \ldots, n)$ and ε_i $(i = 1, \ldots, n)$ small enough in the recursive design procedure. This together with (100) means that $V(e)$ ultimately converges to a region $V(e) \leq \alpha_V(\mu_{11}/c_{11})$. It can be derived that x_1 ultimately converges to the region $|x_1| \leq \mu_{11}/c_{11}$. With c_{11} chosen close to one in the constructive design procedure, the state x_1 ultimately converges to the region close to $|x_1| \leq \mu_{11}$.

Without a detailed proof, we present a theorem on the robust measurement feedback control for system (11)–(13).

Theorem 3 *Consider system* (11)–(13). *Under Assumptions* 2, 3 *and* 6, *the closed-loop signals are bounded, and in particular, the state* x_1 *can be steered to within a region arbitrarily close to* $|x_1| \leq \mu_{11}$ *with the measurement feedback control law in* (97)–(98).

7 Summary and future work

This paper has developed a new tool for quantized nonlinear control design. The essential strategy is to introduce a change of coordinates so that the closed-loop quantized system is a large-scale system of ISS subsystems described by differential inclusions. The recently developed cyclic-small-gain theorem is employed to guarantee the stability property of the closed-loop quantized system and to construct an ISS-Lyapunov function to evaluate the influence of quantization errors. One significance of this result is that an n-dimensional strict-feedback nonlinear system with measurement and actuator quantization can be globally stabilized by a quantized controller with $2n$ three-level dynamic quantizers.

In spite of the obtained results, several related problems should be addressed in the future research:

- Quantized control is closely related to other network control problems such as sampled-data control, control with time-delays. How to deal with more complex network behaviors in a systematic way, in particular those hybrid/switching systems satisfying only a weak semigroup property (see [45]), should be studied in greater details.
- The cyclic-small-gain theorem was originally developed for large-scale systems. It is thus very natural to ask whether decentralized quantized controllers can be developed for a class of large-scale nonlinear systems.
- Controllers are expected to possess adaptive capabilities to cope with "large" system uncertainties. A further extension of the presented methodology to quantized adaptive control is of practical interest for engineering applications.

Appendix: Proofs of technical Lemmas

A Proof of Lemma 1

We simply use S_k instead of $S_k(\bar{x}_k, \bar{\mu}_{k1}, \bar{\mu}_{k2})$ for $k = 1, \ldots, i-1$.

We only consider the case of $e_i > 0$.

Consider the recursive definition of S_k's in (36). For $k = 1, \ldots, i-1$, the strictly decreasing property of the κ_k's implies

$$\max S_k = \kappa_k(x_k - \max S_{k-1} - \max\{c_{k1}|e_k|, \mu_{k1}\}) + \mu_{k2}, \tag{101}$$

$$\min S_k = \kappa_k(x_k - \min S_{k-1} + \max\{c_{k1}|e_k|, \mu_{k1}\}) - \mu_{k2}. \tag{102}$$

From the iteration type definition of e_k's for $k = 1, \ldots, i-1$, e_{i-1} is continuous and differentiable almost everywhere with respect to $\bar{x}_{i-1}, \bar{\mu}_{(i-2)1}, \bar{\mu}_{(i-2)2}$.

Since κ_k's are continuously differentiable for $k = 1, \ldots, i-1$, using (101), we can see $\max S_{i-1}$ is continuously differentiable almost everywhere with respect to $\bar{x}_{i-1}, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2}$.

Considering the definition of e_i in (37), when $e_i > 0$, we can represent the e_i-subsystem with a differential equation

$$\dot{e}_i = x_{i+1} + \Delta_i(\bar{x}_i, z) - \nabla \max S_{i-1}[\dot{\bar{x}}_{i-1}, 0_{(i-1)}, 0_{(i-1)}]^{\mathrm{T}} \tag{103}$$

with $0_{(i-1)}$ being the vector composed of $i-1$ zero elements, wherever $\max S_{i-1}$ is continuously differentiable, or equivalently, $\nabla \max S_{i-1}$ exists. Because $\max S_{i-1}$ is continuously differentiable almost everywhere, $\nabla \max S_{i-1}$ is discontinuous and thus the e_i-subsystem is a discontinuous system. We represent the e_i-subsystem with a differential inclusion by embedding the discontinuous $\nabla \max S_{i-1}$ into a set-valued map

$$\partial \max S_{i-1} = \bigcap_{\varepsilon > 0} \bigcap_{\tau(\tilde{\mathscr{M}})=0} \overline{\mathrm{co}} \left\{ \nabla \max S_{i-1}(\mathscr{B}_\varepsilon(\zeta_{i-1}) \backslash \tilde{\mathscr{M}}) \right\} \tag{104}$$

where $\mathscr{B}_\varepsilon(\zeta_{i-1})$ is an open ball of radius ε around $\zeta_{i-1} := [\bar{x}_{i-1}^{\mathrm{T}}, \bar{\mu}_{(i-1)1}^{\mathrm{T}}, \bar{\mu}_{(i-1)2}^{\mathrm{T}}]^{\mathrm{T}}$, and $\tilde{\mathscr{M}}$ represents all sets of zero measure (i.e., $\tau(\tilde{\mathscr{M}}) = 0$). Then, $\partial \max S_{i-1}$ is convex, compact and upper semi-continuous (see [46, Chapter 1] for a tutorial and [33,34] for recent results on such properties of discontinuous systems).

