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ORIGINAL ARTICLE

Lyapunov functions for time-varying systems satisfying generalized conditions of Matrosov theorem

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Abstract The classical Matrosov theorem concludes uniform asymptotic stability of time-varying systems via a *weak Lyapunov function* (positive definite, decrescent, with *negative semi-definite derivative* along solutions) and another auxiliary function with derivative that is strictly nonzero where the derivative of the Lyapunov function is zero (Mastrosov in J Appl Math Mech 26:1337–1353, 1962). Recently, several generalizations of the classical Matrosov theorem have been reported in Loria et al. (IEEE Trans Autom Control 50:183–198, 2005). None of these results provides a construction of a *strong Lyapunov function* (positive definite, decrescent, with *negative definite derivative* along solutions) which is a very useful analysis and controller design tool for nonlinear systems. Inspired by generalized Matrosov conditions in Loria et al. (IEEE Trans Autom Control 50:183–198, 2005), we provide a construction of a strong Lyapunov function via an appropriate weak Lyapunov function and a set of Lyapunov-like functions whose derivatives along solutions of the system satisfy inequalities that have a particular triangular structure. Our results will be very useful in a range of situations where strong Lyapunov functions are needed, such as robustness analysis and Lyapunov function-based controller redesign. We illustrate our results by constructing a strong Lyapunov function for a simple Euler-Lagrange system controlled by an adaptive controller and use this result to determine an ISS controller.

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1 Introduction

Lyapunov's second method is ubiquitous in stability and robustness analysis of nonlinear systems. In recent years, its different versions were used for controller design, e.g. control Lyapunov functions, nonlinear damping, backstepping, forwarding, and so on [4,6,14,21,25]. Although it is often useful to obtain a *strong Lyapunov function* (positive definite, decrescent, with *negative definite derivative* along solutions) to analyze robustness or redesign the given controller, it is often the case that only a *weak Lyapunov function* (positive definite, decrescent, with *negative semi-definite derivative* along solutions) can be constructed for a problem at hand [1,2,5,8,9,18]. For example, controller design methods that are based on the passivity property typically require the use of the La Salle invariance principle [8] which exploits weak Lyapunov functions to conclude asymptotic stability.

The La Salle Theorem in its original form applies only to time-invariant systems. On the other hand, the Matrosov Theorem [15] concludes uniform asymptotic stability of time-varying systems via a *weak Lyapunov function* and another auxiliary function with derivative that is strictly nonzero where the derivative of the Lyapunov function is zero [15]. Different generalizations of the Matrosov theorem that use an arbitrary number of auxiliary functions to conclude uniform asymptotic stability have been recently reported in [11]. Moreover, results in [11] make use of the recently proposed notion of uniform δ persistency of excitation (uδ-PE condition) [13] that allows to further relax the original Matrosov conditions. The proofs presented in [11,15] do not provide a construction of a strong Lyapunov function and they conclude uniform asymptotic stability by considering directly the behavior of the trajectories of the system.

The main purpose of this paper is to construct strong Lyapunov functions using appropriate generalized Matrosov conditions that are inspired by main results in [11]. In particular, each of our results assumes existence of an appropriate weak Lyapunov function and a set of Lyapunov-like functions, similar to [11], to provide explicit formulas for constructing a strong Lyapunov function. Moreover, our results parallel the main results in [11] and we present a construction that exploits the uδ-PE condition. Constructions provided in this paper will be useful in a range of situations when the knowledge of a strong Lyapunov function is useful, such as robustness analysis and Lyapunov-based controller redesign. Observe, in particular, that an ISS Lyapunov characterization was obtained in [26] and that strong Lyapunov functions have been used to design stabilizing feedback laws that render asymptotically controllable systems ISS (as defined in [23]) to actuator errors and small observation noise (see [24]). Such control laws are expressed in terms of gradients of Lyapunov functions, and therefore require explicit strong Lyapunov functions to be implemented. We illustrate our main results by constructing a strong Lyapunov function for the pendulum equations controlled by an adaptive controller and, in a second step, by using this Lyapunov function to determine a feedback rendering the closed-loop system globally ISS with respect to an additive disturbance in the input. We note that our results can also be applied to a class of nonholonomic systems studied, for instance, in [19] and [11] and to a class of Euler-Lagrange systems (e.g. robotic manipulators) controlled by the well-known Li-Slotine model reference adaptive controller (see [10], [22]). Moreover, our results provide an alternative construction of a strong Lyapunov function to that presented in [17] for time-invariant systems, and a special case of our results also generalizes the constructions of strong Lyapunov functions given in [16] and [14].

The paper is organized as follows. In Sect. 2 we present mathematical preliminaries and assumptions that are needed in the sequel. Section 3 is devoted to the case where the assumptions of the classical Matrosov theorem are satisfied. Section 4 contains main results. An illustration of our main results is presented in Sect. 5, and the proofs of all main results are given in Sect. 6. Conclusions and some auxiliary results are given, respectively, in the last section and the appendix.

2 Preliminaries

Unless otherwise stated, we assume throughout the paper that the functions encountered are sufficiently smooth. We often omit arguments of functions to simplify notation. Throughout this paper, $|\cdot|$ stands for the Euclidean norm vectors and induced norm matrices. A continuous function $k : \mathbf{R}_{\geq 0} \to \mathbf{R}_{\geq 0}$ is said to be of class K if $k(0) = 0$ and k is increasing. It is said to belong to class K_∞ if it is unbounded. A function $\beta : \mathbf{R}_{\geq 0} \times \mathbf{R}_{\geq 0} \to \mathbf{R}_{\geq 0}$ is said to be of class KL if for each fixed s, the mapping $\beta(r, s)$ belongs to class K with respect to r, and for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\lim_{s\to+\infty} \beta(r, s) = 0$. A continuous function $V : \mathbf{R}^n \to \mathbf{R}$ is positive semi-definite if $V(0) = 0$ and $V(x) \geq 0$ for all $x \in \mathbf{R}^n$. It is positive definite if $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$. It is negative semi-definite (definite) if $-V$ is positive semi-definite (definite).

Consider the time-varying system:

$$\dot{x} = f(t, x) \tag{1}$$

with $t \in \mathbf{R}, x \in \mathbf{R}^n$ and assume that it is locally Lipschitz uniformly in t. For all $x_0 \in \mathbf{R}^n$ and $t_0 \in \mathbf{R}$, we will denote by $x(t; t_0, x_0)$, or simply by $x(t)$, the unique solution of (1) that satisfies $x(t_0; t_0, x_0) = x_0$. In order to simplify the notation, we use the following notation:

$$DV := \frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) f(t, x),$$

where $V : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$.

We need the definitions and assumptions given below. The following definition is a slightly modified version of [20, Definition 5.14].

Definition 1 A continuous function $\phi(t, x) : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^p$ is decrescent in norm if, there exists a function $\Upsilon(\cdot)$ of class K, such that for all $x \in \mathbf{R}^n$ and all $t \in \mathbf{R}$ the following holds

$$|\phi(t,x)| \le \Upsilon(|x|). \tag{2}$$

Definition 2 The system (1) is *uniformly globally asymptotically stable* provided there exists $\beta \in KL$ such that $|x(t;t_0,x_0)| \le \beta(|x_0|, t-t_0)$ for all $x_0 \in \mathbf{R}^n$, $t_0 \ge 0$, and $t \ge t_0$.

Definition 3 Suppose that there exist functions $V : \mathbf{R}\times\mathbf{R}^n \to \mathbf{R}$, $\alpha_1, \alpha_2, \alpha_4 \in K_\infty$ and $\alpha_3 : \mathbf{R}^n \to \mathbf{R}$ such that for all $x \in \mathbf{R}^n$ and all $t \in \mathbf{R}$, the following holds:

$$\alpha_1(|x|) \le V(t,x) \le \alpha_2(|x|), \tag{3}$$

$$DV \le -\alpha_3(x), \tag{4}$$

$$\left|\frac{\partial V}{\partial x}(t,x)\right| \le \alpha_4(|x|). \tag{5}$$

If the function α_3 is positive semi-definite, then we say that V is a *weak* Lyapunov function for the system (1). If, on the other hand, α_3 is positive definite, then V is referred to as a *strong* Lyapunov function for the system (1).

Assumption 1 *The function f in (1) is locally Lipschitz uniformly in t, $f(t,0)=0$ for all $t \in \mathbf{R}$, and a weak Lyapunov function V_1 for the system (1) is known. Besides two functions $\alpha_1, \alpha_2 \in K_\infty$ such that for all $x \in \mathbf{R}^n$ and all $t \in \mathbf{R}$,*

$$\alpha_1(|x|) \le V_1(t,x) \le \alpha_2(|x|), \tag{6}$$

are known.

