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ORIGINAL ARTICLE

Well-posedness, regularity and exact controllability of the SCOLE model

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Abstract The SCOLE model is a coupled system consisting of a flexible beam (modelled as an Euler–Bernoulli equation) with one end clamped and the other end linked to a rigid body. Its inputs are the force and the torque acting on the rigid body. It is well-known that the SCOLE model is not exactly controllable with L^2 input signals in the natural energy state space H^c, because the control operator is bounded from the input space $\mathbb{C}^2$ to H^c, and hence compact. We regard the velocity and the angular velocity of the rigid body as the output signals of this system. Using the theory of coupled linear systems (one infinite-dimensional and one finite-dimensional) developed by us recently in another paper, we show that the SCOLE model is well-posed, regular and exactly controllable in arbitrarily short time when using a certain smoother state space $\mathcal{X} \subset H^c$.

Keywords SCOLE model · Coupled system · Well-posedness · Regularity · Exact controllability · Interpolation space · Boundary control system

1 Introduction

This paper investigates the exact controllability of the SCOLE (NASA Spacecraft Control Laboratory Experiment) model with L^2 input signals as well as its well-posedness and regularity with the state space that makes it exactly controllable. The SCOLE

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system models a flexible beam with one end clamped and the other end linked to a rigid body. The vibrations of the beam are described by the Euler–Bernoulli equation, while the rigid body obeys the Newton–Euler equations. The inputs of the system are the force and the torque acting on the rigid body, while the outputs are the velocity and the angular velocity of the rigid body. The importance of the SCOLE model stems from it being used to model the vibrations of a flexible mast holding an antenna on a spacecraft, see Littman and Markus [7,8]. More recently, it is used also to model the vibrations of the tower of a wind turbine holding a heavy nacelle, in the plane of the turbine axis.

Assuming that the beam is uniform and moves only in one plane, the model is

$$\begin{cases} \rho w_{tt}(x,t) + EIw_{xxxx}(x,t) = 0, \qquad (x,t) \in (0,l) \times [0,\infty), \\ w(0,t) = 0, \quad w_x(0,t) = 0, \\ mw_{tt}(l,t) - EIw_{xxx}(l,t) = f(t), \\ Jw_{xtt}(l,t) + EIw_{xx}(l,t) \;=\; v(t), \end{cases} \tag{1.1}$$

where the subscripts t and x denote derivatives with respect to the time t and the position x, respectively. l is the length of the beam, w stands for the transverse displacement of the beam, and EI and ρ are the flexural rigidity and the mass density of the beam (EI and ρ are positive constants). m and J are the mass and the moment of inertia of the rigid body (again positive constants). f and v are the force input and the torque input acting on the rigid body. $-EIw_{xxxx}(x,t)\mathrm{d}x$ is the total lateral force acting on a slice of the beam of length $\mathrm{d}x$, located at the position x and the time t. $EIw_{xxx}(l,t)$ and -$EIw_{xx}(l,t)$ are the force and the torque acting on the rigid body from the beam at the time t. We define the input and output signals of the SCOLE model as follows:

$$u_e = \begin{bmatrix} u_{e1} \\ u_{e2} \end{bmatrix} = \begin{bmatrix} f \\ v \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} w_t(l,\cdot) \\ w_{xt}(l,\cdot) \end{bmatrix}. \tag{1.2}$$

The natural state and state space of the SCOLE model are

$$\begin{aligned} z^c(t) &= \left[w(\cdot,t) \; w_t(\cdot,t) \; w_t(l,t) \; w_{xt}(l,t) \right]^T, \\ H^c &= \mathcal{H}_l^2(0,l) \times L^2[0,l] \times \mathbb{C}^2, \end{aligned} \tag{1.3}$$

where $\mathcal{H}_l^2(0,l) = \{h \in \mathcal{H}^2(0,l) \mid h(0) = 0,\; h_x(0) = 0\}$. The natural norm on H^c is

$$\|z^c(t)\|^2 = EI\|w(\cdot,t)\|^2_{\mathcal{H}_l^2} + \rho\|w_t(\cdot,t)\|^2_{L^2} + m|w_t(l,t)|^2 + J|w_{xt}(l,t)|^2,$$

which represents twice the physical energy. In [19], we have shown that the SCOLE model is an impedance conservative well-posed system with state space H^c.

The exact controllability of the SCOLE model has been investigated by several researchers. It is well-known that the SCOLE model is not exactly controllable with the natural energy state space H^c using L^2 inputs, since the control operator is bounded from the input space $\mathbb{C}^2$ to H^c, and hence compact. Exact controllability can be achieved either by expanding the input signal space (bringing in distributions) or by shrinking the state space. Here are some results obtained by expanding the input signal space. Using the Hilbert Uniqueness Method, Rao [11] obtained the exact controllability of the uniform SCOLE model with the state space H^c by means of a singular input signal. He considered $f \in L^2[0, T]$ but allowed the torque input v to be in in the dual of $\mathcal{H}^1(0, T)$, where $T > 0$ is an arbitrarily short time. He also proved the exact controllability in arbitrarily short time of the SCOLE model with the state space H^c by singular torque input (and zero force input) if $l < 3$, where all the constants (EI, ρ, m, J) are one. Guo and Ivanov [5] removed this length limitation and they allowed the SCOLE model to be non-uniform.

Smaller state spaces have been investigated in at least three papers. The null-controllability of the SCOLE model with a state space of type $\mathcal{H}^6(0, l) \times \mathcal{H}^4(0, l)$ (with boundary conditions) was proved in Littman and Markus [7] based on the theory of semi-infinite beams. Using a constructive cutoff approach, they proved the existence of smooth torque and force inputs for the finite beam leading to the final state zero. Using the Riesz basis approach, Guo [4] proved that the non-uniform SCOLE model with only torque input is exactly controllable with the state space $\mathcal{D}(A^c)$. Here A^c is the generator of the SCOLE system with the state space H^c,

$$\mathcal{D}(A^c) = \left\{ \begin{bmatrix} z \\ q \end{bmatrix} \in \left(\mathcal{H}^4(0, l) \cap \mathcal{H}_l^2(0, l)\right) \times \mathcal{H}_l^2(0, l) \times \mathbb{C}^2 \;\middle|\; \begin{matrix} q_1 = z_2(l) \\ q_2 = z_{2x}(l) \end{matrix} \right\}. \tag{1.4}$$

Guo and Ivanov [5] have shown that for the non-uniform SCOLE model the space $\mathcal{D}(|A^c|^{\frac{1}{2}})$ is reachable using only force control. The definition of $\mathcal{D}(|A^c|^{\frac{1}{2}})$ will be given in Sect. 2. So far $\mathcal{D}(|A^c|^{\frac{1}{2}})$ is the largest known reachable space using L^2 inputs. An explicit description of $\mathcal{D}(|A^c|^{\frac{1}{2}})$ like in (1.4) has not been given in [5], nor were there well-posedness results in case we use this space as the state space.

In this paper, using a new approach to the well-posedness and exact controllability of coupled system developed in Weiss and Zhao [18], we show that the SCOLE model described by (1.1) and (1.2) is well-posed, regular (in the sense of [13, 17]) and exactly controllable in any time $T > 0$ with the state space

$$\mathcal{X} = \left\{ \begin{bmatrix} z \\ q \end{bmatrix} \in [\mathcal{H}^3(0, l) \cap \mathcal{H}_l^2(0, l)] \times \mathcal{H}_l^1(0, l) \times \mathbb{C}^2 \;\middle|\; z_2(l) = q_1 \right\}$$

using both torque and force control in L^2. Here, $\mathcal{H}_l^1(0, l) = \{h \in \mathcal{H}^1(0, l) \mid h(0) = 0\}$. The system remains regular with $y = \begin{bmatrix} -EIw_{xxx}(l,\cdot) \\ EIw_{xx}(l,\cdot) \end{bmatrix}$ as an additional output. We suspect that $\mathcal{X} = \mathcal{D}(|A^c|^{\frac{1}{2}})$, but we did not verify this. It would be nice to prove exact controllability in $\mathcal{X}$ using only force control, but we were not able to do this.

2 Background on controllability and coupled systems

For the background on admissible control and observation operators and controllability of infinite-dimensional systems, we refer to Tucsnak and Weiss [16] (which has many relevant references), and for the background on coupled systems that is needed here, we refer to Weiss and Zhao [18]. For easy reference we reproduce below several well-known results which can be found, e.g., in [16].

We need some preliminaries. Let A be the generator of a strongly continuous semigroup $\mathbb{T}$ on a Hilbert space X. Then A determines several additional Hilbert spaces: X_1 is $\mathcal{D}(A)$ with the norm $\|z\|_1 = \|(\beta I - A)z\|$, X_2 is $\mathcal{D}(A^2)$ with the norm $\|z\|_2 = \|(\beta I - A)^2 z\|$, and X_{-1} is the completion of X with respect to the norm $\|z\|_{-1} = \|(\beta I - A)^{-1} z\|$, where $\beta \in \rho(A)$ is fixed. The spaces X_1, X_2 and X_{-1} are independent of the choice of β, since different values of β lead to equivalent norms on X_1, X_2 and X_{-1}. We have $X_2 \subset X_1 \subset X \subset X_{-1}$, densely and with continuous embeddings. We can continuously extend A to a bounded operator from X to X_{-1}, still denoted by A. The semigroup generated by this extended A is the extension of $\mathbb{T}$ to X_{-1}, still denoted by $\mathbb{T}$. If $X_1^d = \mathcal{D}(A^*)$ with the norm $\|z\|_1^d = \|(\overline{\beta} I - A^*)z\|$, then X_{-1} may be regarded as the dual of X_1^d.

Proposition 2.1 *Let H be a Hilbert space and let $A_0 : \mathcal{D}(A_0) \to H$ be strictly positive. Denote $H_{\frac{1}{2}} = \mathcal{D}(A_0^{\frac{1}{2}})$ with the graph norm. $H_{-\frac{1}{2}}$ is the dual of $H_{\frac{1}{2}}$ with respect to the pivot space H. We define another Hilbert space $X = H_{\frac{1}{2}} \times H$ with the inner product*

$$\left\langle \begin{bmatrix} w_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} w_2 \\ v_2 \end{bmatrix} \right\rangle_X = \langle A_0^{\frac{1}{2}} w_1, A_0^{\frac{1}{2}} w_2 \rangle + \langle v_1, v_2 \rangle,$$

and another operator A by

$$A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}, \quad \mathcal{D}(A) = \mathcal{D}(A_0) \times \mathcal{D}(A_0^{\frac{1}{2}}).$$

Then A is skew-adjoint on X and $0 \in \rho(A)$. Furthermore

$$X_1 = H_1 \times H_{\frac{1}{2}}, \quad X_{-1} = H \times H_{-\frac{1}{2}}.$$

Proposition 2.2 *Using the notation in Proposition 2.1, $\phi = \left[\begin{smallmatrix} \varphi \\ \psi \end{smallmatrix}\right] \in \mathcal{D}(A)$ is an eigenvector of A, corresponding to the eigenvalue $i\mu$ (where $\mu \in \mathbb{R}$), if and only if φ is an eigenvector of A_0, corresponding to the eigenvalue μ^2 and $\psi = i\mu\varphi$.*

We denote by $\mathbb{Z}^*$ the set of all the non-zero integers. Assume that A_0 is diagonalisable, with an orthonormal basis $(\varphi_k)_{k\in\mathbb{N}}$ in H formed of eigenvectors of A_0. Denote by $\lambda_k > 0$ the eigenvalue corresponding to φ_k and $\mu_k = \sqrt{\lambda_k}$. For all $k \in \mathbb{N}$ we define $\varphi_{-k} = -\varphi_k$ and $\mu_{-k} = -\mu_k$. Then A is diagonalisable, with the eigenvalues

$i\mu_k$ corresponding to the orthonormal basis of eigenvectors

$$\phi_k = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{i\mu_k}\varphi_k \\ \varphi_k \end{bmatrix} \qquad \forall\, k \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}.$$

The following result is a version of [16, Corollary 6.9.6].

Proposition 2.3 *Let X, U be Hilbert spaces. Let A be an skew-adjoint operator with compact resolvents, which hence generates a unitary group $\mathbb{T}$ on X. Let Λ be a subset of $\mathbb{Z}$. Assume that A has simple eigenvalues $i\mu_k$ $(k \in \Lambda)$, $\mu_k \in \mathbb{R}$, which are ordered such that the sequence (μ_k) is strictly increasing. Let $(\phi_k)(k \in \Lambda)$ be an orthonormal basis in X formed of eigenvectors of A corresponding to $(i\mu_k)$.*

Let $B \in \mathcal{L}(U, X_{-1})$ and define $b_k \in \mathcal{L}(U, \mathbb{C})$ by

$$b_k \mathrm{v} = \langle B\mathrm{v}, \phi_k \rangle_{X_{-1}, X_1^d} \qquad \forall\, \mathrm{v} \in U,\ k \in \Lambda.$$

If there exists $\delta > 0$, $m > 0$ such that $\mu_{k+1} - \mu_k \geq \delta$ and $\|b_k\| \leq m$ for all $k \in \Lambda$, then B is an admissible control operator for $\mathbb{T}$.

If moreover (μ_k) satisfies that $\mu_{k+1} - \mu_k \to \infty$ and for some $\varepsilon > 0$,

$$\|b_k\| \geq \varepsilon \qquad \forall\, k \in \Lambda,$$

then (A, B) is exactly controllable in any time $\tau > 0$.