Then, in the case of $e_i > 0$, the e_i-subsystem can be represented with a differential inclusion as

$$\begin{aligned}\dot{e}_i &\in \{x_{i+1} + \Delta_i(\bar{x}_i, z) - \varphi_i : \varphi_i \in \partial \max S_{i-1}[\dot{\bar{x}}_{i-1}^{\mathrm{T}}, 0_{(i-1)}, 0_{(i-1)}]^{\mathrm{T}}\} \\ &:= \{x_{i+1} + \phi_i : \phi_i \in \Phi_i(\bar{x}_i, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2}, z)\}\end{aligned} \tag{105}$$

where

$$\begin{aligned}&\Phi_i(\bar{x}_i, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2}, z) \\ &\quad = \{\Delta_i(\bar{x}_i, z) - \varphi_i : \varphi_i \in \partial \max S_{i-1}[\dot{\bar{x}}_{i-1}^{\mathrm{T}}, 0_{(i-1)}, 0_{(i-1)}]^{\mathrm{T}}\}.\end{aligned} \tag{106}$$

Because Δ_i and $\dot{\bar{x}}_i$ are locally Lipschitz and $\partial \max S_{i-1}$ is convex, compact and upper semi-continuous, Φ_i is convex, compact and upper semi-continuous. Considering the definition of $\partial \max S_{i-1}$, one can find a continuous function $\bar{s}_{i-1}$ such that for all $\bar{x}_{i-1}, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2}$, any $s_{i-1} \in \partial \max S_{i-1}$ satisfies $|s_{i-1}| \leq \bar{s}_{i-1}(\bar{x}_{i-1}, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2})$. Thus, for all $\bar{x}_{i-1}, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2}, z$, any $\phi_i \in \Phi_i(\bar{x}_i, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2}, z)$ satisfies

$$|\phi_i| \leq |\Delta_i(\bar{x}_i, z)| + \bar{s}_{i-1}(\bar{x}_{i-1}, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2})|\dot{\bar{x}}_{i-1}|. \tag{107}$$

From (12)–(13) and Assumption 2, $\Delta_i(\bar{x}_i, z)$ is bounded by a $\mathscr{K}_\infty$ function of $(\bar{x}_i, z)$ and $\dot{\bar{x}}_{i-1}$ is bounded by a $\mathscr{K}_\infty$ function of $(\bar{x}_i, z)$. Thus, there exists a $\lambda_{\Phi_i}^0 \in \mathscr{K}_\infty$ such that for any $\phi_i \in \Phi_i(\bar{x}_i, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2}, z)$, it holds that

$$|\phi_i| \leq \lambda_{\Phi_i}^0(|(\bar{x}_i, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2}, z)|). \tag{108}$$

For the purpose of (39), for each $k = 1, \ldots, i-1$, we find a $\lambda_{x_{k+1}} \in \mathscr{K}_\infty$ such that $|x_{k+1}| \le \lambda_{x_{k+1}}(|(\bar{e}_{k+1}, \bar{\mu}_{k1}, \bar{\mu}_{k2})|)$. For $k = 1, \ldots, i-1$, from the definitions of x_{k+1} in (37), we have $\min S_k \le x_{k+1} - e_{k+1} \le \max S_k$ and thus

$$|x_{k+1}| \le \max\{|\max S_k|, |\min S_k|\} + |e_{k+1}|. \tag{109}$$

For each $k = 1, \ldots, i-1$, define $\kappa_k^o(s) = \kappa_k(|s|)$ for $s \in \mathbb{R}_+$. Because κ_k is odd, strictly decreasing and radially unbounded, $\kappa_k^o \in \mathscr{K}_\infty$. From (101), we have

$$\begin{aligned} |\max S_k| &\le \kappa_k^o(|x_k - \max S_{k-1} - \max\{c_{k1}|e_k|, \mu_{k1}\}|) + \mu_{k2} \\ &\le \kappa_k^o(|x_k - \max S_{k-1}| + \max\{c_{k1}|e_k|, \mu_{k1}\}) + \mu_{k2} \\ &\le \kappa_k^o(|\max S_{k-1}| + |\min S_{k-1}| + |e_k| + \max\{c_{k1}|e_k|, \mu_{k1}\}) + \mu_{k2}. \end{aligned} \tag{110}$$

In (110), we used the fact that $\min S_{k-1} \le x_k - e_k \le \max S_{k-1}$ and thus $\min S_{k-1} - \max S_{k-1} + e_k \le x_k - \max S_{k-1} \le e_k$, to arrive at $|x_k - \max S_{k-1}| \le |\max S_{k-1}| + |\min S_{k-1}| + |e_k|$. Similarly, we obtain:

$$|\min S_k| \le \kappa_k^o(|\max S_{k-1}| + |\min S_{k-1}| + |e_k| + \max\{c_{k1}|e_k|, \mu_{k1}\}) + \mu_{k2}. \tag{111}$$

