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ORIGINAL ARTICLE

Input-to-state stability for a class of hybrid dynamical systems via averaging

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Abstract Input-to-state stability (ISS) properties for a class of time-varying hybrid dynamical systems via averaging method are considered. Two definitions of averages, strong average and weak average, are used to approximate the time-varying hybrid systems with time-invariant hybrid systems. Closeness of solutions between the time-varying system and solutions of its weak or strong average on compact time domains is given under the assumption of forward completeness for the average system. We also show that ISS of the strong average implies semi-global practical (SGP)-ISS of the actual system. In a similar fashion, ISS of the weak average implies semi-global practical derivative ISS (SGP-DISS) of the actual system. Through a power converter example, we show that the main results can be used in a framework for a systematic design of hybrid feedbacks for pulse-width modulated control systems.

Keywords Averaging · Hybrid systems · Input-to-state stability · Semi-global practical ISS · Semi-global practical derivative ISS

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1 Introduction

Hybrid systems comprise a rich class of systems with interacting continuous and discrete dynamics, and they have been used extensively to describe a wide range of applications including robotics, automotive electronics, manufacturing, automated highway systems, air traffic management systems, integrated circuit design, and chemical processes [14,24,32]. Recently, a modeling framework was proposed and a range of analysis tools developed for general hybrid systems, such as conditions for existence of solutions, Lyapunov stability, robust stability and so on [1,5,7,8,15,16,25].

The averaging method was developed for continuous-time systems, discrete-time systems, and differential inclusions [3,12,13,31], but there exist only a few results for special classes of hybrid systems. For example, results on averaging of switched linear systems and dither systems were considered in [17,18,30]. Note that all aforementioned results on averaging for hybrid systems approximate the time-varying hybrid systems by a non-hybrid average system. On the other hand, it is sometimes appropriate to average the time-varying hybrid system by a time-invariant hybrid system. For instance, systems controlled with hybrid feedback controllers that are implemented by pulse-width modulation (PWM) require this type of results.

Pulse-width modulation is a technique in which the width of a train of voltage (or current) pulses is adjusted (modulated) by rapidly turning the switch between the supply and load on and off. This technology is used extensively in power electronics and finds wide applications in industry [11,19,22,29,33,34]. Robustness of PWM control systems has been analyzed via the averaging method in [36], where the nonsmooth nature of PWM systems is accommodated by working with upper semicontinuous set-valued maps. For PWM hybrid feedback systems, it is desirable to prove that the PWM implementation produces a closed-loop behavior that is similar to the behavior that would be achieved by implementing the hybrid feedback directly to the system without PWM implementation [37].

The present paper extends the averaging results of [37] by analyzing robustness of PWM systems to exogenous disturbances. In particular, we revisit two kinds of average definitions (the strong and the weak average) defined in [28] for continuous-time systems with disturbances. First, we present results on closeness of solutions between the strong (weak) average and the actual hybrid system on compact time domains assuming that the average system is forward complete. Second, under the stronger assumption of input-to-state stability (ISS) of the strong (weak) average, we show that this condition implies appropriate semi-global practical ISS (derivative ISS) properties for the actual system. Note that while the class of systems for which strong averages exist is smaller, we can conclude stronger ISS properties of the actual systems via strong averages than via weak averages.

The paper is organized as follows: Some definitions in the hybrid system setting are reviewed in Sect. 2. The class of time-varying hybrid systems with disturbances that we will consider is introduced in Sect. 3. Section 4 contains the main results and Sect. 5 illustrates how our results can be applied to a PWM control example. Conclusions are provided in Sect. 6.

2 Preliminaries

$\mathbb{R}_{\geq 0} := [0, +\infty)$, $\mathbb{Z}_{\geq 0} := \{0, 1, 2, \ldots\}$, $\mathbb{B}$ is the closed unit ball in an Euclidean space, the dimension of which should be clear from the context and $|\cdot|$ refers to the Euclidean norm. We also use $|w|$ for hybrid signals w and the definition in this case is given in (4). A set-valued mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semi-continuous at $x \in \mathbb{R}^n$ if for all sequences $x_i \to x$ and $y_i \in M(x_i)$ such that $y_i \to y$ we have $y \in M(x)$, and M is outer semi-continuous (OSC) if it is outer semi-continuous at each $x \in \mathbb{R}^n$. A set-valued mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is locally bounded if for any compact set $A \subset \mathbb{R}^n$ there exists $r > 0$ such that $M(A) := \bigcup_{x\in A} M(x) \subset r\mathbb{B}$; if M is OSC and locally bounded, then $M(A)$ is compact for any compact set A. A function $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is locally absolutely continuous if its derivative is defined almost everywhere and we have $x(t) - x(t_0) = \int_{t_0}^{t} \dot{x}(s)\mathrm{d}s$ for all $t \geq t_0 \geq 0$. Given a compact set $A \subset \mathbb{R}^n$, a function $\chi : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be a proper indicator function for A on $\mathbb{R}^n$ if χ is continuous, $\chi(x) = 0$ if and only if $x \in A$, and $\chi(x) \to \infty$ when $|x| \to \infty$. Given a set S, $\overline{con}\, S$ denotes its convex hull. Given a compact set $A \subset \mathbb{R}^n$ and a $x \in \mathbb{R}^n$, define $|x|_A := \min_{y\in A} |x - y|$. Given a measurable function $\tilde{w}(\cdot)$, we define its infinity norm $||\tilde{w}||_\infty := \text{ess sup}_{t\geq 0} |\tilde{w}(t)|$. If we have $||\tilde{w}||_\infty < \infty$, then we write $\tilde{w} \in \mathcal{L}_\infty$. A continuous function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{L}$ if it is non-increasing and converging to zero as its argument grows unbounded. A continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{G}$ if it is zero at zero and non-decreasing. It is of class-$\mathcal{K}$ if it is of class-$\mathcal{G}$ and strictly increasing. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ if it is of class-$\mathcal{K}$ in its first argument and class of $\mathcal{L}$ in its second argument.

Consider a time-invariant hybrid system

$$\mathcal{H} \quad \begin{array}{ll} \dot{\xi} = F(\xi, w) & (\xi, w) \in C \\ \xi^+ \in G(\xi, w) & (\xi, w) \in D, \end{array} \tag{1}$$

with $\xi \in \mathbb{R}^n$, $w \in \mathcal{W} \subset \mathbb{R}^m$. For any $\Omega \geq 0$, consider a hybrid inclusion

$$\mathcal{H}_\Omega \quad \begin{array}{ll} \dot{\xi} \in F_\Omega(\xi) & \xi \in C_\Omega \\ \xi^+ \in G_\Omega(\xi) & \xi \in D_\Omega \end{array} \tag{2}$$

that is extended from system $\mathcal{H}$ in (1) with the data $(F_\Omega, G_\Omega, C_\Omega, D_\Omega)$ being defined as

$$\begin{aligned} F_\Omega(\xi) &:= \{v \in \mathbb{R}^n : v = F(\xi, w), w \in \mathcal{W} \cap \Omega\mathbb{B} \text{ and } (\xi, w) \in C\}, \\ G_\Omega(\xi) &:= \{v \in \mathbb{R}^n : v \in G(\xi, w), w \in \mathcal{W} \cap \Omega\mathbb{B} \text{ and } (\xi, w) \in D\}, \\ C_\Omega &:= \{\xi : \exists w \in \mathcal{W} \cap \Omega\mathbb{B} \text{ such that } (\xi, w) \in C\}, \\ D_\Omega &:= \{\xi : \exists w \in \mathcal{W} \cap \Omega\mathbb{B} \text{ such that } (\xi, w) \in D\}. \end{aligned} \tag{3}$$

In order to exploit recent results in the literature on robustness for hybrid systems, we make the following assumptions:

Assumption 1 *The sets $C \subset \mathbb{R}^n \times \mathbb{R}^m$, $D \subset \mathbb{R}^n \times \mathbb{R}^m$ and $\mathcal{W} \subset \mathbb{R}^m$ are closed; $F : C \to \mathbb{R}^n$ is continuous, for each $\Omega \geq 0$ and $\xi \in C_\Omega$, the set $F_\Omega(\xi)$ is convex;*

$G : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ *is outer semi-continuous and locally bounded, and for each* $(\xi, w) \in D$, $G(\xi, w)$ *is nonempty.*

The convexity condition in Assumption 1 is used to guarantee robustness to perturbations for hybrid systems and our results are based on robustness properties of the hybrid system. An example mentioned in [5, Remark 3] illustrates that forward completeness, which is weaker than stability, of a hybrid system without convexity assumption for the flow mapping F with respect to disturbances may not be preserved under a small perturbation.

A set $S \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is called a compact hybrid time domain if $S = \bigcup_{j=0}^{J-1}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_J$. The set S is a hybrid time domain if for all $(T, J) \in S$, $S \cap ([0, T] \times \{0, 1, \ldots, J\})$ is a compact hybrid time domain. We need the following definitions of solutions defined on a hybrid time domain [6].

Definition 1 A hybrid signal is a function defined on a hybrid time domain. $w :$ dom $w \to \mathcal{W}$ is called a hybrid input if $w(\cdot, j)$ is Lebesgue measurable and locally essentially bounded for each j. A hybrid signal $\xi :$ dom $\xi \mapsto \mathbb{R}^n$ is called a hybrid arc if $\xi(\cdot, j)$ is locally absolutely continuous for each j. A hybrid arc $\xi :$ dom $\xi \mapsto \mathbb{R}^n$ is a solution to the hybrid inclusion $\mathcal{H}_\Omega$ in (2) if $\xi(0, 0) \in C_\Omega \cup D_\Omega$ and

1. for all $j \in \mathbb{Z}_{\geq 0}$ and almost all t such that $(t, j) \in$ dom ξ, $\xi(t, j) \in C_\Omega$ and $\dot{\xi}(t, j) \in F_\Omega(\xi(t, j))$;
2. for all $(t, j) \in$ dom ξ such that $(t, j+1) \in$ dom ξ, $\xi(t, j) \in D_\Omega$ and $\xi(t, j+1) \in G_\Omega(\xi(t, j))$.

A hybrid arc $\xi :$ dom $\xi \mapsto \mathbb{R}^n$ and a hybrid input $w :$ dom $w \mapsto \mathcal{W}$ form a solution pair to system $\mathcal{H}$ in (1) if dom $\xi =$ dom w, $(\xi(0, 0), w(0, 0)) \in C \cup D$ and

1. for all $j \in \mathbb{Z}_{\geq 0}$ and almost all t such that $(t, j) \in$ dom ξ, $(\xi(t, j), w(t, j)) \in C$ and $\dot{\xi}(t, j) = F(\xi(t, j), w(t, j))$;
2. for all $(t, j) \in$ dom ξ such that $(t, j + 1) \in$ dom ξ, $(\xi(t, j), w(t, j)) \in D$ and $\xi(t, j + 1) \in G(\xi(t, j), w(t, j))$.

A solution or a solution pair is maximal if it cannot be extended.

Given any hybrid signal $w :$ dom $w \mapsto \mathcal{W}$, let $\Gamma(w)$ denote the set of $(t, j) \in$ dom w such that $(t, j + 1) \in$ dom w, and define

$$|w| := \max \left\{ \operatorname*{ess\,sup}_{(t,j)\in \text{dom } w \setminus \Gamma(w)} |w(t, j)|, \quad \sup_{(t,j)\in\Gamma(w)} |w(t, j)| \right\}. \tag{4}$$

Note that for each j, the set of t's such that $(t, j) \in \Gamma(w)$ and $t \in I_j := \{t \in \mathbb{R} : (t, j) \in \text{dom } w\}$ has one-dimensional zero measure and thus we can define $\dot{w}(t, j) = 0$ for all $(t, j) \in$ dom w without affecting w. With this convention, the definition $|\dot{w}(t, j)|$ reduces to

$$|\dot{w}| := \operatorname*{ess\,sup}_{(t,j)\in\text{dom } w} |\dot{w}(t, j)| .$$

Let $\mathcal{L}_{\mathcal{W}}$ be a given subset of hybrid signals w : dom $w \to \mathcal{W}$. The definitions of equi-essential boundedness and local equi-uniform Lipschitz continuity for a set of hybrid signals are given as follows:

Definition 2 The set $\mathcal{L}_{\mathcal{W}}$ is called equi-essentially bounded if there exists $\Omega > 0$ such that $|w| \leq \Omega$ for all $w \in \mathcal{L}_{\mathcal{W}}$.

Definition 3 The set $\mathcal{L}_{\mathcal{W}}$ is called locally equi-uniformly Lipschitz continuous if there exists $L > 0$ such that, for all $w \in \mathcal{L}_{\mathcal{W}}$ and $(t, j), (s, j) \in \text{dom } w$, the following holds:

$$|w(t, j) - w(s, j)| \leq L|t - s|.$$

A sufficient condition for $\mathcal{L}_{\mathcal{W}}$ to be locally equi-uniformly Lipschitz continuous is that there exists a strictly positive real number Ω_1 such that, for each $w \in \mathcal{L}_{\mathcal{W}}$, $w(\cdot, j)$ is locally absolutely continuous for each j and for all $(t, j) \in \text{dom } w$ such that $|\dot{w}| \leq \Omega_1$.

3 Definitions of strong and weak averages

In this section, the class of time-varying hybrid systems we consider is presented and definitions of weak and strong average for such systems are given. In addition, we introduce functions that are used in a coordinate transformation that facilitates establishing the averaging results. Basic requirements of these functions are established in claims and lemmas.

Consider a class of time-varying hybrid systems $\mathcal{H}_\epsilon$ that depends on a small parameter $\varepsilon > 0$:

$$\mathcal{H}_\epsilon \quad \left.\begin{aligned} \dot{x} &= f_\varepsilon(x, w, \tau) \\ \dot{\tau} &= \tfrac{1}{\varepsilon} \end{aligned}\right\}, \quad ((x, w), \tau) \in C \times \mathbb{R}_{\geq 0}, \qquad \left.\begin{aligned} x^+ &\in G(x, w) \\ \tau^+ &\in H(x, w, \tau) \end{aligned}\right\}, \quad ((x, w), \tau) \in D \times \mathbb{R}_{\geq 0}, \tag{5}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, $f_\varepsilon : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $G : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ and $\mathcal{H} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_{\geq 0} \rightrightarrows \mathbb{R}_{\geq 0}$.

Assumption 2 *Suppose that* (G, C, D) *satisfy Assumption* 1*;* $\tau \mapsto f_\varepsilon(x, w, \tau)$ *is measurable for each* $(x, w) \in C$*; and for each* $\delta > 0$ *and compact set* $K \subset \mathbb{R}^n \times \mathbb{R}^m$ *there exist* $M(K) > 0$ *and* $\varepsilon^*(K, \delta) > 0$ *such that*

$$\begin{aligned} |f_0(x, w, \tau)| &\leq M, \quad \forall((x, w), \tau) \in (C \cap K) \times \mathbb{R}_{\geq 0}, \\ |f_\varepsilon(x, w, \tau) - f_0(x, w, \tau)| &\leq \frac{\delta}{3}, \quad \forall((x, w), \tau, \varepsilon) \in (C \cap K) \times \mathbb{R}_{\geq 0} \times (0, \varepsilon^*]. \end{aligned} \tag{6}$$

We next define weak and strong averages that are taken from [28] for the flow mapping $f_0 : C \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ on the flow set C. For simplicity, the following average definitions are defined on the time line t instead of hybrid time domain (t, j).