Assumption 2 *The following functions are known: $V_i : \mathbf{R}\times\mathbf{R}^n \to \mathbf{R}$, $i = 2, 3, \ldots, j$, such that V_i and $\frac{\partial V_i}{\partial x}(t,x)$ are decrescent in norm; positive semi-definite functions $N_i : \mathbf{R}\times\mathbf{R}^n \to \mathbf{R}$ for $i = 2, \ldots, j$, decrescent in norm; a function M_b of class K_∞; continuous functions $\chi_i : \mathbf{R}\times\mathbf{R}^n\times\mathbf{R}^{i-2} \to \mathbf{R}$, for $i = 3, \ldots, j$, continuous and positive semi-definite functions $\chi_{*i} : \mathbf{R}^n \times \mathbf{R}^{i-2} \to \mathbf{R}$ for $i = 3, \ldots, j$, such that, for all $x \in \mathbf{R}^n$, $t \in \mathbf{R}$ and $r_2 \ge 0, \ldots, r_{i-1} \ge 0$,*

$$|\chi_i(t,x,r_2,\ldots,r_{i-1})| \le \chi_{*i}(x,r_2,\ldots,r_{i-1}) \tag{7}$$

and for all $x \in \mathbf{R}^n$,

$$\chi_{*i}(x,0,\ldots,0) = 0. \tag{8}$$

Moreover, for all $t \in \mathbf{R}$ and all $x \in \mathbf{R}^n$, we have:

$$\begin{aligned} DV_2 &\leq -N_2, \\ DV_3 &\leq -N_3 + \chi_3(t, x, N_2), \\ DV_4 &\leq -N_4 + \chi_4(t, x, N_2, N_3), \\ &\vdots \; \vdots \; \vdots \\ DV_j &\leq -N_j + \chi_j(t, x, N_2, \ldots, N_{j-1}) \end{aligned} \tag{9}$$

and

$$\sum_{i=2}^{j} N_i(t, x) + \sum_{i=1}^{j} |V_i(t, x)| \leq M_b(|x|). \tag{10}$$

Remark 1 Since Assumption 2 ensures that the functions N_i, V_i are decrescent in norm, the requirement (10) is not restrictive at all: when functions given by explicit expressions are decrescent in norm, it is in general very easy, from a technical point of view, to determine the explicit expression of such a function.

Remark 2 We note that it is often the case that V_1 in Assumption 1 is the same as the function V_2 in Assumption 2, that is $V_1(t, x) = V_2(t, x), \forall (t, x) \in \mathbf{R} \times \mathbf{R}^n$.

Remark 3 According to [20, Theorem 5.16], when the vector field of the system (1) is locally Lipschitz uniformly in t, satisfies $f(t, 0) = 0$ for all $t \in \mathbf{R}$ and admits a strong Lyapunov function, then it admits the origin as a uniformly globally asymptotically stable equilibrium point.

Remark 4 All our main results will be using Assumptions 1 and 2, as well as some other conditions. We note that Assumption 1 assumes existence of a weak Lyapunov function, whereas Assumption 2 assumes existence of a set of auxiliary functions. We note that these auxiliary functions do not have to be positive definite in general. Moreover, we note that the references [10–13] present a range of different situations where Assumptions 1 and 2 hold. Moreover, the functions V_i are constructed for the cases of model reference control [10,11], classical Matrosov theorem [12], a class of nonholonomic systems [11] and systems satisfying appropriate uniform observability conditions [11].

3 Basic result

The objective of this section is to familiarize the reader with the technique used thoughout our work. We explicitly construct a family of strong Lyapunov functions in the simple case where the system (1) satisfies the conditions of the classical Mastrosov theorem. In this specific case, Assumption 2 is satisfied with only two auxiliary functions, V_2, V_3 and, in addition, $V_1 = V_2$ (see Remark 2). This construction is the first construction of a strong Lyapunov function under the conditions of the Matrosov theorem. Due to its introductory interest, we give it in this section, instead of putting its proof in Sect. 6.

Theorem 3 *Consider the system* (1) *and suppose that Assumptions* 1 *and* 2 *hold with* $j = 3$ *and that* $V_1 = V_2$. *Suppose also that, for all* $t \in \mathbf{R}$ *and all* $x \in \mathbf{R}^n$, *we have:*

$$N_2(t, x) + N_3(t, x) \geq \omega(x) \tag{11}$$

where ω *is a known positive definite function. Then, one can determine two nonnegative functions* p_1, p_3 *such that the following function:*

$$W(t, x) = p_1(V_1(t, x))V_1(t, x) + p_3(V_1(t, x))V_3(t, x) \tag{12}$$

is a strong Lyapunov function for system (1).

Proof Let

$$S_a(t, x) = V_1(t, x) + V_3(t, x). \tag{13}$$

From Assumption 2, we deduce that

$$DS_a = DV_1 + DV_3 \leq -N_2 - N_3 + \chi_3(t, x, N_2). \tag{14}$$

Using the inequality (7) in Assumption 2 and Lemma 6, one can determine the explicit expressions of a function ϕ, of class K_∞ and of a positive and nondecreasing function ρ such that

$$|\chi_3(t, x, N_2)| \leq \phi(N_2)\rho(|x|). \tag{15}$$

This inequality and (11) yield

$$DS_a \leq -\omega(x) + \phi(N_2)\rho(|x|). \tag{16}$$

Let

$$S_b(t, x) = p_3(V_1(t, x))S_a(t, x) \tag{17}$$

where p_3 is a positive definite function to be specified later. A simple calculation yields

$$DS_b \leq -p_3(V_1)\omega(x) + p_3(V_1)\phi(N_2)\rho(x) + p_3'(V_1)S_a DV_1. \tag{18}$$

Let us distinguish between two cases:
First case: $N_2 \le p_3(V_1)$. Since ϕ is increasing, the inequality

$$p_3(V_1)\phi(N_2)\rho(|x|) \le p_3(V_1)\phi(p_3(V_1))\rho(|x|) \tag{19}$$

is satisfied.
Second case: $N_2 \ge p_3(V_1)$. Then the inequality

$$p_3(V_1)\phi(N_2)\rho(|x|) \le N_2\phi(N_2)\rho(|x|) \tag{20}$$

is satisfied. It follows that, for all $x \in \mathbf{R}^n$, $t \in \mathbf{R}$,

$$p_3(V_1)\phi(N_2)\rho(|x|) \le N_2\phi(N_2)\rho(|x|) + p_3(V_1)\phi(p_3(V_1))\rho(|x|). \tag{21}$$

From Lemma 3, we deduce that one can construct a positive definite function p_3 such that

$$\begin{aligned} p_3(V_1) &\le \inf\left\{\phi^{-1}\left(\tfrac{\omega(x)}{2\rho(|x|)}\right), \tfrac{\alpha_1(|x|)}{2M_b(|x|)+1}\right\}, \\ |p_3'(V_1)| &\le \tfrac{1}{2(M_b(|x|)+1)}, \end{aligned} \tag{22}$$

where α_1 is the function provided by Assumption 1 and M_b is the function satisfying (10). For such a choice, it follows from (21) that the inequality

$$p_3(V_1)\phi(N_2)\rho(|x|) \le N_2\phi(N_2)\rho(|x|) + \frac{1}{2}p_3(V_1)\omega(x) \tag{23}$$

is satisfied. Combining (18) and (23), we obtain

$$DS_b \le -\frac{1}{2}p_3(V_1)\omega(x) + N_2\phi(N_2)\rho(|x|) + p_3'(V_1)S_a DV_1. \tag{24}$$

From (22) and (10), we deduce that

$$DS_b \le -\frac{1}{2}p_3(V_1)\omega(x) + N_2\phi(M_b(|x|)+1)\rho(|x|) + |DV_1|. \tag{25}$$

Since $DV_1 = DV_2 \le -N_2$, we obtain

$$DS_b \le -\frac{1}{2}p_3(V_1)\omega(x) + [\phi(M_b(|x|)+1)\rho(|x|)+1]|DV_1|. \tag{26}$$

Thanks to Lemma 4 one can determine a function Γ_1, positive and nondecreasing, such that

$$\phi(M_b(|x|)+1)\rho(|x|) + 1 \le \Gamma_1(|x|) \tag{27}$$

which, in combination with (6), implies that

$$DS_b \leq -\frac{1}{2}p_3(V_1)\omega(x) + \Gamma_1(\alpha_1^{-1}(V_1))|DV_1|. \tag{28}$$

Using again Lemma 4, one can determine a function Γ_2, positive and nondecreasing, of class C^N where N is a positive integer, such that, for all $r \geq 0$,

$$\max\left\{2, \Gamma_1(\alpha_1^{-1}(r))\right\} \leq \Gamma_2(r). \tag{29}$$

Hence, we obtain the inequality

$$DS_b \leq -\frac{1}{2}p_3(V_1)\omega(x) + \Gamma_2(V_1)|DV_1|. \tag{30}$$

Now, observe that the expression of the function W given in (12) with $p_1(r) = \frac{1}{r}\int_0^r \Gamma_2(l)\mathrm{d}l + p_3(r)$ and p_3 satisfying (22) is

$$\begin{aligned} W(t,x) &= \left[\frac{1}{V_1(t,x)}\int_0^{V_1(t,x)} \Gamma_2(l)\mathrm{d}l + p_3(V_1(t,x))\right]V_1(t,x) \\ &\quad + p_3(V_1(t,x))V_3(t,x) \\ &= \int_0^{V_1(t,x)} \Gamma_2(l)\mathrm{d}l + p_3(V_1(t,x))V_1(t,x) + p_3(V_1(t,x))V_3(t,x) \\ &= \int_0^{V_1(t,x)} \Gamma_2(l)\mathrm{d}l + S_b(t,x) \end{aligned} \tag{31}$$

and therefore (30) implies that

$$DW \leq -\frac{1}{2}p_3(V_1)\omega(x). \tag{32}$$

We deduce from (6) that there exists a positive definite function γ_3 such that $\frac{1}{2}p_3(V_1)\omega(x) \geq \gamma_3(x)$. Therefore the requirement (4) is satisfied by W. Besides, W satisfies, for all $t \in \mathbf{R}$ and all $x \in \mathbf{R}^n$,

$$W(t,x) \geq \Gamma_2(0)V_1(t,x) + p_3(V_1(t,x))V_3(t,x) \tag{33}$$

Using (10) and (6), we obtain

$$W(t,x) \geq \Gamma_2(0)\alpha_1(|x|) - p_3(V_1(t,x))M_b(|x|). \tag{34}$$

From (22) and (29), we deduce that

$$W(t, x) \geq \alpha_1(|x|). \tag{35}$$

Moreover the functions $\frac{\partial V_1}{\partial x}(t, x)$, $\frac{\partial V_3}{\partial x}(t, x)$ and W are decrescent in norm. Therefore W also satisfies the requirement (3) and (5). It follows that W is a strong Lyapunov function for system (1).