For each x_{k+1} ($k = 1, \ldots, i-1$), using (109), (110) and (111), one can find a $\lambda_{x_{k+1}} \in \mathscr{K}_\infty$ such that $|x_{k+1}| \le \lambda_{x_{k+1}}(|(\bar{e}_{k+1}, \bar{\mu}_{k1}, \bar{\mu}_{k2})|)$. This together with (108) guarantees that there exists a $\lambda_{\Phi_i} \in \mathscr{K}_\infty$ such that for all $(\bar{x}_i, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2}, z)$, any $\phi_i \in \Phi_i(\bar{x}_i, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2}, z)$ satisfies

$$|\phi_i| \le \lambda_{\Phi_i}(|(\bar{e}_i, z, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2})|) \tag{112}$$

where $\bar{e}_i := [e_1, \ldots, e_i]^{\mathrm{T}}$.

Define

$$\Phi_i^*(e_{i+1}, \bar{x}_i, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2}, z) = \{\phi_i + e_{i+1} : \phi_i \in \Phi_i(\bar{x}_i, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2}, z)\}. \tag{113}$$

From (37), $x_{i+1} - e_{i+1} \in S_i(\bar{x}_i, \bar{\mu}_{i1}, \bar{\mu}_{i2})$. Then, equation (105) can be rewritten as (38).

From (112), we can find a $\lambda_{\Phi_i^*} \in \mathscr{K}_\infty$ such that for all $(e_{i+1}, \bar{x}_i, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2}, z)$, any $\phi_i^* \in \Phi_i^*(e_{i+1}, \bar{x}_i, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2}, z)$ satisfies (39).

By also considering the cases of $e_i = 0$ and $e_i < 0$, Lemma 1 can be proved. This ends the proof.

B Proof of Lemma 2

Note that $e_0 = z$. With (39) satisfied, one can find $\lambda_{\Phi_i^*}^{e_k} \in \mathscr{K}_\infty$ for $k = 0, \ldots, i+1$ and $\lambda_{\Phi_i^*}^{\mu_{k1}}, \lambda_{\Phi_i^*}^{\mu_{k2}} \in \mathscr{K}_\infty$ for $k = 1, \ldots, i-1$ such that for any $\phi_i^* \in \Phi_i^*(e_{i+1}, \bar{x}_i, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2}, z)$, it holds that

$$|\phi_i^*| \leq \sum_{k=1}^{i+1} \lambda_{\Phi_i^*}^{e_k}(|e_k|) + \sum_{k=1}^{i-1} \left(\lambda_{\Phi_i^*}^{\mu_{k1}}(\mu_{k1}) + \lambda_{\Phi_i^*}^{\mu_{k2}}(\mu_{k2})\right) \tag{114}$$

By convenience, let $\gamma_{e_i}^{e_i} = \mathrm{Id}$. Define

$$\begin{aligned}\Pi_i(s) = \ & \lambda_{\Phi_i^*}^{e_0} \circ \underline{\alpha}_0^{-1} \circ \left(\gamma_{e_i}^{e_0}\right)^{-1} \circ \alpha_V(s) + \sum_{k=1}^{i+1} \lambda_{\Phi_i^*}^{e_k} \circ \alpha_V^{-1} \circ \left(\gamma_{e_i}^{e_k}\right)^{-1} \circ \alpha_V(s) \\ & + \sum_{k=1}^{i-1} \lambda_{\Phi_i^*}^{\mu_{k1}} \circ \left(\gamma_{e_i}^{\mu_{k1}}\right)^{-1} \circ \alpha_V(s) + \sum_{k=1}^{i-1} \lambda_{\Phi_i^*}^{\mu_{k2}} \circ \left(\gamma_{e_i}^{\mu_{k2}}\right)^{-1} \circ \alpha_V(s) + \frac{\iota_i}{2} s\end{aligned} \tag{115}$$

for $s \in \mathbb{R}_+$. Then, $\Pi_i \in \mathscr{K}_\infty$.

From Lemma 1 in [41], for any $0 < c_{i1}, c_{i2} < 1$, $\varepsilon_i > 0$, one can find a $\nu_i : \mathbb{R}_+ \to \mathbb{R}_+$ positive, nondecreasing and continuously differentiable on $(0, \infty)$ and satisfying

$$(1 - c_{i2})(1 - c_{i1})\nu_i\left((1 - c_{i1})s\right)s \geq \Pi_i(s) \tag{116}$$

for $s \geq \sqrt{2\varepsilon_i}$. With the ν_i satisfying (116), define $\kappa_i(r) = -\nu_i(|r|)r$ for $r \in \mathbb{R}$. Noticing that $\lim_{t\to 0^+} \frac{d\kappa_i(r)}{dr} = \lim_{t\to 0^-} \frac{d\kappa_i(r)}{dr}$, κ_i is continuously differentiable, odd, strictly decreasing and radially unbounded.