Definition 4 (*Weak Average*) For a function $f_0 : C \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, the function $f_{\mathrm{wa}} : C \to \mathbb{R}^n$ is said to be a weak average on C if for each compact set $K \subset \mathbb{R}^n \times \mathbb{R}^m$ there exists a class-$\mathcal{L}$ function σ_K such that, for all $((x, w), \tau, T) \in (C \cap K) \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$:

$$\left| \frac{1}{T} \int_{\tau}^{\tau+T} [f_{\mathrm{wa}}(x, w) - f_0(x, w, s)]\mathrm{d}s \right| \leq \sigma_K(T).$$

The strong average is defined for a subset of input signals. Noting that the average definition is based on the time domain t, we need the notation for sets of input signals defined on t. Let $\tilde{\mathcal{L}}_{\mathcal{W}}$ be a set of input signals $\tilde{w} \in \mathcal{L}_\infty : \mathbb{R}_{\geq 0} \to \mathcal{W}$. We have the following strong average definition:

Definition 5 (*Strong Average*) For a function $f_0 : C_1 \times \mathcal{W} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, the function $f_{\mathrm{sa}} : C_1 \times \mathcal{W} \to \mathbb{R}^n$ is said to be a strong average on $C_1 \times \mathcal{W}$ if for each compact set $K \subset \mathbb{R}^n \times \mathbb{R}^m$ there exists a class-$\mathcal{L}$ function σ_K such that, for all $\tilde{w} \in \tilde{\mathcal{L}}_{\mathcal{W}}$ with $((x, \tilde{w}(s)), \tau, T) \in ((C_1 \times \mathcal{W}) \cap K) \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ for all $s \geq 0$, the following holds:

$$\left| \frac{1}{T} \int_{\tau}^{\tau+T} [f_{\mathrm{sa}}(x, \tilde{w}(s)) - f_0(x, \tilde{w}(s), s)]\mathrm{d}s \right| \leq \sigma_K(T).$$

Let f_{wa} come from Definition 4 and (G, C, D) from (5). The weak average $\mathcal{H}_{\mathrm{wa}}$ of system $\mathcal{H}_\varepsilon$ is

$$\mathcal{H}_{\mathrm{wa}} \quad \begin{array}{ll} \dot{\xi} = f_{\mathrm{wa}}(\xi, w), & (\xi, w) \in C, \\ \xi^+ \in G(\xi, w), & (\xi, w) \in D. \end{array} \tag{7}$$

Similarly, for the case where $C = C_1 \times \mathcal{W}$, the strong average $\mathcal{H}_{\mathrm{sa}}$ of system $\mathcal{H}_\varepsilon$ is

$$\mathcal{H}_{\mathrm{sa}} \quad \begin{array}{ll} \dot{\xi} = f_{\mathrm{sa}}(\xi, w), & (\xi, w) \in C, \\ \xi^+ \in G(\xi, w), & (\xi, w) \in D, \end{array} \tag{8}$$

where f_{sa} comes from Definition 5. Note that the averaging technique is only applied to simplify the time-varying flow dynamics and the jump mapping G in the average systems is identical to the jump mapping G in the actual hybrid system $\mathcal{H}_\epsilon$ in (5); this is motivated by systems like PWM hybrid feedback control systems; see Sect. 5.

To employ a coordinate transformation, a continuous function that reflects accumulating errors between the actual system and its average is usually constructed to facilitate averaging techniques [20,37]. We next define the functions η_{wa} and η_{sa} used in a coordinate transformation to facilitate the averaging results for weak average and strong average case, respectively. Let f_{wa} and f_{sa} come from the definitions of weak average and strong average. For each $((x, w), \tau, \mu) \in C \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ and $\tau_0 \in [0, \tau]$, let

$$\eta_{\mathrm{wa}}(x, w, \tau, \tau_0, \mu) := \int_{\tau_0}^{\tau} \exp(\mu(s-\tau))[f_0(x, w, s) - f_{\mathrm{wa}}(x, w)]\mathrm{d}s. \tag{9}$$

Let $0 \leq \tau_0 \leq \tau_1$ and $\tilde{w} : [\tau_0, \tau_1] \to \mathcal{W}$ be given. For each $\tau \in [\tau_0, \tau_1]$ and $(x, \mu) \in C_1 \times \mathbb{R}_{\geq 0}$, let

$$\eta_{\mathrm{sa}}(x, \tilde{w}, \tau, \tau_0, \mu) := \int_{\tau_0}^{\tau} \exp(\mu(s-\tau))[f_0(x, \tilde{w}(s), s) - f_{\mathrm{sa}}(x, \tilde{w}(s))]\mathrm{d}s. \tag{10}$$

The following lemmas describing the properties of functions η_{wa} and η_{sa} are helpful to prove the results of closeness of solutions in Theorems 1 and 2. The proof of Lemma 2 is given in Appendix A and the proof of Lemma 1 is omitted since it is nearly identical to the proof of Lemma 2.

Lemma 1 *For a function f_0 defined on $C \times \mathbb{R}_{\geq 0}$, suppose f_{wa} is a continuous function that is a weak average of f_0 on C. Then, for each compact set $K \subset \mathbb{R}^n \times \mathbb{R}^m$, there exists a function α_K of class-$\mathcal{G}$ such that for all $((x, w), \mu, \tau) \in (C \cap K) \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ and $\tau_0 \in [0, \tau]$:*

$$\mu|\eta_{\mathrm{wa}}(x, w, \tau, \tau_0, \mu)| \leq \alpha_K(\mu).$$

Lemma 2 *For a function f_0 defined on $C_1 \times \mathcal{W} \times \mathbb{R}_{\geq 0}$, where $C \subset C_1 \times \mathcal{W}$, suppose f_{sa} is a continuous function that is a strong average of f_0 on $C_1 \times \mathcal{W}$. Then, for each compact set $K \subset \mathbb{R}^n \times \mathbb{R}^m$, there exists a function α_K of class-$\mathcal{G}$ such that for all $0 \leq \tau_0 \leq \tau_1$, $\tilde{w} : [\tau_0, \tau_1] \to \mathcal{W}$ and $((x, \tilde{w}(s)), \mu, \tau) \in ((C_1 \times \mathcal{W}) \cap K) \times \mathbb{R}_{\geq 0} \times [\tau_0, \tau_1]$ for all $s \in [\tau_0, \tau_1]$, the following holds:*

$$\mu|\eta_{\mathrm{sa}}(x, \tilde{w}, \tau, \tau_0, \mu)| \leq \alpha_K(\mu).$$

We assume that when $\mu = 0$, η_{sa} and η_{wa} are locally Lipschitz, uniformly in τ and τ_0, as stated below in Assumptions 3 and 4. These assumptions may hold even when f is not periodic in τ nor continuous in (x, w). The pulse-width modulated system in Sect. 5 illustrates this situation. Let $\bar{N} := \{1, \ldots, n\}$. For each $i \in \bar{N}$, η^i_{sa} represents the ith component of η_{sa}, and similarly for η^i_{wa}.

Assumption 3 *For a function f_0 defined on $C \times \mathbb{R}_{\geq 0}$, f_{wa} is a continuous function that is a weak average of f_0 on C and, for each compact set $K \subset \mathbb{R}^n \times \mathbb{R}^m$, there exists $L(K)$ such that, for all $i \in \bar{N}$, $((x_1, w_1), \tau_a), ((x_2, w_2), \tau_b) \in (C \cap K) \times \mathbb{R}_{\geq 0}$ and $\tau_0 \in [0, \min\{\tau_a, \tau_b\}]$:*

$$\left|\eta^i_{\mathrm{wa}}(x_1, w_1, \tau_a, \tau_0, 0) - \eta^i_{\mathrm{wa}}(x_2, w_2, \tau_b, \tau_0, 0)\right| \\ \leq L(|x_1 - x_2| + |w_1 - w_2| + |\tau_a - \tau_b|).$$

Assumption 4 *For a function f_0 defined on $C_1 \times \mathcal{W} \times \mathbb{R}_{\geq 0}$, where $C \subset C_1 \times \mathcal{W}$, $f_{\rm sa}$ is a continuous function that is a strong average of f_0 on $C_1 \times \mathcal{W}$ and, for each compact set $K \subset \mathbb{R}^n \times \mathbb{R}^m$, there exists $L(K)$ such that, for all $i \in \bar{N}$, $0 \leq \tau_0 \leq \tau_1$, $\tilde{w} : [\tau_0, \tau_1] \to \mathcal{W}$ and $((x_1, \tilde{w}(s)), \tau_a), ((x_2, \tilde{w}(s)), \tau_b) \in ((C_1 \times \mathcal{W}) \cap K) \times [\tau_0, \tau_1]$ for all $s \in [\tau_0, \tau_1]$:*

$$\left|\eta^i_{\rm sa}(x_1, \tilde{w}, \tau_a, \tau_0, 0) - \eta^i_{\rm sa}(x_2, \tilde{w}, \tau_b, \tau_0, 0)\right| \leq L(|x_1 - x_2| + |\tau_a - \tau_b|).$$

The following claims show that the Lipschitz property in Assumptions 3 and 4 implies a similar Lipschitz condition for $\eta_{\rm wa}$ and $\eta_{\rm sa}$ for values $\mu > 0$. The proof of Claim 2 is given in the Appendix B and the proof of Claim 1 is omitted due to its similarity to the proof of Claim 2.

Claim 1 Under Assumption 3, for each compact set $K \subset \mathbb{R}^n \times \mathbb{R}^m$ there exists $L(K)$ such that, for each $i \in \bar{N}$, $\mu > 0$, $((x_1, w_1), \tau_a), ((x_2, w_2), \tau_b) \in (C \cap K) \times \mathbb{R}_{\geq 0}$ and $\tau_0 \in [0, \min\{\tau_a, \tau_b\}]$:

$$\begin{aligned}&\left|\eta^i_{\rm wa}(x_1, w_1, \tau_a, \tau_0, \mu) - \eta^i_{\rm wa}(x_2, w_2, \tau_b, \tau_0, \mu)\right|\\ &\quad \leq 2L(|x_1 - x_2| + |w_1 - w_2| + |\tau_a - \tau_b|).\end{aligned}$$

Claim 2 Under Assumption 4, for each compact set $K \subset \mathbb{R}^n \times \mathbb{R}^m$ there exists $L(K)$ such that, for each $i \in \bar{N}$, $\mu > 0$, $0 \leq \tau_0 \leq \tau_1$, $\tilde{w} : [\tau_0, \tau_1] \to \mathcal{W}$ and $((x_1, \tilde{w}(s)), \tau_a), ((x_2, \tilde{w}(s)), \tau_b) \in ((C_1 \times \mathcal{W}) \cap K) \times [\tau_0, \tau_1]$ for all $s \in [\tau_0, \tau_1]$, the following holds:

$$\left|\eta^i_{\rm sa}(x_1, \tilde{w}, \tau_a, \tau_0, \mu) - \eta^i_{\rm sa}(x_2, \tilde{w}, \tau_b, \tau_0, \mu)\right| \leq 2L(|x_1 - x_2| + |\tau_a - \tau_b|).$$

4 Main results

In this section, we present the results on closeness between solutions of hybrid system $\mathcal{H}_\epsilon$ and solutions of its averages systems. We also state results on semiglobal practical ISS (DISS) properties of $\mathcal{H}_\epsilon$ assuming ISS of its strong average (weak average). The following concept of (T, ρ)-closeness that defines graphical convergence of hybrid arcs, see details in [16, Section 4], is required:

Definition 6 (*Closeness of hybrid signals*) Two hybrid signals $\xi_1 : \text{dom}\, \xi_1 \mapsto \mathbb{R}^n$ and $\xi_2 : \text{dom}\, \xi_2 \mapsto \mathbb{R}^n$ are said to be (T, ρ)-close if

1. for each $(t, j) \in \text{dom}\, \xi_1$ with $t + j \leq T$ there exists s such that $(s, j) \in \text{dom}\, \xi_2$, with $|t - s| \leq \rho$ and $|\xi_1(t, j) - \xi_2(s, j)| \leq \rho$,
2. for each $(t, j) \in \text{dom}\, \xi_2$ with $t + j \leq T$ there exists s such that $(s, j) \in \text{dom}\, \xi_1$, with $|t - s| \leq \rho$ and $|\xi_2(t, j) - \xi_1(s, j)| \leq \rho$.

The results on closeness of solutions are derived under the assumption that the average system is forward pre-complete.

Definition 7 (*Forward completeness*) A hybrid solution pair is said to be forward complete if its domain is unbounded. A hybrid solution pair is said to be forward pre-complete if its domain is compact or unbounded. System $\mathcal{H}$ in (1) is said to be forward pre-complete from a compact set $K_0 \subset \mathbb{R}^n$ with a disturbance bound $\Omega \geq 0$ if all maximal solution pairs (ξ, w) with $\xi(0, 0) \in K_0$ and w with $|w| \leq \Omega$ are forward pre-complete.

We are now ready to state Theorems 1 and 2 that demonstrate closeness of solutions for the time-varying system $\mathcal{H}_\epsilon$ and solutions of its weak and strong average systems. Proofs of Theorems 1 and 2 are given in Appendices C.2 and C.3, respectively.

Theorem 1 (Weak average) *Suppose that the set $\mathcal{L}_\mathcal{W}$ is equi-essentially bounded and locally equi-uniformly Lipschitz continuous, system $\mathcal{H}_\epsilon$ in (5) satisfies Assumptions 2 and 3, and its weak average system $\mathcal{H}_{\rm wa}$ satisfies Assumption 1 and is forward pre-complete from a compact set $K_0 \subset \mathbb{R}^n$ with a disturbance bound $\Omega \geq 0$. Then, for each $T \geq 0$ and $\rho > 0$, there exists $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*]$ and $w \in \mathcal{L}_\mathcal{W}$, each solution pair (x, w) to $\mathcal{H}_\epsilon$ with $x(0, 0) \in K_0$ there exists some solution pair (ξ, w_1) to $\mathcal{H}_{\rm wa}$ with $\xi(0, 0) \in K_0$ and $|w_1| \leq |w|$ such that x and ξ are (T, ρ)-close.*

Theorem 2 (Strong average) *Suppose that the set $\mathcal{L}_\mathcal{W}$ is equi-essentially bounded, system $\mathcal{H}_\epsilon$ in (5) satisfies Assumptions 2 and 4, and its strong average system $\mathcal{H}_{\rm sa}$ satisfies Assumption 1 and is forward pre-complete from a compact set $K_0 \subset \mathbb{R}^n$ with a disturbance bound $\Omega \geq 0$. Then, for each $T \geq 0$ and $\rho > 0$, there exists $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*]$ and $w \in \mathcal{L}_\mathcal{W}$, each solution pair (x, w) to $\mathcal{H}_\epsilon$ with $x(0, 0) \in K_0$ there exists some solution pair (ξ, w_1) to $\mathcal{H}_{\rm sa}$ with $\xi(0, 0) \in K_0$ and $|w_1| \leq |w|$ such that x and ξ are (T, ρ)-close.*

We next study robust stability properties for the class of time-varying hybrid systems $\mathcal{H}_\epsilon$ with the assumption that its strong (respectively, weak) average system is ISS. The definition of input-to-state stability for the time-invariant hybrid system $\mathcal{H}$ in (1), see [6], is first reviewed. Then, the definitions of semi-global practical-ISS and semi-global practical derivative-ISS for the time-varying hybrid system $\mathcal{H}_\epsilon$ in (5) are given, respectively. To study stability concepts with respect to a certain measure instead of a vector norm, in the following we let $A \subset \mathbb{R}^n$ be nonempty and compact and let $\chi : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a proper indicator for A.