4 Main results

In this section, we establish main results of this paper that are summarized in Theorems 4, 5, and 6. Each of these results provides a construction of a strong Lyapunov function using an existing weak Lyapunov function from Assumption 1, a set of Lyapunov-like functions from Assumption 2 and other appropriate conditions.

The first result of this section is an extention of Theorem 3 to the case where, instead of only one auxiliary function, several auxiliary functions are available.

Theorem 4 *Consider the system* (1) *and suppose that Assumptions* 1 *and* 2 *hold and that, for all* $x \in \mathbf{R}^n$ *and* $t \in \mathbf{R}$,

$$\sum_{i=2}^{j} N_i(t, x) \geq \omega(x) \tag{36}$$

where $\omega(x)$ *is a positive definite function. Then, one can determine nonnegative functions* p_i *such that the following function:*

$$W(t, x) = \sum_{i=1}^{j} p_i(V_1(t, x))V_i(t, x) \tag{37}$$

is a strong Lyapunov function for system (1).

Remark 5 We note that a construction of the functions p_i in (37) is provided in the proof of Theorem 4. Moreover, we emphasize that there is some flexibility in terms of choosing functions p_i in (37). This flexibility can be seen from the proof of Theorem 4. As illustrated by the example studied in Sect. 5, this flexibility can be frequently used to simplify the design of the functions p_i. The same comment applies to all the results of this section.

To state the second main result, we will suppose that the system (1) admits the decomposition:

$$\dot{x}_1 = f_1(t, x), \quad \dot{x}_2 = f_2(t, x) \tag{38}$$

with $x_1 \in \mathbf{R}^{n_1}$, $x_2 \in \mathbf{R}^{n_2}$, $n_1 + n_2 = n$. Note that we allow for the cases when either $n_1 = n$ or $n_2 = n$ that correspond to $x_1 = x$ and $x_2 = x$, respectively.

Theorem 5 *Consider the system* (38) *and suppose that Assumptions* 1 *and* 2 *hold. Suppose also that the following conditions hold:*

C1. *There exist a positive definite real-valued function* ω*, and a positive semi-definite continuously differentiable function* $M : \mathbf{R} \times \mathbf{R}^{n_2} \to \mathbf{R}$ *such that* $M(t, x_2)$ *and* $\frac{\partial M}{\partial x_2}(t, x_2)$ *are decrescent in norm and the following holds for all* $x \in \mathbf{R}^n$ *and* $t \in \mathbf{R}$*,*

$$\sum_{i=2}^{j} N_i(t, x) \geq \omega(|x_1|) + M(t, x_2) \tag{39}$$

and

$$|f_2(t, x)| \leq \chi_f(t, x, N_2, N_3, \ldots, N_{j-1}), \tag{40}$$

where χ_f *is so that, for all* $x \in \mathbf{R}^n$ *and* $t \in \mathbf{R}$*,*

$$0 \leq \chi_f(t, x, N_2, \ldots, N_{j-1}) \leq \lambda_{f*}(x, N_2, \ldots, N_{j-1}) \tag{41}$$

where the function λ_{f*} *is positive semi-definite and such that, for all* $x \in \mathbf{R}^n$

$$\lambda_{f*}(x, 0, \ldots, 0) = 0. \tag{42}$$

C2. *There exist a differentiable function* $\theta : \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ *and a positive definite function* $\gamma : \mathbf{R} \to \mathbf{R}$ *such that for all* $(t, x_2) \neq (t, 0)$*, we have:*

$$\int_{t-\theta(|x_2|^2)}^{t} M(s, x_2)\mathrm{d}s \geq \gamma(|x_2|). \tag{43}$$

Then, one can determine nonnegative functions p_i *and a positive definite function* δ *such that the following function:*

$$W(t, x) = \sum_{i=1}^{j} p_i(V_1(t, x))V_i(t, x) + p_{j+1}(V_1(t, x))\delta(|x_2|^2)A(t, x_2) \tag{44}$$

with

$$A(t, x_2) = \int_{t-\theta(|x_2|^2)}^{t} \left(\int_{s}^{t} M(l, x_2)\mathrm{d}l \right) \mathrm{d}s \tag{45}$$

when $x_2 \neq 0$ *and*

$$A(t,0) = 0, \quad \forall t \tag{46}$$

is continuously differentiable and is a strong Lyapunov function for system (38).

Remark 6 Conditions of Theorem 5 can be regarded as generalized Matrosov theorem conditions and they are directly related to conditions used in [11, Theorem 1]. Indeed, our Assumption 1 corresponds to [11, Assumption 1]. Our Assumption 2 corresponds to [11, Assumptions 2 and 3], and so on. In particular, our condition **C2** corresponds to the so-called $u\delta$-PE condition introduced in [13]. Note, however, that our conditions are stronger and we assume that we know all the bounding functions because they are required in the construction of the strong Lyapunov function W. For instance, we assume that we know the functions θ and γ in the condition **C2** of Theorem 5, whereas this is not needed in the main results of [11]. This is the main difference between our conditions and those given in [11]. A consequence of our stronger assumptions is that we construct a strong Lyapunov function W, which was not done in [11].

It is possible to strengthen the persistency condition (43) and at the same time relax the condition (40) to provide a similar Lyapunov function construction that is presented in the next corollary. Observe that the strong Lyapunov functions we obtain are given by expressions slightly simpler than (44).

Theorem 6 *Consider the system* (1) *and suppose that Assumptions* 1 *and* 2 *hold. Suppose also that the following holds for all* $x \in \mathbf{R}^n$ *and* $t \in \mathbf{R}$

$$\sum_{i=2}^{j} N_i(t,x) \geq M(t,x) = \overline{p}(t)\mu(x) \tag{47}$$

where μ *is a positive definite function and* $\overline{p}(t)$ *is a nonnegative function such that, for all* $t \in \mathbf{R}$,

$$\int_{t-\tau}^{t} \overline{p}(l)dl \geq p_m, \quad \overline{p}(t) \leq p_M \tag{48}$$

where $\tau > 0$, $p_m > 0$, $p_M > 0$. *Then, one can determine nonnegative functions* p_i *such that the following function:*

$$W(t,x) = \sum_{i=1}^{j} p_i(V_1(t,x))V_i(t,x) + p_{j+1}(V_1(t,x))\left(\int_{t-\tau}^{t}\left(\int_{s}^{t}\overline{p}(l)\mathrm{d}l\right)\mathrm{d}s\right) \tag{49}$$

is a strong Lyapunov function for system (1).

5 Illustration

In this section, we illustrate our main results by means of a system resulting from an adaptive tracking control problem for the well-known pendulum equations with an unknown friction coefficient (see [6, Sect. 1.1.1]). First, we recall how an adaptive control law can be constructed, using a classical approach, which relies on the construction of a weak Lyapunov function. In a second step, we use Theorem 6 to determine a strong Lyapunov function. The construction we will carry out is slightly different from the one used to prove Theorem 6, and hence our example also illustrates the flexibility of the approach. Finally we exploit this strong Lyapunov function to obtain a control law which renders the system ISS with respect to additive disturbance in the input. Observe that the ISS property, introduced by Sontag in [23] plays a central role in modern nonlinear control analysis, controller design, and robustness analysis.

The system we consider is given by the equations

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\frac{g}{l}\sin(x_1) - \theta x_2 - \frac{1}{ml^2}(T + d), \end{cases} \tag{50}$$

with θ unknown but constant, with g, m, l known, where T is the input and where d is a disturbance. The controller will be designed to track the trajectory

$$x_1^*(t) = \frac{1}{2}\sin(t), \quad x_2^*(t) = \frac{1}{2}\cos(t). \tag{51}$$

When $d \equiv 0$, this simple adaptive control problem can be solved by classical Lyapunov-based design techniques, presented for instance in [7]. The proof relies on a dynamic extension and the construction of a weak Lyapunov function, which ensures convergence of the state variables to the reference trajectory (51), when $d \equiv 0$. But the construction of a strong Lyapunov function is still an open problem and therefore the problem of constructing a control law is such that the corresponding closed-loop system is globally ISS with respect to the additive disturbance d. This absence of strong Lyapunov function in the broad literature devoted to mechanical systems and adaptive control, and the advantages inherent to the knowledge of strong Lyapunov functions, such as the possibility of constructing a robust control law, are motivations for our choice of illustrating this example.

Step 1 Solution of the adaptive problem when $d \equiv 0$.

First, we briefly recall a solution to the adaptive control problem in the absence of disturbance d, based on the Lyapunov technique of [7, Sect. 4.3].