Recall $V_k(e_k) = \alpha_V(|e_k|) = \frac{1}{2}e_k^2$ for $k = 1, \ldots, n$. We use V_k instead of $V_k(e_k)$ for $k = 1, \ldots, n$. Consider the case of

$$V_i \geq \max_{k=1,\ldots,i-1} \left\{ \begin{array}{l} \gamma_{e_i}^{e_0}(V_0), \gamma_{e_i}^{e_k}(V_k), \gamma_{e_i}^{e_{i+1}}(V_{i+1}), \\ \gamma_{e_i}^{\mu_{k1}}(\mu_{k1}), \gamma_{e_i}^{\mu_{k2}}(\mu_{k2}), \gamma_{e_i}^{\mu_{i1}}(\mu_{i1}), \gamma_{e_i}^{\mu_{i2}}(\mu_{i2}), \varepsilon_i \end{array} \right\}. \tag{117}$$

In this case, we have

$$\Pi_i(|e_i|) - \frac{\iota_i}{2}|e_i| \geq \phi_i^* \tag{118}$$

for all $\phi_i^* \in \Phi_i^*(e_{i+1}, \bar{x}_i, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2}, z)$. And it also holds that

$$\mu_{i1} \leq c_{i1}|e_i|; \quad \mu_{i2} \leq c_{i2}\bar{\kappa}_i\left((1 - c_{i1})|e_i|\right)|e_i|; \quad |e_i| \geq \sqrt{2\varepsilon_i}. \tag{119}$$

When $e_i \neq 0$, with $0 < c_{i1} < 1$, for $\varsigma_{i-1} \in S_{i-1}$ and $|b_{i1}| \leq \max\{c_{i1}|e_i|, \mu_{i1}\} = c_{i1}|e_i|$, we have

$$|x_i - \varsigma_{i-1} + b_{i1}| \geq (1 - c_{i1})|e_i| \tag{120}$$

$$\operatorname{sgn}(x_i - \varsigma_{i-1} + b_{i1}) = \operatorname{sgn}(e_i) \tag{121}$$

and thus

$$\nu_i(|x_i - \varsigma_{i-1} + b_{i1}|)|x_i - \varsigma_{i-1} + b_{i1}| \geq (1 - c_{i1})\nu_i((1 - c_{i1})|e_i|)|e_i|. \tag{122}$$

In the case of (117), for any $\phi_i^* \in \Phi_i^*(e_{i+1}, \bar{x}_i, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2}, z)$, with $\varsigma_{i-1} \in S_{i-1}$, $|b_{i1}| \leq \max\{c_{i1}|e_i|, \mu_{i1}\}$ and $|b_{i2}| \leq \mu_{i2}$, using (118)–(122), we have

$$\begin{aligned}
&\nabla V_i \left(\kappa_i(x_i - \varsigma_{i-1} + b_{i1}) + b_{i2} + \phi_i^*\right)\\
&= e_i \left(-\nu_i(|x_i - \varsigma_{i-1} + b_{i1}|)(x_i - \varsigma_{i-1} + b_{i1}) + b_{i2} + \phi_i^*\right)\\
&\leq -\nu_i(|x_i - \varsigma_{i-1} + b_{i1}|)|x_i - \varsigma_{i-1} + b_{i1}||e_i| + |b_{i2}||e_i| + |\phi_i^*||e_i|\\
&\leq -(1 - c_{i2})(1 - c_{i1})\nu_i((1 - c_{i1})|e_i|)|e_i|^2 + \Pi_i(|e_i|)|e_i| - \frac{\iota_i}{2}|e_i|^2\\
&\leq -\frac{\iota_i}{2}|e_i|^2 = -\iota_i \alpha_V(|e_i|)
\end{aligned} \tag{123}$$

which implies (43). This ends the proof.

C Proof of Lemma 3

For convenience of notations, define $v_n = u$. Following the discussions in Remark 9, we have $v_0 \in S_0(\bar{x}_0, \bar{\mu}_{01}, \bar{\mu}_{02})$. Suppose that $v_{i-1} \in S_{i-1}(\bar{x}_{i-1}, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2})$. We will find ς_{i-1}, b_{i1} and b_{i2} satisfying $\varsigma_{i-1} \in S_{i-1}(\bar{x}_{i-1}, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2})$, $|b_{i1}| \leq \max\{c_{i1}|e_i|, \mu_{i1}\}$ and $|b_{i2}| \leq \mu_{i2}$, respectively, such that

$$v_i = \kappa_i(x_i - \varsigma_{i-1} + b_{i1}) + b_{i2} \in S_i(\bar{x}_i, \bar{\mu}_{i1}, \bar{\mu}_{i2}). \tag{124}$$

By applying this reasoning repeatedly, property (50) can be proved. We consider only the case of $e_i \geq 0$. The proof for the case of $e_i < 0$ is similar. We study the following cases (A) and (B).