If (ϕ_k) $(k \in \Lambda$, Λ countable) is a Riesz basis in the Hilbert space X, we denote by $(\tilde{\phi}_k)$ $(k \in \Lambda)$ the biorthogonal sequence to (ϕ_k). Every $z \in X$ can be represented as $z = \sum_{k\in\Lambda} z_k \phi_k$, where $z_k = \langle z, \tilde{\phi}_k \rangle$ and $(z_k) \in l^2(\Lambda)$. Let $\mathbb{T}$ be a diagonalisable semigroup on X with generator A. This means that there exists a Riesz basis (ϕ_k) $(k \in \Lambda)$ in X such that

$$\mathbb{T}_t z = \sum_{k\in\Lambda} e^{\lambda_k t} z_k \phi_k. \tag{2.1}$$

The generator of $\mathbb{T}$ is given by

$$Az = \sum_{k\in\Lambda} \lambda_k z_k \phi_k, \quad \mathcal{D}(A) = \left\{ z \in X \;\middle|\; \sum_{k\in\Lambda} |\lambda_k z_k|^2 < \infty \right\}.$$

For $\alpha \geq 0$ we define

$$|A|^\alpha : \mathcal{D}(|A|^\alpha) \to X$$

by

$$|A|^\alpha z = \sum_{k\in\Lambda} |\lambda_k|^\alpha z_k \phi_k, \quad \mathcal{D}(|A|^\alpha) = X_\alpha = \left\{ z \in X \;\middle|\; \sum_{k\in\Lambda} |\lambda_k|^{2\alpha} |z_k|^2 < \infty \right\}.$$

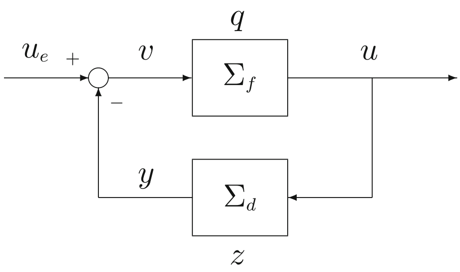

Fig. 1 A coupled system Σ_{cs} consisting of an infinite-dimensional system Σ_d and a finite-dimensional system Σ_f, connected in feedback

The space X_α is a Hilbert space with the norm

$$\|z\|_\alpha = \|(I + |A|)^\alpha z\|. \tag{2.2}$$

We define $X_{-\alpha}$ as the dual of X_α with respect to the pivot space X. Note that for $\alpha = 1$ we obtain X_1 and X_{-1} as defined earlier and $|A|^\alpha$ commutes with $\mathbb{T}_t$. It is clear that $\mathbb{T}$ can be extended (or restricted) to X_α for any $\alpha \in \mathbb{R}$. The formula (2.1) for $\mathbb{T}$ remains the same, with $(|\lambda_k|^\alpha z_k) \in l^2(\Lambda)$. The generator of $\mathbb{T}$ acting on X_α is an extension (or restriction) of A with $\mathcal{D}(A) = X_{\alpha+1}$ and $\mathcal{D}(A^2) = X_{\alpha+2}$.

Proposition 2.4 *Let $\mathbb{T}$ be a diagonalisable semigroup on the state space X, and let $B \in \mathcal{L}(U, X_{-1})$. Then B is admissible (or exactly controllable) for $\mathbb{T}$ on X if and only if $(I + |A|)^{-\alpha} B$ is admissible (or exactly controllable) for $\mathbb{T}$ on X_α ($\alpha \in \mathbb{R}$).*

Proof We denote by Φ_τ the input map of (A, B) at time τ:

$$\Phi_\tau u = \int_0^\tau \mathbb{T}_{\tau-\sigma} B u(\sigma) \mathrm{d}\sigma. \tag{2.3}$$

The proposition follows from the following factorization:

$$\Phi_\tau u = (I + |A|)^\alpha \int_0^\tau \mathbb{T}_{\tau-\sigma} (I + |A|)^{-\alpha} B u(\sigma) \mathrm{d}\sigma.$$

□

In the sequel we recall some results about coupled systems from our paper [18].

Consider a coupled system Σ_c, in which an infinite-dimensional system Σ_d is connected to a finite-dimensional system Σ_f as shown in Fig. 1. The external world interacts with the coupled system Σ_c via the finite-dimensional part Σ_f, which receives the input $v = u_e - y$, where u_e is the input of Σ_c and the signal y comes from Σ_d. The system Σ_f sends out the output u, which is also the output of the coupled system Σ_c. The equations of Σ_f are

$$\begin{cases} \dot{q}(t) = aq(t) + bu_e(t) - by(t), & (2.4) \\ u(t) = cq(t), & (2.5) \end{cases}$$

where $a \in \mathbb{C}^{n\times n}$, $b \in \mathbb{C}^{n\times m}$, $c \in \mathbb{C}^{m\times n}$, and $q(t) \in \mathbb{C}^n$ is the state of the finite-dimensional subsystem at the time t.

Let p be a function defined on some domain in $\mathbb{C}$ that contains a right half-plane, with values in a normed space. We say that p is *strictly proper* if

$$\lim_{\mathrm{Re}\, s\to\infty} \|p(s)\| = 0, \quad \text{uniformly with respect to } \mathrm{Im}\, s.$$

A linear system is called *strictly proper* if its transfer function is strictly proper.

We assume that Σ_d belongs to an abstract class of infinite-dimensional systems called *strictly proper with an integrator* (SPI) systems, introduced in [18]. SPI systems do not fit into any of the known classes of linear infinite-dimensional systems (they are not well-posed, they are not even a system node or a resolvent linear system). An informal way to characterize an SPI system Σ_d is to say that an integrator in cascade with Σ_d is a well-posed and strictly proper system.

Definition 2.5 An SPI system Σ_d with input space U, state space X and output space Y (all Hilbert spaces) is determined by three operators A, B, C and a transfer function $\mathbf{G}$, which satisfy the following assumptions:

(a) A is the generator of a strongly continuous semigroup $\mathbb{T}$ on X. The spaces X_1, X_2 and X_{-1} are as introduced at the beginning of this section.
(b) $B \in \mathcal{L}(U, X_{-1})$ is an admissible control operator for $\mathbb{T}$.
(c) $X_2 \subset \mathcal{D}(C) \subset X_1$ and $C : \mathcal{D}(C) \to Y$ is such that its restriction to $\mathcal{D}(A^2)$ is in $\mathcal{L}(X_2, Y)$ and it is an admissible observation operator for $\mathbb{T}$ restricted to X_1.
(d) For some (hence, for every) $s, \beta \in \rho(A)$ we have

$$(sI - A)^{-1}(\beta I - A)^{-1} BU \subset \mathcal{D}(C).$$

(e) We have $\mathbf{G} : \rho(A) \to \mathcal{L}(U, Y)$. For every $s, \beta \in \rho(A)$ we have

$$\mathbf{G}(s) - \mathbf{G}(\beta) = C[(sI - A)^{-1} - (\beta I - A)^{-1}]B.$$

(f) The function $\frac{1}{s}\mathbf{G}(s)$ is strictly proper.
The operators A, B, C are called the *semigroup generator*, the *control operator* and the *observation operator* of Σ_d. $\mathbf{G}$ is called the *transfer function* of Σ_d.

We make some simple comments on SPI systems. The dynamic behavior of Σ_d is assumed to be described similarly as for a system node (as defined in Staffans [13]):

$$\dot{z}(t) = Az(t) + Bu(t), \quad y(t) = C\&D \begin{bmatrix} z(t) \\ u(t) \end{bmatrix}. \tag{2.6}$$

Here $C\&D$ is defined similarly as for a system node: for some $\beta \in \rho(A)$,

$$C\&D \begin{bmatrix} x \\ u \end{bmatrix} = C[x - (\beta I - A)^{-1} Bu] + \mathbf{G}(\beta)u, \tag{2.7}$$

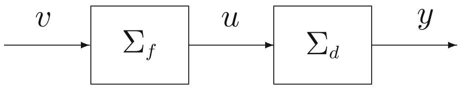

Fig. 2 A cascaded system Σ_{casc} consisting of an infinite-dimensional system Σ_d and a finite-dimensional system $\Sigma_f = (a, b, c)$

with the domain

$$\mathcal{D}(C\&D) = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in X \times U \;\middle|\; x - (\beta I - A)^{-1} B u \in \mathcal{D}(C) \right\}. \tag{2.8}$$

For a system node, we would have X_1 in place of $\mathcal{D}(C)$ in (2.8), so that for an SPI system, $\mathcal{D}(C\&D)$ is smaller. It is easy to see that $C\&D$ (and its domain) is independent of the choice of β appearing in the formulas. It is also easy to see that we have the following relation between $C\&D$ and $\mathbf{G}$:

$$\mathbf{G}(s) = C\&D \begin{bmatrix} (sI - A)^{-1} B \\ I \end{bmatrix} \qquad \forall\, s \in \rho(A). \tag{2.9}$$

Equations (2.6) have classical solutions if u is of class $\mathcal{H}^2_{\text{loc}}$ and the initial conditions of z and u are compatible. In this case, y is continuous. For the proof and for more details about SPI systems we refer to [18].

Now consider the situation when Σ_d is an SPI system with input and output space $\mathbb{C}^m$ and state space H^d, semigroup $\mathbb{T}$ and transfer function $\mathbf{G}$. We can consider the coupled system Σ_c as a cascaded system Σ_{casc} (the open loop system in Fig. 2) with a feedback. The input of Σ_{casc} is v from Fig. 1, and its outputs are u and y. The system Σ_{casc} is described by:

$$\begin{cases} \dot{q}(t) = a q(t) + b v(t), & (2.10) \\ u(t) = c q(t), & (2.11) \\ \dot{z}(t) = A z(t) + B u(t), & (2.12) \\ y(t) = C\&D \begin{bmatrix} z(t) \\ u(t) \end{bmatrix}. & (2.13) \end{cases}$$

Here $z(t)$ is the state of Σ_d, so that $z(t) \in H^d$.

It is easy to show that Eqs. (2.10)–(2.12) give rise to a strongly continuous semigroup $\mathcal{S}$ on the state space $H^d \times \mathbb{C}^n$, whose generator $\mathcal{A}$ is given by

$$\mathcal{A} = \begin{bmatrix} A & Bc \\ 0 & a \end{bmatrix}, \quad \mathcal{D}(\mathcal{A}) = \left\{ \begin{bmatrix} z \\ q \end{bmatrix} \in H^d \times \mathbb{C}^n \;\middle|\; Az + Bcq \in H^d \right\}. \tag{2.14}$$

In fact, $H^d \times \mathbb{C}^n$ is not a good choice for the state space of Σ_{casc}, because it is too large and the system may not be well-posed with this state space. However, we have shown in [18] that Σ_{casc} (and also Σ_c) is a well-posed system with the state space $\mathcal{X} = \mathcal{D}(\mathcal{A})$, which is a Hilbert space with the graph norm of $\mathcal{A}$.

For the well-posedness and controllability properties of the coupled system Σ_c, we have the following theorem from [18]:

Theorem 2.6 *Let Σ_d be an SPI system described by (2.12)–(2.13), with input space $\mathbb{C}^m$, state space H^d, output space $\mathbb{C}^m$, semigroup generator A, control operator B, observation operator C and transfer function* **G**. *Let a, b, c be matrices as in (2.4)–(2.5). Then the coupled system Σ_c from Fig. 1 described by (2.4), (2.5), (2.12) and (2.13), with input u_e, state $\left[\begin{smallmatrix} z \\ q \end{smallmatrix}\right]$ and output u, is well-posed with the state space $\mathcal{X} = \mathcal{D}(\mathcal{A})$ from (2.14). The coupled system remains well-posed also with y as an additional output. Moreover, Σ_c is regular, with feed-through operator zero.*

The semigroup of Σ_c, denoted by $\mathcal{S}^c$, is generated by

$$\mathcal{A}^c \begin{bmatrix} z \\ q \end{bmatrix} = \begin{bmatrix} Az + Bcq \\ aq - b[C\&D]\left[\begin{smallmatrix} z \\ cq \end{smallmatrix}\right] \end{bmatrix}, \quad \mathcal{D}(\mathcal{A}^c) = \left\{ \begin{bmatrix} z \\ q \end{bmatrix} \in \mathcal{X} \;\middle|\; \mathcal{A}^c \begin{bmatrix} z \\ q \end{bmatrix} \in \mathcal{X} \right\}.$$

If $C \in \mathcal{L}(H_1^d, \mathbb{C}^m)$, then the operators $\mathcal{S}_t^c$ can be extended to form a strongly continuous semigroup on the space $H^d \times \mathbb{C}^n$. The generator of this extension, denoted by $\tilde{\mathcal{A}}^c$, is given by the same formula as $\mathcal{A}^c$ but it has the larger domain $\mathcal{D}(\tilde{\mathcal{A}}^c) = \mathcal{X}$.

Now assume additionally the following:

(i) *(A, B) is exactly controllable in time T_0*;
(ii) *(a, b) is controllable*;
(iii) *$cb \in \mathbb{C}^{m\times m}$ is invertible*;
(iv) *Denote $a^\times(\beta) = a + b(cb)^{-1}c(\beta I - a)$. There exists $\beta \in \rho(A)$ such that A^* and $a^\times(\beta)^*$ have no common eigenvalue.*

Then Σ_c is exactly controllable in any time $T > T_0$ (on the state space $\mathcal{X}$).

3 Some background on boundary control systems

This section is an introduction to boundary control systems, without any well-posedness assumptions. The general theory of such systems started with Fattorini [2] and it was significantly developed by Salamon [12].

In finite-dimensional linear linear systems theory, the equations of a system are usually given in terms of four matrices A, B, C, D in the form

$$\dot{z}(t) = Az(t) + Bu(t), \quad y(t) = Cz(t) + Du(t), \tag{3.1}$$

where u is the input function, z is the state trajectory and y is the output function. This formulation and the associated control theory extends easily to infinite-dimensional systems where A is a generator and B, C, D are bounded operators, see Curtain and Zwart [1]. It is desirable to describe also other linear infinite-dimensional systems in a form resembling (3.1).

Systems described by linear partial differential equations with non-homogeneous boundary conditions often appear in the following, quite different looking form:

$$\dot{z}(t) = Lz(t), \quad Gz(t) = u(t), \quad y(t) = Kz(t). \tag{3.2}$$

Often (but not necessarily) L is a differential operator and G is a boundary trace operator. We assume that U, Z, X and Y are complex Hilbert spaces such that

$$Z \subset X ,$$

with continuous embedding. We shall call U the *input space*, Z the *solution space*, X the *state space* and Y the *output space*.