Lemma 1 *Consider the system (50) with, for all $t \in \mathbf{R}$, $d(t) = 0$ and the adaptive controller*

$$\begin{aligned} T(t, x_1, x_2, \hat{\theta}) &= -mlg\sin(x_1) + ml^2\left[e_1 + e_2 + \tfrac{1}{2}\sin(t) - \hat{\theta}x_2\right] \\ \dot{\hat{\theta}} &= -x_2[2e_2 + e_1] \end{aligned} \tag{52}$$

with

$$e_1 = x_1 - x_1^*(t), \quad e_2 = x_2 - x_2^*(t). \tag{53}$$

Then this adaptive controller guarantees that global asymptotic tracking is achieved:

$$\lim_{t \to +\infty} [x_1(t) - x_1^*(t)] = 0, \quad \lim_{t \to +\infty} [x_2(t) - x_2^*(t)] = 0. \tag{54}$$

Besides,

$$\lim_{t \to +\infty} [\theta - \hat{\theta}(t)] = 0. \tag{55}$$

Proof Using the error variables e_1, e_2 and the expression of the control law in (52), we obtain

$$\dot{e}_1 = e_2 \tag{56}$$

and

$$\begin{aligned} \dot{e}_2 &= -\tfrac{g}{l}\sin(x_1) - \theta x_2 - \dot{x}_2^*(t) \\ &\quad -\tfrac{1}{ml^2}[-mlg\sin(x_1) + ml^2[e_1 + e_2 + \tfrac{1}{2}\sin(t) - \hat{\theta}x_2] \\ &= -e_1 - e_2 - \theta(e_2 + x_2^*(t)) + \hat{\theta}(e_2 + x_2^*(t)). \end{aligned} \tag{57}$$

Hence, using the notation $\tilde{\theta} = \hat{\theta} - \theta$, we obtain the system

$$\begin{cases} \dot{e}_1 = e_2, \\ \dot{e}_2 = -e_1 - e_2 + \tilde{\theta}(e_2 + x_2^*(t)), \\ \dot{\tilde{\theta}} = -(e_2 + x_2^*(t))[2e_2 + e_1]. \end{cases} \tag{58}$$

To simplify the notations, let $Z = (e_1, e_2, \tilde{\theta})$. The derivative of the positive definite and radially unbounded function

$$V_1(Z) = e_1^2 + e_2^2 + e_1 e_2 + \frac{1}{2}\tilde{\theta}^2 \tag{59}$$

along the trajectories of (58) satisfies

$$\begin{aligned} DV_1 &= 2e_1e_2 + (2e_2 + e_1)[-e_1 - e_2 + \tilde{\theta}(e_2 + x_2^*(t))] + e_2^2 - \tilde{\theta}(e_2 + x_2^*(t))[2e_2 + e_1] \\ &= -e_1^2 - e_1e_2 - e_2^2 \end{aligned} \tag{60}$$

and therefore $DV_1 < 0$ when $(e_1, e_2) \neq (0, 0)$. Since the system (58) is periodic in time, the LaSalle Invariance Principle applies and ensures that (54), (55) are satisfied. Observe that V_1 is a weak Lyapunov function for the system (58).

Step 2 Construction of a strong Lyapunov function.

By using Theorem 6, we now construct a strong Lyapunov function for the system (58). Since V_1 is a weak Lyapunov function, a natural choice for V_2 and N_2 is $V_2 = V_1$ and

$$N_2(Z) = e_1^2 + e_1 e_2 + e_2^2. \tag{61}$$

We select as auxiliary function V_3 the function

$$V_3(t, Z) = -\frac{1}{2}\tilde{\theta}\cos(t)e_2 \tag{62}$$

because its derivative along the trajectories of (58) satisfies

$$\begin{aligned} DV_3 &= -\frac{1}{2}\tilde{\theta}\cos(t)\left[-e_1 - e_2 + \tilde{\theta}\left(e_2 + \frac{1}{2}\cos(t)\right)\right] \\ &\quad + \frac{1}{2}\left[\left(e_2 + \frac{1}{2}\cos(t)\right)(2e_2 + e_1)\cos(t) + \tilde{\theta}\sin(t)\right]e_2 \\ &= -N_3(t, Z) + \chi_3(t, Z) \end{aligned} \tag{63}$$

with

$$N_3(t, Z) = \frac{1}{4}\cos^2(t)\tilde{\theta}^2 \geq 0 \tag{64}$$

and

$$\begin{aligned} \chi_3(t, Z) &= \frac{1}{2}\tilde{\theta}\cos(t)e_1 + \frac{1}{2}\tilde{\theta}\cos(t)e_2 - \frac{1}{2}\cos(t)\tilde{\theta}^2 e_2 \\ &\quad + \frac{1}{2}\left[\left(e_2 + \frac{1}{2}\cos(t)\right)(2e_2 + e_1)\cos(t) + \tilde{\theta}\sin(t)\right]e_2. \end{aligned} \tag{65}$$

Observe that $DV_3 < 0$ when $N_2(Z) = 0$ and $\tilde{\theta} \neq 0$, $\cos(t) \neq 0$. More precisely, one can check that, with our choice of functions V_1, V_2, V_3, Theorem 6 applies:

1. i) Assumption 1 is satisfied because V_1 defined in (59) is a weak Lyapunov function for the system (58).
2. One can easily prove that

$$\begin{aligned} |\chi_3(t, Z)| &\leq \frac{1}{2}|\tilde{\theta}e_1| + |\tilde{\theta}e_2| + \frac{1}{2}\tilde{\theta}^2|e_2| + |e_2|^3 + \frac{1}{2}|e_1 e_2^2| + \frac{1}{2}e_2^2 \\ &\quad + \frac{1}{4}|e_1 e_2|. \end{aligned} \tag{66}$$

Using successively the inequality $N_2(Z) \geq \frac{1}{2}[e_1^2+e_2^2]$ and the inequality $V_1(Z) \geq \frac{1}{2}\left[e_1^2 + e_2^2 + \tilde{\theta}^2\right]$, we deduce that

$$|\chi_3(t, Z)| \leq \frac{3}{2}|\tilde{\theta}|\sqrt{2N_2} + \frac{1}{2}\tilde{\theta}^2\sqrt{2N_2} + \left[\frac{5}{2}|e_2| + \frac{5}{4}\right] N_2 \leq \chi_{*3}(Z, N_2) \quad (67)$$

with

$$\begin{aligned}\chi_{*3}(Z, N_2) &= \left[\frac{3}{\sqrt{2}} + \sqrt{V_1}\right] |\tilde{\theta}|\sqrt{N_2} + \frac{5}{2}\left[1 + V_1)\right] N_2 \\ &\leq \left[\left(\frac{3}{\sqrt{2}} + \sqrt{V_1}\right)|\tilde{\theta}| + \frac{5}{2}(1 + V_1)\right]\left[\sqrt{N_2} + N_2\right].\end{aligned} \quad (68)$$

Moreover, one can easily find a function M_b of class K so that, for the choices we made, the inequality (10) is satisfied. It follows that Assumption 2 is satisfied.

3. The inequality

$$N_2(Z) + N_3(t, Z) \leq \overline{p}(t)\mu(Z) \quad (69)$$

is satisfied with

$$\overline{p}(t) = \frac{1}{4}\cos^2(t), \quad \mu(Z) = e_1^2 + e_2^2 + \tilde{\theta}^2 \quad (70)$$

and $\int_{t-\pi}^{t} \overline{p}(l)\mathrm{d}l = \frac{1}{8}\pi, 0 \leq \overline{p}(l) \leq \frac{1}{4}$.

Hence, all the conditions of Theorem 6 are satisfied and therefore one can construct a strong Lyapunov function for the system (58). By performing explicitly this construction, we obtain the following result:

Lemma 2 *The function*

$$W(t, Z) = \frac{\pi}{2}\left[\frac{\sin(2t)}{4} + \frac{\pi}{4} + 79\right] V_1(Z) + \frac{21\pi}{4} V_1(Z)^2 + \pi V_3(t, Z) \quad (71)$$

where V_1 is the function defined in (59) *and V_3 is the function defined in* (62)*, is a strong Lyapunov function of the system* (58)*. Its derivative along the trajectories of this system satisfies*

$$DW \leq -\frac{\pi}{8}V_1(Z). \quad (72)$$

Proof Observe that the function μ defined in (70) satisfies

$$\mu(Z) \geq \frac{1}{2}V_1(Z). \quad (73)$$

This inequality and the proof of Theorem 6 lead us to consider the function

$$C(t, Z) = \frac{1}{2}\left(\int_{t-\pi}^{t}\left(\int_{s}^{t}\cos^2(l)\mathrm{d}l\right)\mathrm{d}s\right)V_1(Z) = \frac{\pi}{8}\left[\sin(2t)+\pi\right]V_1(Z) \tag{74}$$

whose derivative along the trajectories of (58) is

$$DC = \frac{\pi}{4}\cos(2t)V_1(Z) + \frac{\pi}{8}\left[\sin(2t)+\pi\right]DV_1. \tag{75}$$

Since $DV_1 \leq 0$, $[\sin(2t)+\pi] \geq 0$ and $\cos(2t) = 2\cos^2(t) - 1$, it follows that

$$DC \leq -\frac{\pi}{4}V_1(Z) + \frac{\pi}{2}\cos^2(t)V_1(Z). \tag{76}$$

Moreover, using (60) and (63), one can prove easily that derivative along the trajectories of (58) of $V_1 + V_3$ satisfies the inequality

$$DV_1 + DV_3 \leq -\frac{1}{2}\cos^2(t)V_1(Z) + \chi_3(t, Z), \tag{77}$$

where χ_3 is the function defined in (65). It follows that the derivative of

$$V_4(t, Z) = C(t, Z) + \pi V_1(Z) + \pi V_3(t, Z) \tag{78}$$

along the trajectories of (58) satisfies

$$DV_4 \leq -\frac{\pi}{4}V_1(Z) + \pi\chi_3(t, Z). \tag{79}$$

By using (67), the expression of χ_{*3} in (68) and the triangular inequality, we deduce that

$$\begin{aligned}\chi_{*3}(Z, N_2) &\leq \frac{1}{16}\tilde{\theta}^2 + 4\left(\frac{3}{\sqrt{2}} + \sqrt{V_1}\right)^2 + \frac{5}{1}(1+V_1)N_2\\ &\leq \frac{1}{16}\tilde{\theta}^2 + \frac{77}{2}N_2 + \frac{21}{2}V_1N_2.\end{aligned} \tag{80}$$

Combining (79) and (80), we obtain

$$DV_4 \leq -\frac{\pi}{8}V_1 + \frac{77\pi}{2}N_2 + \frac{21\pi}{2}V_1N_2. \tag{81}$$

Since $N_2 = -DV_1$, it follows that the derivative of the function (71) along the trajectories of (58) satisfies (72). By using the fact that V_1 is a weak Lyapunov function and that the functions V_3 and $\frac{\partial V_3}{\partial x}$ are decrescent in norm, and that $W \geq V_1$, one can check easily that W is a strong Lyapunov function.