(A) $|\kappa_i(q_{i1}(x_i - v_{i-1}, \mu_{i1}))| \leq M_{i2}\mu_{i2}$.

With Assumption 4 satisfied, one can find a $|b_{i2}| \leq \mu_{i2}$ such that

$$q_{i2}(\kappa_i(q_{i1}(x_i - v_{i-1}, \mu_{i1})), \mu_{i2}) = \kappa_i(q_{i1}(x_i - v_{i-1}, \mu_{i1})) + b_{i2}. \tag{125}$$

(A1) $|x_i - v_{i-1}| \le M_{i1}\mu_{i1}$.

In this case, Assumption 4 implies that there exists a $|b_{i1}| \le \mu_{i1}$ such that

$$q_{i1}(x_i - v_{i-1}, \mu_{i1}) = x_i - v_{i-1} + b_{i1} \tag{126}$$

Choose $\varsigma_{i-1} = v_{i-1}$. Then, $\varsigma_{i-1} \in S_{i-1}(\bar{x}_{i-1}, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2})$ and

$$q_{i1}(x_i - v_{i-1}, \mu_{i1}) = x_i - \varsigma_{i-1} + b_{i1}. \tag{127}$$

(A2) $|x_i - v_{i-1}| > M_{i1}\mu_{i1}$.

In this case, Assumption 4 implies that there exists a $|b_{i1}| \le \mu_{i1}$ such that

$$q_{i1}(x_i - v_{i-1}, \mu_{i1}) = \mathrm{sgn}(x_i - v_{i-1})M_{i1}\mu_{i1} + b_{i1} \tag{128}$$

We study the following two cases.

- $e_i > 0$. Recall (37) and (48). In this case, we have $x_i > v_{i-1}$ and

$$x_i - v_{i-1} > M_{i1}\mu_{i1} \ge e_i = x_i - \max S_{i-1}(\bar{x}_{i-1}, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2}). \tag{129}$$

  One can find a $\varsigma_{i-1} \in [v_{i-1}, \max S_{i-1}(\bar{x}_{i-1}, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2})]$ such that $x_i - \varsigma_{i-1} = M_{i1}\mu_{i1}$ and thus

$$q_{i1}(x_i - v_{i-1}, \mu_{i1}) = x_i - \varsigma_{i-1} + b_{i1}. \tag{130}$$

- $e_i = 0$. In this case, using (37), we have $x_i \in S_{i-1}(\bar{x}_{i-1}, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2})$ and can directly find a $\varsigma_{i-1} \in S_{i-1}(\bar{x}_{i-1}, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2})$, which is closer to x_i than v_{i-1} such that $x_i - \varsigma_{i-1} = \mathrm{sgn}(x_i - v_{i-1})M_{i1}\mu_{i1}$ and thus

$$q_{i1}(x_i - v_{i-1}, \mu_{i1}) = x_i - \varsigma_{i-1} + b_{i1}. \tag{131}$$

From (127) and (131), in the case of $|\kappa_i(q_{i1}(x_i - v_{i-1}))| \le M_{i2}\mu_{i2}$, we can find $\varsigma_{i-1} \in S_{i-1}(\bar{x}_{i-1})$, $|b_{i1}| \le \mu_{i1}$ and $|b_{i2}| \le \mu_{i2}$ such that

$$v_i = q_{i2}(\kappa_i(q_{i1}(x_i - v_{i-1}))) = \kappa_i(x_i - \varsigma_{i-1} + b_{i1}) + b_{i2}. \tag{132}$$

(B) $|\kappa_i(q_{i1}(x_i - v_{i-1}, \mu_{i1}))| > M_{i2}\mu_{i2}$.

Before the discussions, we give the following lemma.

Lemma 9 *Under the conditions of Lemma 3, if* $|\kappa_i(q_{i1}(x_i - v_{i-1}, \mu_{i1}))| > M_{i2}\mu_{i2}$, *then*

$$\mathrm{sgn}(x_i - v_{i-1}) = \mathrm{sgn}(q_{i1}(x_i - v_{i-1}, \mu_{i1})). \tag{133}$$

The proof of Lemma 9 is in Appendix C.1

Note that κ_i is an odd and strictly decreasing function. We have

$$\operatorname{sgn}(\kappa_i(q_{i1}(x_i - v_{i-1}, \mu_{i1}))) = -\operatorname{sgn}(q_{i1}(x_i - v_{i-1}, \mu_{i1})) = -\operatorname{sgn}(x_i - v_{i-1}). \tag{134}$$

Under Assumption 4, using Lemma 9 one can find a $|b_{i2}| \leq \mu_{i2}$ such that

$$\begin{aligned} & q_{i2}(\kappa_i(q_{i1}(x_i - v_{i-1}, \mu_{i1})), \mu_{i2}) \\ & \quad = \operatorname{sgn}(\kappa_i(q_{i1}(x_i - v_{i-1}, \mu_{i1})))M_{i2}\mu_{i2} + b_{i2} \\ & \quad = -\operatorname{sgn}(x_i - v_{i-1})M_{i2}\mu_{i2} + b_{i2}. \end{aligned} \tag{135}$$

(B1) $e_i > 0$.