Definition 8 System $\mathcal{H}$ in (1) is called ISS with respect to (χ, β, γ) with $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{G}$ if each solution pair (ξ, w) satisfies

$$\chi(\xi(t, j)) \leq \max\{\beta(\chi(\xi_0), t + j), \gamma(|w|)\}, \quad \forall (t, j) \in \text{dom}\xi. \tag{11}$$

Definition 9 System $\mathcal{H}_\epsilon$ in (5) is called semi-globally practically ISS (SGP-ISS) with respect to (χ, β, γ) with $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{G}$ if, for each compact set $K_0 \subset \mathbb{R}^n$ and any positive real numbers Ω and ν there exists $\varepsilon^* > 0$ such that for each $\varepsilon \in (0, \varepsilon^*]$, each solution pair (x, w) with $x_0 := x(0, 0) \in K_0$ and $|w| \leq \Omega$ satisfies

$$\chi(x(t, j)) \leq \max\{\beta(\chi(x_0), t + j), \gamma(|w|)\} + \nu, \quad \forall (t, j) \in \text{dom } x.$$

Definition 10 System $\mathcal{H}_\epsilon$ in (5) is called semi-globally practically derivative ISS (SGP-DISS) with respect to (χ, β, γ) with $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{G}$ if, for each compact set $K_0 \subset \mathbb{R}^n$ and each triple of positive real numbers (Ω, Ω_1, ν), there exists $\varepsilon^* > 0$ such that for each $\varepsilon \in (0, \varepsilon^*]$, each solution pair (x, w), where $w(\cdot, j)$ is locally absolutely continuous, with $x_0 := x(0, 0) \in K_0$, $|w| \leq \Omega$ and $|\dot{w}| \leq \Omega_1$ satisfies

$$\chi(x(t, j)) \leq \max\{\beta(\chi(x_0), t + j), \gamma(|w|)\} + \nu, \quad \forall (t, j) \in \text{dom } x.$$

With the assumption that the weak average system is ISS with functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{G}$ for the class of bounded disturbances that have bounded derivatives, we have Theorem 3 implying that the actual system is SGP-DISS with the same β and γ, for which the proof is listed in Appendix C.4.

Theorem 3 *Suppose that the set $\mathcal{L}_\mathcal{W}$ is equi-essentially bounded and locally equi-uniformly Lipschitz continuous, system $\mathcal{H}_\epsilon$ in (5) satisfies Assumptions 2 and 3 and its weak average system $\mathcal{H}_{\text{wa}}$ satisfies Assumption 1 and is ISS with respect to (χ, β, γ). Then, system $\mathcal{H}_\epsilon$ is SGP-DISS with respect to (χ, β, γ).*

We can also obtain similar conclusions for system $\mathcal{H}_\epsilon$ for bounded input signals if its strong average system is ISS; see Theorem 4. The proof is omitted due to having identical steps as the proof of Theorem 3.

Theorem 4 *Suppose that the set $\mathcal{L}_\mathcal{W}$ is equi-essentially bounded, system $\mathcal{H}_\epsilon$ in (5) satisfies Assumptions 2 and 4 and its strong average system $\mathcal{H}_{\text{sa}}$ satisfies Assumption 1 and is ISS with respect to (χ, β, γ). Then, system $\mathcal{H}_\epsilon$ is SGP-ISS with respect to (χ, β, γ).*

Compared with strong averages, weak averages exist for a larger class of systems, but using them we can only state weaker results; compare Theorems 1 and 3 using weak averages, to Theorems 2 and 4 for strong averages. The structure of periodic continuous-time nonlinear systems that allow for strong average is characterized in [28]. Using the result in [28], for a function $f(x, w, \tau)$ that is periodic in τ and for all measurable disturbances, there exists a strong average $f_{\text{sa}}(x, w)$ for $f(x, w, \tau)$ if and only if f has the following form:

$$f(x, w, \tau) = \tilde{f}(x, \tau) + \tilde{g}(x, w), \tag{12}$$

where $\tilde{f}(x, \tau)$ has a well-defined average $\tilde{f}_{av}$ in the sense of Definition 4 or 5. Moreover, we have $f_{\text{sa}}(x, w) = \tilde{f}_{av}(x) + \tilde{g}(x, w)$.

Note that Theorem 4 using the strong average guarantees SGP-ISS to the actual system $\mathcal{H}_\epsilon$, whereas Theorem 3 implies SGP-ISS via a weak average. In the following example, we can see for bounded disturbances that do not have bounded derivatives, ISS of the weak average system does not guarantee robustness to disturbances for the original system.

Example 1 Consider a hybrid system of the form

$$\begin{aligned} &\left.\begin{aligned} \dot{x} &= f(x, w, \tau) \\ \dot{\tau} &= \tfrac{1}{\varepsilon} \end{aligned}\right\} \quad (x, w, \tau) \in C \times \mathbb{R} \times \mathbb{R}_{\geq 0} \\ &\left.\begin{aligned} x^+ &= g(x) \\ \tau^+ &= 0 \end{aligned}\right\} \quad (x, w, \tau) \in D \times \mathbb{R} \times \mathbb{R}_{\geq 0} \end{aligned} \tag{13}$$

where the constraint sets $C := \mathbb{R}_{\geq 0}$, $D := \mathbb{R}_{\leq 0}$, and

$$\begin{aligned} f(x, w, \tau) &:= -kx^3 + \cos(\tau)x^3 w \\ g(x) &:= -x \end{aligned} \tag{14}$$

with the parameter $k \in (0, 0.5)$. The flow dynamics of hybrid system (13) agree with the continuous-time system $\dot{x} = -kx^3 + \cos\left(\frac{t}{\varepsilon}\right) x^3 w$ for $x \in C$ considered in [28, Example 1]. It is showed in [28, Example 1] that there does not exist a strong average for the function $f : C \times \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ in (14) and its weak average is $f_{\mathrm{wa}}(x) = -kx^3$. Then, from (7), we have that the weak average of system (13) is

$$\begin{aligned} \dot{y} &= f_{\mathrm{wa}}(y) \quad y \in C \\ y^+ &= g(y) \qquad y \in D, \end{aligned} \tag{15}$$

where sets C, D and $g : \mathbb{R} \to \mathbb{R}$ come from (13) and (14), respectively.

Let $\mathcal{A} := \{0\}$. Consider a Lyapunov function candidate $V(y) = \frac{1}{2}y^2$. Note that this $V : \mathbb{R} \to \mathbb{R}_{\geq 0}$ satisfies

$$\begin{aligned} u_C(y) &< 0 \quad \forall y \in C \backslash \mathcal{A} \\ u_D(y) &= 0 \quad \forall y \in D, \end{aligned} \tag{16}$$

where $u_C(y) := \langle \nabla V(y), f_{\mathrm{wa}}(y) \rangle = -ky^4$ and $u_D(y) := V(g(y)) - V(y)$. For any $\mu > 0$, noting that the set $V^{-1}(\mu) \cap \{y \in \mathbb{R} | u_C(y) = u_D(y) = 0\}$ is empty and using the LaSalle's Principle for hybrid systems [15, Theorem 23], we can get that set $\mathcal{A}$ is globally asymptotically stable. In other words, the weak average system is ISS with zero disturbance gain. We next show that the actual system (13) is not SGP-ISS. Similar calculations were also presented in [28, Exmaple 1]. In fact, the original system exhibits finite time escapes under the action of bounded signals.

Consider a bounded continuous input signal $w(\tau) = \cos(\tau)$ that can be rewritten as $w_\varepsilon(t) = \cos\left(\frac{t}{\varepsilon}\right)$ on the t time domain. Note that $|w_\varepsilon| = 1$ for any ε, but $|\dot{w}_\varepsilon| = \frac{1}{\varepsilon}$ that becomes arbitrarily large when ε is sufficiently small. Thus, the signal w is not locally equi-uniformly Lipschitz continuous. Recall that

$$\int_t^T \cos^2(s) = 0.5T + 0.25(\sin(2t + 2T) - \sin(2t)),$$

By direct integration of $\dot{x} = -kx^3 + \cos\left(\frac{t}{\varepsilon}\right)x^3 w$ with the input signal $w_\varepsilon(t) = \cos\left(\frac{t}{\varepsilon}\right)$, we have

$$\int_{x(t_0)}^{x(t)} \frac{\mathrm{d}x}{x^3} = \int_{t_0}^{t} \left(\cos^2\left(\frac{s}{\varepsilon}\right)\right) \mathrm{d}s$$

and

$$x^2(t) = \frac{x^2(t_0)}{1 - 2\psi(\varepsilon, t, t_0)x^2(t_0)}, \tag{17}$$

where

$$\psi(\varepsilon, t, t_0) = (0.5 - k)(t - t_0) + 0.25\varepsilon\left(\sin\left(\frac{2t}{\varepsilon}\right) - \sin\left(\frac{2t_0}{\varepsilon}\right)\right).$$

Fix $t_0 \geq 0, \varepsilon > 0$ and let $x(t_0) := x(t_0, 0) = 1$. Considering (17), we know that there exists some $t_1 \geq t_0$ such that $\psi(\varepsilon, t_1, t_0) = 0.5$ since $(0.5 - k) \in (0, 0.5)$. Moreover, we have that $(t_1, 0) \in \operatorname{dom} x$ for the solution x of actual hybrid system (13) as the solution $x(t, 0)$ with the initial condition $x(t_0, 0) = 1$ will stay in the set C ($x(t, 0) \in \mathbb{R}_{\geq 0}$) and keep flowing when $t_0 \leq t \leq t_1$. Then, there are finite escape times for such a maximal solution x and the actual hybrid system (13) is not semi-globally practically ISS.

5 Pulse-width modulated control example

Pulse-width-modulated (PWM) control strategy is useful for systems controlled by on-off switches, which are commonly utilized to model switching power electronic systems, and find wide application in industry [11,22,29,33,34]. In this section, we take PWM hybrid feedback control systems as an example to show how to apply the results presented above.

In particular, Sect. 5.1 illustrates how to model a hybrid feedback controlled PWM power converter as hybrid systems of the form (5). In Sect. 5.2, we consider strong and weak averages for the PWM hybrid feedback control system and apply the results given in Sect. 4 to analyze its ISS properties. Moreover, we revisit the power converter example to show that we can design a hybrid controller based on the simpler average model such that the actual converter system can be stabilized using the same controller.

5.1 Models

To illustrate how to apply the main results given in last section, we first consider a single rate PWM boost power converter example, see Fig. 1. For this PWM power converter, the open-loop model considered in [21] and closed-loop model with a continuous feedback controller presented in [22, Section 4] are first given. We also present

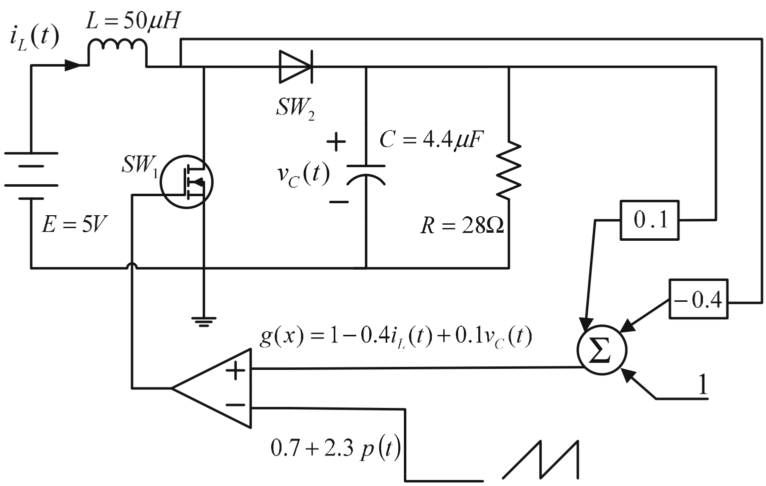

Fig. 1 Continuous-time feedback control boost converter [22]

the closed-loop system when this power converter is controlled by a hybrid controller. After that, we consider a general continuous-time plant with hybrid feedbacks that are implemented via multi-rate PWM. We show that the class of hybrid systems (5) include this general multi-rate PWM hybrid model as a special case.

Example 2 Suppose that the boost converter in Fig. 1 operates in the continuous conduction mode; in this case there are two configuration modes for the converter system corresponding to the on/off state of the switches. Namely, the mode q_1 corresponds to the switch SW_1 on and SW_2 off and the mode q_2 corresponds to SW_1 off and SW_2 on.

Let ξ_1 denote the instantaneous value of the inductor current i_L and $\xi_2 := v_C$ be the capacitor voltage. Let $\xi := (\xi_1, \xi_2)$. Considering the circuit in Fig. 1, we have that dynamics of states ξ agree with $\dot{\xi} = A_{q_i}\xi + B_{q_i}$ on the q_i configuration for $i = 1, 2$ [22], where

$$A_{q_1} := \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix}, \quad A_{q_2} := \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}, \quad B_{q_1} = B_{q_2} := \begin{bmatrix} \frac{E}{L} \\ 0 \end{bmatrix}.$$

Noting that the point of equilibrium of the converter system can be moved to the origin using a coordinate transformation, one can consider stabilization of the origin for the closed-loop converter system. For the converter in Fig. 1 that was also considered in [22], note that the triangle switched signal is denoted by $0.7 + 2.3\,p(t)$, where $p(t)$ is periodic in t satisfying $p(t) = \frac{t}{T}$ for $t \in [0, T)$ and $T > 0$. Then, we have the closed-loop model of this converter from [22]:

$$\dot{\xi} = A_{q_2}\xi + B_{q_1} + \left(A_{q_1}\xi - A_{q_2}\xi\right) u(d(\xi) - p(\tau)), \tag{18}$$

where the duty ratio function $d(\xi) := \frac{g(\xi)-0.7}{3-0.7}$, with $g(\xi) = 1 - 0.4\xi_1 + 0.1\xi_2$ is scaled using the minimum and the maximum values of the triangle signal so that $d(\xi)$

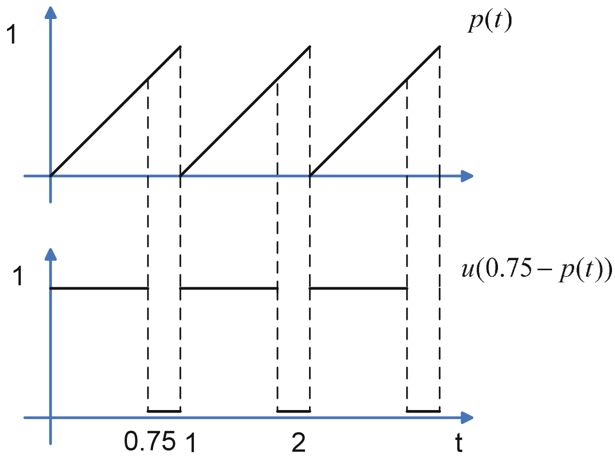

Fig. 2 A typical triangle switching signal $p(t)$ in PWM control systems and $u(d - p(t))$ for $d = 0.75$

takes values in [0, 1]; $u : \mathbb{R} \to [0, 1]$ is the unit step function with $u(s) = 1$ for $s \geq 0$ and $u(s) = 0$ for $s < 0$. Figure 2 is an example of $u(d - p(t))$ for $d = 0.75$ and $T = 1$.

The open-loop model for this converter system is given in [21]:

$$\dot{\xi} = A_{q_2}\xi + B_{q_1} + \left(A_{q_1}\xi - A_{q_2}\xi\right) u(d - p(\tau)), \tag{19}$$

where $d \in [0, 1]$ is the duty cycle for the open-loop PWM operation.