Definition 3.1 A *boundary control system* on U, Z, X and Y is a triple of operators $\Sigma_b = (L, G, K)$, where

$$L \in \mathcal{L}(Z, X), \quad G \in \mathcal{L}(Z, U), \quad K \in \mathcal{L}(Z, Y),$$

if there exists a $\beta \in \mathbb{C}$ such that the following properties hold:

(i) G is onto,
(ii) Ker G is dense in X,
(iii) $\beta I - L$ restricted to Ker G is onto,
(iv) Ker $(\beta I - L) \cap$ Ker $G = \{0\}$.

We think of the three operators in this definition as determining a system via Eqs. (3.2). Broadly, our aim is to translate these equations into another form, which resembles (3.1). Our exposition follows the ideas of [12], but in a more concise form. Interesting recent papers on passive and conservative boundary control systems are Malinen and Staffans [9,10].

With the assumptions of the last definition, we introduce the Hilbert space X_1 and the operator A by

$$X_1 = \text{Ker } G, \quad A = L|_{X_1}. \tag{3.3}$$

Obviously, X_1 is a closed subspace of Z and $A \in \mathcal{L}(X_1, X)$. Condition (iii) means that $\beta I - A$ is onto. Condition (iv) means that Ker $(\beta I - A) = \{0\}$. Thus, (iii) and (iv) together imply that $\beta \in \rho(A)$, so that $(\beta I - A)^{-1} \in \mathcal{L}(X)$. In fact, $(\beta I - A)^{-1} \in \mathcal{L}(X, X_1)$, so that the norm on X_1 is equivalent to the norm

$$\|z\|_1 = \|(\beta I - A)z\|,$$

which in turn is equivalent to the graph norm of A. We define the Hilbert space X_{-1} as the completion of X with respect to the norm

$$\|z\|_{-1} = \|(\beta I - A)^{-1}z\|.$$

It is easy to see that this space is independent of the choice of $\beta \in \rho(A)$. This paragraph looks like the beginning of Sect. 2, but the context is different, since A is not assumed to be a generator.

Proposition 3.2 *Let $\Sigma_b = (L, G, K)$ be a boundary control system on U, Z, X and Y. Let A and X_{-1} be as introduced earlier. Then there exists a unique operator $B \in \mathcal{L}(U, X_{-1})$ such that*

$$L = A + BG, \tag{3.4}$$

where A is regarded as an operator from X to X_{-1}. For every $\beta \in \rho(A)$ we have $(\beta I - A)^{-1}B \in \mathcal{L}(U, Z)$ and

$$G(\beta I - A)^{-1}B = I, \tag{3.5}$$

so that in particular, B is bounded from below.

For the proof see Tucsnak and Weiss [16, Proposition 10.1.2].

Remark 3.3 The following fact is an easy consequence of Proposition 3.2: For every $v \in U$ and every $\beta \in \rho(A)$, the vector $z = (\beta I - A)^{-1}Bv$ is the unique solution of the "abstract elliptic problem"

$$Lz = \beta z, \quad Gz = v.$$

Remark 3.4 It follows from (3.5) that B is *strictly unbounded* with respect to X, meaning that $X \cap BU = \{0\}$. Another consequence of (3.5) is that we have

$$Z = X_1 + (\beta I - A)^{-1}BU.$$

Indeed, for each $z \in Z$, denoting $v = Gz$, we have $z = z_1 + (\beta I - A)^{-1}Bv$, where $z_1 \in X_1$ (because $Gz_1 = 0$). The converse inclusion is trivial.

Definition 3.5 With the notation of Definition 3.1 and Proposition 3.2, we define $C \in \mathcal{L}(X_1, Y)$ as the restriction of K to X_1. Then the *generating triple* of Σ_b is (A, B, C). The *transfer function* of Σ_b is the $\mathcal{L}(U, Y)$-valued function $\mathbf{G}$ defined on $\rho(A)$ by the formula

$$\mathbf{G}(s) = K(sI - A)^{-1}B. \tag{3.6}$$

By the resolvent identity, for any $s, \beta \in \rho(A)$, the difference $(sI - A)^{-1} - (\beta I - A)^{-1}$ maps X_{-1} into X_1, so that (3.6) implies

$$\mathbf{G}(s) - \mathbf{G}(\beta) = C\left[(sI - A)^{-1} - (\beta I - A)^{-1}\right]B. \tag{3.7}$$

It is now clear that if A is the generator of a strongly continuous semigroup on X, then A, B, C and $\mathbf{G}$ determine a system node in the sense of Staffans [13].

Remark 3.6 As a consequence of Proposition 3.2, the first two equations in (3.2) can be rewritten equivalently as a single equation, namely

$$\dot{z}(t) = Az(t) + Bu(t), \quad \text{with } \dot{z}(t) \in X. \tag{3.8}$$

Indeed, the transformation from (3.2) to (3.8) is obvious from (3.4). Conversely, if (3.8) holds, then applying $G(\beta I - A)^{-1}$ to both sides we obtain with (3.5) that $Gz(t) = u(t)$. Now from (3.4) it follows that $\dot{z}(t) = Lz(t)$.

4 The beam subsystem on the energy state space

To obtain the well-posedness and exact controllability results for the SCOLE model Σ_c described by (1.1) and (1.2), we follow the framework of Theorem 2.6. We decompose Σ_c into an infinite-dimensional system Σ_d (the clamped flexible beam) coupled with a finite-dimensional system Σ_f (the rigid body). We model and analyse the beam subsystem first.

The clamped flexible beam Σ_d that we extract from Σ_c is described by the following Euler–Bernoulli equation with boundary control and boundary observation:

$$\begin{cases} \rho w_{tt}(x,t) + EIw_{xxxx}(x,t) = 0, & (x,t) \in (0,l) \times [0,\infty), \\ w(0,t) = 0, \quad w_x(0,t) = 0, \\ w_t(l,t) = u_1(t), \quad w_{xt}(l,t) = u_2(t), \\ y_1(t) = -EIw_{xxx}(l,t), \quad y_2(t) = EIw_{xx}(l,t), \end{cases} \tag{4.1}$$

where $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is the input of Σ_d (the transverse velocity and angular velocity of the nacelle). $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is the output of Σ_d (the force and the torque at the top of the tower). The other notation is as in (1.1).

In order to reformulate the system Σ_d as a boundary control system like (3.2), we introduce two functions, which are the first two components of z^c in (1.3):

$$z_1(x,t) = w(x,t), \quad z_2(x,t) = w_t(x,t). \tag{4.2}$$

Then (4.1) can be written as:

$$\begin{cases} \dot{z}_1(x,t) = z_2(x,t), \\ \dot{z}_2(x,t) = -\frac{EI}{\rho} z_{1xxxx}(x,t), \\ z_1(0,t) = 0, \quad z_{1x}(0,t) = 0, \\ z_2(l,t) = u_1(t), \quad z_{2x}(l,t) = u_2(t), \\ y_1(t) = -EIz_{1xxx}(l,t), \quad y_2(t) = EIz_{1xx}(l,t). \end{cases} \tag{4.3}$$

We denote $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$, and similarly for u and y. The natural state space of Σ_d is

$$X = \mathcal{H}_l^2(0,l) \times L^2[0,l],$$

where $\mathcal{H}_l^2(0,l)$ is defined as after (1.3). We define the norm on X as follows:

$$\|z\|^2 = EI\|z_1\|^2_{\mathcal{H}_l^2} + \rho\|z_2\|^2_{L^2}, \tag{4.4}$$

where

$$\|z_1\|^2_{\mathcal{H}^2_l} = \int_0^l |z_{1xx}|^2 \mathrm{d}x, \quad \|z_2\|^2_{L^2} = \int_0^l |z_2|^2 \mathrm{d}x. \tag{4.5}$$

The physical energy in the system Σ_d is $\frac{1}{2}\|z\|^2$.

We introduce the space $Z \subset X$ by

$$Z = \left[\mathcal{H}^4(0,l) \cap \mathcal{H}^2_l(0,l)\right] \times \mathcal{H}^2_l(0,l). \tag{4.6}$$

We define the operators $L : Z \to X$, G, $K : Z \to \mathbb{C}^2$ by

$$L = \begin{bmatrix} 0 & I \\ -\frac{EI}{\rho}\frac{d^4}{dx^4} & 0 \end{bmatrix}, \quad G\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} z_2(l) \\ z_{2x}(l) \end{bmatrix}, \quad K\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -EIz_{1xxx}(l) \\ EIz_{1xx}(l) \end{bmatrix}.$$

With the above notation, (4.3) can be written as follows:

$$\dot{z} = Lz, \quad Gz = u, \quad y = Kz. \tag{4.7}$$

Such equations determine a boundary control system if L, G and K satisfy certain conditions, see Sect. 3. Now we prove that this is indeed the case. Before we do this, we introduce the system operator by $A = L|_{\mathrm{Ker}\ G}$. It is easy to verify that

$$\mathcal{D}(A) = \mathrm{Ker}\ G = \left[\mathcal{H}^4(0,l) \cap \mathcal{H}^2_l(0,l)\right] \times \mathcal{H}^2_0(0,l) \tag{4.8}$$

where $\mathcal{H}^2_0(0,l) = \{h \in \mathcal{H}^2(0,l) \mid h(0) = h(l) = 0,\ h_x(0) = h_x(l) = 0\}$. The norm on $\mathcal{H}^2_0(0,l)$ is defined by $\|f\|_{\mathcal{H}^2_0} = \|f''\|_{L^2}$.

Proposition 4.1 *The beam system (L, G, K) is a boundary control system.*

Proof It is clear that G is onto. The space Ker G is dense in X because $\mathcal{H}^4(0,l) \cap \mathcal{H}^2_l(0,l)$ is dense in $\mathcal{H}^2_l(0,l)$, and $\mathcal{H}^2_0(0,l)$ is dense in $L^2[0,l]$. The last two conditions in the definition of a boundary control system are equivalent to the fact that $sI - A$ is invertible for some $s \in \mathbb{C}$. We show that for every $s > 0$, $sI - A$ is invertible, or equivalently, for every $q \in X$, the following equation has a unique solution $z \in \mathcal{D}(A)$:

$$(sI - A)z = q.$$

The above equation is equivalent to

$$\begin{cases} \frac{EI}{\rho} z_{1xxxx} + s^2 z_1 = sq_1 + q_2, \\ z_1(0) = 0, \quad z_{1x}(0) = 0, \\ z_1(l) = \frac{1}{s}q_1(l), \quad z_{1x}(l) = \frac{1}{s}q_{1x}(l), \\ z_2 = sz_1 - q_1. \end{cases} \tag{4.9}$$

Remember that $s > 0$. First we show that the corresponding homogeneous equation, where we replace $sq_1 + q_2$ in the first equation of (4.9) with zero but leave the other equations unchanged, has a unique solution $z_h = \left[\begin{smallmatrix} z_{h1} \\ z_{h2} \end{smallmatrix}\right] \in \mathcal{D}(A)$. Solving this homogeneous equation, we get $z_{h2} = sz_{h1} - q_1$ and

$$z_{h1}(x) = c_1 \cosh mx \sin mx - c_1 \sinh mx \cos mx + c_2 \sinh mx \sin mx,$$

where

$$\begin{aligned} m &= \frac{s}{2}\sqrt{\frac{\rho}{EI}}, \quad c_1 = \frac{d \cdot q_{1x}(l) - bm \cdot q_1(l)}{(ad - bc)ms}, \quad c_2 = \frac{am \cdot q_1(l) - c \cdot q_{1x}(l)}{(ad - bc)ms}, \\ a &= 2 \sinh ml \sin ml, \qquad b = \sinh ml \cos ml + \cosh ml \sin ml, \\ c &= \cosh ml \sin ml - \sinh ml \cos ml, \qquad d = \sinh ml \sin ml. \end{aligned}$$

The above solution only makes sense if $ad - bc \neq 0$. Since $ad - bc = \sinh^2(ml) - \sin^2(ml)$ and $\sinh \alpha > |\sin \alpha|$ for any $\alpha > 0$, we obtain that $ad - bc > 0$, so that z_h exists and it is unique.

Similarly it can be shown that the non-homogeneous equation corresponding to (4.9), where we replace $q_1(l)$ and $q_{1x}(l)$ with zero, has a solution $z_n \in \mathcal{D}(A)$. Hence $z = z_h + z_n$ is a solution of (4.9). z is unique because if (4.9) had another solution $\tilde{z}$, then $z - \tilde{z}$ would be a solution of the homogeneous equation with zero boundary conditions (which is 0), hence $z - \tilde{z} = 0$. Therefore $sI - A$ is invertible for $s > 0$.

□

Remark 4.2 Using the techniques of Le Gorrec et al. [3] (in particular, their Theorem 4.4) it can be shown that (L, G, K) is an impedance conservative boundary control system. For this, we would have to use $\xi_1 = z_{1xx}$ and $\xi_2 = z_2$ as state functions. Here we prefer to give direct proofs of the facts that are needed, to avoid the formalism of [3].

Since $sI - A$ is invertible for $s > 0$, we can introduce the space X_{-1} as the completion of X with respect to the norm $\|x\|_{-1} = \|(sI - A)^{-1}x\|$. We can extend A to a bounded operator from X to X_{-1}, still denoted by A. We know from Sect. 3 that there exists a unique $B : \mathbb{C}^2 \to X_{-1}$ such that $L = A + BG$. According to Remark 3.6, the state trajectories of Σ_d from (4.1) or (4.7) satisfy (3.8).