Step 3 Robust control law.

In this part we use our Lyapunov construction to robustify our controller and obtain the desirable ISS property.

Theorem 7 *Consider the system* (50) *with the adaptive controller*

$$\begin{aligned} T(t, x_1, x_2, \hat{\theta}) &= -mlg\sin(x_1) + ml^2\left[e_1 + e_2 + \tfrac{1}{2}\sin(t) - \hat{\theta}x_2\right] \\ &\quad + \left[\tfrac{\pi\sin(2t)+\pi^2}{16} + \tfrac{79\pi}{4} + \tfrac{21\pi}{4}V_1(e_1, e_2, \tilde{\theta})\right]\left[e_2 + \tfrac{e_1}{2}\right] \\ &\quad - \tfrac{\pi}{8}\tilde{\theta}\cos(t) \\ \dot{\hat{\theta}} &= -x_2[2e_2 + e_1] \end{aligned} \tag{82}$$

where V_1 is the function defined in (59), *e_1, e_2 defined in* (53), *$\tilde{\theta} = \hat{\theta} - \theta$, $Z = (e_1, e_2, \tilde{\theta})$. Then this adaptive controller guarantees that there are a function β of class KL and a function γ of class K (see the preliminaries for the definitions of functions of class K and class KL) such that, for all $t_0 \in R$, $Z_0 \in \mathbf{R}^n$, $t \geq t_0$,*

$$|Z(t; t_0, Z_0)| = \beta(|Z_0|, t - t_0) + \gamma\left(\sup_{\{s\in[t_0,t]\}} |d(s)|\right). \tag{83}$$

Proof We deduce directly from Lemma 2 and its proof that, when the adaptive controller is

$$T(t, x_1, x_2, \hat{\theta}) = -mlg\sin(x_1) + ml^2[e_1 + e_2 - \dot{x}_2^*(t) - \hat{\theta}(e_2 + x_2^*(t))] + v, \tag{84}$$

where v is an input to be specified later, and when there is a disturbance d, then

$$DW \leq -\frac{\pi}{8}V_1(Z) - \frac{1}{ml^2}\frac{\partial W}{\partial e_2}(t, Z)(v + d). \tag{85}$$

The choice

$$v = \frac{1}{4}\frac{\partial W}{\partial e_2}(t, Z) \tag{86}$$

gives

$$DW \leq -\frac{\pi}{8}V_1(Z) - \frac{1}{4ml^2}\left(\frac{\partial W}{\partial e_2}(t, Z)\right)^2 - \frac{1}{ml^2}\frac{\partial W}{\partial e_2}(t, Z)d. \tag{87}$$

Thanks to the triangular inequality, we deduce that

$$DW \leq -\frac{\pi}{8}V_1(Z) + \frac{1}{ml^2}d^2. \tag{88}$$

Since V_1 is a weak Lyapunov function, it follows that W is a ISS Lyapunov function for (58) (see [3,26] for the definition of ISS Lyapunov function). From the results of [24] or [26] (see also [6, Theorem 5.2]), one can deduce that the closed-loop system is ISS.

To conclude, one can prove after lengthy but simple calculations that the function T given in (84) with v defined in (86) admits the expression (82).

6 Proofs of main results

Proof of Theorem 4 We prove this result by induction on the number of the auxiliary functions in Assumption 2. The result of Theorem 4 holds in the case where its assumptions are satisfied with only one auxiliary function, i.e. when $j = 2$ because in that case $N_2 = \omega(x)$ and one can construct a strong Lyapunov function by following the proof of Theorem 3. Assume that the result of Theorem 4 holds when its assumptions are satisfied with $j - 1$ auxiliary functions with $j \geq 2$. Let us prove that it holds as well when the assumptions are satisfied with j auxiliary functions. To prove this, let us consider a system (1) satisfying the assumptions of Theorem 4 with j auxiliary functions, with $j \geq 2$, and let us construct a new set of $j - 1$ auxiliary functions for which the assumptions of Theorem 4 are satisfied.

Let us define

$$S_a(t, x) := \sum_{i=2}^{j+1} V_i(t, x). \tag{89}$$

Then, according to Assumption 2 and (36),

$$\begin{aligned} DS_a &\leq -\sum_{i=2}^{j+1} N_i + \sum_{i=3}^{j+1} \chi_i(t, x, N_2, \dots, N_{i-1}) \\ &\leq -\omega(x) + \sum_{i=3}^{j+1} \chi_i(t, x, N_2, \dots, N_{i-1}). \end{aligned} \tag{90}$$

Using the inequality (7) in Assumption 2 and Lemma 6, one can determine the explicit expression of a function ϕ, of class K_∞ and of a nondecreasing and positive function ρ such that

$$\left| \sum_{i=3}^{j+1} \chi_i(t, x, N_2, \dots, N_{i-1}) \right| \leq \phi\left(\sum_{i=2}^{j} N_i \right) \rho(|x|). \tag{91}$$

It follows that

$$DS_a \leq -\omega(x) + \phi\left(\sum_{i=2}^{j} N_i \right) \rho(|x|). \tag{92}$$

By following verbatim the proof of Theorem 3 from (18) to (30), one can determine a positive definite function p_* and a function Γ_a, positive and nondecreasing, such that the derivative of the function

$$S_b(t,x) = p_*(V_1(t,x))S_a(t,x) \tag{93}$$

along the trajectories of (1) satisfies

$$DS_b \leq -\frac{1}{2}p_*(V_1)\omega(x) + \frac{1}{2}\left(\sum_{i=2}^{j} N_i\right)\Gamma_a(V_1) - \frac{1}{2}\Gamma_a(V_1)DV_1. \tag{94}$$

Let

$$\nu_a(t,x) = S_b(t,x) + \frac{1}{2}\Gamma_a(V_1(t,x))V_j(t,x). \tag{95}$$

Simple calculations yield

$$\begin{aligned} D\nu_a \leq & -\frac{1}{2}p_*(V_1)\omega(x) + \frac{1}{2}\left(\sum_{i=2}^{j} N_i\right)\Gamma_a(V_1) - \frac{1}{2}\Gamma_a(V_1)DV_1 \\ & + \frac{1}{2}\Gamma_a'(V_1)V_jDV_1 + \frac{1}{2}\Gamma_a(V_1)DV_j. \end{aligned} \tag{96}$$

Thanks to (10) and (6), one can determine a function Γ_b, positive and nondecreasing, such that

$$\left|-\frac{1}{2}\Gamma_a(V_1) + \frac{1}{2}\Gamma_a'(V_1)V_j\right| \leq \Gamma_b(V_1). \tag{97}$$

It follows that the derivative of the function

$$\nu_b(t,x) = \nu_a(t,x) + \int_0^{V_1(t,x)} \Gamma_b(l)\mathrm{d}l \tag{98}$$

along the trajectories of (1) satisfies

$$D\nu_b \leq -\frac{1}{2}p_*(V_1)\omega(x) + \frac{1}{2}\left(\sum_{i=2}^{j} N_i\right)\Gamma_a(V_1) + \frac{1}{2}\Gamma_a(V_1)DV_j. \tag{99}$$

Using Assumption 2, we deduce that

$$D\nu_b \leq -\tfrac{1}{2}p_*(V_1)\omega(x) + \tfrac{1}{2}\left(\sum_{i=2}^{j-1} N_i\right)\Gamma_a(V_1) \\ +\tfrac{1}{2}\Gamma(V_1)\chi_j(t, x, N_2, \ldots, N_{j-1}). \tag{100}$$

One can easily prove that ν_b is decrescent in norm and determine a function M_{bn} of class K_∞ such that

$$\sum_{i=2}^{j} N_i(t,x) + \frac{1}{2}p_*(V_1(t,x))\omega(x) + \sum_{i=1}^{j}|V_i(t,x)| + |\nu_b(t,x)| \leq M_{bn}(|x|). \tag{101}$$

It follows that the system (1) satisfies the assumptions of Theorem 4 with $j-1$ auxiliary functions, $V_2,\ldots,V_{j-1}$, ν_b. According to our induction assumption, it follows that one can construct explicitly a strong Lyapunov function. Consequently, our induction assumption is satisfied at the step j.

Proof of Theorem 5 The proof of this result consists in constructing a function V_{j+1} such that the condition (36) of Theorem 4 is satisfied.

The function $A(t, x_2)$, defined in (45), is continuously differentiable, except at $x_2 = 0$. This function is not necessarily bounded but the condition C1 ensures that both $M(t, x_2)$ and $\frac{\partial M}{\partial x_2}(t, x_2)$ are decrescent in norm which guarantees the existence of a function σ_2 of class K_∞ such that, for all $t \in \mathbf{R}$, $x_2 \in \mathbf{R}^{n_2}$,

$$|M(t, x_2)| \leq \sigma_2(|x_2|), \quad \left|\frac{\partial M}{\partial x_2}(t, x_2)\right| \leq \sigma_2(|x_2|) \tag{102}$$

and therefore, for all $(t, x_2) \neq (t, 0)$,

$$0 \leq A(t, x_2) \leq \theta(|x_2|^2)^2\sigma_2(|x_2|). \tag{103}$$

On the other hand, the derivative of $A(t, x_2)$ along the trajectories of (38) satisfies, when $x_2 \neq 0$,

$$DA = \theta(|x_2|^2)M(t, x_2) - \left[\int_{t-\theta(|x_2|^2)}^{t} M(l, x)\mathrm{d}l\right]\left[1 - 2\theta'(|x_2|^2)x_2^\top f_2(t, x)\right] \\ + \int_{t-\theta(|x_2|^2)}^{t}\left(\int_{s}^{t}\frac{\partial M}{\partial x_2}(l, x_2) f_2(t, x)\mathrm{d}l\right)\mathrm{d}s. \tag{104}$$