Note that there are situations when certain closed-loop performance specifications cannot be achieved with any linear feedback controller whereas they are achievable with a hybrid controller, see [2]. A switched controller designed via Lyapunov approach in [4] is employed to control a power converter system in [23] and it was shown to provide better performance on transient and steady dynamics than continuous PID controllers. More details can be found in the survey of hybrid control techniques for power converter systems [9,26]. This observation provides a partial motivation for developing averaging techniques for hybrid systems that can be used to analyze a class of hybrid PWM systems in general and power converter systems in particular.

We next consider the same converter but instead of the continuous controller $d(\xi)$ in Fig. 1 we want to apply a hybrid feedback controller, see Fig. 3. Suppose that hybrid controller $h : \bar{C} \times \bar{D} \to [0, 1]$ that was designed to satisfy given performance specifications is given as

$$\begin{aligned} &\dot{\eta} = R(\eta, \xi) \quad (\xi, \eta) \in \bar{C} \\ &\eta^{+} \in S(\eta, \xi) \quad (\xi, \eta) \in \bar{D} \\ &h := h(\eta, \xi), \end{aligned} \tag{20}$$

where $\eta \in \mathbb{R}^n$; $\bar{C}$, $\bar{D}$ are the constraint sets that allow flows and jumps for η; $R : \bar{C} \to \mathbb{R}^n$ is a flow mapping and $S : \bar{D} \rightrightarrows \mathbb{R}^n$ is a set-valued mapping. Note that states η

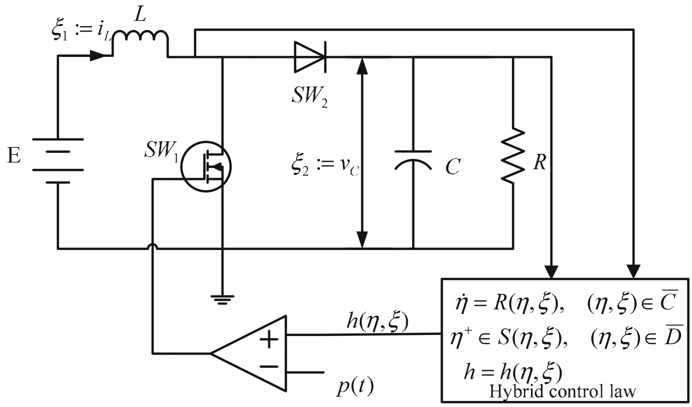

Fig. 3 Hybrid feedback control boost converter

may include physical variables together with logic variables or operation modes that are used to describe the hybrid feedback control law.

Applying this $h(\eta, \xi)$ to the open-loop converter system (19), we have that the closed-loop model of the converter system in Fig. 3 is

$$\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A_{q_2}\xi + B_{q_1} \\ R(\eta, \xi) \end{bmatrix} + \begin{bmatrix} A_{q_1}\xi - A_{q_2}\xi \\ 0 \end{bmatrix} \cdot u(h(\eta, \xi) - p(\tau)) \quad (\xi, \eta) \in \bar{C}$$
$$\begin{bmatrix} \xi^+ \\ \eta^+ \end{bmatrix} \in \begin{bmatrix} \xi \\ S(\eta, \xi) \end{bmatrix} \qquad (\xi, \eta) \in \bar{D}. \tag{21}$$

Note that the average model of open-loop converter system (19) is given in [21]:

$$\dot{\xi} = A_{q_2}\xi + B_{q_1} + d\left(A_{q_1} - A_{q_2}\right)\xi, \tag{22}$$

where the duty cycle $d \in [0, 1]$ can be taken as a control signal. Suppose a controller $h : \bar{C} \cup \bar{D} \to [0, 1]$ is designed to stabilize the open-loop average system (22) by letting $d := h$. Then, our results can be used to analyze the stability properties of the PWM converter system (21) through stability of the closed-loop of system (22) using the same controller h.

We assume that the controller h and the triangle signal p in Example 2 satisfy $h(\eta, \xi) : \bar{C} \times \bar{D} \to [0, 1]$ and $p : \mathbb{R}_{\geq 0} \to [0, 1]$, respectively. The following remark shows that we may consider h and p only with their images in [0, 1] without loss of generality.

Remark 1 Suppose that we need a controller $\tilde{U} = \tilde{U}(\xi, \eta)$ that takes values in $[a, b]$ to stabilize the plant and achieve appropriate performance. To implement this controller via PWM we need to get an average ranging from a to b using a step function $\tilde{u}(\cdot)$ that satisfies $\tilde{u}(s) = a$ for $s < 0$ and $\tilde{u}(s) = b$ for $s \geq 0$. Suppose also that we want to use a triangle signal $\hat{p}(\cdot) = c + kp(\cdot) \in [c, c+k]$, with $k > 0$ to implement this controller, where $p(\cdot)$ is the triangle wave defined earlier. Then, we need a duty cycle function $\hat{U}(\xi, \eta)$, generated from $\tilde{U}(\xi, \eta)$, but taking values in $[c, c+k]$. In particular, we take $\hat{U}(\xi, \eta) := c + k\frac{\tilde{U}(\xi,\eta)-a}{b-a}$. The PWM control that we need to implement is then

$$\tilde{u}(\hat{U}(\xi, \eta) - \hat{p}(\tau)),$$

which can be written in other ways and it is also equal to

$$a + (b-a)u(h(\xi, \eta) - p(\tau)),$$

where

$$h(\xi, \eta) := \frac{\tilde{U}(\xi, \eta) - a}{b - a},$$
$$u(s) := \frac{\tilde{u}(s) - a}{b - a},$$
$$p(\tau) := \frac{\hat{p}(\tau) - c}{k}.$$

Note that our results pertain to a more general class of PWM systems that is presented next. We consider a general continuous-time plant with disturbances controlled by hybrid feedbacks that are implemented via multi-rate PWM. Consider a continuous-time plant with states $\xi \in \mathbb{R}^n$, disturbances $w \in \mathcal{W} \subset \mathbb{R}^m$ and outputs $y \in \mathbb{R}^l$:

$$\begin{aligned} \dot{\xi} &= O(\xi, w) + \sum_{i}^{k} P_i(\xi, w)h_i, \\ y &= Q(\xi, w). \end{aligned} \tag{23}$$

For this continuous-time plant, the hybrid feedback controllers h_i are given through the following auxiliary hybrid system with states $\eta \in \mathbb{R}^h$:

$$\begin{aligned} \dot{\eta} &= R(\eta, y) \quad (\eta, y) \in C_1 \\ \eta^+ &\in S(\eta, y) \quad (\eta, y) \in D_1 \\ h_i &= h_i(\eta, y), \end{aligned}$$

where $C_1, D_1 \in \mathbb{R}^h \times \mathbb{R}^l$ are the constraint sets that allow flows and jumps for η; $S : \mathbb{R}^h \rightrightarrows \mathbb{R}^h$ is a set-valued mapping that is outer semi-continuous, locally bounded and for each $(\eta, y) \in D_1$, $S(\eta, y)$ is nonempty; functions $O : \mathbb{R}^n \times \mathcal{W} \to \mathbb{R}^n$ and $R : C_1 \to \mathbb{R}^h$ are continuous while $P_i : \mathbb{R}^n \times \mathcal{W} \to \mathbb{R}^n$, $Q : \mathbb{R}^n \times \mathcal{W} \to \mathbb{R}^l$ and $h_i : \mathbb{R}^h \times \mathbb{R}^l \to [0, 1]$ are locally Lipschitz.

Let $C := C_1 \times \mathcal{W}$, $D := D_1 \times \mathcal{W}$, $h_i(x, w) := h_i(\eta, Q(\xi, w))$, $x := [\xi, \eta]^T$ and

$$\psi_i(x, w) := \begin{bmatrix} P_i(\xi, w) \\ 0 \end{bmatrix}, \quad \mathcal{F}_0(x, w) := \begin{bmatrix} O(\xi, w) \\ R(\eta, Q(\xi, w)) \end{bmatrix},$$
$$\mathcal{G}(x, w) := \begin{bmatrix} \xi \\ S(\eta, Q(\xi, w)) \end{bmatrix}.$$

In the case when feedback controllers h_i are implemented by multi-rate PWM, the closed-loop of system (23) becomes

$$\left.\begin{aligned} \dot{x} &= \mathcal{F}(x, w, \tau) \\ \dot{\tau} &= \tfrac{1}{\varepsilon} \end{aligned}\right\} \quad ((x, w), \tau) \in C \times \mathbb{R}_{\geq 0}$$
$$\left.\begin{aligned} x^+ &\in \mathcal{G}(x, w) \\ \tau^+ &= \tau \end{aligned}\right\} \quad ((x, w), \tau) \in D \times \mathbb{R}_{\geq 0}, \tag{24}$$

where $\mathcal{G} : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is the jump mapping, C and $D \subset \mathbb{R}^n \times \mathbb{R}^m$ are given sets that allow for flow and jump for the designed hybrid feedback controller and

$$\mathcal{F}(x, w, \tau) := \mathcal{F}_0(x, w) + \sum_{i=1}^{m} g_i(x, w) u(h_i(x, w) - p_i(\tau)). \tag{25}$$

The second term of $\mathcal{F}$ in (25) is used to model a multi-rate implementation of a PWM hybrid controller. As $p_i(\tau)$ are the only time-varying terms, the small parameter $\varepsilon > 0$ in (24) is used to guarantee that the switching signals p_i change fast compared with state ξ and so the effect of p_i can be averaged.

5.2 Averaging analysis

We next consider the PWM hybrid feedback control system with disturbances in (24) to illustrate how our results can be applied so that the ISS properties of the actual closed loop of system (24) can be studied through its time-invariant average system.

First, we show that there exists a weak average for function $\mathcal{F} : C \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ in (25) on the set C and Assumption 3 holds for the PWM control system in (24). Given $T > \max\{T_1, \ldots, T_n\}$, let $k_i = k_i(T) \in \mathbb{Z}_{\geq 0}$ and $\tilde{T}_i \in [0, T_i)$ satisfying $T = k_i T_i + \tilde{T}_i$. Note that $k_i(T) \to \infty$ when the given T approaches infinity. For all $(x, w) \in C$, we get

$$\frac{1}{T} \int_{\tau}^{\tau+T} \left\{ \mathcal{F}_0(x, w) + \sum_{i=1}^{m} g_i(x, w) u(h_i(x, w) - p_i(s)) \right\} \mathrm{d}s$$
$$= \mathcal{F}_0(x, w) + \sum_{i=1}^{m} g_i(x, w) \frac{1}{T} \left\{ \int_{\tau}^{\tau+k_i T_i} u(h_i(x, w) - p_i(s)) \mathrm{d}s \right.$$

$$+ \int_{\tau+k_iT_i}^{\tau+k_iT_i+\tilde{T}_i} u(h_i(x,w)-p_i(s))\mathrm{d}s \Bigg\},$$

$$= \mathcal{F}_0(x,w) + \sum_{i=1}^{m} g_i(x,w)\left(\frac{k_iT_i}{k_iT_i+\tilde{T}_i}h_i(x,w) + \frac{v_i(x,w,\tilde{T}_i)}{k_iT_i+\tilde{T}_i}\right),$$

where $v_i(x,w,\tilde{T}_i) := \int_{\tau}^{\tau+\tilde{T}_i} u(h_i(x,w)-p_i(s))\mathrm{d}s$ satisfies $|v_i(x,w,\tilde{T}_i)| \le \tilde{T}_i$. Let $f_{\mathrm{wa}}(x,w) := \mathcal{F}_0(x,w) + \sum_{i=1}^{m} g_i(x,w)h_i(x,w)$. Note that for any compact set $K \subset \mathbb{R}^n \times \mathbb{R}^m$, there exists $r > 0$ such that, for all $(x,w) \in C \cap K$

$$\left| f_{\mathrm{wa}}(x,w) - \frac{1}{T}\int_{\tau}^{\tau+T}\left\{\mathcal{F}_0(x,w) + \sum_{i=1}^{m} g_i(x,w)u(h_i(x,w)-p_i(s))\right\}\mathrm{d}s \right|$$

$$\le \frac{1}{T}\sum_{i=1}^{m} |g_i(x,w)v_i(x,w,\tilde{T}_i)| \le \frac{r}{T+1} := \sigma_K(T)\ ,$$

which shows that f_{wa} agrees with Definition 4. Let $\mathcal{G}, C, D$ come from (24). Then, the hybrid system $\mathcal{H}_{\mathrm{wa}}$ with data $\{f_{\mathrm{wa}}, \mathcal{G}, C, D\}$ formed as (7) is the weak average for the PWM closed-loop control system.

Next we verify the Assumption 3. Considering the definition of η_{wa} in (9), it follows for each $\tau \in (0, \min_i\{T_i\})$ and $\tau_0 \in [0,\tau]$ that

$$\begin{aligned}
&\eta_{\mathrm{wa}}(x,w,\tau,\tau_0,0)\\
&= \int_{\tau_0}^{\tau}\left(\mathcal{F}_0(x,w) + \sum_{i=1}^{m} g_i(x,w)u_i(h_i(x,w)-p_i(s)) - f_{\mathrm{wa}}(x,w)\right)\mathrm{d}s\\
&= \sum_{i=1}^{m} g_i(x,w)\int_{\tau_0}^{\tau}[u_i(h_i(x,w)-p_i(s)) - h_i(x,w)]\mathrm{d}s\\
&= \sum_{i=1}^{m} g_i(x,w)(\min\{(\tau-\tau_0), h_i(x,w)\} - (\tau-\tau_0)h_i(x,w)),
\end{aligned} \tag{26}$$

which is bounded for any (x,w) in a compact set and locally Lipschitz as functions g_i and h_i are locally Lipschitz. Then, Assumption 3 holds for the function $\eta_{\mathrm{wa}}(x,w,\tau,\tau_0,0)$.

For PWM control system (24), noting the structure of f allows for strong average in (12), there exists a strong average if $g_i(x,w)$ and $h_i(x,w)$ are independent of w, i.e., $g_i(x,w) := g_i(x)$ and $h_i(x,w) := h_i(x)$. In this case, following the calculations used to establish the weak average, we get that $f_{\mathrm{sa}}(x,w) := f_0(x,w) + \sum_{i=1}^{m} g_i(x)h_i(x)$ on the set C, at least when C has the form $C = C_1 \times \mathcal{W}$, and the strong average of system (24) is system $\mathcal{H}_{\mathrm{sa}}$ in (8) with data $\{f_{\mathrm{sa}}, \mathcal{G}, C, D\}$. Using the definition of η_{sa} in (10), we have

$$\eta_{\text{sa}}(x, w, \tau, \tau_0, 0) = \int_{\tau_0}^{\tau} \left(\mathcal{F}_0(x, w) + \sum_{i=1}^{m} g_i(x) u_i(h_i(x) - p_i(s)) - f_{\text{sa}}(x, w) \right) \mathrm{d}s$$
$$= \sum_{i=1}^{m} g_i(x) \int_{\tau_0}^{\tau} (u_i(h_i(x) - p_i(s)) - h_i(x)) \mathrm{d}s.$$

Noting (26), it follows that Assumption 4 holds for the function $\eta_{\text{sa}}(x, w, \tau, \tau_0, 0)$.