We decompose the state space X into 2 parts: the null-space of A, X_n, and its orthogonal complement X_r. By a simple computation, we get

$$X_n = \text{Ker } A = \left\{ \begin{bmatrix} ax^3 + bx^2 \\ 0 \end{bmatrix} \middle| \, a, b \in \mathbb{C} \right\}. \tag{4.10}$$

Now we determine $X_r = X_n^{\perp}$. Let $z = \left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] \in X_r$, then $z_1 \in \mathcal{H}_l^2(0, l)$, $z_2 \in L^2[0, l]$. The condition $\langle q, z \rangle = 0$ for all $q \in X_n$ is equivalent to $\langle q_1, z_1 \rangle_{\mathcal{H}_l^2} = 0$ for all q_1 of the form

$$q_1(x) = ax^3 + bx^2, \quad \text{where } a, b \in \mathbb{C}. \tag{4.11}$$

For q_1 as above and for every $h \in \mathcal{H}_l^2(0,l)$ we have, using twice integration by parts,

$$\langle q_1, h\rangle_{\mathcal{H}_l^2} = q_{1xx}(l)\cdot \overline{h}_x(l) - \left[q_{1xxx}\cdot \overline{h}\right]_0^l + \int_0^l q_{1xxxx}\cdot \overline{h}\mathrm{d}x.$$

Using that $q_{1xxxx} = 0$ and $h(0) = 0$, we get

$$\langle q_1, h\rangle_{\mathcal{H}_l^2} = q_{1xx}(l)\cdot \overline{h}_x(l) - q_{1xxx}(l)\cdot \overline{h}(l).$$

Therefore we have for z_1 in place of h, $q_{1xx}(l)\cdot\overline{z}_{1x}(l) - q_{1xxx}(l)\cdot\overline{z}_1(l) = 0$. Clearly $q_{1xx}(l)$ and $q_{1xxx}(l)$ can be any complex numbers (in fact $q_{1xx}(l) = 6al + 2b$ and $q_{1xxx} = 6a$). Thus $\langle q_1, z_1\rangle = 0$ for all q_1 as in (4.11) is equivalent to

$$z_1(l) = 0, \quad z_{1x}(l) = 0.$$

Therefore $z_1 \in \mathcal{H}_0^2(0,l)$, where $\mathcal{H}_0^2(0,l)$ is defined as after (4.8). Thus we get

$$X_r = \mathcal{H}_0^2(0,l)\times L^2[0,l].$$

We denote by A_r the restriction of A to X_r. Then

$$\mathcal{D}(A_r) = \left[\mathcal{H}^4(0,l)\cap\mathcal{H}_0^2(0,l)\right]\times\mathcal{H}_0^2(0,l).$$

It is easy to see that X_r is invariant under A, or equivalently, $A_r z \in X_r$, $\forall z \in \mathcal{D}(A_r)$. We can decompose

$$A_r = \begin{bmatrix} 0 & I \\ -A_0 & 0\end{bmatrix}, \tag{4.12}$$

where

$$A_0 h = \frac{EI}{\rho} h_{xxxx}, \quad \mathcal{D}(A_0) = \mathcal{H}^4(0,l)\cap\mathcal{H}_0^2(0,l). \tag{4.13}$$

Note that A_r corresponds to the equations of a beam clamped at both ends.

Proposition 4.3 *A_0 is a strictly positive densely defined operator on $H = L^2[0,l]$, with compact resolvents. We have $\mathcal{D}(A_0^{\frac{1}{2}}) = \mathcal{H}_0^2(0,l)$.*

For a proof see, e.g., [16, Example 3.4.13]. This implies that $\sigma(A_0)$ consists of isolated positive eigenvalues, which converge to ∞. Moreover, there exists in H an orthonormal basis consisting of eigenvectors of A_0 (see, e.g., [16, Proposition 3.2.12]).

Proposition 4.4 *A_r is skew-adjoint on X_r and A is skew-adjoint on X.*

Proof As $A_0 > 0$, according to Proposition 2.1 A_r is skew-adjoint on X_r and $0 \in \rho(A_r)$. According to the decomposition $X = X_n \oplus X_r$ into $A-$invariant subspaces, it follows that A is skew-adjoint on X. □

As remarked after Definition 3.5, the above proposition (together with Proposition 4.1) implies that the beam system Σ_d is a system node with state space X. This system node is not well-posed (see the end of the next section).

We define $C = K|_{\text{Ker } G}$, so that $C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$, where

$$C_1 h = -EIh_{1xxx}(l), \quad C_2 h = EIh_{1xx}(l) \qquad \forall\, h = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \in \text{Ker } G.$$

Proposition 4.5 $B^* = C$.

Proof From the theory of boundary control systems in Sect. 3, we know that $A+BG = L$. Here $A : X \to X_{-1}$. Since G is onto, proving that $B^* = C$ is equivalent to proving that $A + C^*G = L$, which is equivalent to

$$\left\langle z, A^*\eta \right\rangle + \langle Gz, C\eta\rangle = \langle Lz, \eta\rangle \qquad \forall\, z \in Z,\ \eta \in \mathcal{D}(A^*).$$

Using $A^* = -A$ (see Proposition 4.4) and $\mathcal{D}(A^*) = \mathcal{D}(A)$, this becomes

$$-\langle z, A\eta\rangle + \langle Gz, C\eta\rangle = \langle Lz, \eta\rangle \qquad \forall\, z \in Z,\ \eta \in \mathcal{D}(A). \tag{4.14}$$

Now we prove (4.14). We denote by $left$ and $right$ the left-hand side and the right-hand side of (4.14), respectively. Then (using (4.4))

$$\begin{aligned} left &= -\left\langle \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \begin{bmatrix} \eta_2 \\ -\frac{EI}{\rho}\eta_{1xxxx} \end{bmatrix} \right\rangle_X + \left\langle \begin{bmatrix} z_2(l) \\ z_{2x}(l) \end{bmatrix}, \begin{bmatrix} -EI\eta_{1xxx}(l) \\ EI\eta_{1xx}(l) \end{bmatrix} \right\rangle \\ &= -EI\,\langle z_1, \eta_2\rangle_{\mathcal{H}_l^2} - \rho\left\langle z_2, -\frac{EI}{\rho}\eta_{1xxxx}\right\rangle_{L^2} - EIz_2(l)\overline{\eta_{1xxx}(l)} \\ &\quad + EIx_{2x}(l)\overline{\eta_{1xx}(l)}. \end{aligned}$$

Using twice integration by parts, we get

$$left = EI\int_0^l z_{2xx}\overline{\eta_{1xx}}\mathrm{d}x - EI\int_0^l z_{1xx}\overline{\eta_{2xx}}\mathrm{d}x. \tag{4.15}$$

Now we determine $right$:

$$right = \left\langle \begin{bmatrix} z_2 \\ -\frac{EI}{\rho} z_{1xxxx} \end{bmatrix}, \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \right\rangle = EI\,\langle z_2, \eta_1\rangle_{\mathcal{H}_l^2} + \rho\left\langle -\frac{EI}{\rho} z_{1xxxx}, \eta_2\right\rangle_{L^2}.$$

Using twice integration by parts, we get from here and (4.15) that $left = right$, so that $B^* = C$. □

In the sequel, for any $\alpha \in \mathbb{R}$, we denote by $\mathbb{C}_\alpha$ the right half-plane where $\operatorname{Re} s > \alpha$.

Remark 4.6 If $v, \omega \in \mathbb{C}$ and $s \in \mathbb{C}_0$, then the vector $\begin{bmatrix} g \\ f \end{bmatrix} = (sI - A)^{-1} B \begin{bmatrix} v \\ \omega \end{bmatrix}$ is the unique solution of the "abstract elliptic problem" from Remark 3.3:

$$(sI - L)\begin{bmatrix} g \\ f \end{bmatrix} = 0, \quad G\begin{bmatrix} g \\ f \end{bmatrix} = \begin{bmatrix} v \\ \omega \end{bmatrix}.$$

More explicitly, $f = sg$, while g is the unique solution of the differential equation

$$\frac{EI}{\rho} g_{xxxx} + s^2 g = 0,$$

with the boundary conditions $g(0) = 0$, $g_x(0) = 0$, $sg(l) = v$, $sg_x(l) = \omega$. According to our discussion after (4.9), the solution is of the form

$$g(x) = c_1 \cosh mx \sin mx - c_1 \sinh mx \cos mx + c_2 \sinh mx \sin mx,$$

where m, c_1 and c_2 (which are functions of s) are as in the formulas after (4.9), but with v, ω in place of $q_1(l), q_{1x}(l)$. From here, the transfer function $\mathbf{G}(s) = K(sI - A)^{-1} B$ of the beam system can be computed explicitly.

5 The diagonal representation of A and the corresponding infinite matrix form of B

Remember from Sect. 4 that the beam system is a system node with state space $X = \mathcal{H}_l^2(0, l) \times L^2[0, l]$, skew-adjoint semigroup generator A, control operator B and observation operator $C = B^*$. In this section we derive an asymptotic formula for an orthonormal basis (ϕ_k) in X formed of eigenvectors of A. Here $k \in \mathbb{M}$, an index set to be specified later. When representing A in this basis, it becomes an infinite diagonal matrix. The corresponding representation of B is an infinite matrix with two columns, with entries b_k^j ($j \in \{1, 2\}, k \in \mathbb{M}$), that we shall approximate.

Recall the operator $A_0 : \mathcal{D}(A_0) \to H,\ A_0 > 0$ from Proposition 4.3. The norm on $H = L^2[0, l]$ is defined by $\|f\|_H = \sqrt{\rho}\|f\|_{L^2}$, to make it fit with (4.4).

We order the eigenvalues of A_0 from (4.13) such that $0 < \lambda_1 < \lambda_2 < \lambda_3 \ldots$. We denote by φ_k a normalized (in H) eigenvector of A_0 corresponding to λ_k. We shall see below that φ_k is unique up to multiplication with a constant of absolute value one. Then it follows from what we said after Proposition 4.3 that (φ_k) $(k \in \mathbb{N})$ is an orthonormal basis in H. The functions φ_k satisfy

$$\begin{cases} \frac{EI}{\rho}\varphi_{kxxxx} = \lambda_k \varphi_k, \\ \varphi_k(0) = 0, \quad \varphi_{kx}(0) = 0, \\ \varphi_k(l) = 0, \quad \varphi_{kx}(l) = 0. \end{cases} \tag{5.1}$$

By solving (5.1), with the boundary conditions at 0 only, we see that

$$\varphi_k(x) = p_{1k}\,[\cos\alpha_k x - \cosh\alpha_k x] + p_{2k}\,[\sin\alpha_k x - \sinh\alpha_k x], \tag{5.2}$$

where

$$\alpha_k = \left(\frac{\rho\lambda_k}{EI}\right)^{\frac{1}{4}} \quad \forall k \in \mathbb{N}, \tag{5.3}$$

and p_{1k} and p_{2k} are arbitrary constants. The boundary conditions at l imply that a certain 2×2 determinant is zero, which reduces to

$$\cos\alpha_k l \cosh\alpha_k l = 1. \tag{5.4}$$

If this is the case, then

$$p_{1k} = M_k\,(\cos\alpha_k l - \cosh\alpha_k l)\,, \quad p_{2k} = M_k\,(\sin\alpha_k l + \sinh\alpha_k l), \tag{5.5}$$

so that φ_k is unique up to multiplication with a constant. We choose $M_k > 0$ such that $\|\varphi_k\|_H = 1$. Elementary considerations (looking at the graphs of the functions $\cos\alpha$ and $\frac{1}{\cosh\alpha}$ for $\alpha > 0$) show that the positive solutions of (5.4) are

$$\alpha_k = \left(k - \frac{1}{2}\right)\frac{\pi}{l} + (-1)^k\varepsilon_k \quad \left(k \in \mathbb{N}\right), \tag{5.6}$$

where $0 < \varepsilon_k < \frac{\pi}{2l}$ and

$$\varepsilon_k \approx \frac{2}{l}e^{-\left(k-\frac{1}{2}\right)\pi} \quad (\text{hence } \varepsilon_k \to 0).$$

By $\varepsilon_k \approx E_k$, where (E_k) is some sequence, we mean that $\lim \frac{\varepsilon_k}{E_k} = 1$.

Now we show that for all $k \in \mathbb{N}$, $0 < \varepsilon_k < \frac{\pi}{4l}$ (we need this later). We know that $\varepsilon_k > 0$. Let us show the second inequality. From (5.4) and (5.6) we know that

$$(-1)^{k+1}\cos\left(\frac{1}{2}\pi + (-1)^k\varepsilon_k l\right) = (\cosh\alpha_k l)^{-1}.$$

It is easy to see that the sequence $((\cosh\alpha_k l)^{-1})$ is positive, decreasing and converges to zero. This implies that $(\varepsilon_k l)$ is decreasing. Thus, to prove $\varepsilon_k < \frac{\pi}{4l}$, it is enough to show that $\varepsilon_1 < \frac{\pi}{4l}$, which follows from $\cos(\frac{\pi}{4}) > (\cosh\frac{\pi}{4})^{-1}$. Thus,

$$0 < \varepsilon_k < \frac{\pi}{4l} \qquad \forall\, k \in \mathbb{N}. \tag{5.7}$$

From (5.6) we get that for all $k \in \mathbb{N}$,

$$\lambda_k = \frac{EI}{\rho}\left(\left(k-\frac{1}{2}\right)\frac{\pi}{l} + (-1)^k \varepsilon_k\right)^4 = \frac{EI}{\rho}\left(k-\frac{1}{2}\right)^4\left(\frac{\pi}{l}\right)^4 + (-1)^k \delta_k, \tag{5.8}$$

where, using the symbol $\approx$ introduced a little earlier,

$$\delta_k \approx \frac{8EI}{\rho l^4}\left(k-\frac{1}{2}\right)^3 \pi^3 e^{-(k-\frac{1}{2})\pi}.$$

Rearranging (5.2) we have

$$\varphi_k(x) = p_{1k}\cos\alpha_k x + p_{2k}\sin\alpha_k x - \frac{1}{2}(p_{1k}+p_{2k})e^{\alpha_k x} + \frac{1}{2}(p_{2k}-p_{1k})e^{-\alpha_k x}. \tag{5.9}$$

It can be easily verified that for large $k \in \mathbb{N}$, $p_{1k} \approx -\frac{1}{2}M_k e^{\alpha_k l}$, $p_{2k} \approx \frac{1}{2}M_k e^{\alpha_k l}$, $p_{1k}+p_{2k} = M_k(\sin\alpha_k l + \cos\alpha_k l - e^{-\alpha_k l}) \approx -(-1)^k M_k$ and $p_{2k}-p_{1k} \approx M_k e^{\alpha_k l}$. Thus, we get that for every $x \in [0, l]$,

$$\varphi_k(x) \approx -\frac{1}{2}M_k e^{\alpha_k l}\left[\cos\alpha_k x - \sin\alpha_k x - e^{-\alpha_k x} - (-1)^k e^{-\alpha_k(l-x)}\right]. \tag{5.10}$$