Using (102) and (40) in the condition C1, we deduce that, when $x_2 \neq 0$,

$$
\begin{aligned}
DA \leq\ & \theta(|x_2|^2) M(t, x_2) - \left[\int_{t-\theta(|x_2|^2)}^{t} M(l, x_2)\mathrm{d}l \right] \\
& + 2\theta(|x_2|^2)\sigma_2(|x_2|)|\theta'(|x_2|^2)||x_2|\chi_f(t, x, N_2, N_3, \ldots, N_{j-1}) \\
& + \theta(|x_2|^2)^2\sigma_2(|x_2|)\chi_f(t, x, N_2, N_3, \ldots, N_{j-1}). \qquad (105)
\end{aligned}
$$

Grouping the terms and using the inequality (43) in the condition C2, we obtain, when $x_2 \neq 0$,

$$
DA \leq \theta(|x_2|^2)M(t, x_2) - \gamma(|x_2|) + \vartheta(|x_2|)\chi_f(t, x, N_2, N_3, \ldots, N_{j-1}) \qquad (106)
$$

with, for all $m > 0$,

$$
\vartheta(m) = \theta(m^2)\sigma_2(m)\left[2|\theta'(m^2)||x_2| + \theta(m^2)\right]. \qquad (107)
$$

We define now a function B as follows

$$
\begin{aligned}
B(t, 0) &= 0, \quad \forall t \qquad (108)\\
B(t, x_2) &= \delta(|x_2|^2)A(t, x_2), \quad \forall (t, x_2) \neq (t, 0) \qquad (109)
\end{aligned}
$$

where δ is a positive definite function such that, for all $s \geq 0$,

$$
\begin{aligned}
0 \leq \delta(s) &\leq \frac{s}{\sqrt{1+s^2}} \min\left\{ \frac{1}{\theta(s)[\sigma_2(\sqrt{s})+1]}, \frac{1}{2\sigma_2(\sqrt{s})\left[2\theta(s)|\theta'(s)|\sqrt{s} + \theta(s)^2\right]} \right\}, \\
|\delta'(s)| &\leq \frac{s}{\sqrt{1+s^2}} \frac{1}{4\sqrt{s}\theta(s)^2\sigma_2(\sqrt{s})}. \qquad (110)
\end{aligned}
$$

Observe that such a function δ can be obtained by using Lemma 3. The inequalities (103) and (110) imply that, for all $(t, x_2) \in \mathbf{R} \times \mathbf{R}^{n_2}$,

$$
0 \leq B(t, x_2) \leq \delta(|x_2|^2)\theta(|x_2|^2)^2\sigma_2(|x_2|) \leq \frac{|x_2|^2}{\sqrt{1 + |x_2|^4}}. \qquad (111)
$$

This inequality and (46) imply that B is continuous. Moreover, by taking advantage of (110) and (111), one can show that $B(t, x_2)$ is continuously differentiable on $\mathbf{R} \times \mathbf{R}^{n_2}$ by showing that, for any $(t, x_2) \in \mathbf{R} \times \mathbf{R}^{n_2}$,

$$
\lim_{l \to 0} \frac{B(t, lx_2) - B(t, 0)}{l} = 0
$$

and, for all $(t, x_2) \neq (t, 0)$,

$$\left|\frac{\partial B}{\partial t}(t, x_2)\right|^2 + \left|\frac{\partial B}{\partial x_2}(t, x_2)\right|^2 \leq (4 + n_2)\frac{|x_2|^4}{1 + |x_2|^4}. \tag{112}$$

The derivative of $B(t, x_2)$ along the along the trajectories of (38) is given by

$$DB = \delta(|x_2|^2)DA + \delta'(|x_2|^2)2x_2^\top f_2(t, x)A(t, x_2) \tag{113}$$

when $x_2 \neq 0$. From (106), (103) and (40) in the condition C1 we deduce that, when $x_2 \neq 0$,

$$\begin{aligned} DB \leq\ & \delta(|x_2|^2)\theta(|x_2|^2)M(t, x_2) - \delta(|x_2|^2)\gamma(|x_2|) \\ & + \varsigma(|x_2|)\chi_f(t, x, N_2, N_3, \ldots, N_{j-1}) \\ & + \delta'(|x_2|^2)2|x_2|\chi_f(t, x, N_2, N_3, \ldots, N_{j-1})\theta(|x_2|^2)^2\sigma_2(|x_2|) \end{aligned} \tag{114}$$

with, for all $m > 0$,

$$\varsigma(m) = \delta(m^2)\theta(m^2)\sigma_2(m)\left[2|\theta'(m^2)|m + \theta(m^2)\right]. \tag{115}$$

We deduce from the inequalities (110) that, for all (t, x_2),

$$DB \leq M(t, x_2) - \delta(|x_2|^2)\gamma(|x_2|) + \chi_f(t, x, N_2, N_3, \ldots, N_{j-1}). \tag{116}$$

We define now the following function

$$V_{j+1}(t, x) = \sum_{i=2}^{j} V_i(t, x) + B(t, x_2). \tag{117}$$

Using Assumption 2 and (112), one can conclude that $\frac{\partial V_{j+1}}{\partial x}$ is decrescent in norm. Then, from Assumption 2 and (116), it follows that

$$\begin{aligned} DV_{j+1} \leq & -\sum_{i=2}^{j} N_i(t, x) + \sum_{i=2}^{j} \chi_i(t, x, N_2, N_3, \ldots, N_{i-1}) + M(t, x_2) \\ & - \delta(|x_2|^2)\gamma(|x_2|) + \chi_f(t, x, N_2, N_3, \ldots, N_{j-1}). \end{aligned} \tag{118}$$

Thanks to (39) in the condition C1, we deduce that

$$\dot{V}_{j+1} \leq -N_{j+1}(x) + \chi_{j+1}(t, x, N_2, N_3, \ldots, N_{j-1}, N_j) \tag{119}$$

with

$$N_{j+1}(x) = \omega(|x_1|) + \delta(|x_2|^2)\gamma(|x_2|) \tag{120}$$

and

$$\chi_{j+1}(t, x, N_2, \ldots, N_{j-1}, N_j) = \sum_{i=2}^{j} \chi_i(t, x, N_2, \ldots, N_{i-1}) + \chi_f(t, x, N_2, \ldots, N_{j-1}). \tag{121}$$

The function N_{j+1} is positive definite and using (10), (120), (110), (111), one can easily determine a function M_{bn} of class K_∞ such that

$$\sum_{i=2}^{j+1} N_i(t, x) + \sum_{i=1}^{j+1} |V_i(t, x)| \leq M_{bn}(|x|). \tag{122}$$

One can check readily that Theorem 4 applies. This theorem provides a strong Lyapunov function for the system (1) with the features of (44).

Proof of Theorem 6 The proof of this result consits in constructing a function V_{j+1} such that the condition (36) of Theorem 4 is satisfied. The function μ is positive definite. Therefore, by using Lemma 3, one can determine a positive definite real-valued function γ of class C^1 such that, for all $(t, x) \in \mathbf{R} \times \mathbf{R}^n$,

$$\mu(x) \geq \gamma(V_1(t, x)), \quad |\gamma'(V_1(t, x))| \leq 1. \tag{123}$$

Next, let us consider the function

$$C(t, x) = \left(\int_{t-\tau}^{t} \left(\int_{s}^{t} \overline{p}(l) \mathrm{d}l \right) \mathrm{d}s \right) \gamma(V_1(t, x)). \tag{124}$$

This function and $\frac{\partial C}{\partial x}$ are decrescent in norm and the derivative of C along (1) satisfies

$$DC = \tau \overline{p}(t) \gamma(V_1) - \left(\int_{t-\tau}^{t} \overline{p}(l) \mathrm{d}l \right) \gamma(V_1) + \left(\int_{t-\tau}^{t} \left(\int_{s}^{t} \overline{p}(l) \mathrm{d}l \right) \mathrm{d}s \right) \gamma'(V_1) DV_1. \tag{125}$$

Thanks to (48) and (123), we deduce that

$$DC \leq \tau \overline{p}(t) \mu(x) - p_m \gamma(V_1(t, x)) + \tau^2 p_M |DV_1|. \tag{126}$$

Consider now the function

$$V_{j+1}(t, x) := C(t, x) + \tau^2 p_M V_1(t, x) + \tau \sum_{i=2}^{j} V_i(t, x) \tag{127}$$

which is decrescent in norm as long as $\frac{\partial V_{j+1}}{\partial x}$. From (126), we deduce that its derivative along (1) satisfies

$$DV_{j+1} \leq \tau \overline{p}(t)\mu(x) - p_m\gamma(V_1) + \tau^2 p_M|DV_1| + \tau^2 p_M DV_1 \\ + \tau \sum_{i=2}^{j} DV_i. \tag{128}$$

Using the fact that DV_1 is nonpositive and Assumption 2, we deduce that

$$DV_{j+1} \leq \tau \overline{p}(t)\mu(x) - p_m\gamma(V_1) - \tau \sum_{i=2}^{j} N_j + \tau \sum_{i=3}^{j} \chi_i(t, x, N_2, \ldots, N_{i-1}). \tag{129}$$

Using (47) we obtain

$$DV_{j+1} \leq -N_{j+1} + \tau \sum_{i=3}^{j} \chi_i(t, x, N_2, \ldots, N_{i-1}), \quad N_{j+1} = p_m\gamma(V_1). \tag{130}$$

Moreover, using (123), (48) and (10), one can easily determine a positive nondecreasing function M_{bn} such that

$$\sum_{i=2}^{j+1} N_i(t, x) + \sum_{i=1}^{j+1} |V_i(t, x)| \leq M_{bn}(|x|). \tag{131}$$

One can check readily now that Theorem 4 applies. This theorem provides with a strong Lyapunov function for the system (1) with the features of (49).