With the above analysis that Assumptions 3 and 4 hold with the assumption that functions h_i and g_i are locally Lipschitz, and noting that only local boundedness but no continuity condition is required for the flow mapping of the actual hybrid systems, we can get that the main results apply to PWM hybrid feedback control systems under some mild regular conditions. The following corollaries come directly from Theorems 3–4 and with which we can consider the robust stability of the time-varying PWM control system (24) based on its weak or strong average system:

Corollary 1 *Suppose that the set $\mathcal{L}_{\mathcal{W}}$ is equi-essentially bounded and locally equi-uniformly Lipschitz continuous, the PWM hybrid control system in (24) satisfies Assumptions 2 and its weak average system $\mathcal{H}_{\text{wa}}$ satisfies Assumption 1 and is ISS with respect to (χ, β, γ). Then, the PWM hybrid control system in (24) is SGP-DISS with respect to (χ, β, γ).*

Corollary 2 *Suppose that the set $\mathcal{L}_{\mathcal{W}}$ is equi-essentially bounded, the PWM hybrid control system in (24) satisfies Assumptions 2 and its strong average system $\mathcal{H}_{\text{sa}}$ satisfies Assumption 1 and is ISS with respect to (χ, β, γ). Then, the PWM hybrid control system in (24) is SGP-ISS with respect to (χ, β, γ).*

6 Conclusions

We considered ISS properties for a class of time-varying hybrid dynamical systems via the averaging method. Using the notions of strong and weak average, the time-varying hybrid system is approximated by a time-invariant hybrid system. We showed that the solutions of the actual time-varying hybrid system and its weak or strong average can be made arbitrarily close on compact time domains by reducing the parameter ε if the average system is forward pre-complete. Our main results also showed that ISS of the strong (weak) average implies SGP-ISS (SGP-DISS) of the actual system. An example in PWM control was used to illustrate our results.

Appendix A: Proof of Lemma 2

The proof uses the same technical method as [37, Lemma 1] and follows the calculation of [20, p. 415]. Let the compact set $K \subset \mathbb{R}^n \times \mathbb{R}^m$ be given. From the definitions of the strong average, for any $0 \leq \tau_0 \leq \tau_1$, τ, $(\tau + T) \in [\tau_0, \tau_1]$, $\tilde{w} : [\tau_0, \tau_1] \to \mathcal{W}$ and $(x, \tilde{w}(s)) \in (C_1 \times \mathcal{W}) \cap K$ for all $s \in [\tau_0, \tau_1]$, the following holds:

$$
\begin{aligned}
&|\eta_{\text{sa}}(x, \tilde{w}, \tau + T, \tau_0, 0) - \eta_{\text{sa}}(x, \tilde{w}, \tau, \tau_0, 0)| \\
&\quad = \left| \int_{\tau}^{\tau+T} [f_0(x, \tilde{w}(s), s) - f_{\text{sa}}(x, \tilde{w}(s))]\mathrm{d}s \right| \leq T\sigma_K(T).
\end{aligned} \tag{27}
$$

Integrating by parts in the definition of η_{sa}, we have

$$
\begin{aligned}
&\eta_{\text{sa}}(x, \tilde{w}, \tau, \tau_0, \mu) \\
&= \left[\exp(\mu(s - \tau)) \int_{\tau_0}^{s} (f_0(x, \tilde{w}(r), r) - f_{\text{sa}}(x, \tilde{w}(r)))dr \right]_{\tau_0}^{\tau} \\
&\quad - \mu \int_{\tau_0}^{\tau} \exp(\mu(s - \tau)) \int_{\tau_0}^{s} (f_0(x, \tilde{w}(r), r) - f_{\text{sa}}(x, \tilde{w}(r)))\mathrm{d}r\mathrm{d}s, \\
&= \eta_{\text{sa}}(x, \tilde{w}, \tau, \tau_0, 0) - \mu \int_{\tau_0}^{\tau} \exp(\mu(s - \tau))\eta_{\text{sa}}(x, \tilde{w}, s, \tau_0, 0)\mathrm{d}s.
\end{aligned} \tag{28}
$$

Then, adding and subtracting $\mu\eta_{\text{sa}}(x, \tilde{w}, \tau, \tau_0, 0) \int_{\tau_0}^{\tau} \exp(\mu(s - \tau))\mathrm{d}s$ to the right-hand side of (28), we obtain

$$
\begin{aligned}
&\eta_{\text{sa}}(x, \tilde{w}, \tau, \tau_0, \mu) \\
&\quad = \exp(-\mu(\tau - \tau_0))\eta_{\text{sa}}(x, \tilde{w}, \tau, \tau_0, 0) \\
&\qquad + \mu \int_{\tau_0}^{\tau} \exp(-\mu(\tau - s))[\eta_{\text{sa}}(x, \tilde{w}, \tau, \tau_0, 0) - \eta_{\text{sa}}(x, \tilde{w}, s, \tau_0, 0)]\mathrm{d}s.
\end{aligned}
$$

Let $\hat{\tau} := \tau - \tau_0$. Using the fact $\eta_{\text{sa}}(x, \tilde{w}, \tau_0, \tau_0, 0) = 0$ and (27), it follows that

$$
\begin{aligned}
\mu|\eta_{\text{sa}}(x, \tilde{w}, \tau, \tau_0, \mu)| &\leq \exp(-\mu(\tau - \tau_0))\mu(\tau - \tau_0)\sigma_K(\tau - \tau_0) \\
&\quad + \mu^2 \int_{\tau_0}^{\tau} \exp(-\mu(\tau - s))(\tau - s)\sigma_K(\tau - s)\mathrm{d}s \\
&= \exp(-\mu\hat{\tau})\mu\hat{\tau}\sigma_K(\hat{\tau}) + \mu^2 \int_{0}^{\hat{\tau}} \exp(-\mu r)r\sigma_K(r)\mathrm{d}r \\
&= \exp(-\mu\hat{\tau})\mu\hat{\tau}\sigma_K(\hat{\tau}) + \int_{0}^{\mu\hat{\tau}} \exp(-z)z\sigma_K\left(\frac{z}{\mu}\right)\mathrm{d}z.
\end{aligned}
$$

There are two possibilities for $\mu\tilde{\tau}$: $\mu\tilde{\tau} \leq \sqrt{\mu}$ and $\mu\tilde{\tau} \geq \sqrt{\mu}$. In the first case, we have

$$\exp(-\mu\hat{\tau})\mu\hat{\tau}\sigma_K(\hat{\tau}) + \int_0^{\mu\hat{\tau}} \exp(-z)z\sigma_K\left(\frac{z}{\mu}\right)\mathrm{d}z \leq \sqrt{\mu}\sigma_K(0) + \frac{\mu}{2}\sigma_K(0).$$

For the second case, using $\eta\exp(-\eta) \leq \exp(-1)$ for all $\eta \geq 0$ and $\int_0^\infty \exp(-z)z\mathrm{d}z = 1$, and then

$$\begin{aligned}
&\exp(-\mu\hat{\tau})\mu\hat{\tau}\sigma_K(\hat{\tau}) + \int_0^{\mu\hat{\tau}} \exp(-z)z\sigma_K\left(\frac{z}{\mu}\right)\mathrm{d}z \\
&\leq \exp(-1)\sigma_K\left(\frac{1}{\sqrt{\mu}}\right) + \sigma_K(0)\int_0^{\sqrt{\mu}} z\mathrm{d}z + \sigma_K\left(\frac{1}{\sqrt{\mu}}\right)\int_{\sqrt{\mu}}^{\infty} z\exp(-z)\mathrm{d}z \\
&\leq (\exp(-1)+1)\sigma_K\left(\frac{1}{\sqrt{\mu}}\right) + \frac{\mu}{2}\sigma_K(0).
\end{aligned}$$

Then, let

$$\alpha_K(\mu) := \frac{\mu}{2}\sigma_K(0) + \max\left\{\sqrt{\mu}\sigma_K(0), \sigma_K\left(\frac{1}{\sqrt{\mu}}\right)(\exp(-1)+1)\right\}.$$

Since σ_K is of class-$\mathcal{L}$, it follows that α_K is of class-$\mathcal{G}$. □

Appendix B: Proof of Claim 2

Let the compact $K \subset \mathbb{R}^n \times \mathbb{R}^m$ be given. Similarly, like the proof of Lemma 2, integrate by parts in the definition of η_{sa} to get (28). Then, for each $i \in \bar{N}$, $0 \leq \tau_0 \leq \tau_1$, $\tilde{w} : [\tau_0, \tau_1] \to \mathcal{W}$ and $((x_1, \tilde{w}(s)), \tau_a), ((x_2, \tilde{w}(s)), \tau_b) \in ((C_1 \times \mathcal{W}) \cap K) \times [\tau_0, \tau_1]$ for all $s \in [\tau_0, \tau_1]$ (without loss of generality, let $\tau_b \geq \tau_a$), it follows from Assumption 4 that

$$\begin{aligned}
&\left|\eta^i_{\mathrm{sa}}(x_1, \tilde{w}, \tau_a, \tau_0, \mu) - \eta^i_{\mathrm{sa}}(x_2, \tilde{w}, \tau_b, \tau_0, \mu)\right| \\
&\leq \left|\eta^i_{\mathrm{sa}}(x_1, \tilde{w}, \tau_a, \tau_0, 0) - \eta^i_{\mathrm{sa}}(x_2, \tilde{w}, \tau_b, \tau_0, 0)\right| \\
&\quad + \mu\int_{\tau_0}^{\tau_a} \exp(\mu(s-\tau_a))|\eta^i_{\mathrm{sa}}(x_1, \tilde{w}, s, \tau_0, 0) - \eta^i_{\mathrm{sa}}(x_2, \tilde{w}, s+\tau_b-\tau_a, \tau_0, 0)|\mathrm{d}s,
\end{aligned}$$

$$\leq L(|x_1 - x_2| + |\tau_a - \tau_b|)\left(1 + \mu \int_{\tau_0}^{\tau_a} \exp(\mu(s - \tau_a))\mathrm{d}s\right),$$

$$\leq 2L(|x_1 - x_2| + |\tau_a - \tau_b|), \tag{29}$$

where the last inequality in (29) follows from the fact $\mu \int_{\tau_0}^{\tau_a} \exp(\mu(s - \tau_a))\mathrm{d}s \leq 1$ for any $\mu, \tau_0, \tau_a \geq 0$. □

Appendix C: Proofs of Theorems 1–3

We need some technical results considering the robust properties to small perturbations for general hybrid systems $\mathcal{H}$ in (1) to show Theorems 1 and 2. To present these technical results in Sect. C.1, consider the following hybrid system $\mathcal{H}_\delta$ inflated from system $\mathcal{H}$:

$$\mathcal{H}_\delta \quad \begin{array}{ll} \dot{\bar{x}} \in F_\delta(\bar{x}, w) & (\bar{x}, w) \in C_\delta \\ \bar{x}^+ \in G_\delta(\bar{x}, w) & (\bar{x}, w) \in D_\delta, \end{array} \tag{30}$$

with $\bar{x} \in \mathbb{R}^n$, $w \in \mathcal{W} \subset \mathbb{R}^m$. For a parameter $\delta > 0$, the data $\{F_\delta, G_\delta, C_\delta, D_\delta\}$ are defined as

$$\begin{aligned} F_\delta(\bar{x}, w) &:= \overline{con} F((\bar{x} + \delta\mathbb{B}, w) \cap C) + \delta\mathbb{B} \\ G_\delta(\bar{x}, w) &:= G((\bar{x} + \delta\mathbb{B}, w) \cap D) + \delta\mathbb{B} \\ C_\delta &:= \{(\bar{x}, w) : (\bar{x} + \delta\mathbb{B}, w) \cap C \neq \emptyset\} \\ D_\delta &:= \{(\bar{x}, w) : (\bar{x} + \delta\mathbb{B}, w) \cap D \neq \emptyset\}. \end{aligned}$$

C.1 Technical results

Before we give Propositions 1–3 on properties of system $\mathcal{H}_\delta$ based on the assumption of system $\mathcal{H}$, we need the following claim; also see [6, Claim 3.7].

Claim 3 The hybrid arc ξ is a solution to the hybrid inclusion

$$\mathcal{H}_\Omega \quad \begin{array}{ll} \dot{\xi} \in F_\Omega(\xi) & \xi \in C_\Omega \\ \xi^+ \in G_\Omega(\xi) & \xi \in D_\Omega \end{array} \tag{31}$$

that is extended from system $\mathcal{H}$ in (1) for some $\Omega \geq 0$ with the data $(F_\Omega, G_\Omega, C_\Omega, D_\Omega)$ being defined as

$$\begin{aligned} F_\Omega(\xi) &:= \{v \in \mathbb{R}^n : v = F(\xi, w), w \in \mathcal{W} \cap \Omega\mathbb{B} \text{ and } (\xi, w) \in C\} \\ G_\Omega(\xi) &:= \{v \in \mathbb{R}^n : v \in G(\xi, w), w \in \mathcal{W} \cap \Omega\mathbb{B} \text{ and } (\xi, w) \in D\} \\ C_\Omega &:= \{\xi : \exists w \in \mathcal{W} \cap \Omega\mathbb{B} \text{ such that } (\xi, w) \in C\} \\ D_\Omega &:= \{\xi : \exists w \in \mathcal{W} \cap \Omega\mathbb{B} \text{ such that } (\xi, w) \in D\}, \end{aligned} \tag{32}$$

if and only if there exists a hybrid input w_1 such that (ξ, w_1) is a solution pair to system $\mathcal{H}$ in (1) with $|w_1| \leq \Omega$.

Proposition 1 *Suppose that system $\mathcal{H}$ in (1) satisfies Assumption 1, and it is forward pre-complete from a compact set $K_0 \subset \mathbb{R}^n$ with a disturbance bound $\Omega \geq 0$. Then, for each $\rho > 0$ and $T \geq 0$ there exists $\delta^* > 0$ such that for all $\delta \in (0, \delta^*]$, each solution pair $(\bar{x}, w)$ of system $\mathcal{H}_\delta$ in (30) with $\bar{x}(0,0) \in (K_0 + \delta\mathbb{B})$ and $|w| \leq \Omega$ there exists a solution pair (ξ, w_1) to system $\mathcal{H}$ with $\xi(0,0) \in K_0$ and $|w_1| \leq |w|$ such that $\bar{x}$ and ξ are (T, ρ)-close.*

Proof of Proposition 1 Let the compact set K_0 and $\Omega \geq 0$ be given. For some $\delta > 0$, let $(\bar{x}, w)$ be a solution pair to system $\mathcal{H}_\delta$ with $\bar{x}(0,0) \in (K_0 + \delta\mathbb{B})$ and w with $|w| \leq \Omega$. Let $\Omega_1 := |w| \in [0, \Omega]$. Consider a hybrid inclusion $\mathcal{H}_{(\Omega_1,\delta)} := \{F_{(\Omega_1,\delta)}, G_{(\Omega_1,\delta)}, C_{(\Omega_1,\delta)}, D_{(\Omega_1,\delta)}\}$ formed as (31) with its data being constructed from the system $\mathcal{H}_\delta := \{F_\delta, G_\delta, C_\delta, D_\delta\}$ in (30):

$$
\begin{aligned}
F_{(\Omega_1,\delta)}(\bar{x}) &:= \{v \in \mathbb{R}^n : v \in F_\delta(\bar{x}, w), w \in \mathcal{W} \cap \Omega_1\mathbb{B} \text{ and } (\bar{x}, w) \in C_\delta\} \\
G_{(\Omega_1,\delta)}(\bar{x}) &:= \{v \in \mathbb{R}^n : v \in G_\delta(\bar{x}, w), w \in \mathcal{W} \cap \Omega_1\mathbb{B} \text{ and } (\bar{x}, w) \in D_\delta\} \\
C_{(\Omega_1,\delta)} &:= \{\bar{x} : \exists w \in \mathcal{W} \cap \Omega_1\mathbb{B} \text{ such that } (\bar{x}, w) \in C_\delta\} \\
D_{(\Omega_1,\delta)} &:= \{\bar{x} : \exists w \in \mathcal{W} \cap \Omega_1\mathbb{B} \text{ such that } (\bar{x}, w) \in D_\delta\}.
\end{aligned} \tag{33}
$$

Note that the data $\{F_{(\Omega_1,\delta)}, G_{(\Omega_1,\delta)}, C_{(\Omega_1,\delta)}, D_{(\Omega_1,\delta)}\}$ in (33) satisfies

$$
\begin{aligned}
F_{(\Omega_1,\delta)}(\bar{x}) &= \overline{con}\, F_{\Omega_1}((\bar{x} + \delta\mathbb{B}) \cap C_{\Omega_1})) + \delta\mathbb{B} \\
G_{(\Omega_1,\delta)}(\bar{x}) &= G_{\Omega_1}((\bar{x} + \delta\mathbb{B}) \cap D_{\Omega_1}) + \delta\mathbb{B} \\
C_{(\Omega_1,\delta)} &= \{\bar{x} : (\bar{x} + \delta\mathbb{B}) \cap C_{\Omega_1} \neq \emptyset\} \\
D_{(\Omega_1,\delta)} &= \{\bar{x} : (\bar{x} + \delta\mathbb{B}) \cap D_{\Omega_1} \neq \emptyset\},
\end{aligned} \tag{34}
$$

with $\{F_{\Omega_1}, G_{\Omega_1}, C_{\Omega_1}, D_{\Omega_1}\}$ defined as (32). From (34), it is straightforward that $\mathcal{H}_{(\Omega_1,\delta)}$ is an inclusion inflated from $\mathcal{H}_{\Omega_1} := \{F_{\Omega_1}, G_{\Omega_1}, C_{\Omega_1}, D_{\Omega_1}\}$.