In this case, using (37), we have $x_i > v_{i-1}$ and thus

$$\kappa_i(q_{i1}(x_i - v_{i-1}, \mu_{i1})) < 0, \tag{136}$$

$$q_{i2}(\kappa_i(q_{i1}(x_i - v_{i-1}, \mu_{i1}))) = -M_{i2}\mu_{i2} + b_{i2}. \tag{137}$$

With $|\kappa_i(q_{i1}(x_i - v_{i-1}, \mu_{i1}))| > M_{i2}\mu_{i2}$, property (136) implies

$$\kappa_i(q_{i1}(x_i - v_{i-1}, \mu_{i1})) < -M_{i2}\mu_{i2}. \tag{138}$$

Consider the following two cases.

- $x_i - v_{i-1} \leq M_{i1}\mu_{i1}$. In this case, under Assumption 4, one can find a $|b'_{i1}| \leq \mu_{i1}$ such that

$$\kappa_i(x_i - v_{i-1} + b'_{i1}) = \kappa_i(q_{i1}(x_i - v_{i-1}, \mu_{i1})) < -M_{i2}\mu_{i2}. \tag{139}$$

- $x_1 - v_{i-1} > M_{i1}\mu_{i1}$. In this case, under Assumption 4, one can find a $|b'_{i1}| \leq \mu_{i1}$ such that

$$\kappa_i(M_{i1}\mu_{i1} + b'_{i1}) = \kappa_i(q_{i1}(x_i - v_{i-1}, \mu_{i1})) < -M_{i2}\mu_{i2}. \tag{140}$$

Using the strictly decreasing property of κ_i, we have

$$\kappa_i(x_i - v_{i-1} + b'_{i1}) < \kappa_i(M_{i1}\mu_{i1} + b'_{i1}) < -M_{i2}\mu_{i2}. \tag{141}$$

Thus, in both the cases above, one can find a $|b'_{i1}| \leq \mu_{i1}$ such that

$$\kappa_i(x_i - v_{i-1} + b'_{i1}) < \kappa_i(M_{i1}\mu_{i1} + b'_{i1}) < -M_{i2}\mu_{i2}. \tag{142}$$

From (49), we have

$$\bar{\kappa}_i((1 - c_{i1})|e_i|) < M_{i2}\mu_{i2}. \tag{143}$$

From the definition of e_i, using $e_i > 0$, we have

$$\kappa_i(x_i - \max S_{i-1}(\bar{x}_{i-1}, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2}) - c_{i1}e_i) > -M_{i2}\mu_{i2}. \tag{144}$$

From (142) and (144), using the continuity of κ_i, one can find a $\varsigma_{i-1} \in [v_{i-1}, \max S_{i-1}(\bar{x}_{i-1}, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2})]$ and a $b_{i1} \in [-c_{i1}e_i, b'_{i1}]$ such that

$$\kappa_i(x_i - \varsigma_{i-1} + b_{i1}) = -M_{i2}\mu_{i2}. \tag{145}$$

Recall (137). We have

$$v_i = q_{i2}(\kappa_i(q_{i1}(x_i - v_{i-1}, \mu_{i1}))) = \kappa_i(x_i - \varsigma_{i-1} + b_{i1}) + b_{i2}. \tag{146}$$

(B2) $e_i = 0$.

In this case, $x_i \in S_{i-1}(\bar{x}_{i-1}, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2})$. From Lemma 9, we have $x_i - v_{i-1} \neq 0$. Consider the following two cases.

– $|x_i - v_{i-1}| \leq M_{i1}\mu_{i1}$. In this case, define $\varsigma'_{i-1} = v_{i-1}$. With Assumption 4, one can find a $|b'_{i1}| \leq \mu_{i1}$ such that

$$\kappa_i(x_i - \varsigma'_{i-1} + b'_{i1}) = \kappa_i(q_{i1}(x_i - v_{i-1}, \mu_{i1})) \begin{cases} > M_{i2}\mu_{i2}, & \text{if } x_i < \varsigma'_{i-1} \\ < -M_{i2}\mu_{i2}, & \text{if } x_i > \varsigma'_{i-1}. \end{cases} \tag{147}$$

We used $|\kappa_i(q_{i1}(x_i - v_{i-1}, \mu_{i1}))| > M_{i2}\mu_{i2}$ and (134) for the last part of (147).