From $\sqrt{\rho}\|\varphi_k(x)\| = 1$ we get

$$M_k e^{\alpha_k l} \approx \frac{2}{\sqrt{l\rho}}. \tag{5.11}$$

Now we determine an orthonormal basis (ϕ_k) in X_r formed of eigenvectors of A_r. We denote $\mu_k = \sqrt{\lambda_k}$, so that

$$\begin{aligned}\mu_k &= \sqrt{\frac{EI}{\rho}}\left(\left(k-\frac{1}{2}\right)\frac{\pi}{l} + (-1)^k\varepsilon_k\right)^2 \\ &= \sqrt{\frac{EI}{\rho}}\left(k-\frac{1}{2}\right)^2\left(\frac{\pi}{l}\right)^2 + (-1)^k\sigma_k \qquad \forall\, k \in \mathbb{N},\end{aligned} \tag{5.12}$$

where

$$\sigma_k \approx 4\sqrt{\frac{EI}{\rho}}\left(k-\frac{1}{2}\right)\frac{\pi}{l^2}e^{-(k-\frac{1}{2})\pi}.$$

Denote

$$\mu_{-k} = -\mu_k, \quad \varphi_{-k} = -\varphi_k \qquad \forall\, k \in \mathbb{N}. \tag{5.13}$$

Denote again by $\mathbb{Z}^*$ the set of all the non-zero integers. According to Proposition 2.2, A_r is diagonalisable, its eigenvalues are $i\mu_k$ ($k \in \mathbb{Z}^*$) and the corresponding orthonormal basis in X_r formed of eigenvectors of A_r is

$$\phi_k = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{i\mu_k}\varphi_k \\ \varphi_k \end{bmatrix} \qquad \forall\, k \in \mathbb{Z}^*. \tag{5.14}$$

Our next step is to determine the orthonormal basis in X_n, described in (4.10). It is clear that $\dim X_n = 2$. We use the index set $\{(0, 1),\ (0, 2)\}$ for an orthonormal basis in X_n. We choose the basis $\{\phi_{(0,1)}, \phi_{(0,2)}\}$, where

$$\phi_{(0,1)} = \begin{bmatrix} -\frac{1}{3l}\sqrt{\frac{3}{EI\,l}}\, x^3 + \frac{1}{2}\sqrt{\frac{3}{EI\,l}}\, x^2 \\ 0 \end{bmatrix}, \quad \phi_{(0,2)} = \begin{bmatrix} \frac{1}{2}\sqrt{\frac{1}{EI\,l}}\, x^2 \\ 0 \end{bmatrix}. \tag{5.15}$$

From our results so far, it is clear that A is diagonalisable. Let $\mu_k = 0$ for $k \in \{(0, 1), (0, 2)\}$. Let

$$\mathbb{M} = \mathbb{Z}^* \cup \{(0, 1),\ (0, 2)\}.$$

Then the set $\{\phi_k \mid k \in \mathbb{M}\}$ is an orthonomal basis in X formed of eigenvectors of A, with the corresponding eigenvalues $i\mu_k (k \in \mathbb{M})$.

Recall the duality between X_1^d and X_{-1}, mentioned at the beginning of Sect. 2. For our particular system we have $X_1^d = X_1$.

Proposition 5.1 *Decompose* $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$, *so that* $B_1, B_2 \in X_{-1}$. *Define*

$$b_k^1 = \langle B_1, \phi_k \rangle_{X_{-1}, X_1^d}, \quad b_k^2 = \langle B_2, \phi_k \rangle_{X_{-1}, X_1^d} \qquad \forall\, k \in \mathbb{M}.$$

Then for large $|k|$, $k \in \mathbb{Z}^*$ *we have*

$$b_k^1 \approx i(-1)^{k+1}\sqrt{\frac{2EI}{l}}\left(k - \frac{1}{2}\right)\frac{\pi}{l}, \quad b_k^2 \approx i(-1)^k\sqrt{\frac{2EI}{l}}, \tag{5.16}$$

and

$$b_k^1 \neq 0, \quad b_k^2 \neq 0 \qquad \forall\, k \in \mathbb{Z}^*. \tag{5.17}$$

Proof From (5.13) and (5.14), it is clear that the above proposition holds if it holds for $k > 0$. Thus in the sequel we consider $k > 0$. From Proposition 4.5,

$$b_k^1 = \langle B_1, \phi_k \rangle_{X_{-1}, X_1^d} = \overline{C_1 \phi_k} = -i \frac{EI}{\sqrt{2}\mu_k} \varphi_{kxxx}(l). \tag{5.18}$$

Now we compute $\varphi_{kxxx}(l)$, using (5.2) and (5.5):

$$\begin{aligned}\varphi_{kxxx}(l) &= p_{1k}\alpha_k^3(\sin\alpha_k l - \cosh\alpha_k l) + p_{2k}\alpha_k^3(-\cos\alpha_k l - \sinh\alpha_k l)\\ &= \alpha_k^3 M_k(\cos\alpha_k l - \cosh\alpha_k l)(\sin\alpha_k l - \cosh\alpha_k l)\\ &\quad -\alpha_k^3 M_k(\sin\alpha_k l + \sinh\alpha_k l)(\cos\alpha_k l + \sinh\alpha_k l)\\ &= \alpha_k^3 M_k\left(1 - (\cos\alpha_k l + \sin\alpha_k l)e^{\alpha_k l}\right).\end{aligned} \tag{5.19}$$

Remember that $\mu_k = \sqrt{\lambda_k}$. Combining this with (5.3), we have

$$\mu_k = \sqrt{\frac{EI}{\rho}}\alpha_k^2. \tag{5.20}$$

Substituting this and (5.19) into (5.18), we get

$$b_k^1 = -i\sqrt{\frac{EI\rho}{2}}\alpha_k M_k\left(1 - (\cos\alpha_k l + \sin\alpha_k l)e^{\alpha_k l}\right). \tag{5.21}$$

Now we show that $b_k^1 \neq 0$ for all $k \in \mathbb{N}$. It is clear from (5.21) that if $b_k^1 = 0$, then $\cos\alpha_k l + \sin\alpha_k l = e^{-\alpha_k l}$. Since $\alpha_k l > 0$, we get

$$0 < \cos\alpha_k l + \sin\alpha_k l < 1. \tag{5.22}$$

This implies $\cos\alpha_k l \sin\alpha_k l < 0$, which is true if and only if k is even. Then for any positive even number k, from (5.6) we have

$$\cos\alpha_k l + \sin\alpha_k l = \cos\left(\frac{3}{2}\pi + \varepsilon_k l\right) + \sin\left(\frac{3}{2}\pi + \varepsilon_k l\right).$$

From (5.7) we know that $0 \le \varepsilon_k l < \frac{\pi}{4}$. Thus we have

$$\frac{3}{2}\pi \le \left(\frac{3}{2}\pi + \varepsilon_k l\right) < \frac{3}{2}\pi + \frac{1}{4}\pi.$$

It is easy to verify (by looking at the derivative) that $f(\alpha) = \cos\alpha + \sin\alpha$ is an increasing function for $\frac{3}{2}\pi < \alpha < \frac{3}{2}\pi + \frac{1}{4}\pi$. This implies

$$-1 \le \cos\alpha_k l + \sin\alpha_k l < 0$$

for any even $k \in \mathbb{N}$, which contradicts (5.22). Therefore $b_k^1 \neq 0$ for all $k \in \mathbb{N}$.

Now we prove the first part of (5.16). For large k, from (5.21) and (5.6), we have

$$b_k^1 \approx i(-1)^{k+1}\sqrt{\frac{EI\rho}{2}}\left(k-\frac{1}{2}\right)\frac{\pi}{l}M_k e^{\alpha_k l}.$$

Using here (5.11), the desired approximation for b_k^1 follows.

Now we prove that $b_k^2 \neq 0$ for all $k \in \mathbb{N}$. We have

$$b_k^2 = \langle B_2, \phi_k\rangle_{X_{-1},X_1^d} = \overline{C_2\phi_k} = i\frac{EI}{\sqrt{2}\mu_k}\varphi_{kxx}(l). \tag{5.23}$$

We compute $\varphi_{kxx}(l)$ from (5.2) and (5.5):

$$\begin{aligned}\varphi_{kxx}(l) &= p_{1k}\alpha_k^2(-\cos\alpha_k l - \cosh\alpha_k l) + p_{2k}\alpha_k^2(-\sin\alpha_k l - \sinh\alpha_k l)\\ &= -\alpha_k^2 M_k(\cos\alpha_k l - \cosh\alpha_k l)(\cos\alpha_k l + \cosh\alpha_k l)\\ &\quad -\alpha_k^2 M_k(\sin\alpha_k l + \sinh\alpha_k l)^2\\ &= -2\alpha_k^2 M_k \sin\alpha_k l \sinh\alpha_k l.\end{aligned}$$

If we substitute this into (5.23), we get $b_k^2 = -i\frac{\sqrt{2}EI}{\mu_k}\alpha_k^2 M_k \sin\alpha_k l \sinh\alpha_k l$. Using (5.20), this becomes

$$b_k^2 = -i\sqrt{2\rho EI}M_k \sin\alpha_k l \sinh\alpha_k l. \tag{5.24}$$

Since $\alpha_k > 0$, we have $\sinh\alpha_k l > 0$. Thus, from (5.24) it follows that $b_k^2 = 0$ if and only if $\sin\alpha_k l = 0$. But this would imply $|\cos\alpha_k l| = 1$, whence, by (5.4), $|\cosh\alpha_k l| = 1$, which is impossible for $\alpha_k > 0$. Therefore $b_k^2 \neq 0$ for all $k \in \mathbb{N}$.

Finally, we prove the second part of (5.16). From (5.24) and (5.6) we see that

$$b_k^2 \approx i(-1)^k\frac{1}{2}\sqrt{2\rho EI}M_k e^{\alpha_k l}.$$

Substituting (5.11) into this formula, the desired approximation for b_k^2 follows. □

Remark 5.2 From Proposition 5.1 and the definition of X_α at (2.2), we see that $B : \mathbb{C}^2 \to X_{-\frac{3}{4}-\varepsilon}$ for every $\varepsilon > 0$. Using the duality result $C = B^*$ (Proposition 4.5) we see that $C \in \mathcal{L}(X_{\frac{3}{4}+\varepsilon}, \mathbb{C}^2)$ for every $\varepsilon > 0$.

6 The spaces $X_{\frac{1}{2}}$ and $X_{-\frac{1}{2}}$

Recall that we denote by $\mathcal{H}^m(0,l)$ and $\mathcal{H}_0^m(0,l)$ the standard Sobolev spaces over the interval $(0,l)$. In the sequel, we suppress the notation $(0,l)$.

We need three theorems about interpolation spaces. The following two theorems are taken from Lions and Magenes [6, p. 43, 64]:

Theorem 6.1 *Let* $s_1 > s_2$, $s_1 > 0$, $0 < \theta < 1$. *We have* (*with equivalent norms*)

$$[\mathcal{H}^{s_1}, \mathcal{H}^{s_2}]_\theta = \mathcal{H}^{(1-\theta)s_1+\theta s_2}.$$

Here, $[X, Y]_\theta$ denotes the θ-interpolation space of X and Y (see [6]).

Theorem 6.2 *Let* $s_1 > s_2 \geq 0$, s_1 *and* $s_2 \neq$ *integer* $+\frac{1}{2}$. *If* $(1-\theta)s_1 + \theta s_2 \neq$ *integer* $+\frac{1}{2}$, *then*

$$[\mathcal{H}_0^{s_1}, \mathcal{H}_0^{s_2}]_\theta = \mathcal{H}_0^{(1-\theta)s_1+\theta s_2}$$

(*with equivalent norms*).

Actually, in [6] the above results are given for a more general n-dimensional domain. The following theorem follows from a result in Triebel [14, p. 118].

Theorem 6.3 *Let* Z_a, Z_b *be Banach spaces such that* $\{Z_a, Z_b\}$ *is an interpolation couple. Let* V *be a complemented subspace of* $Z_a + Z_b$ *whose projection* P *restricted to* Z_a *is a bounded operator on* Z_a, *and similarly on* Z_b.

Then $\{Z_a \cap V, Z_b \cap V\}$ *is also an interpolation couple, and for every* $0 < \theta < 1$,

$$[Z_a \cap V, Z_b \cap V]_\theta = [Z_a, Z_b]_\theta \cap V.$$

Recall from (4.13) that $A_0 : \mathcal{D}(A_0) \to H$ is a strictly positive operator on $H = L^2[0, l]$. Denote $H_\alpha = \mathcal{D}(A_0^\alpha)$ $(\alpha \geq 0)$ with the graph norm. $H_{-\alpha}$ is the dual of H_α with respect to the pivot space H. From (4.12) and Propositions 4.3 and 2.1 we know that

$$\begin{aligned} H_1 &= \mathcal{H}^4 \cap \mathcal{H}_0^2, \quad H_{\frac{1}{2}} = \mathcal{H}_0^2, \quad X_r = H_{\frac{1}{2}} \times H, \\ (X_r)_1 &= \mathcal{D}(A_r) = \mathcal{D}(A_0) \times \mathcal{D}(A_0^{\frac{1}{2}}) = H_1 \times H_{\frac{1}{2}}, \\ (X_r)_{-1} &= H \times H_{-\frac{1}{2}}. \end{aligned}$$

According to Theorem 6.2 with $s_1 = 2$, $s_2 = 0$, $\theta = \frac{1}{2}$, we have

$$H_{\frac{1}{4}} = [H, H_{\frac{1}{2}}]_{\frac{1}{2}} = [L_2, \mathcal{H}_0^2]_{\frac{1}{2}} = \mathcal{H}_0^1. \tag{6.1}$$

Note that (by definition) the dual of $\mathcal{H}_0^1$ with respect to L^2 is $\mathcal{H}^{-1}$, i.e.

$$H_{-\frac{1}{4}} = \mathcal{H}^{-1}. \tag{6.2}$$

Recall from Sect. 4 that $X = \mathcal{H}_l^2 \times L^2 = X_n \oplus X_r$, where $\dim X_n = 2$, the spaces $X_\alpha = \mathcal{D}(|A|^\alpha)$ (for $\alpha > 0$) were introduced before (2.2), $X_{-\alpha}$ is the dual of X_α with respect to the pivot space X, and $\mathcal{H}_l^1$ is defined as at the end of Sect. 1.