Consider arbitrary $\rho > 0$ and $T \geq 0$. Note that forward pre-completeness of $\mathcal{H}_{\Omega_1}$ on the set K_0 comes from Claim 3 and the assumption that $\mathcal{H}$ is forward pre-complete from K_0. Noting that for each $\xi \in C_{\Omega_1}$, $F_{\Omega_1}(\xi)$ is convex in Assumption 1, we have $F_{\Omega_1}(\xi) = \overline{con}\ F_{\Omega_1}(\xi)$ for each $\xi \in C_{\Omega_1}$. Using the results of [16, Corollary 5.2] and [16, Theorem 5.4], we have that there exists a $\delta^* > 0$ such that for all $\delta \in (0, \delta^*]$ and for each solution $\bar{x}$ of $\mathcal{H}_{(\Omega_1,\delta)}$ with $\bar{x}(0,0) \in K_0 + \delta\mathbb{B}$ there exists a solution ξ to $\mathcal{H}_{\Omega_1}$ with $\xi(0,0) \in K_0$ such that $\bar{x}$ and ξ are (T, ρ)-close. Consider $\mathcal{H}_\delta$ in (30) and $\mathcal{H}_{(\Omega_1,\delta)}$ in (33) with $\delta \in (0, \delta^*]$. Note that for each solution pair $(\bar{x}, w)$ of system $\mathcal{H}_\delta$ there exists a solution $\bar{x}$ to the inclusion $\mathcal{H}_{(\Omega_1,\delta)}$. Considering any solution pair $(\bar{x}, w)$ of system $\mathcal{H}_\delta$ with $\bar{x}(0,0) \in (K_0 + \delta\mathbb{B})$ and w with $|w| = \Omega_1 \in [0, \Omega]$, noting the closeness of solutions $\bar{x}$ to $\mathcal{H}_{(\Omega_1,\delta)}$ and ξ to $\mathcal{H}_{\Omega_1}$ and applying Claim 3 completes the proof.

Proposition 2 *Suppose that system $\mathcal{H}$ in (1) satisfies Assumption 1 and it is forward pre-complete from a compact set $K_0 \subset \mathbb{R}^n$ with a disturbance bound $\Omega \geq 0$. Then, for each $T \geq 0$ the reachable set*

$$R_T(K_0, \Omega) := \{\xi(t, j) : (\xi, w) \in S(K_0), t + j \leq T, |w| \leq \Omega\} \tag{35}$$

is compact, where $S(K_0)$ *denotes the set of maximal solution pairs* (ξ, w) *to system* $\mathcal{H}$ *in* (1) *with* $\xi(0, 0) \in K_0$.

Proof of Proposition 2 The result follows using Claim 3 and the result of [16, Corollary 4.7].

Proposition 3 *Suppose that system* $\mathcal{H}$ *in* (1) *satisfies Assumption 1, and it is ISS with respect to* (χ, β, γ). *Then, for each compact set* $K_0 \subset \mathbb{R}^n$ *and each pair of* $(\Omega, \nu) \geq 0$ *there exists* $\delta > 0$ *such that for each solution pair* $(\bar{x}, w)$ *of system* $\mathcal{H}_\delta$ *in* (30) *with* $\bar{x}_0 := \bar{x}(0, 0) \in (K_0 + \delta\mathbb{B})$ *and* $|w| \leq \Omega$, *the following holds:*

$$\chi(\bar{x}(t, j)) \leq \max\{\beta(\chi(\bar{x}_0), t + j), \gamma(|w|)\} + \nu \quad \forall (t, j) \in \text{dom}\, \bar{x}. \tag{36}$$

Proof of Proposition 3 The proof is based on the trajectory method used in [35]. Let the compact set K_0 be given. Let $\Omega, \nu \geq 0$ be arbitrary. Due to the compactness of K_0 and continuity of γ, there exist $m > \nu + \gamma(\Omega)$ such that $K_0 + \mathbb{B}$ is contained in a compact set $\{\bar{x} \in \mathbb{R}^n : \chi(\bar{x}) \leq m\}$. Pick large enough $T \geq 0$ so that $\beta(m, r) \leq \frac{\nu}{2}$ for $r \geq T$.

For the compact set K_0 and Ω, let K be the reachable set defined as (35) for system $\mathcal{H}$ for any $(t, j) \in \text{dom}\, \xi$ with $t + j \leq 2T$, which is compact from Proposition 2 with forward pre-completeness of system $\mathcal{H}$ on K_0, thanks to the assumed ISS property. Let $M \geq 0$ be such that $\max_{\xi \in K_0} \chi(\xi) \leq M$. Using the continuity of χ and β, and the fact that $\beta(s, l)$ approaches zero as $l \geq 0$ tends to infinity, let $\rho_1^* > 0$ be small enough such that

$$\beta(s, l - \rho_1^*) - \beta(s, l) \leq \frac{\nu}{6} \quad \forall s \leq M, \;\; l \geq 0. \tag{37}$$

Let ρ_2^* be sufficiently small such that, for all $\xi \in K$ and $\bar{x} \in (K + \rho_2^*\mathbb{B})$ satisfying $|\xi - \bar{x}| \leq \rho_2^*$, we have

$$\begin{aligned} \chi(\bar{x}) &\leq \chi(\xi) + \frac{\nu}{6} \\ \beta(\chi(\xi), l) &\leq \beta(\chi(\bar{x}), l) + \frac{\nu}{6}, \quad \forall\, l \geq 0. \end{aligned} \tag{38}$$

Let $\rho = \min\{\rho_1^*, \rho_2^*\}$ and $\xi_0 := \xi(0, 0)$. Let Proposition 1 with $(2T, \rho, \Omega)$ and the set K_0 generate a $\delta^* > 0$. Consider $\delta \in (0, \delta^*]$ and without loss of generality assume that $\delta < 1$. From Proposition 1, we know that for each solution pair $(\bar{x}, w)$ of system $\mathcal{H}_\delta$ with $\bar{x}_0 \in (K_0 + \delta\mathbb{B})$ there exists some solution pair (ξ, w_1) of system $\mathcal{H}$ with $\xi_0 \in K_0$ and $|w_1| \leq |w|$ such that $\bar{x}$ and ξ are $(2T, \rho)$-close. Then, with ISS property of $\mathcal{H}$ and the definitions of ρ^* in (37) and (38), we have for all $(t, j) \in \text{dom}\, \bar{x}$ with $0 \leq t + j \leq 2T$, all solution pairs $(\bar{x}, w)$ of system $\mathcal{H}_\delta$ with $\bar{x}_0 \in (K_0 + \delta\mathbb{B})$ satisfy

$$\begin{aligned}\chi(\bar{x}(t,j)) &\leq \chi(\xi(s,j)) + \frac{\nu}{6}\\ &\leq \max\{\beta(\chi(\xi_0), t+j-\rho), \gamma(|w_1|)\} + \frac{\nu}{6}\\ &\leq \max\{\beta(\chi(\xi_0), t+j), \gamma(|w_1|)\} + \frac{\nu}{3}\\ &\leq \max\{\beta(\chi(\bar{x}_0), t+j), \gamma(|w|)\} + \frac{\nu}{2}\\ &\leq \max\{\beta(m, t+j), \gamma(|w|)\} + \frac{\nu}{2}. \end{aligned} \tag{39}$$

In particular, from the choice of T, (39) shows that $\chi(\bar{x}(t,j)) \leq \max\left\{\frac{\nu}{2}, \gamma(|w|)\right\} + \frac{\nu}{2} \leq \gamma(|w|) + \nu$ for all $(t,j) \in \operatorname{dom} \bar{x}$ with $T \leq t+j \leq 2T$.

Let $\bar{x}_T := \bar{x}(s,i)$ and inputs $\bar{w}(\cdot,\cdot) := w(s+\cdot, i+\cdot)$ for each (s,i) such that $(s,i) \in \operatorname{dom} \bar{x}$ and $s+i = T$. For $(t,j) \in \operatorname{dom} \bar{x}$ satisfying $2T \leq t+j \leq 3T$, using $m > \gamma(\Omega) + \nu$, (39) implies

$$\begin{aligned}\chi(\bar{x}(t,j)) &\leq \max\{\beta(\chi(\bar{x}_T), t+j), \gamma(|\bar{w}|)\} + \frac{\nu}{2},\\ &\leq \max\{\beta(\gamma(|w|)+\nu, t+j), \gamma(|w|)\} + \frac{\nu}{2}\\ &\leq \gamma(|w|) + \nu. \end{aligned}$$

Using this fact recursively shows that $\chi(\bar{x}(t,j)) \leq \gamma(|w|) + \nu$ for all $(t,j) \in \operatorname{dom} \bar{x}$ with $t+j \geq T$. This bound and (39) establish the bound in (36). □

We also require the following claim, a Lipschitz extension theorem based on [27, Theorem 1], that is useful in proving Theorems 1 and 2.

Claim 4 Let $J \subset \mathbb{R}^n$ be compact, $L > 0$, and $M > 0$. For a vector-valued function $f := (f_1, \ldots, f_n)$ where $f_i : J \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ are real-valued functions, define

$$\tilde{g}_i(x,\tau) := \sup_{z \in J}\{f_i(z,\tau) - L|x-z|\}. \tag{40}$$

Let

$$\operatorname{sat}(s) := \frac{Ms}{\max\{M, |s|\}}, \tag{41}$$

and $g(x,\tau) := \operatorname{sat}(\tilde{g}(x,\tau))$ with $\tilde{g} := (\tilde{g}_1, \ldots, \tilde{g}_n)$. If, for all $i \in \{1, \ldots, n\}$, $x, y \in J$, $\tau, \tau_1 \in \mathbb{R}_{\geq 0}$, $|f(x,\tau)| \leq M$ and $|f_i(x,\tau) - f_i(y,\tau_1)| \leq L(|x-y| + |\tau-\tau_1|)$, then $g(x,\tau) = f(x,\tau)$ for all $x \in J$, $\tau \in \mathbb{R}_{\geq 0}$, and satisfies the following properties:

1. $|g(x,\tau)| \leq M$ for all $x \in \mathbb{R}^n$ and $\tau \in \mathbb{R}_{\geq 0}$,
2. $|g(x,\tau) - g(y,\tau_1)| \leq \sqrt{n}L(|x-y| + |\tau-\tau_1|)$ for all $x, y \in \mathbb{R}^n$, $\tau, \tau_1 \in \mathbb{R}_{\geq 0}$.

Proof of Claim 4 Let $M > 0$ and $L > 0$ be such that

$$|f(x,\tau)| \le M \quad \forall\, x \in J \text{ and } \tau \in \mathbb{R}_{\ge 0}, \tag{42}$$

$$|f_i(x,\tau) - f_i(y,\tau)| \le L(|x-y| + |\tau - \tau_1|) \qquad \forall\, i \in \bar{N},\; x, y \in J, \text{ and } \tau, \tau_1 \in \mathbb{R}_{\ge 0}. \tag{43}$$

Using (43), for all $\tau \in \mathbb{R}_{\ge 0}$ and for $x \in J$, we have

$$\begin{aligned} f_i(x,\tau) &\le \sup_{z\in J}\{f_i(z,\tau) - L|x-z|\} \\ &= \tilde{g}_i(x,\tau) = \sup_{z\in J}\{f_i(z,\tau) - f_i(x,\tau) + f_i(x,\tau) - L|x-z|\} \le f_i(x,\tau)\,. \end{aligned}$$

Noting the construction of g_i in (40), we have that $g_i(x,\tau) \ge \operatorname{sat}(f_i(x,\tau))$ with letting $z = x$ for all $x \in J$. With (42) and the fact that

$$\operatorname{sat}\left(\sup_{z\in J}\{f_i(z,\tau) - f_i(x,\tau) + f_i(x,\tau) - L|x-z|\}\right) \le \operatorname{sat}(f_i(x,\tau)) = f_i(x,\tau)\,,$$

it follows that $g(x,\tau) = \operatorname{sat}(f(x,\tau)) = f(x,\tau)$ when $x \in J$ and $\tau \in \mathbb{R}_{\ge 0}$.