7 Conclusion

We provided several constructions of strong Lyapunov functions for time-varying systems that satisfy generalized conditions of the Matrosov theorem. We expect that our results will have significant implications in several areas of nonlinear control, especially in the areas of tracking and adaptive control. We will address these issues in our future work.

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Appendix: Technical lemmas

Lemma 3 *Let $w_i : \mathbf{R}^n \to \mathbf{R}$ $i = 1, 2$ be two positive definite functions; $V : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ and γ_1, γ_2 of class K_∞ such that, for all $(t, x) \in \mathbf{R} \times \mathbf{R}^n$, we have:*

$$\gamma_1(|x|) \leq V(t, x) \leq \gamma_2(|x|). \tag{132}$$

Then, one can construct a real-valued function L of class C^N, where $N \geq 1$ is an integer, such that $L(0) = 0$, $L(s) > 0$ for all $s > 0$ and for all $(t, x) \in \mathbf{R} \times \mathbf{R}^n$,

$$L(V(t,x)) \leq w_1(x), \tag{133}$$
$$|L'(V(t,x))| \leq w_2(x). \tag{134}$$

Proof We will prove at the end of this proof that one can construct a function ρ, positive, increasing and of class C^N, and a function α of class K_∞ and of class C^N such that

$$\alpha(V(t,x)) \leq w_1(x)\rho(V(t,x)), \tag{135}$$
$$\alpha(V(t,x)) \leq w_2(x)\rho(V(t,x)). \tag{136}$$

For the time being, we assume that these functions are known and introduce now the following function

$$L(s) := \int_{\frac{s}{2}}^{s} \frac{\alpha(l)}{2(1+l^2)(1+\rho(2l)^2)} \mathrm{d}l. \tag{137}$$

Then $L(0) = 0$, $L(s) > 0$ for all $s > 0$, L is of class C^N and, because both α and ρ are increasing, for all $s \geq 0$,

$$L(s) \leq \int_{\frac{s}{2}}^{s} \frac{\alpha(s)}{2\left(1+\left(\frac{s}{2}\right)^2\right)(1+\rho(s)^2)} \mathrm{d}l \leq \frac{\alpha(s)}{4(1+\rho(s)^2)} \leq \frac{\alpha(s)}{\rho(s)}. \tag{138}$$

It follows that

$$L(V(t,x)) \leq \tfrac{\alpha(V(t,x))}{\rho(V(t,x))} \leq w_1(x). \tag{139}$$

Therefore (133) is satisfied. On the other hand, the first derivative of L is

$$L'(s) = \tfrac{\alpha(s)}{2(1+s^2)(1+\rho(2s)^2)} - \tfrac{1}{2} \tfrac{\alpha\left(\frac{s}{2}\right)}{2\left(1+\left(\frac{s}{2}\right)^2\right)(1+\rho(s)^2)}. \tag{140}$$

Since both α and ρ are increasing, it follows that

$$\begin{aligned} |L'(s)| &\leq \frac{\alpha(s)}{2(1+s^2)(1+\rho(2s)^2)} + \frac{1}{2} \frac{\alpha\left(\frac{s}{2}\right)}{2\left(1+\left(\frac{s}{2}\right)^2\right)(1+\rho(s)^2)} \\ &\leq \frac{\alpha(s)}{2(1+\rho(s)^2)} + \frac{\alpha(s)}{4(1+\rho(s)^2)} \\ &\leq \frac{\alpha(s)}{\rho(s)}. \end{aligned} \tag{141}$$

Consequently, the inequalities

$$|L'(V(t,x))| \leq \frac{\alpha(V(t,x))}{\rho(V(t,x))} \leq w_2(x) \tag{142}$$

are satisfied and therefore (134) is satisfied.

We end this proof by constructing a function ρ, positive, increasing and of class C^N, and a function α of class K_∞ and of class C^N such that (135) and (136) are satisfied. We introduce the constant

$$W_f = \inf_{\{z:|z|=1\}} w(z) \tag{143}$$

and define four functions:

$$w(x) = \inf\{w_1(x), w_2(x)\}, \tag{144}$$

$$\delta_l(r) = \begin{cases} \inf\limits_{\{z:|z|\in[1,r]\}} w(z) & \text{if} \quad r \geq 1, \\ W_f & \text{if} \quad r \in [0,1], \end{cases} \tag{145}$$

$$\delta_s(r) = \begin{cases} \inf\limits_{\{z:|z|\in[r,1]\}} w(z) & \text{if} \quad r \in [0,1], \\ W_f & \text{if} \quad r \geq 1, \end{cases} \tag{146}$$

$$\delta(r) = \frac{1}{W_f}\delta_s(r)\delta_l(r). \tag{147}$$

Observe that

1. If $|x| \leq 1$, then $\delta(|x|) = \frac{1}{W_f}\delta_s(|x|)\delta_l(|x|) = \delta_s(|x|) = \inf\limits_{\{z:|z|\in[|x|,1]\}} w(z) \leq w(x)$.
2. If $|x| \geq 1$, then $\delta(|x|) = \frac{1}{W_f}\delta_s(|x|)\delta_l(|x|) = \delta_l(|x|) = \inf\limits_{\{z:|z|\in[1,|x|]\}} w(z) \leq w(x)$.

It follows that, for all $x \in \mathbf{R}^n$,

$$w(x) \geq \delta(|x|) = \frac{1}{W_f}\delta_s(|x|)\delta_l(|x|). \tag{148}$$

Since w is a positive definite function, δ_l is a positive function on $\mathbf{R}_{\geq 0}$. Therefore, from (148), it follows that, for all $x \in \mathbf{R}^n$,

$$\delta_s(|x|) \leq w(x)\frac{W_f}{\delta_l(|x|)}. \tag{149}$$

We introduce two functions

$$\alpha_a(r) = r\delta_s(r), \quad \rho_a(r) = \frac{W_f(1+r)}{\delta_l(r)}, \quad \forall r \geq 0. \tag{150}$$

Then, from (149), we deduce that, for all $x \in \mathbf{R}^n$,

$$\alpha_a(|x|) \leq w(x)\rho_a(|x|). \tag{151}$$

Since w is positive definite and at least continuous, one can prove easily that $\delta_s(0) = 0$, $\delta_s(r) > 0$ if $r > 0$ and δ_s is nondecreasing and continuous. It follows that α_a is of class K_∞. For similar reasons, δ_l is continuous, positive and nonincreasing. It follows that ρ_a is well defined, positive and increasing. Using these properties of α_a and ρ_a and (132), we deduce that, for all $(t, x) \in \mathbf{R} \times \mathbf{R}^n$,

$$\alpha_a(\gamma_2^{-1}(V(t,x)) \leq w(x)\rho_a(\gamma_1^{-1}(V(t,x))). \tag{152}$$

As an immediate consequence, we have

$$V(t,x)^N \alpha_b(V(t,x)) \leq w(x)[V(t,x)+1]^N \rho_b(V(t,x)) \tag{153}$$

with

$$\alpha_b(r) = \alpha_a(\gamma_2^{-1}(r)), \quad \rho_b(r) = \rho_a(\gamma_1^{-1}(r)), \quad \forall r \geq 0. \tag{154}$$

We define now, for all $r \geq 0$, two functions

$$\alpha(r) = \int_0^r \left(\int_0^{s_1} \cdots \int_0^{s_{N-1}} \alpha_b(s_N)\mathrm{d}s_N \right) \cdots \mathrm{d}s_1, \tag{155}$$

$$\rho(r) = \int_0^{r+1} \left(\int_0^{s_1+1} \cdots \int_0^{s_{N-1}+1} (s_N+1)^N \rho_b(s_N)\mathrm{d}s_N \right) \cdots \mathrm{d}s_1. \tag{156}$$

Observe that, for all $r \geq 0$,

$$\alpha(r) \leq r^N \alpha_b(r), \quad \rho(r) \geq (r+1)^N \rho_b(r). \tag{157}$$

These inequalities and (153) yield

$$\alpha(V(t,x)) \leq w(x)\rho(V(t,x)). \tag{158}$$

Since $0 \leq w_1(x) \leq w(x)$ and $0 \leq w_2(x) \leq w(x)$, we deduce that (135) and (136) are satisfied. One can check readily that ρ is positive, increasing and of class C^N, and α is of class K_∞ and of class C^N.

Lemma 4 *Let $\Omega : \mathbf{R}^n \to \mathbf{R}$ be a continuous function. Then, the function $\zeta : \mathbf{R}_{\geq 0} \to \mathbf{R}$ defined by*

$$\zeta(r) = 1 + \int_0^{r+1} \left(\int_0^{s_1+1} \cdots \int_0^{s_{N-1}+1} \left[\sup_{\{z \in \mathbf{R}^n : |z| \le s_N\}} |\Omega(z)| \right] \mathrm{d}s_N \right) \cdots \mathrm{d}s_1 \tag{159}$$

is positive, of class C^N, nondecreasing and so that, for all $x \in \mathbf{R}^n$,

$$|\Omega(x)| \le \zeta(|x|). \tag{160}$$

Proof From the definition of ζ, it follows immediately that ζ is positive, nondecreasing and of class C^N. To simplify the notations, we define the function

$$\Omega_s(r) = \sup_{\{z \in \mathbf{R}^n : |z| \le r\}} |\Omega(z)|. \tag{161}$$

The function Ω_s is nondecreasing on $\mathbf{R}_{\ge 0}$. Therefore, for all $s_{N-1} \ge 0$,

$$\int_0^{s_{N-1}+1} \Omega_s(s_N)\mathrm{d}s_N \ge \int_{s_{N-1}}^{s_{N-1}+1} \Omega_s(s_N)\mathrm{d}s_N \ge \Omega_s(s_{N-1}). \tag{162}$$