Proposition 6.4 $X_{-\frac{1}{2}} = \mathcal{H}_l^1 \times \mathcal{H}^{-1}$.

Proof Let ϕ_k be the eigenvectors of A_r (see (5.14)), then

$$\begin{bmatrix} A_0^{\frac{1}{2}} & 0 \\ 0 & A_0^{\frac{1}{2}} \end{bmatrix} \phi_k = \lambda_k^{\frac{1}{2}} \phi_k = |\mu_k| \phi_k = |A_r| \phi_k \qquad \forall\, k \in \mathbb{Z}^*.$$

Thus we get

$$|A_r| = \begin{bmatrix} A_0^{\frac{1}{2}} & 0 \\ 0 & A_0^{\frac{1}{2}} \end{bmatrix}, \quad \text{hence} \quad |A_r|^{\frac{1}{2}} = \begin{bmatrix} A_0^{\frac{1}{4}} & 0 \\ 0 & A_0^{\frac{1}{4}} \end{bmatrix}.$$

Therefore

$$(X_r)_{\frac{1}{2}} = \left\{ z \in X_r \;\middle|\; |A_r|^{\frac{1}{2}} z \in X_r \right\} = H_{\frac{3}{4}} \times H_{\frac{1}{4}}. \tag{6.3}$$

By definition $(X_r)_{-\frac{1}{2}}$ is the dual of $(X_r)_{\frac{1}{2}} = H_{\frac{3}{4}} \times H_{\frac{1}{4}}$ with respect to $X_r = H_{\frac{1}{2}} \times H$. Combining this fact with Eqs. (6.1) and (6.2), we have

$$(X_r)_{-\frac{1}{2}} = H_{\frac{1}{4}} \times H_{-\frac{1}{4}} = \mathcal{H}_0^1 \times \mathcal{H}^{-1}.$$

Therefore we have

$$X_{-\frac{1}{2}} = X_n \oplus (X_r)_{-\frac{1}{2}} = \left\{ \begin{bmatrix} ax^3 + bx^2 \\ 0 \end{bmatrix} \;\middle|\; a, b \in \mathbb{C} \right\} \oplus \left(\mathcal{H}_0^1 \times \mathcal{H}^{-1} \right) = \mathcal{H}_l^1 \times \mathcal{H}^{-1}.$$

□

Proposition 6.5 $X_{\frac{1}{2}} = (\mathcal{H}^3 \cap \mathcal{H}_l^2) \times \mathcal{H}_0^1$.

Proof From Theorem 6.1 with $s_1 = 4$, $s_2 = 2$ and $\theta = \frac{1}{2}$ we know that

$$[\mathcal{H}^4, \mathcal{H}^2]_{\frac{1}{2}} = \mathcal{H}^3.$$

From this and Theorem 6.3 (with $Z_a = \mathcal{H}^4$, $Z_b = \mathcal{H}^2$ and $V = \mathcal{H}_0^2$), we get

$$[\mathcal{H}^4 \cap \mathcal{H}_0^2, \mathcal{H}_0^2]_{\frac{1}{2}} = [\mathcal{H}^4, \mathcal{H}^2]_{\frac{1}{2}} \cap \mathcal{H}_0^2 = \mathcal{H}^3 \cap \mathcal{H}_0^2.$$

Therefore

$$H_{\frac{3}{4}} = [H_1, H_{\frac{1}{2}}]_{\frac{1}{2}} = [\mathcal{H}^4 \cap \mathcal{H}_0^2, \mathcal{H}_0^2]_{\frac{1}{2}} = \mathcal{H}^3 \cap \mathcal{H}_0^2.$$

Substituting this and (6.1) into (6.3), we get

$$(X_r)_{\frac{1}{2}} = (\mathcal{H}^3 \cap \mathcal{H}_0^2) \times \mathcal{H}_0^1.$$

Therefore

$$X_{\frac{1}{2}} = X_n \oplus (X_r)_{\frac{1}{2}} = \left\{ \begin{bmatrix} ax^3 + bx^2 \\ 0 \end{bmatrix} \middle| \, a, b \in \mathbb{C} \right\} \oplus \left((\mathcal{H}^3 \cap \mathcal{H}_0^2) \times \mathcal{H}_0^1 \right).$$

A simple reasoning shows that by adding functions of the form $ax^3 + bx^2$ to $\mathcal{H}^3 \cap \mathcal{H}_0^2$, we get $\mathcal{H}^3 \cap \mathcal{H}_l^2$. From here, the proposition follows. □

Proposition 6.6 *Let $\mathbb{T}$ be the semigroup generated by A on X, as introduced in Sect.* 4. *If we extend $\mathbb{T}$ to $X_{-\frac{1}{2}}$, then its generator is an extension of A (still denoted by A) with $\mathcal{D}(A) = X_{\frac{1}{2}}$ and $\mathcal{D}(A^2) = X_{\frac{3}{2}}$.*

Indeed, this follows from what we said after (2.2).

7 The beam subsystem with state space $X_{-\frac{1}{2}}$

In Sect. 4 we have seen that the beam system Σ_d is a boundary control system with skew-adjoint generator (and hence also a system node) with the state space $X = \mathcal{H}_l^2 \times L^2$. In this section we show that if instead we take the larger space $H^d = X_{-\frac{1}{2}} = \mathcal{H}_l^1 \times \mathcal{H}^{-1}$ as the state space, then Σ_d is an SPI system and this system is exactly controllable in any positive time.

Recall from Sects. 4 and 6 that we have the decompositions

$$X = X_n \oplus X_r, \quad X_{-\frac{1}{2}} = X_n \oplus (X_r)_{-\frac{1}{2}},$$

where $X_n = \operatorname{Ker} A$. We denote by $\mathbb{T}$ the unitary semigroup generated by A on X, and also its extension to a unitary semigroup on $X_{-\frac{1}{2}}$. In Sect. 5 we have introduced an orthonormal basis in X, denoted $(\phi_k)_{k\in\mathbb{M}}$, formed of eigenvectors of A. The eigenvectors ϕ_k with $k \in \{(0, 1), (0, 2)\}$ span X_n, while those with $k \in \mathbb{Z}^*$ span X_r and also $(X_r)_{-\frac{1}{2}}$. Recall from Proposition 5.1 that b_k^1 $(k \in \mathbb{M})$ are the entries of the first column in the matrix representation of B, i.e., they represent the control operator B_1 for the velocity input u_1. Similarly, the entries b_k^2 $(k \in \mathbb{M})$ represent the control operator B_2 for the angular velocity input u_2. We have $b_k^1 = \langle B_1, \phi_k \rangle$ for $k \in \mathbb{M}$ (this is a duality pairing between X_{-1} and $X_1^d = X_1$), and similarly for b_k^2.

The numbers b_k^1, b_k^2 for $k \in \mathbb{Z}^*$ have been estimated in Proposition 5.1. If we compute b_k^1 and b_k^2 for $k \in \{(0, 1), (0, 2)\}$, as in the proof of Proposition 5.1, from (5.15) we get

$$b^1_{(0,1)} = \frac{2}{l}\sqrt{\frac{3EI}{l}}, \quad b^1_{(0,2)} = 0, \tag{7.1}$$

$$b^2_{(0,1)} = -\sqrt{\frac{3EI}{l}}, \quad b^2_{(0,2)} = \sqrt{\frac{EI}{l}}. \tag{7.2}$$

Proposition 7.1 *B is admissible for $\mathbb{T}$ on the state space $X_{-\frac{1}{2}}$ and (using this state space) and the pair (A, B) is exactly controllable in any time $T_0 > 0$.*

Proof Let $\mathcal{P}_1$ be the orthogonal projection from X onto X_n an let $\mathcal{P}_2 = I - \mathcal{P}_1$ be the projection onto X_r. These projections have bounded extensions to X_{-1}, where (as seen in Sect. 4) the range of B lies. The restriction of A to X_n is 0, obviously. It is easy to see from (7.1)–(7.2) that Ran $\mathcal{P}_1 B = X_n$, so that the two-dimensional system $(0, \mathcal{P}_1 B)$ is controllable (on the state space X_n).

Let $\mathbb{T}^r$ be the restriction of $\mathbb{T}$ to X_r, whose generator is A_r from (4.12). It is easy to see that the entries of $\mathcal{P}_2 B_1$ are b^1_k, for all $k \in \mathbb{Z}^*$. From the first part of (5.16) it is clear that $|b^1_k| \to \infty$ as $|k| \to \infty$, and this implies that $\mathcal{P}_2 B_1$ is not admissible for $\mathbb{T}^r$ on X_r. However, we show that

$$\mathcal{B} = (I + |A_r|)^{-\frac{1}{2}}\,\mathcal{P}_2 B_1$$

is admissible for $\mathbb{T}^r$ on X_r and $(A_r, \mathcal{B})$ is exactly controllable on X_r. Using the notation $\mu_k = \sqrt{\lambda_k}$ introduced at (5.12), the entries of $\mathcal{B}$ are

$$(1 + |\mu_k|)^{-\frac{1}{2}} b^1_k \approx i(-1)^{k+1} l^{\frac{1}{2}} (\rho EI)^{\frac{1}{4}} \quad (k \in \mathbb{Z}^*), \tag{7.3}$$

which is a bounded sequence without any subsequence converging to zero. From (5.17) in Proposition 5.1 we know that $(1 + |\mu_k|)^{-\frac{1}{2}} b^1_k \neq 0$ for all $k \in \mathbb{Z}^*$. It follows that there exists $\varepsilon > 0$ such that $(1 + |\mu_k|)^{-\frac{1}{2}} |b^1_k| \geq \varepsilon$ for all $k \in \mathbb{Z}^*$. It can be verified easily that $\mu_{k+1} - \mu_k \to \infty$ as $|k| \to \infty$. According to Proposition 2.3, $\mathcal{B}$ is admissible for $\mathbb{T}_r$ and $(A_r, \mathcal{B})$ is exactly controllable in any time $\tau > 0$.

By a similar argument, we can check that $(I + |A_r|)^{-\frac{1}{2}} \mathcal{P}_2 B_2$ is also admissible for $\mathbb{T}^r$ (but A_r with this control operator is not exactly controllable). Putting the two columns together, we obtain that $(I + |A_r|)^{-\frac{1}{2}} \mathcal{P}_2 B$ is admissible for $\mathbb{T}^r$ (on X_r) and $(A_r, (I + |A_r|)^{-\frac{1}{2}} \mathcal{P}_2 B)$ is exactly controllable in any time $\tau > 0$.

Putting the two orthogonal components of X together, we see that $(I + |A|)^{-\frac{1}{2}} B$ is admissible for $\mathbb{T}$ on the state space X. Moreover, according to the simultaneous exact controllability result from [15] (see also [16, Corollary 11.3.3]), $(A, (I + |A|)^{-\frac{1}{2}} B)$ is exactly controllable (on X) in any time $T_0 > 0$.

From Proposition 2.4 it follows that B is admissible for $\mathbb{T}$ on the state space $X_{-\frac{1}{2}}$ and (A, B) is exact controllable in any time $T_0 > 0$ (on $X_{-\frac{1}{2}}$). □

We can see from the above proof that by using only B_1, i.e., only velocity control, the beam system would be "almost" exactly controllable: its reachable space would be $X_{-\frac{1}{2}}$ except for a one-dimensional subspace of Ker A.

Remark 7.2 The beam system Σ_d (described by Eqs. (4.1)) with state space $H^d = X_{-\frac{1}{2}}$ satisfies assumptions (a)–(e) in the definition of an SPI system (Definition 2.5). (We have to use $X_{-\frac{1}{2}}$ and $X_{\frac{1}{2}}$ in place of what is called X and X_1 in Definition 2.5.) Indeed, assumption (a) follows from what we said after (2.2). Assumption (b) (and more) follows from Proposition 7.1. Assumption (c) follows from Remark 5.2. Assumption (d) holds because (using again Remark 5.2) we have $(sI - A)^{-1}(\beta I - A)^{-1}B\mathbb{C}^2 \subset X_{\frac{5}{4}-\varepsilon}$, for every $\varepsilon > 0$. Assumption (e) holds because Σ_d is a boundary control system with state space X (see Proposition 4.1) and hence (3.7) holds. The remainder of this section is devoted to proving that also assumption (f) (the strict properness) holds. This is far more difficult.

Let $\mathbf{G} = \begin{bmatrix} \mathbf{G}^{11} & \mathbf{G}^{12} \\ \mathbf{G}^{21} & \mathbf{G}^{22} \end{bmatrix}$ denote the transfer function of the beam (with inputs velocity and angular velocity, and with outputs force and torque). The most problematic (the largest at ∞) is the component $\mathbf{G}^{11}$. Using our estimates for b_k^1 in Proposition 5.1, we can check that $\mathbf{G}^{11}$ is not proper.

Proposition 7.3 *The function $s \mapsto \frac{1}{s}\mathbf{G}^{11}(s)$ is strictly proper.*

There are two possible approaches to proving this. One is to compute $\mathbf{G}^{11}(s)$ as a closed formula, as outlined in Remark 4.6, and then find estimates for it. The other approach is to use the diagonal representation of A and the infinite matrix representation of B, both derived in Sect. 5, get from here a partial fractions representation for $\mathbf{G}^{11}$ (this is an infinite series) and then use the dominated convergence theorem to derive estimates. We do not know which approach is simpler, but we shall work with the second approach. We need the following lemma:

Lemma 7.4 *Let (ω_k), (β_k), $(\hat{\omega}_k)$, $(\hat{\beta}_k)$ be sequences of real numbers such that $\beta_k \approx \hat{\beta}_k$, $\hat{\beta}_k \neq 0$, $\omega_k \to \infty$ and $\omega_k^2 - \hat{\omega}_k^2$ is bounded. Assume that the series*

$$\hat{p}(s) = \sum_{k=1}^{\infty} \left| \frac{\hat{\beta}_k}{s^2 + \hat{\omega}_k^2} \right|$$

is convergent for some (hence, for every) $s \in \mathbb{C}_0$. Then also the series

$$p(s) = \sum_{k=1}^{\infty} \left| \frac{\beta_k}{s^2 + \omega_k^2} \right|$$

is convergent for some (hence, for every) $s \in \mathbb{C}_0$. Moreover, there exists $M > 0$ such that

$$p(s) \leq M\hat{p}(s) \qquad \forall\, s \in \mathbb{C}_1.$$

In particular, if $\hat{p}$ is (strictly) proper, then so is p.