– $|x_i - v_{i-1}| > M_{i1}\mu_{i1}$. In this case, under Assumption 4, one can find a $|b'_{i1}| \leq \mu_{i1}$ such that

$$\begin{aligned} &\kappa_i(\mathrm{sgn}(x_i - v_{i-1})M_{i1}\mu_{i1} + b'_{i1}) \\ &\quad = \kappa_i(q_{i1}(x_i - v_{i-1}, \mu_{i1})) \begin{cases} > M_{i2}\mu_{i2}, & \text{if } x_i < v_{i-1} \\ < -M_{i2}\mu_{i2}, & \text{if } x_i > v_{i-1}. \end{cases} \end{aligned} \tag{148}$$

In the case of $|x_i - v_{i-1}| > M_{i1}\mu_{i1}$, one can find a $\varsigma'_{i-1} \in [x_i, v_{i-1}]$ satisfying $\mathrm{sgn}(x_i - \varsigma'_{i-1}) = \mathrm{sgn}(x_i - v_{i-1})$ and $\mathrm{sgn}(x_i - v_{i-1})M_{i1}\mu_{i1} = x_i - \varsigma'_{i-1}$. In this way, we achieve

$$\kappa_i(x_i - \varsigma'_{i-1} + b'_{i1}) \begin{cases} > M_{i2}\mu_{i2}, & \text{if } x_i < \varsigma'_{i-1} \\ < -M_{i2}\mu_{i2}, & \text{if } x_i > \varsigma'_{i-1}. \end{cases} \tag{149}$$

Note that $\kappa_i(x_i - x_i + 0) = \kappa_i(0) = 0$. By using the continuity of κ_i, one can find a $\varsigma_{i-1} \in [x_i, \varsigma'_{i-1}]$ and a $b_{i1} \in [0, b'_{i1}]$ such that

$$\mathrm{sgn}(x_i - \varsigma_{i-1}) = \mathrm{sgn}(x_i - \varsigma'_{i-1}) = \mathrm{sgn}(x_i - v_{i-1}) \tag{150}$$

$$\kappa_i(x_i - \varsigma_{i-1} + b_{i1}) = -\mathrm{sgn}(x_i - v_{i-1})M_{i2}\mu_{i2}. \tag{151}$$

Clearly, $\varsigma_{i-1} \in S_{i-1}(\bar{x}_{i-1}, \bar{\mu}_{(i-1)1}, \bar{\mu}_{(i-1)2})$ and $|b_{i1}| \le \mu_{i1}$. Recall (135). We have

$$v_i = q_{i2}(\kappa_i(q_{i1}(x_i - v_{i-1}, \mu_{i1}))) = \kappa_i(x_i - \varsigma_{i-1} + b_{i1}) + b_{i2}. \quad (152)$$

Considering both cases (A) and (B), the proof of Lemma 3 is concluded.

C.1 Proof of Lemma 9

Consider the following two cases.

– $|x_i - v_{i-1}| > M_{i1}\mu_{i1}$. In this case, under Assumption 4, one can find a $|b_{i1}| \le \mu_{i1}$ such that

$$q_{i1}(x_i - v_{i-1}, \mu_{i1}) = \operatorname{sgn}(x_i - v_{i-1})M_{i1}\mu_{i1} + b_{i1} \quad (153)$$

Note that $M_{i1} > 2$. Thus,

$$\operatorname{sgn}(x_i - v_{i-1}) = \operatorname{sgn}(q_{i1}(x_i - v_{i-1}, \mu_{i1})). \quad (154)$$

– $|x_i - v_{i-1}| \le M_{i1}\mu_{i1}$. In this case, under Assumption 4, one can find a $|b_{i1}| \le \mu_{i1}$ such that

$$q_{i1}(x_i - v_{i-1}, \mu_{i1}) = x_i - v_{i-1} + b_{i1}. \quad (155)$$

Condition $|\kappa_i(q_{i1}(x_i - v_{i-1}, \mu_{i1}))| > M_{i2}\mu_{i2}$ implies $q_{i1}(x_i - v_{i-1}, \mu_{i1}) \ne 0$. If $\operatorname{sgn}(x_i - v_{i-1}) \ne \operatorname{sgn}(q_{i1}(x_i - v_{i-1}, \mu_{i1}))$, then $\operatorname{sgn}(b_{i1}) = \operatorname{sgn}(q_{i1}(x_i - v_{i-1}, \mu_{i1}))$ and $|b_{i1}| > |x_i - v_{i-1}|$. Thus, $|x_i - v_{i-1} + b_{i1}| \le |b_{i1}| \le \mu_{i1}$. Note that $\frac{1}{M_{i1}} < c_{i1} < 0.5$. Then, we can derive

$$\begin{aligned} |\kappa_i(q_{i1}(x_i - v_{i-1}, \mu_{i1}))| &\le \bar{\kappa}_i(\mu_{i1}) \le \bar{\kappa}_i\left(\frac{1 - c_{i1}}{c_{i1}}\mu_{i1}\right) < \bar{\kappa}_i((1 - c_{i1})M_{i1}\mu_{i1}) \\ &= M_{i2}\mu_{i2} \end{aligned} \quad (156)$$

which leads to a contradiction with $|\kappa_i(q_{i1}(x_i - v_{i-1}))| > M_{i2}\mu_{i2}$. We used (60) for the last equality in (156). This ends the proof of Lemma 9.