From $g(x,\tau) = \operatorname{sat}(\tilde{g}(x,\tau))$ and (41), it is straightforward that the first property is satisfied. Noting (40), (43) and the fact that for each y and $z \in \mathbb{R}^n$:

$$\sup_{z\in J}\{f_i(z,\tau) - L|y-z|\} - \sup_{z\in J}\{f_i(z,\tau_1) - L|y-z|\} \le \sup_{z\in J}\{f_i(z,\tau) - f_i(z,\tau_1)\},$$

we have

$$\tilde{g}_i(y,\tau) - \tilde{g}_i(y,\tau_1) \le \sup\nolimits_{z\in J}\{f_i(z,\tau) - f_i(z,\tau_1)\} \le L|\tau - \tau_1|.$$

Since this inequality holds for arbitrarily τ and $\tau_1 \in \mathbb{R}_{\ge 0}$, one gets $\tilde{g}_i(y,\tau_1) - \tilde{g}_i(y,\tau) \le L|\tau - \tau_1|$, which gives

$$|\tilde{g}_i(y,\tau) - \tilde{g}_i(y,\tau_1)| \le L|\tau - \tau_1| \quad \forall y \in \mathbb{R}^n \tag{44}$$

Let $\bar{N} = \{1, \ldots, n\}$. Let $k \in \bar{N}$ satisfy $|\tilde{g}_k(x,\tau) - \tilde{g}_k(y,\tau)| = \max_{i\in\bar{N}} |\tilde{g}_i(x,\tau) - \tilde{g}_i(y,\tau)|$. Without loss of generality, assume $\tilde{g}_k(x,\tau) \ge \tilde{g}_k(y,\tau)$. With (44), the fact $|\operatorname{sat}(\xi) - \operatorname{sat}(\psi)| \le |\xi - \psi|$ for all $\xi, \psi \in \mathbb{R}^n$, the extended function g satisfies

$$
\begin{aligned}
&|g(x,\tau)-g(y,\tau_1)| \\
&\quad=|\mathrm{sat}(\tilde g(x,\tau))-\mathrm{sat}(\tilde g(y,\tau_1))| \\
&\quad\le|\tilde g(x,\tau)-\tilde g(y,\tau)|+|\tilde g(y,\tau)-\tilde g(y,\tau_1)| \\
&\quad=\left(\sum_{i=1}^{n}|\tilde g_i(x,\tau)-\tilde g_i(y,\tau)|^2\right)^{\frac12}+\left(\sum_{i=1}^{n}|\tilde g_i(y,\tau)-\tilde g_i(y,\tau_1)|^2\right)^{\frac12} \\
&\quad\le\left(n\cdot|\tilde g_k(x,\tau)-\tilde g_k(y,\tau)|^2\right)^{\frac12}+\left(\sum_{i=1}^{n}L^2|\tau-\tau_1|^2\right)^{\frac12} \\
&\quad\le\left(n\cdot\sup_{a\in J}|L|a-x|-L|a-y||^2\right)^{\frac12}+\sqrt{n}L|\tau-\tau_1| \\
&\quad\le\left(nL^2\cdot\sup_{a\in J}|a-x-a+y|^2\right)^{\frac12}+\sqrt{n}L|\tau-\tau_1|=\sqrt{n}L(|x-y|+|\tau-\tau_1|)
\end{aligned}
$$

for all $x, y \in \mathbb{R}^n$ and $\tau, \tau_1 \in \mathbb{R}_{\geq 0}$. □

C.2 Proof of Theorem 1

Let $\Omega, \Omega_1 > 0$ come from the definitions of equi-essential boundedness and local equi-uniform Lipschitz continuity respectively. Let the compact set K_0 be given. Let $T \geq 0$ and $\rho > 0$ be given. Apply Proposition 1 with the set K_0 and (T, ρ, Ω) to generate a $\delta^* > 0$ such that for all $\delta \in (0, \delta^*]$ and the system $\mathcal{H}_\delta$ inflated from the weak average system $\mathcal{H}_{\mathrm{wa}}$, for each solution pair $(\bar{x}, w)$ to $\mathcal{H}_\delta$ with $\bar{x}(0,0) \in (K_0 + \delta\mathbb{B})$ there exists a solution pair (ξ, w_1) to system $\mathcal{H}_{\mathrm{wa}}$ with $\xi(0,0) \in K_0$ and $|w_1| \leq |w|$ such that the solutions $\bar{x}$ and ξ are $(T, \frac{\rho}{2})$-close. Without loss of generality, assume that $\delta < 1$ and $\rho < 1$.

Let $S_{\mathrm{wa}}(K_0)$ denote the set of maximum solution pairs to the weak average system $\mathcal{H}_{\mathrm{wa}}$ with $\xi(0,0) \in K_0$ and define the set K as

$$
\begin{aligned}
R_T(K_0,\Omega) &:= \{\xi(t,j):\ (\xi,w)\in S_{\mathrm{wa}}(K_0), t+j\le T, |w|\le\Omega\}, \\
K_1 &:= R_T(K_0)+\mathbb{B}, \\
K &:= K_1\cup G((K_1\times\Omega\mathbb{B})\cap D),
\end{aligned}
\tag{45}
$$

where K_1 is compact from Proposition 2. The set K is also compact as G is outer semi-continuous and locally bounded.

Set $\bar{K} := K \times \Omega\mathbb{B} \subset \mathbb{R}^n \times \mathbb{R}^m$. Let $\eta_{\mathrm{wa}}(x, w, \tau, \tau_0, \mu)$ be defined as (9). Let $\bar{K}$ generate $L\left(\bar{K}\right) \geq 1$ such that Assumption 3 holds for all $((x_1, w_1), \tau_a), ((x_2, w_2), \tau_b) \in (C \cap \bar{K}) \times \mathbb{R}_{\geq 0}$ and $\tau_0 \in [0, \min\{\tau_a, \tau_b\}]$ with $L := L\left(\bar{K}\right)$. Let $\bar{K}$ and Lemma 1 generate $\alpha_{\bar{K}}$ and pick $\mu > 0$ such that $\alpha_{\bar{K}}(\mu) \leq \frac{\delta}{3}$. Then, for this μ, for all $((x, w), \tau) \in \left(C \cap \bar{K}\right) \times \mathbb{R}_{\geq 0}$ and $\tau_0 \in [0, \tau]$, we have $|\eta_{\mathrm{wa}}(x, w, \tau, \tau_0, \mu)| \leq \frac{\delta}{3\mu}$.

Let $J := \left(C \cap \bar{K}\right)$. Claim 4 gives us a new function $\tilde{\eta}_{\mathrm{wa}}$ that defined on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$. For the picked μ, the following properties are satisfied for $\tilde{\eta}_{\mathrm{wa}}$:

1. for all $(x, w, \tau) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_{\geq 0}$ and $\tau_0 \in [0, \tau]$:

$$|\tilde{\eta}_{\mathrm{wa}}(x, w, \tau, \tau_0, \mu)| \leq \frac{\delta}{3\mu}, \tag{46}$$

2. $|\tilde{\eta}_{\mathrm{wa}}(x_1, w_1, \tau_a, \tau_0, \mu) - \tilde{\eta}_{\mathrm{wa}}(x_2, w_2, \tau_b, \tau_0, \mu)| \leq 2\sqrt{n}L(|x_1 - x_2| + |w_1 - w_2| + |\tau_a - \tau_b|)$ for each $(x_1, w_1, \tau_a), (x_2, w_2, \tau_b) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_{\geq 0}$ and $\tau_0 \in [0, \min\{\tau_a, \tau_b\}]$,
3. $\tilde{\eta}_{\mathrm{wa}}(x, w, \tau, \tau_0, \mu) = \eta_{\mathrm{wa}}(x, w, \tau, \tau_0, \mu)$ for all $((x, w), \tau) \in (C \cap \bar{K}) \times \mathbb{R}_{\geq 0}$ and $\tau_0 \in [0, \tau]$.

Let Assumption 2, δ and the set $\bar{K}$ generate $M\left(\bar{K}\right) \geq 1$ and ε_1^* such that the bounds (6) hold with $M := M\left(\bar{K}\right)$ and $\varepsilon \in (0, \varepsilon_1^*]$. Let $\varepsilon_2^* = \frac{\delta}{6\sqrt{n}L(M+1+\Omega_1)}$, $\varepsilon_3^* = \frac{3\rho\mu}{2\delta}$, $\varepsilon_4^* = 3\mu$ and $\varepsilon^* = \min\{\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*, \varepsilon_4^*\}$. Consider $\varepsilon \in (0, \varepsilon^*]$.

Let (x, w, τ) be a solution to the system

$$H_K \quad \left.\begin{aligned} \dot{x} &= f_\varepsilon(x, w, \tau) \\ \dot{\tau} &= \tfrac{1}{\varepsilon} \end{aligned}\right\} \quad ((x, w), \tau) \in (C \cap \bar{K}) \times \mathbb{R}_{\geq 0}, \qquad \left.\begin{aligned} x^+ &\in G(x, w) \cap K \\ \tau^+ &\in H(x, w, \tau) \end{aligned}\right\} \quad ((x, w), \tau) \in (D \cap \bar{K}) \times \mathbb{R}_{\geq 0}. \tag{47}$$

Note that the system $\mathcal{H}_K$ agrees with system $\mathcal{H}_\epsilon$ but with G intersected with K and C, D intersected with $K \times \Omega\mathbb{B}$. By construction, for each $(t, j) \in \mathrm{dom}\,(x, w, \tau)$, we have $(x(t, j), w(t, j)) \in \bar{K}$. With (46) and the definitions of ε, we have for all $(t, j) \in \mathrm{dom}\,(x, w, \tau)$:

$$|\varepsilon\tilde{\eta}_{\mathrm{wa}}(x(t, j), w(t, j), \tau(t, j), \tau_0, \mu)| \leq \frac{\varepsilon\delta}{3\mu} \leq \delta \tag{48}$$

holds for all $\tau_0 \in [0, \tau(t, j)]$. For each $(t, j) \in \mathrm{dom}\,(x, w, \tau)$, define

$$\bar{x}(t, j) = x(t, j) - \varepsilon\tilde{\eta}_{\mathrm{wa}}(x(t, j), w(t, j), \tau(t, j), \tau_0, \mu), \tag{49}$$

with $\tau_0 := \tau(t_j, j)$ and $t_j := \min\{t : (t, j) \in \mathrm{dom}\,(x, w, \tau)\}$. It follows that $\bar{x}$ is a hybrid arc. For each $(t, j) \in \mathrm{dom}\,\bar{x}$ such that for all $(t, j+1) \in \mathrm{dom}\,\bar{x}$,

$$(x(t, j), w(t, j)) = (\bar{x}(t, j) + \varepsilon\tilde{\eta}_{\mathrm{wa}}(x(t, j), w(t, j), \tau(t, j), \tau_0, \mu), w(t, j)) \in D \cap \bar{K},$$

which with (48) implies that $(\bar{x}(t, j), w(t, j)) \in D_\delta$ and

$$\begin{aligned} \bar{x}(t, j+1) &= x(t, j+1) - \varepsilon\tilde{\eta}_{\mathrm{wa}}(x(t, j+1), w(t, j+1), \tau(t, j+1), \tau_0, \mu) \\ &\in (G((x(t, j), w(t, j)) \cap D) + \delta\mathbb{B}) \cap K \\ &\subset G((x(t, j), w(t, j)) \cap D) + \delta\mathbb{B} \end{aligned}$$

$$
\begin{aligned}
&= G((\bar{x}(t,j) + \varepsilon\tilde{\eta}_{\mathrm{wa}}(x(t,j), w(t,j), \tau(t,j), \tau_0, \mu), w(t,j)) \cap D) + \delta\mathbb{B} \\
&\subset G((\bar{x}(t,j) + \delta\mathbb{B}, w(t,j)) \cap D) + \delta\mathbb{B} \\
&= G_\delta(\bar{x}(t,j), w(t,j)).
\end{aligned}
$$

Moreover, for each j such that the set $I_j := \{t : (t,j) \in \mathrm{dom}\ \bar{x}\}$ has nonempty interior and for all $t \in I_j$,

$$
(x(t,j), w(t,j)) = (\bar{x}(t,j) + \varepsilon\tilde{\eta}_{\mathrm{wa}}(x(t,j), w(t,j), \tau(t,j), \tau_0, \mu), w(t,j)) \in C \cap \bar{K}
$$

implies that $(\bar{x}(t,j), w(t,j)) \in C_\delta$. Noting $\tilde{\eta}_{\mathrm{wa}}$ is globally Lipschitz continuous, $\bar{x}(\cdot,j)$ is locally absolutely continuous and the set $\mathcal{L}_{\mathcal{W}}$ is locally equi-uniformly Lipschitz continuous, and for almost all $t \in I_j$ we have

$$
\begin{aligned}
&\dot{\bar{x}}(t,j) \\
&\in \dot{x}(t,j) - \varepsilon\frac{\partial\tilde{\eta}_{\mathrm{wa}}(x(t,j), w(t,j), \tau(t,j), \tau_0, \mu)}{\partial x}\dot{x}(t,j) \\
&\quad -\varepsilon\frac{\partial\tilde{\eta}_{\mathrm{wa}}(x(t,j), w(t,j), \tau(t,j), \tau_0, \mu)}{\partial w}\dot{w}(t,j) \\
&\quad -\frac{\partial\tilde{\eta}_{\mathrm{wa}}(x(t,j), w(t,j), \tau(t,j), \tau_0, \mu)}{\partial \tau} \\
&= f_\varepsilon(x(t,j), w(t,j), \tau(t,j)) - \varepsilon\frac{\partial\tilde{\eta}_{\mathrm{wa}}(x(t,j), w(t,j), \tau(t,j), \tau_0, \mu)}{\partial x}\dot{x}(t,j) \\
&\quad -\varepsilon\frac{\partial\tilde{\eta}_{\mathrm{wa}}(x(t,j), w(t,j), \tau(t,j), \tau_0, \mu)}{\partial w}\dot{w}(t,j) - f_0(x(t,j), w(t,j), \tau(t,j)) \\
&\quad + f_{\mathrm{wa}}(x(t,j), w(t,j)) - \mu\tilde{\eta}_{\mathrm{wa}}(x(t,j), w(t,j), \tau(t,j), \tau_0, \mu) \\
&\in f_{\mathrm{wa}}(\bar{x}(t,j) + \varepsilon\tilde{\eta}_{\mathrm{wa}}(x(t,j), w(t,j), \tau(t,j), \tau_0, \mu), w(t,j)) + \frac{\delta\mathbb{B}}{3} \\
&\quad + \varepsilon 2\sqrt{n}L(M + 1 + \Omega_1)\mathbb{B} + \alpha_K(\mu)\mathbb{B} \\
&\in F(\bar{x}(t,j) + \delta\mathbb{B}, w(t,j)) + \delta\mathbb{B} \\
&\subset F_\delta(\bar{x}(t,j), w(t,j)),
\end{aligned}
\tag{50}
$$

where

$$
\left[\frac{\partial\tilde{\eta}_{\mathrm{wa}}(x, w, \tau, \tau_0, \mu)}{\partial x}, \frac{\partial\tilde{\eta}_{\mathrm{wa}}(x, w, \tau, \tau_0, \mu)}{\partial w}, \frac{\partial\tilde{\eta}_{\mathrm{wa}}(x, w, \tau, \tau_0, \mu)}{\partial \tau}\right]
$$

can be considered as generalized Jacobian of $\tilde{\eta}_{\mathrm{wa}}$. The sequence of equalities and inclusions in (50) hold from the results in (Section 2.6, [10]) with Assumption 2, definitions of ε^* and μ. Then, it follows that $(\bar{x}, w)$ is the solution pair of system $\mathcal{H}_\delta$, and we can conclude that for each $(\bar{x}, w)$ there exists some solution pair (ξ, w_1) to system $\mathcal{H}_{\mathrm{wa}}$ such that $\bar{x}$ and ξ are $(T, \frac{\rho}{2})$ close. Moreover, from the definition of $\bar{x}$ in (49) and definition of ε^*, we know that for the solution pair (x, w) to system $\mathcal{H}_K$, x is $(T, \frac{\rho}{2})$ close to $\bar{x}$ and then it is (T, ρ)-close to ξ.

Next, consider solution pairs of system $\mathcal{H}_\epsilon$ that start in K_0. Let $(\tilde{x}, w)$ be a solution pair to system $\mathcal{H}_\epsilon$ with $\tilde{x}(0,0) \in K_0$ and $|w| \leq \Omega$. If $\tilde{x} \in K$ for all $(t, j) \in \text{dom } \tilde{x}$ with $t + j \leq T$, then for each solution pair $(\tilde{x}, w)$ of $\mathcal{H}_\epsilon$, there exists some solution pair (ξ, w_1) of system $\mathcal{H}_{\text{wa}}$ such that $\tilde{x}$ is also (T, ρ) close to ξ. Now, suppose that there exists $(t, j) \in \text{dom } \tilde{x}$ such that $\tilde{x}(s, i) \in K$ satisfying $s + i \leq t + j$ and either

1. $(t, j + 1) \in \text{dom } \tilde{x}$ and $\tilde{x}(t, j + 1) \notin K$ or else,
2. there exist a monotonically decreasing sequences r_i with the limit $\lim_{i\to\infty} r_i = t$ such that $(r_i, j) \in \text{dom } \tilde{x}$ and $\tilde{x}(r_i, j) \notin K$ for each i.