Proof It can be verified by elementary methods that

$$|s^2 + \gamma| \geq 1 \qquad \forall\, \gamma \geq 0,\ s \in \mathbb{C}_1. \tag{7.4}$$

We define r_k, ζ_k as follows:

$$r_k = \frac{\beta_k}{\hat{\beta}_k}, \quad \zeta_k = \omega_k^2 - \hat{\omega}_k^2.$$

Then we have

$$r_k \to 1, \quad (\zeta_k) \text{ is bounded.}$$

We define the following functions on $\mathbb{C}_1$ (for each $k \in \mathbb{N}$):

$$\rho_k(s) = \frac{s^2 + \hat{\omega}_k^2}{s^2 + \omega_k^2} = 1 - \frac{\zeta_k}{s^2 + \omega_k^2}.$$

According to (7.4) we have $|s^2 + \omega_k^2| \geq 1$ for all $s \in \mathbb{C}_1$. Therefore $|\rho_k(s)| \leq 1 + |\zeta_k|$, which shows that there exists $M > 0$ such that

$$|r_k \rho_k(s)| \leq M \qquad \forall\, k \in \mathbb{N},\ s \in \mathbb{C}_1.$$

Thus, for every $s \in \mathbb{C}_1$,

$$p(s) = \sum_{k=1}^{\infty} |r_k \rho_k(s)| \left| \frac{\hat{\beta}_k}{s^2 + \hat{\omega}_k^2} \right| \leq \sum_{k=1}^{\infty} M \left| \frac{\hat{\beta}_k}{s^2 + \hat{\omega}_k^2} \right| = M \hat{p}(s).$$

□

Proof of Proposition 7.3 The transfer function $\mathbf{G}^{11}$ corresponds to the triple (A, B_1, C_1), in the sense that it satisfies $\mathbf{G}^{11}(s) - \mathbf{G}^{11}(\beta) = (\beta - s) C_1 (sI - A)^{-1} (\beta I - A)^{-1} B_1$ for all $s, \beta \in \rho(A)$, or equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}s} \mathbf{G}^{11}(s) = -C_1 (sI - A)^{-2} B_1 \qquad \forall\, s \in \rho(A). \tag{7.5}$$

We represent A as a diagonal operator in the orthonormal basis $\{\phi_k | k \in \mathbb{M}\}$ (see Sect. 5), so that we have $i\mu_k$ on the diagonal (where $k \in \mathbb{M}$). Using the coefficients $c_k^1 = C_1 \phi_k$ and $b_k^1 = \langle B_1, \phi_k \rangle$, we can represent B_1 and C_1 as infinite matrices (B_1 is a column and C_1 is a row). Then we can write (7.5) as a series:

$$-\frac{\mathrm{d}}{\mathrm{d}s} \mathbf{G}^{11}(s) = \frac{|b_{01}^1|^2}{s^2} + \frac{|b_{02}^1|^2}{s^2} + \sum_{k \in \mathbb{Z}^*} \frac{|b_k^1|^2}{(s - i\mu_k)^2} \qquad \forall\, s \in \rho(A).$$

Using the estimates for μ_k and b_k^1 derived in Sect. 5, we see that this series is absolutely convergent. For each $k \in \mathbb{N}$, we group the terms corresponding to k and $-k$, using $b_{-k} = b_k$ and $\mu_{-k} = -\mu_k$, obtaining

$$-\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{G}^{11}(s) = \frac{|b_{01}^1|^2}{s^2} + \frac{|b_{02}^1|^2}{s^2} + \sum_{k\in\mathbb{N}} 2|b_k^1|^2 \frac{s^2 - \mu_k^2}{(s^2 + \mu_k^2)^2} \qquad \forall\, s \in \rho(A).$$

Integrating with respect to s, we obtain

$$\mathbf{G}^{11}(s) = \kappa + \frac{|b_{01}^1|^2}{s} + \frac{|b_{02}^1|^2}{s} + \sum_{k=1}^{\infty} \frac{2s|b_k^1|^2}{s^2 + \mu_k^2},$$

where κ is an unknown integration constant, and this series is again absolutely convergent for every $s \in \rho(A)$. Luckily, the value of κ will not be needed.

We denote

$$\tilde{\mu}_k^2 = \frac{\rho}{EI}\left(\frac{l}{\pi}\right)^4 \mu_k^2, \quad \tilde{p}^{11}(s) = \sum_{k=1}^{\infty} \frac{|b_k^1|^2}{s^2 + \tilde{\mu}_k^2}.$$

Clearly the strict properness of $\tilde{p}^{11}(s)$ is equivalent to the strict properness of $\frac{1}{s}\mathbf{G}^{11}(s)$. Using the fact that $\mu_k = \sqrt{\lambda_k}$ $(k \in \mathbb{N})$ and (5.8), we have

$$\tilde{\mu}_k^2 = \left(k - \frac{1}{2}\right)^4 + \tilde{\delta}_k \qquad \forall\, k \in \mathbb{N}, \tag{7.6}$$

where $\tilde{\delta}_k \to 0$. From Proposition 5.1, we know that

$$|b_k^1|^2 \approx \frac{2\pi^2 EI}{l^3}\left(k - \frac{1}{2}\right)^2.$$

Let

$$p^{11}(s) = \sum_{k=1}^{\infty} \frac{|b_k^1|^2}{|s^2 + \tilde{\mu}_k^2|}, \quad \hat{p}^{11}(s) = \sum_{k=1}^{\infty} \frac{(k - \frac{1}{2})^2}{|s^2 + (k - \frac{1}{2})^4|}. \tag{7.7}$$

It is clear that the strict properness of $\tilde{p}^{11}$ would follow from the strict properness of p^{11}. According to Lemma 7.4, the strict properness of p^{11} would follow from the strict properness of $\hat{p}^{11}$. Thus, to prove the proposition, it will suffice to show that $\hat{p}^{11}$ is strictly proper. We take $s = \eta + i\omega$ with $\eta > 0$ and $\omega \in \mathbb{R}$. Then

$$\hat{p}^{11}(s) = \sum_{k=1}^{\infty} \frac{(k - \frac{1}{2})^2}{\sqrt{(\eta^2 - \omega^2 + (k - \frac{1}{2})^4)^2 + 4\omega^2\eta^2}}. \tag{7.8}$$

It can be verified that, for each fixed ω and k, $f_1(\eta) = (\eta^2 - \omega^2 + (k - \frac{1}{2})^4)^2 + 4\omega^2\eta^2$ is an increasing function of $\eta > 0$ (by checking that its first derivative is strictly positive). This implies that, for each fixed ω, each term of the sum in (7.8) is a decreasing function of $\eta > 0$. Define

$$f(s, x) = \frac{x^2}{\sqrt{(\eta^2 - \omega^2 + x^4)^2 + 4\omega^2\eta^2}}, \quad x \in \left[\frac{1}{2}, \infty\right). \tag{7.9}$$

We study the behaviour of f. By computation,

$$\frac{\partial f}{\partial x}(s, x) = \frac{2x(\omega^2 + \eta^2 + x^4)(\omega^2 + \eta^2 - x^4)}{\left((\eta^2 - \omega^2 + x^4)^2 + 4\omega^2\eta^2\right)^{\frac{3}{2}}}.$$

If $x < (\omega^2 + \eta^2)^{\frac{1}{4}}$, we get $\frac{\partial f}{\partial x}(s, x) > 0$ (f is increasing), while if $x > (\omega^2 + \eta^2)^{\frac{1}{4}}$, we get $\frac{\partial f}{\partial x}(s, x) < 0$ (f is decreasing). Thus, for each fixed $s \in \mathbb{C}_0$, $f(s, x)$ attains its maximum at $x = (\omega^2 + \eta^2)^{\frac{1}{4}} = \sqrt{|s|}$, and this maximum is $f(\sqrt{|s|}) = \frac{1}{2\eta}$.

Let m be the integer part of $(\omega^2 + \eta^2)^{\frac{1}{4}} + \frac{1}{2}$. We have

$$\sum_{k=1}^{m-1} f\left(s, k - \frac{1}{2}\right) < \int_{\frac{1}{2}}^{m-\frac{1}{2}} f(s, x)\mathrm{d}x, \quad \sum_{k=m}^{m+1} f\left(s, k - \frac{1}{2}\right) < \frac{1}{\eta},$$

$$\sum_{k=m+2}^{\infty} f\left(s, k - \frac{1}{2}\right) < \int_{m+\frac{1}{2}}^{\infty} f(s, x)\mathrm{d}x.$$

It follows that

$$\sum_{k=1}^{\infty} \frac{(k - \frac{1}{2})^2}{\sqrt{(\eta^2 - \omega^2 + (k - \frac{1}{2})^4)^2 + 4\omega^2\eta^2}} < \int_{\frac{1}{2}}^{\infty} f(s, x)\mathrm{d}x + \frac{1}{\eta},$$

hence

$$\hat{p}^{11}(s) < \int_{\frac{1}{2}}^{\infty} f(s, x)\mathrm{d}x + \frac{1}{\eta}. \tag{7.10}$$

We have to consider three cases:

Case I Assume $|\omega| < 0.1\eta$. Then from (7.9) and (7.10), eliminating the last term in the denominator of f,

$$\hat{p}^{11}(s) < E_1(\eta) \quad \text{if } \omega < 0.1\eta, \tag{7.11}$$

where

$$E_1(\eta) = \int_{\frac{1}{2}}^{\infty} \frac{x^2}{0.99\eta^2 + x^4}\mathrm{d}x + \frac{1}{\eta}.$$

The functions

$$g(\eta, x) = \frac{x^2}{0.99\eta^2 + x^4}, \quad x \in \left[\frac{1}{2}, \infty\right)$$

are decreasing with respect to η and they are in $L^1[\frac{1}{2}, \infty)$. By the dominated convergence theorem

$$\lim_{\eta\to\infty} E_1(\eta) = \int_{\frac{1}{2}}^{\infty} \lim_{\eta\to\infty} \frac{x^2}{0.99\eta^2 + x^4}\mathrm{d}x = 0.$$

Case II Assume $0.1\eta \leq |\omega| \leq \sqrt{\eta^2 + 2}$. Then from (7.9) and (7.10), replacing the expression $\eta^2 - \omega^2 + x^4$ with zero when $x < 2$, and replacing the expression $\eta^2 - \omega^2$ in the denominator of f with -2 when $x \geq 2$, we have

$$\hat{p}^{11}(s) < \int_{\frac{1}{2}}^{2} \frac{x^2}{\sqrt{0.4\eta^4}}\mathrm{d}x + \int_{2}^{\infty} \frac{x^2}{\sqrt{(x^4 - 2)^2 + 0.4\eta^4}}\mathrm{d}x + \frac{1}{\eta}.$$

We denote

$$E_2(\eta) = \frac{1}{\eta^2\sqrt{0.4}} \int_{\frac{1}{2}}^{2} x^2\mathrm{d}x + \int_{2}^{\infty} \frac{x^2}{\sqrt{(x^4 - 2)^2 + 0.4\eta^4}}\mathrm{d}x + \frac{1}{\eta},$$

so that

$$\hat{p}^{11}(s) < E_2(\eta) \quad \text{if} \quad 0.1\eta \leq \omega \leq \sqrt{\eta^2 + 2}. \tag{7.12}$$

The functions that we integrate with respect to x from 2 to ∞ are decreasing with respect to η and they are in $L^1[2, \infty)$. By the dominated convergence theorem

$$\lim_{\eta\to\infty} E_2(\eta) = 0.$$

Case III Assume $|\omega| > \sqrt{\eta^2 + 2}$. Let $z = x^4$, so that $\mathrm{d}x = \frac{1}{4x^3}\mathrm{d}z$. By changing the integration variable in (7.10) and using (7.9) we get, denoting $\delta = \omega^2 - \eta^2$ (so that $\delta > 2$)

$$
\begin{aligned}
4\hat{p}^{11}(s) &< \int_{(\frac{1}{2})^4}^{\infty} \frac{\mathrm{d}z}{x\sqrt{(\eta^2-\omega^2+z)^2+4\omega^2\eta^2}} + \frac{4}{\eta} \\
&< \int_0^{\delta-1} \frac{\mathrm{d}z}{z^{\frac{1}{4}}\sqrt{(\delta-z)^2+4\eta^4}} + \frac{1}{2\eta^2}\int_{\delta-1}^{\delta+1}\frac{\mathrm{d}z}{z^{\frac{1}{4}}} + \int_{\delta+1}^{\infty}\frac{\mathrm{d}z}{z^{\frac{1}{4}}\sqrt{(z-\delta)^2+4\eta^4}} + \frac{4}{\eta} \\
&< \int_0^{\delta-1} \frac{\mathrm{d}z}{z^{\frac{1}{4}}\sqrt{(\delta-z)^2+4\eta^4}} + \frac{1}{\eta^2(\delta-1)^{\frac{1}{4}}} + \int_1^{\infty}\frac{\mathrm{d}y}{(y+\delta)^{\frac{1}{4}}\sqrt{y^2+4\eta^4}} + \frac{4}{\eta} \\
&< \int_0^{\delta-1} \frac{\mathrm{d}z}{z^{\frac{1}{4}}\sqrt{(\delta-z)^2+4\eta^4}} + \frac{1}{\eta^2} + \int_1^{\infty}\frac{\mathrm{d}y}{y^{\frac{1}{4}}\sqrt{y^2+4\eta^4}} + \frac{4}{\eta}. \qquad (7.13)
\end{aligned}
$$

We estimate the first integral above:

$$
\int_0^{\delta-1} \frac{\mathrm{d}z}{z^{\frac{1}{4}}\sqrt{(\delta-z)^2+4\eta^4}} < \int_0^{\frac{\delta}{2}} \frac{\mathrm{d}z}{z^{\frac{1}{4}}\sqrt{z^2+4\eta^4}} + \int_{\frac{\delta}{2}}^{\delta-1} \frac{\mathrm{d}z}{(\delta-z)^{\frac{1}{4}}\sqrt{(\delta-z)^2+4\eta^4}}.
$$

In the first integral, we have used that $\delta - z \geq z$ on the integration interval, while in the last integral, we have used that $z \geq \delta - z$ on the integration interval. Denoting $t = \delta - z$, we obtain

$$
\begin{aligned}
\int_0^{\delta-1} \frac{\mathrm{d}z}{z^{\frac{1}{4}}\sqrt{(\delta-z)^2+4\eta^4}} &< \int_0^{\frac{\delta}{2}} \frac{\mathrm{d}z}{z^{\frac{1}{4}}\sqrt{z^2+4\eta^4}} + \int_1^{\frac{\delta}{2}} \frac{\mathrm{d}t}{t^{\frac{1}{4}}\sqrt{t^2+4\eta^4}} \\
&< 2\int_0^{\infty} \frac{\mathrm{d}t}{t^{\frac{1}{4}}\sqrt{t^2+4\eta^4}}.
\end{aligned}
$$

Combining this estimate with (7.13), we obtain

$$
4\hat{p}^{11}(s) < 3\int_0^{\infty} \frac{\mathrm{d}t}{t^{\frac{1}{4}}\sqrt{t^2+4\eta^4}} + \frac{1}{\eta^2} + \frac{4}{\eta} = 4E_3(\eta).
$$

From the dominated convergence theorem we see that the above integral tends to zero as $\eta \to \infty$. Thus, we have

$$
\hat{p}^{11}(s) < E_3(\eta) \quad \text{if } \omega > \sqrt{\eta^2+2}, \qquad (7.14)
$$

where $\lim_{\eta\to\infty} E_3(\eta) = 0$.