D Proof of Lemma 6

Recall that if $\chi_1, \chi_2 \in \mathscr{K}_\infty$ satisfies $\chi_1(s) > \chi_2(s)$ for $s \in \mathbb{R}_+$, then $(\mathrm{Id} - \tilde{\chi}) \circ \chi_1(s) \ge \chi_2(s)$ for $s \in \mathbb{R}_+$ with $\tilde{\chi} := \mathrm{Id} - \chi_2 \circ \chi_1^{-1}$ being continuous and positive definite.

For each $i = 1, \dots, n$, with (47) satisfied, we can find a continuous and positive definite ρ_i such that

$$\sigma_i \circ \alpha_V\left(\frac{1}{c_{i1}}s\right) \le (\mathrm{Id} - \rho_i) \circ \sigma_i \circ \alpha_V(M_{i1}s) \tag{157}$$

$$\sigma_i \circ \alpha_V\left(\frac{1}{1-c_{i1}}\bar{\kappa}_i^{-1}\left(\frac{1}{c_{i2}}s\right)\right) \le (\mathrm{Id} - \rho_i) \circ \sigma_i \circ \alpha_V\left(\frac{1}{1-c_{i1}}\bar{\kappa}_i^{-1}(M_{i2}s)\right) \tag{158}$$

for all $s \in \mathbb{R}_+$. Define $\rho(s) = \min_{i=1,\dots,n}\{\rho_i(s)\}$ for $s \in \mathbb{R}_+$. Then, ρ is continuous and positive definite. Using (58), (59), (64) and (69), we have

$$B_2(\bar{\mu}_{n1}(t), \bar{\mu}_{n2}(t)) \le (\mathrm{Id} - \rho)(\Theta(t)) \tag{159}$$

for any $t \in \mathbb{R}_+$.

Note that the zooming variables $\mu_{n1}(t)$ and $\mu_{n2}(t)$ are constant on $[t_k, t_{k+1})$, that is, $\mu_{n1}(t) = \mu_{n1}(t_k)$ and $\mu_{n2}(t) = \mu_{n2}(t_k)$ for $t \in [t_k, t_{k+1})$.

Suppose (79) holds. We study the following two cases:

(a) $V(e(X(t_{k+1}), \bar{\mu}_{n1}(t_k), \bar{\mu}_{n2}(t_k))) < \max\{(\mathrm{Id} - \rho)(\Theta(t_k)), \theta_0\}$.

(b) $V(e(X(t_{k+1}), \bar{\mu}_{n1}(t_k), \bar{\mu}_{n2}(t_k))) \ge \max\{(\mathrm{Id} - \rho)(\Theta(t_k)), \theta_0\}$. In this case, from (63), (79) and (159), it follows that $V(e(X(t), \bar{\mu}_{n1}(t), \bar{\mu}_{n2}(t)))$ is strictly decreasing for $t \in [t_k, t_{k+1})$ and

$$\max\{(\mathrm{Id} - \rho)(\Theta(t_k)), \theta_0\} \le V(e(X(t), \bar{\mu}_{n1}(t), \bar{\mu}_{n2}(t))) \le \Theta(t_k) \tag{160}$$

for all $t \in [t_k, t_{k+1})$. Using (63), we have

$$\begin{aligned} &V(e(X(t_{k+1}), \bar{\mu}_{n1}(t_k), \bar{\mu}_{n2}(t_k))) \\ &\le V(e(X(t_k), \bar{\mu}_{n1}(t_k), \bar{\mu}_{n2}(t_k))) - \int_{t_k}^{t_{k+1}} \alpha(V(e(X(\tau), \bar{\mu}_{n1}(\tau), \bar{\mu}_{n2}(\tau))))d\tau \\ &\le \Theta(t_k) - t_d \cdot \min_{\max\{(\mathrm{Id}-\rho)(\Theta(t_k)),\theta_0\} \le v \le \Theta(t_k)} \alpha(v) \\ &\le \Theta(t_k) - t_d \cdot \min_{(\mathrm{Id}-\rho)(\Theta(t_k)) \le v \le \Theta(t_k)} \alpha(v) \end{aligned} \tag{161}$$

where $t_d = t_{k+1} - t_k$. Define $\rho'(s) = t_d \cdot \min_{(\mathrm{Id}-\rho)(s) \le v \le s} \alpha(v)$ for $s \in \mathbb{R}_+$. Then, it can be directly verified that ρ' is continuous and positive definite and that

$$V(e(X(t_{k+1}), \bar{\mu}_{n1}(t_k), \bar{\mu}_{n2}(t_k))) \le (\mathrm{Id} - \rho')(\Theta(t_k)). \tag{162}$$

Lemma 6 is proved by finding a continuous and positive definite function $\bar{\rho}$ such that $(\mathrm{Id} - \bar{\rho}) \in \mathscr{K}_\infty$ and $(\mathrm{Id} - \bar{\rho})(s) \ge \max\{(\mathrm{Id} - \rho)(s), (\mathrm{Id} - \rho')(s)\}$ for $s \in \mathbb{R}_+$.

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