The solution pair $(\tilde{x}, w)$ must agree with a solution pair of system $\mathcal{H}_K$ up to time (t, j), and thus must satisfy $\tilde{x} \in R_T(K_0) + \rho\mathbb{B}$. If this follows, by the definition of K in (45) and $\rho < 1$, which implies that $R_T(K_0) + \rho\mathbb{B}$ is contained inside of K, that neither of these two case can occur. This establishes the result. □

C.3 Proof of Theorem 2

The proof of Theorem 2 follows exactly the same steps in the proof of Theorem 1 with following changes. Let $\Omega \geq 0$ come from the definition of equi-essential boundedness and δ be same as the proof of Theorem 1. Let K be defined as (45) for strong average system $\mathcal{H}_{\text{sa}}$. Let the set $\bar{K} := K \times \Omega\mathbb{B}$ and δ generate $M\left(\bar{K}\right) \geq 1$ and ε_1^* such that bounds (6) hold with $M := M\left(\bar{K}\right)$ and $\varepsilon \in (0, \varepsilon_1^*]$.

Let $\eta_{\text{sa}}(x, \tilde{w}, \tau, \tau_0, \mu)$ be defined as (10). Let $\bar{K}$ generate $L\left(\bar{K}\right) \geq 1$ such that Assumption 4 holds with $L := L\left(\bar{K}\right)$. Let the set $\bar{K}$ and Lemma 2 generate $\alpha_{\bar{K}}$ and pick $\mu > 0$ such that $\alpha_{\bar{K}}(\mu) \leq \frac{\delta}{3}$. Then, for this μ, for all $0 \leq \tau_0 \leq \tau_1, \tilde{w} : [\tau_0, \tau_1] \to \mathcal{W}$ and $((x, \tilde{w}(s)), \tau) \in \left((C_1 \times \mathcal{W}) \cap \bar{K}\right) \times [\tau_0, \tau_1]$ for all $s \in [\tau_0, \tau_1]$, we have $|\eta_{\text{sa}}(x, \tilde{w}, \tau, \tau_0, \mu)| \leq \frac{\delta}{3\mu}$.

Let $J := ((C_1 \times \mathcal{W}) \cap \bar{K})$. Using the result in Claim 4, we have the function $\tilde{\eta}_{\text{sa}}$ such that, for the picked μ and for all $0 \leq \tau_0 \leq \tau_1, \tilde{w} : [\tau_0, \tau_1] \to \mathcal{W}$, the following properties are satisfied:

1. for each $(x, \tilde{w}(s), \tau) \in \mathbb{R}^n \times \mathbb{R}^m \times [\tau_0, \tau_1]$ for all $s \in [\tau_0, \tau_1]$:
$$|\tilde{\eta}_{\text{sa}}(x, \tilde{w}, \tau, \tau_0, \mu)| \leq \frac{\delta}{3\mu}, \tag{51}$$
2. $|\tilde{\eta}_{\text{sa}}(x_1, \tilde{w}, \tau_a, \tau_0, \mu) - \tilde{\eta}_{\text{sa}}(x_2, \tilde{w}, \tau_b, \tau_0, \mu)| \leq 2\sqrt{n}L(|x_1 - x_2| + |\tau_a - \tau_b|)$ for all $(x_1, \tilde{w}(s), \tau_a), (x_2, \tilde{w}(s), \tau_b) \in \mathbb{R}^n \times \mathbb{R}^m \times [\tau_0, \tau_1]$ for all $s \in [\tau_0, \tau_1]$,
3. $\tilde{\eta}_{\text{sa}}(x, \tilde{w}, \tau, \tau_0, \mu) = \eta_{\text{sa}}(x, \tilde{w}, \tau, \tau_0, \mu)$ for all $((x, \tilde{w}(s)), \tau) \in ((C_1 \times \mathcal{W}) \cap \bar{K}) \times [\tau_0, \tau_1]$ for all $s \in [\tau_0, \tau_1]$.

Let Assumption 2, δ and the set $\bar{K}$ generate $M\left(\bar{K}\right) \geq 1$ and ε_1^* such that the bounds (6) hold with $M := M\left(\bar{K}\right)$ and $\varepsilon \in (0, \varepsilon_1^*]$. Let $\varepsilon_2^* = \frac{\delta}{6\sqrt{n}L(M+1)}$, $\varepsilon_3^* = \frac{3\rho\mu}{2\delta}$, $\varepsilon_4^* = 3\mu$ and $\varepsilon^* = \min\{\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*, \varepsilon_4^*\}$. Consider $\varepsilon \in (0, \varepsilon^*]$.

Letting (x, w, τ) be a solution to the system $\mathcal{H}_K$ in (47), it follows from the construction of $\mathcal{H}_K$ that $(x(t, j), w(t, j)) \in \bar{K}$ for all $(t, j) \in \text{dom } (x, w, \tau)$. Let $\tau_0 :=$

$\tau\left(t_j^0, j\right)$ and $\tau_1 := \tau\left(t_j^1, j\right)$ with $t_j^0 := \min\{t : (t, j) \in \text{dom}\,(x, w, \tau)\}$ and $t_j^1 := \max\{t : (t, j) \in \text{dom}\,(x, w, \tau)\}$. Let $\tilde{w} : [\tau_0, \tau_1] \to \mathcal{W}$ be defined as $\tilde{w}(\tau(s, j)) := w(s, j)$ for each $s \in \{t : (t, j) \in \text{dom}\,(x, w, \tau)\}$. With (51) and the definition of ε, it follows that for all $(t, j) \in \text{dom}\,(x, w, \tau)$,

$$|\varepsilon\tilde{\eta}_{\text{sa}}(x(t, j), \tilde{w}, \tau(t, j), \tau_0, \mu)| \leq \frac{\varepsilon\delta}{3\mu} \leq \delta. \tag{52}$$

For each $(t, j) \in \text{dom}\,(x, w, \tau)$, define

$$\bar{x}(t, j) = x(t, j) - \varepsilon\tilde{\eta}_{\text{sa}}(x(t, j), \tilde{w}, \tau(t, j), \tau_0, \mu).$$

For each $(t, j) \in \text{dom}\,\bar{x}$ such that for all $(t, j+1) \in \text{dom}\,\bar{x}$,

$$(x(t, j), w(t, j)) = (\bar{x}(t, j) + \varepsilon\tilde{\eta}_{\text{sa}}(x(t, j), \tilde{w}, \tau(t, j), \tau_0, \mu), w(t, j)) \in D \cap \bar{K}$$

with (52) implies that $(\bar{x}(t, j), w(t, j)) \in D_\delta$ and

$$\begin{aligned}
\bar{x}(t, j+1) &= x(t, j+1) - \varepsilon\tilde{\eta}_{\text{sa}}(x(t, j+1), \tilde{w}, \tau(t, j+1), \tau_0, \mu) \\
&\in (G((x(t, j), w(t, j)) \cap D) + \delta\mathbb{B}) \cap K \\
&\subset G((x(t, j), w(t, j)) \cap D) + \delta\mathbb{B} \\
&= G((\bar{x}(t, j) + \varepsilon\tilde{\eta}_{\text{sa}}(x(t, j), \tilde{w}, \tau(t, j), \tau_0, \mu), w(t, j)) \cap D) + \delta\mathbb{B} \\
&\subset G((\bar{x}(t, j) + \delta\mathbb{B}, w(t, j)) \cap D) + \delta\mathbb{B} \\
&= G_\delta(\bar{x}(t, j), w(t, j)).
\end{aligned}$$

Moreover, for each j such that the set $I_j := \{t : (t, j) \in \text{dom}\,\bar{x}\}$ has nonempty interior and for all $t \in I_j$,

$$(x(t, j), w(t, j)) = (\bar{x}(t, j) + \varepsilon\tilde{\eta}_{\text{sa}}(x(t, j), \tilde{w}, \tau(t, j), \tau_0, \mu), w(t, j)) \in C \cap \bar{K}$$

implies that $(\bar{x}(t, j), w(t, j)) \in C_\delta$. Noting the definition of $\tilde{w}$, instead of (50), we have

$$\begin{aligned}
\dot{\bar{x}}(t, j) &\in \dot{x}(t, j) - \varepsilon\frac{\partial\tilde{\eta}_{\text{sa}}(x(t, j), \tilde{w}, \tau(t, j), \tau_0, \mu)}{\partial x}\dot{x}(t, j) \\
&\quad - \frac{\partial\tilde{\eta}_{\text{sa}}(x(t, j), \tilde{w}, \tau(t, j), \tau_0, \mu)}{\partial\tau} \\
&= f_\varepsilon(x(t, j), w(t, j), \tau(t, j)) - f_0(x(t, j), \tilde{w}(\tau(t, j)), \tau(t, j)) \\
&\quad - \mu\tilde{\eta}_{\text{sa}}(x(t, j), \tilde{w}, \tau(t, j), \tau_0, \mu) - \varepsilon\frac{\partial\tilde{\eta}_{\text{sa}}(x(t, j), \tilde{w}, \tau(t, j), \tau_0, \mu)}{\partial x}\dot{x}(t, j) \\
&\quad + f_{\text{sa}}(x(t, j), \tilde{w}(\tau(t, j)))
\end{aligned}$$

$$
\begin{aligned}
&= f_\varepsilon(x(t,j),w(t,j),\tau(t,j)) - f_0(x(t,j),w(t,j),\tau(t,j)) \\
&\quad -\mu\tilde{\eta}_{\rm sa}(x(t,j),\tilde{w},\tau(t,j),\tau_0,\mu) - \varepsilon\frac{\partial\tilde{\eta}_{\rm sa}(x(t,j),\tilde{w},\tau(t,j),\tau_0,\mu)}{\partial x}\dot{x}(t,j) \\
&\quad + f_{\rm sa}(x(t,j),w(t,j)) \\
&\in f_{\rm sa}(\bar{x}(t,j) + \varepsilon\tilde{\eta}_{\rm sa}(x(t,j),\tilde{w},\tau(t,j),\tau_0,\mu),w(t,j)) + \frac{\delta\mathbb{B}}{3} \\
&\quad + \varepsilon 2\sqrt{n}L(M+1)\mathbb{B} + \alpha_K(\mu) \\
&\in F(\bar{x}(t,j)+\delta\mathbb{B},w(t,j)) + \delta\mathbb{B} \subset F_\delta(\bar{x}(t,j),w(t,j)).
\end{aligned}
\tag{53}
$$

Then, it follows that $(\bar{x}, w)$ is the solution pair to system $\mathcal{H}_\delta$, and we can conclude that for any solution pair $(\bar{x}, w)$ there exists some solution pair (ξ, w_1) to system $\mathcal{H}_{\rm sa}$ such that $\bar{x}$ and ξ are $(T, \frac{\rho}{2})$-close. Then, using the same steps in proof of Theorem 1, we can complete the proof. □

C.4 Proof of Theorem 3

Let $\Omega, \Omega_1 > 0$ come from the definitions of equi-essential boundedness and local equi-uniform Lipschitz continuity respectively. Let functions (χ, β, γ) come from the definition of ISS in Def. 8 for system $\mathcal{H}_{\rm wa}$. Let the compact set $K_0 \subset \mathbb{R}^n$ be given, and define

$$
\begin{aligned}
K_1 &:= \left\{x \in \mathbb{R}^n : \chi(x) \le \max\left\{\beta\left(\max_{\bar{x}\in K_0}\chi(\bar{x}), 0\right), \gamma(\Omega)\right\} + 1\right\} \\
K &:= K_1 \cup G((K_1 \times \mathcal{W}) \cap D).
\end{aligned}
\tag{54}
$$

The set K is a compact because of continuity of the proper indicator χ and outer semi-continuity of the set mapping $G : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$.

Let $\nu \in (0, 1)$. From the Proposition 3, the ν, Ω and the compact set K generate a $\delta > 0$ such that each solution pair $(\bar{x}, w)$ of system $\mathcal{H}_\delta$ inflated from $\mathcal{H}_{\rm wa}$ with $\bar{x}_0 := \bar{x}(0, 0) \in K + \delta\mathbb{B}$ satisfies

$$
\chi(\bar{x}(t,j)) \le \max\{\beta(\chi(\bar{x}_0), t+j), \gamma(|w|)\} + \frac{\nu}{3} \quad \forall (t,j) \in \operatorname{dom}\bar{x}. \tag{55}
$$

Without loss of generality, assume that $\delta < 1$. Let $\bar{K} := K \times \Omega\mathbb{B} \subset \mathbb{R}^n \times \mathbb{R}^m$. Let $\bar{K}$ and Lemma 1 generate $\alpha_{\bar{K}}$ and pick $\mu > 0$ such that $\alpha_{\bar{K}} \le \frac{\delta}{3}$. Let $\bar{K}$, δ and Assumption 2 generate $M\left(\bar{K}\right) > 1$ and $\varepsilon_1^* > 0$ such that bounds (6) hold with $M := M\left(\bar{K}\right)$ and $\varepsilon \in (0, \varepsilon_1^*]$. Let Assumption 3 and the set $\bar{K}$ generate $L := L\left(\bar{K}\right) \ge 1$ so that Assumption 3 holds for all $((x_1, w_1), \tau_a), ((x_2, w_2), \tau_b) \in (C \cap \bar{K}) \times \mathbb{R}_{\ge 0}$. Let $\varepsilon_2^* = \frac{\delta}{6\sqrt{n}L(M+1+\Omega_1)}$, $\varepsilon_3^* = 3\mu$.

System $\mathcal{H}_K$ defined in (47) is introduced. With the continuity of the proper indicator χ and class-$\mathcal{KL}$ function β and the fact that $\beta(m, s)$ converges to zero as $s \ge 0$ approaches infinity for all $m \ge 0$, let $\varepsilon_4^* > 0$ be such that for all $x \in K$ and $\bar{x} \in K + \varepsilon_4^* L\mathbb{B}$ satisfying $|x - \bar{x}| \le \varepsilon_4^* L$, the following holds:

$$\begin{aligned} \chi(x) &\leq \chi(\bar{x}) + \frac{\nu}{3} \\ \beta(\chi(\bar{x}), s) &\leq \beta(\chi(x), s) + \frac{\nu}{3}, \quad \forall s \in \mathbb{R}_{\geq 0}. \end{aligned} \tag{56}$$

Letting $\varepsilon^* = \min\{\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*, \varepsilon_4^*\}$, for each $\varepsilon \in (0, \varepsilon^*]$, define $\bar{x}$ as (49) with the same construction method in the proof of Theorem 1. Then, we can show that $(\bar{x}(t, j), w(t, j))$ is a solution pair to the inflated system $\mathcal{H}_\delta$, and then (55) holds. Letting $x_0 := x(0, 0)$ and using (56), for all solution pairs $(x, w) \in K$ to system $\mathcal{H}_K$ and $(t, j) \in \operatorname{dom} x$, we have

$$\begin{aligned} \chi(x(t, j)) &\leq \chi(\bar{x}(t, j)) + \frac{\nu}{3}, \\ &\leq \max\{\beta(\chi(\bar{x}_0), t + j), \gamma(|w|)\} + \frac{2\nu}{3} \\ &\leq \max\{\beta(\chi(x_0), t + j), \gamma(|w|)\} + \nu. \end{aligned} \tag{57}$$

In particular, since $\nu < 1$, each solution pair to system $\mathcal{H}_K$ starting in K_0 remains in the compact set

$$K_\nu := \left\{ x \in \mathbb{R}^n : \ \chi(x) \leq \max\left\{ \beta\left(\max_{\bar{x} \in K_0} \chi(\bar{x}), 0 \right), \gamma(\Omega) \right\} + \nu \right\}.$$

With $\nu < 1$ and continuity of χ, K_ν is contained in K_1 defined in (54). Finally, using the same steps in the proof of Theorem 1, and the bound (57) on the solution pairs of system $\mathcal{H}_K$ to get conclusions about the solutions of system $\mathcal{H}_\epsilon$ with $x_0 \in K_0$, which establishes the result. □

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