Putting together the estimates (7.11), (7.12) and (7.14), we see that no matter what $\omega \in \mathbb{R}$ is, we have

$$\hat{p}^{11}(s) < \max\{E_1(\eta), E_2(\eta), E_3(\eta)\}.$$

We have seen earlier that each of the functions E_1, E_2 and E_3 tends to zero as $\eta \to \infty$. Therefore,

$$\lim_{\eta \to \infty} \hat{p}^{11}(s) = 0,$$

uniformly with respect to $\omega \in \mathbb{R}$, which implies that $\frac{1}{s}\mathbf{G}^{11}(s)$ is strictly proper. □

Proposition 7.5 *The function* $s \mapsto \frac{1}{s}\mathbf{G}(s)$ *is strictly proper.*

Proof Recall that $\mathbf{G} = \begin{bmatrix} \mathbf{G}^{11} & \mathbf{G}^{12} \\ \mathbf{G}^{21} & \mathbf{G}^{22} \end{bmatrix}$. The fact that $\frac{1}{s}\mathbf{G}(s)$ is strictly proper is equivalent to the fact that its four components are strictly proper. We have shown in Proposition 7.3 that $\frac{1}{s}\mathbf{G}^{11}(s)$ is strictly proper. Now we show that $\frac{1}{s}\mathbf{G}^{12}(s)$ is strictly proper. Following similar steps as in the proof of Proposition 7.3, we have

$$\mathbf{G}^{12}(s) = \sum_{k \in \mathbb{M}} \frac{c_k^1 b_k^2}{s - i\mu_k}.$$

For each $k \in \mathbb{N}$, we group the terms corresponding to k and $-k$, using $b_{-k} = b_k$, $c_{-k} = c_k$, $c_k = \overline{b}_k$ and $\mu_{-k} = -\mu_k$, obtaining

$$\mathbf{G}^{12}(s) = \frac{\overline{b_{01}^1} b_{01}^2}{s} + \frac{\overline{b_{02}^1} b_{02}^2}{s} + \sum_{k=1}^{\infty} \frac{2s\overline{b_k^1} b_k^2}{s^2 + \mu_k^2}, \tag{7.15}$$

which is absolutely convergent for every $s \in \mathbb{C}_0$. Let

$$\tilde{p}^{12}(s) = \sum_{k=1}^{\infty} \frac{\overline{b_k^1} b_k^2}{s^2 + \tilde{\mu}_k^2}.$$

Recall from (7.6) that

$$\tilde{\mu}_k^2 = \frac{\rho}{EI}\left(\frac{l}{\pi}\right)^4 \mu_k^2 = \left(k - \frac{1}{2}\right)^4 + \tilde{\delta}_k \qquad \forall\, k \in \mathbb{N},$$

where $\tilde{\delta}_k \to 0$. Clearly the strict properness of $\tilde{p}^{12}(s)$ is equivalent to the strict properness of $\frac{1}{s}\mathbf{G}^{12}(s)$. From Proposition 5.1, we know that

$$\overline{b_k^1} b_k^2 \approx -\frac{2\pi EI}{l^2}\left(k - \frac{1}{2}\right).$$

Let

$$p^{12}(s) = \sum_{k=1}^{\infty} \frac{|\overline{b_k^1} b_k^2|}{|s^2 + \tilde{\mu}_k^2|}, \quad \hat{p}^{12}(s) = \sum_{k=1}^{\infty} \frac{(k - \frac{1}{2})}{|s^2 + (k - \frac{1}{2})^4|}. \tag{7.16}$$

The strict properness of $\tilde{p}^{12}$ would follow from that of p^{12}. According to Lemma 7.4, this would follow from the strict properness of $\hat{p}^{12}$. Clearly $\hat{p}^{12}(s) < \hat{p}^{11}(s)$ (see (7.7)), and we have shown that $\hat{p}^{11}$ is strictly proper. Thus, $\frac{1}{s}\mathbf{G}^{12}(s)$ is strictly proper.

From the expression of $\mathbf{G}^{21}(s)$ similar to (7.15), it is easy to see that $\mathbf{G}^{21}(s) = \mathbf{G}^{12}(s)$. Thus, $\frac{1}{s}\mathbf{G}^{21}(s)$ is also strictly proper. Following a similar reasoning, we can show that $\frac{1}{s}\mathbf{G}^{22}(s)$ is also strictly proper (in fact, $\mathbf{G}^{22}$ is proper). □

Proposition 7.6 *The beam subsystem is an SPI system with state space $H^d = X_{-\frac{1}{2}}$.*

Proof This follows from Remark 7.2 together with Proposition 7.5. □

8 Well-posedness, regularity and exact controllability of the SCOLE model

The rigid body system Σ_f that we extract from the SCOLE model Σ_c (see (1.1) and (1.2)) is described by the following Newton–Euler equations with control and observation:

$$\begin{cases} \dot{q}_1 = -\frac{1}{m} y_1 + \frac{1}{m} f, \\ \dot{q}_2 = -\frac{1}{J} y_2 + \frac{1}{J} v, \\ u_1 = q_1, \qquad u_2 = q_2. \end{cases} \tag{8.1}$$

For this system, the state is $q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} w_t(l,t) \\ w_{xt}(l,t) \end{bmatrix}$, which is the last two components of z^c in (1.3). The inputs are $f - y_1$ and $v - y_2$. $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is the output of both Σ_f and Σ_c. It is easy to see that this system is a particular case of the finite-dimensional subsystem in Theorem 2.6 with $a = 0$, $b = \begin{bmatrix} \frac{1}{m} & 0 \\ 0 & \frac{1}{J} \end{bmatrix}$ (using both torque and force control) and $c = I$. It is clear that (a, b) is controllable.

Theorem 8.1 *The SCOLE model Σ_c described by* (1.1) *and* (1.2) *is well-posed, regular, and exactly controllable in any time $T > 0$ with the state space*

$$\mathcal{X} = \left\{ \begin{bmatrix} z \\ q \end{bmatrix} \in \left[\mathcal{H}^3(0,l) \cap \mathcal{H}_l^2(0,l) \right] \times \mathcal{H}_l^1(0,l) \times \mathbb{C}^2 \;\middle|\; z_2(l) = q_1 \right\}$$

when using both torque and force control in L^2. It remains regular with $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ from (4.1) *as an additional output.*

Proof From Proposition 7.6 we know that the beam subsystem Σ_d is an SPI system with state space $H^d = X_{-\frac{1}{2}}$. From the descriptions of Σ_c, Σ_d and Σ_f, it is clear that they fit into the framework of Theorem 2.6. Therefore, by Theorem 2.6, Σ_c (with input u_e and output $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$) is well-posed and regular with the state space

$$\mathcal{X} = \mathcal{D}(\mathcal{A}) = \left\{ \begin{bmatrix} z \\ q \end{bmatrix} \in X_{-\frac{1}{2}} \times \mathbb{C}^2 \;\middle|\; Az + Bcq \in X_{-\frac{1}{2}} \right\},$$

where $\mathcal{A}$ is the generator of the cascaded system in the state space $X_{-\frac{1}{2}} \times \mathbb{C}^2$.

From Proposition 7.1 we also know that Σ_d is exactly controllable in any time $T_0 > 0$ with the state space $H^d = X_{-\frac{1}{2}}$ using both velocity and angular velocity control. Thus assumption (i) of Theorem 2.6 is satisfied. From the beginning of this section, we know that (a, b) is controllable, so that assumption (ii) of Theorem 2.6 is satisfied. Since $cb = \begin{bmatrix} \frac{1}{m} & 0 \\ 0 & \frac{1}{J} \end{bmatrix}$ is invertible, assumption (iii) is also satisfied. As

$$a^{\times}(\beta) = \beta I, \quad \beta \in \rho(A),$$

we know that $a^{\times}(\beta)^*$ and A^* have no common eigenvalues, which is assumption (iv). So far all the assumptions of Theorem 2.6 are satisfied. Thus the coupled system Σ_c is exactly controllable in any time $T > 0$ with the state space $\mathcal{X}$.

Now we determine $\mathcal{X}$. Recall that $c = I$. Take $z \in X_{-\frac{1}{2}}$ and $q \in \mathbb{C}^2$. The fact that $Az + Bq \in X_{-\frac{1}{2}}$ is equivalent to $(A - I)z + Bq \in X_{-\frac{1}{2}}$, which is equivalent to

$$z - (I - A)^{-1} Bq \in X_{\frac{1}{2}}.$$

Thus,

$$\mathcal{X} = \left\{ \begin{bmatrix} z \\ q \end{bmatrix} \in X_{-\frac{1}{2}} \times \mathbb{C}^2 \;\middle|\; z - (I - A)^{-1} Bq \in X_{\frac{1}{2}} \right\}. \tag{8.2}$$

Take $\begin{bmatrix} z \\ q \end{bmatrix} \in \mathcal{X}$. Let $\gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \in X_{\frac{1}{2}}$ be such that

$$z = \gamma + (I - A)^{-1} Bq.$$

Define $\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = (I - A)^{-1} B \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$. It is clear that $\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \in X$, which means that $g_1(0) = 0$ and $g_{1x}(0) = 0$. According to Remark 4.6, g_1, g_2 are the solution of

$$\begin{cases} g_1 - g_2 = 0, & (8.3) \\ g_{1xxxx} + g_2 = 0, & (8.4) \\ g_2(l) = q_1, \qquad g_{2x}(l) = q_2, & (8.5) \end{cases}$$

which is equivalent to $g_2 = g_1$ and $g_{1xxxx} + g_1 = 0$ subject to $g_1(l) = q_1, g_{1x}(l) = q_2$, $g_1(0) = 0$ and $g_{1x}(0) = 0$. Thus, g_1 is the solution of a fourth order ODE with four boundary conditions. It is easy to see that

$$g_1 = g_2 \in C^\infty[0, l] \subset \mathcal{H}^3.$$

Combining this fact, the boundary conditions of g_1 and Proposition 6.5, we get that

$$z \in \left[\mathcal{H}^3(0, l) \cap \mathcal{H}_l^2(0, l)\right] \times \mathcal{H}_l^1.$$

From Proposition 6.5 we know that $\gamma_2(l) = 0$. Hence, from equation (8.5) $z_2(l) = g_2(l) = q_1$. Thus, we have proved that

$$\mathcal{X} \subset \left\{ \begin{bmatrix} z \\ q \end{bmatrix} \in [\mathcal{H}^3(0, l) \cap \mathcal{H}_l^2(0, l)] \times \mathcal{H}_l^1(0, l) \times \mathbb{C}^2 \;\middle|\; z_2(l) = q_1 \right\}. \tag{8.6}$$

Now we prove the reversed inclusion. Take

$$\begin{bmatrix} z \\ q \end{bmatrix} \in \left\{ \begin{bmatrix} z \\ q \end{bmatrix} \in [\mathcal{H}^3(0, l) \cap \mathcal{H}_l^2(0, l)] \times \mathcal{H}_l^1(0, l) \times \mathbb{C}^2 \;\middle|\; z_2(l) = q_1 \right\}.$$

Consider $z - (I - A)^{-1} Bq = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} - \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$, where $\begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$ is the solution of (8.3)–(8.5). So $g_1 = g_2 \in C^\infty[0, l] \subset \mathcal{H}^3$. We also know that $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in [\mathcal{H}^3(0, l) \cap \mathcal{H}_l^2(0, l)] \times \mathcal{H}_l^1(0, l)$ and $z_2(l) = q_1$. Combining these facts with Eq. (8.5), we get

$$\left(z - (I - A)^{-1} Bq\right) \in [\mathcal{H}^3(0, l) \cap \mathcal{H}_l^2(0, l)] \times \mathcal{H}_0^1(0, l) = X_{\frac{1}{2}}.$$

We know that $[\mathcal{H}^3(0, l) \cap \mathcal{H}_l^2(0, l)] \times \mathcal{H}_l^1(0, l) \subset X_{-\frac{1}{2}}$. From (8.2) it is now clear that $\begin{bmatrix} z \\ q \end{bmatrix} \in \mathcal{X}$, i.e, the reversed inclusion of (8.6) holds. □

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