We deduce that, for all $r \ge 0$,

$$\int_0^{r+1} \left(\int_0^{s_1+1} \cdots \int_0^{s_{N-1}+1} \Omega_s(s_N)\mathrm{d}s_N \right) \cdots \mathrm{d}s_1 \ge \Omega_s(r). \tag{163}$$

It follows that, for all $x \in \mathbf{R}^n$,

$$\zeta(|x|) \ge \Omega_s(|x|) = \sup_{\{z \in \mathbf{R}^n : |z| \le |x|\}} |\Omega(z)| \ge |\Omega(x)|. \tag{164}$$

Lemma 5 *Let $F : \mathbf{R}_{\ge 0} \times \mathbf{R}_{\ge 0} \to \mathbf{R}$ be a continuous nonnegative function such that, for all $a \ge 0$,*

$$F(a, 0) = 0 \tag{165}$$

and, for all $(a, b) \in \mathbf{R}_{\ge 0} \times \mathbf{R}_{\ge 0}$,

$$F(a, b) \le \Theta(a)\Theta(b) \tag{166}$$

where Θ is a positive nondecreasing continuous function on $\mathbf{R}_{\ge 0}$. Then the function defined by

$$Z(b) = \sup_{\alpha \ge 0} \frac{F(\alpha, b)}{(\alpha^2 + 1)\Theta(\alpha)}, \quad \forall b \ge 0 \tag{167}$$

is well defined, nonnegative and continuous. Moreover, $Z(0) = 0$ *and, for all* $(a, b) \in \mathbf{R}_{\geq 0} \times \mathbf{R}_{\geq 0}$,

$$F(a, b) \leq (a^2 + 1)\Theta(a)Z(b). \tag{168}$$

Proof Let us prove that Z is well-defined on $\mathbf{R}_{\geq 0}$. To simplify the notation, let us introduce the following function:

$$\overline{F}(a, b) = \frac{F(a, b)}{(a^2 + 1)\Theta(a)}. \tag{169}$$

Since (166) is satisfied, then

$$\overline{F}(\alpha, b) \leq \frac{\Theta(\alpha)}{(\alpha^2 + 1)\Theta(\alpha)}\Theta(b) \leq \Theta(b). \tag{170}$$

It follows that, for all $b \geq 0$, $\sup_{\{\alpha \in \mathbf{R}_{\geq 0}\}} \overline{F}(\alpha, b)$ is a finite nonnegative real number. Therefore the function Z is well defined on $\mathbf{R}_{\geq 0}$. From the definition of Z, (165) and (166), we deduce easily that Z is nonnegative, (168) is satisfied and $Z(0) = 0$. Let us prove now that this function is continuous. Let b_c be a positive real number. Let ε be a positive real number. For all $b \geq 0$,

$$Z(b) = \max\left\{ \sup_{\{\alpha \in [0, \alpha^*]\}} \overline{F}(\alpha, b), \sup_{\{\alpha \geq \alpha^*\}} \overline{F}(\alpha, b) \right\} \tag{171}$$

with $\alpha^* = \sqrt{\frac{2}{\varepsilon}\Theta(b_c + 1)}$. From (166), it follows that, for all $b \geq 0$,

$$\begin{aligned} \sup_{\{\alpha \geq \alpha^*\}} \overline{F}(\alpha, b) &\leq \sup_{\{\alpha \geq \alpha^*\}} \frac{\Theta(\alpha)\Theta(b)}{(\alpha^2 + 1)\Theta(\alpha)} \\ &\leq \sup_{\{\alpha \geq \alpha^*\}} \frac{\Theta(b)}{(\alpha^2 + 1)} \\ &\leq \frac{\Theta(b)}{\left(\sqrt{\frac{2}{\varepsilon}\Theta(b_c + 1)}\right)^2 + 1}. \end{aligned} \tag{172}$$

Since Θ is nondecreasing, it follows that, for all $b \in [\max\{0, b_c - 1\}, b_c + 1]$,

$$\sup_{\{\alpha \geq \alpha^*\}} \overline{F}(\alpha, b) \leq \frac{\varepsilon}{2}. \tag{173}$$

We deduce easily that, for all $b \in [\max\{0, b_c - 1\}, b_c + 1]$,

$$\sup_{\{\alpha \in [0, \alpha^*]\}} \overline{F}(\alpha, b) \leq Z(b) \leq \sup_{\{\alpha \in [0, \alpha^*]\}} \overline{F}(\alpha, b) + \frac{\varepsilon}{2}. \tag{174}$$

In particular, for $b = b_c$,

$$\sup_{\{\alpha \in [0,\alpha^*]\}} \overline{F}(\alpha, b_c) \leq Z(b_c) \leq \sup_{\{\alpha \in [0,\alpha^*]\}} \overline{F}(\alpha, b_c) + \frac{\varepsilon}{2}. \tag{175}$$

From (174) and (175), we deduce that

$$|Z(b) - Z(b_c)| \leq \left| \sup_{\{\alpha \in [0,\alpha^*]\}} \overline{F}(\alpha, b) - \sup_{\{\alpha \in [0,\alpha^*]\}} \overline{F}(\alpha, b_c) \right| + \frac{\varepsilon}{2}. \tag{176}$$

The function $\sup_{\{\alpha \in [0,\alpha^*]\}} \overline{F}(\alpha, b)$ is continuous because $[0, \alpha^*]$ is a compact set. It follows that there exists $\delta \in (0, 1]$ such that, for all $b \in [\max\{0, b_c - \delta\}, b_c + \delta]$,

$$\left| \sup_{\{\alpha \in [0,\alpha^*]\}} \overline{F}(\alpha, b) - \sup_{\{\alpha \in [0,\alpha^*]\}} \overline{F}(\alpha, b_c) \right| \leq \frac{\varepsilon}{2}. \tag{177}$$

From (176) and (177), it follows that, for all $b \in [\max\{0, b_c - \delta\}, b_c + \delta]$,

$$|Z(b) - Z(b_c)| \leq \varepsilon. \tag{178}$$

We deduce that Z is continuous on $[0, +\infty)$.

Lemma 6 *Let $\chi_* : \mathbf{R}^{n+q-1} \to \mathbf{R}$, with $n \geq 1, q \geq 2$, be a nonnegative continuous function such that, for all $x \in \mathbf{R}^n$,*

$$\chi_*(x, 0, \ldots, 0) = 0. \tag{179}$$

Then, one can determine a continuous, positive and nondecreasing function ρ_* and a function ϕ_* of class K_∞, such that, for all $x \in \mathbf{R}^n$, $r_2 \in \mathbf{R}_{\geq 0}, \ldots, r_q \in \mathbf{R}_{\geq 0}$,

$$\chi_*(x, r_2, \ldots, r_q) \leq \phi_*\left(\sum_{k=2}^{q} r_k\right) \rho_*(|x|). \tag{180}$$

Proof The function χ_* satisfies, for all $x \in \mathbf{R}^n$, $r_2 \geq 0, \ldots, r_q \geq 0$,

$$\chi_*(x, r_2, \ldots, r_q) \leq F_*\left(|x|, \sum_{k=2}^{q} r_k\right) \tag{181}$$

where F_*, is defined by

$$F_*(s, R) = \sup_{\{(z,l_2,\ldots,l_q) \in E(s,R)\}} \chi_*(z, l_2, \ldots, l_q), \quad \forall s \geq 0, \mathbf{R} \geq 0 \tag{182}$$

with

$$E(s, R) = \left\{(z, l_2, \ldots, l_q) \in \mathbf{R}^n \times \mathbf{R}_{\geq 0}^{q-1} : |z| \leq s, l_k \in [0, R], k = 2, \ldots, q\right\}.$$

For all $s \geq 0$,

$$F_*(s, 0) = \sup_{\{(z,l_2,\ldots,l_q) \in E(s,0)\}} \chi_*(z, l_2, \ldots, l_q) = \sup_{\{z \in \mathbf{R}^n : |z| \leq s\}} \chi_*(z, 0, \ldots, 0). \quad (183)$$

According to (179), it follows that, for all $s \in \mathbf{R}_{\geq 0}$, $F_*(s, 0) = 0$. Moreover, F_* is nonnegative and nondecreasing with respect to each of its arguments which implies that, for all $s \in \mathbf{R}_{\geq 0}$, $R \in \mathbf{R}_{\geq 0}$,

$$F_*(s, R) \leq [F_*(s, s) + 1][F_*(R, R) + 1]. \quad (184)$$

It follows that Lemma 5 applies to the function F_* and provides a function Z, nonnegative, zero at zero, continuous and such that, for all $s \geq 0$, $R \geq 0$,

$$F_*(s, R) \leq (s^2 + 1)[F_*(s, s) + 1]Z(R). \quad (185)$$

From (181), it follows that, for all $x \in \mathbf{R}^n$, $r_2 \geq 0,\ldots,r_q \geq 0$,

$$|\chi_*(x, r_2, \ldots, r_q)| \leq Z\left(\sum_{k=2}^{q} r_k\right)\rho_*(|x|) \quad (186)$$

with ρ_* defined by

$$\rho_*(s) = (s^2 + 1)[F_*(s, s) + 1], \quad \forall s \geq 0. \quad (187)$$

This function is positive and nondecreasing on $\mathbf{R}_{\geq 0}$ and such that for all $x \in \mathbf{R}^n$, $r_2 \in \mathbf{R}_{\geq 0},\ldots,r_q \in \mathbf{R}_{\geq 0}$,

$$|\chi_*(x, r_2, \ldots, r_q)| \leq \phi_*\left(\sum_{k=2}^{q} r_k\right)\rho_*(|x|) \quad (188)$$

with, ϕ_* defined by

$$\phi_*(s) = s + \sup_{\{l \in [0,s]\}} Z(l), \ \forall s \geq 0. \quad (189)$$

One can prove easily that this function ϕ_* is of class K_∞.

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