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ORIGINAL ARTICLE

Higher order geodesics in Lie groups

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Abstract For all $n > 2$, we study nth order generalisations of Riemannian cubics, which are second-order variational curves used for interpolation in semi-Riemannian manifolds M. After finding two scalar constants of motion, one for all M, the other when M is locally symmetric, we take M to be a Lie group G with bi-invariant semi-Riemannian metric. The Euler–Lagrange equation is reduced to a system consisting of a *linking equation* and an equation in the Lie algebra. A Lax pair form of the second equation is found, as is an additional vector constant of motion, and a *duality* theory, based on the invariance of the Euler–Lagrange equation under group inversion, is developed. When G is *semisimple*, these results allow the linking equation to be solved by quadrature using methods of two recent papers; the solution is presented in the case of the rotation group $SO(3)$, which is important in rigid body motion planning.

Keywords Riemannian cubic · Riemannian polynomial · Geodesic · Lie quadratic · Lax equation · Lie group

1 Introduction

Problems of interpolation by variational curves in semi-Riemannian manifolds, and Lie groups in particular, arise in numerous applications. An important problem is that of trajectory planning for rigid body motion, where interpolation is

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done in either the group $SO(3)$ of real 3×3 orthogonal matrices of determinant 1 (rotations of Euclidean 3-space) or, more generally, the group

$$SE(3) := \left\{ \begin{bmatrix} \Theta & p \\ \mathbf{0} & 1 \end{bmatrix} : \Theta \in SO(3) \text{ and } p \in \mathbb{R}^3 \right\}$$

of rigid body motions. In such applications, velocities, accelerations and possibly higher order derivatives of the interpolant need to be moderated; this can be achieved by imposing a suitable variational condition. The trajectory planning problem motivated the introduction in [11] and [25] (independently) of a class of variational curves called *Riemannian cubics*. The present paper is concerned with certain higher order generalisations of Riemannian cubics, defined below.

We assume the reader is familiar with basic definitions and results of semi-Riemannian geometry; some references are [10,20]. Let M be a connected m-dimensional C^∞ semi-Riemannian manifold, $m \geq 1$, with metric $\langle \cdot, \cdot \rangle$ and Levi–Civita covariant derivative ∇. Choose the sign convention

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for the Riemannian curvature R, which is often defined with the opposite sign, as in [10,20]. Denote the covariant derivative of a C^∞ vector field u along a C^∞ curve $t \mapsto x(t) \in M$ by $\nabla_{\mathrm{d}/\mathrm{d}t} u$. Let $\nabla^k_{\mathrm{d}/\mathrm{d}t} u$ denote the k-fold covariant derivative of u along x, where $k \geq 2$, and set $\nabla^0_{\mathrm{d}/\mathrm{d}t} u := u$ and $\nabla^1_{\mathrm{d}/\mathrm{d}t} u := \nabla_{\mathrm{d}/\mathrm{d}t} u$. Let $x^{(1)}$ denote the velocity vector field of x. Similarly, denote the kth derivative of a real or vector-valued function of a scalar variable by a superscript (k). For an integer $n \geq 2$, consider minimising the functional Φ_n defined by

$$\Phi_n(x) := \int_0^1 \langle \nabla^{n-1}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t), \nabla^{n-1}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t) \rangle \mathrm{d}t$$

over the set $\mathcal{C}^{n-1}_{v_0,v_1}$ of all C^{2n-3} curves $x : [0,1] \to M$ that satisfy, for some fixed points $p_0, \ldots, p_\nu \in M$, fixed vectors $v_i^k \in T_{p_i}M$, where $i \in \{0, \nu\}$ and $k \in \{1, \ldots, n-1\}$, and fixed parameter values $t_0, \ldots, t_\nu \in [0,1]$ with $0 = t_0 < t_1 < \cdots < t_{\nu-1} < t_\nu = 1$, the conditions

(i) $x(t_i) = p_i$ for all $i \in \{0, \ldots, \nu\}$,
(ii) $\nabla^{k-1}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t_i) = v_i^k$ for all $k \in \{1, \ldots, n-1\}$ and $i \in \{0, \nu\}$,
(iii) the restriction of x to the interval $[t_i, t_{i+1}]$ is C^∞ for all $i \in \{0, \ldots, \nu - 1\}$.

This variational problem was introduced by Camarinha et al. [5] in the case where the metric on M is Riemannian, namely, the inner product on each tangent space is positive-definite. The aforementioned authors derived necessary conditions for a curve $x \in \mathcal{C}^{n-1}_{v_0,v_1}$ to minimise Φ_n. Their proof extends naturally to the general semi-Riemannian case, giving the following theorem.

Theorem 1 (Camarinha et al. [5]) *$x \in \mathcal{C}^{n-1}_{\nu_0,\nu_1}$ is a critical point of Φ_n if and only if x is C^{2n-2} and satisfies the differential equation*

$$\nabla^{2n-1}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t) + \sum_{j=2}^{n} (-1)^j R\left(\nabla^{2n-1-j}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t), \nabla^{j-2}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t)\right) x^{(1)}(t) = \mathbf{0} \qquad (1)$$

on each of the intervals $[t_i, t_{i+1}]$, where $i \in \{0, \ldots, \nu - 1\}$.

In [5], the critical points of Φ_n are called C^{2n-2} *geometric splines*. In the present paper, we investigate mathematical properties of solutions of (1), but do not explicitly consider piecing these curves together into splines. In [14], solutions of (1) are called *polynomial curves of order* $2n - 1$. Although solutions of (1) are indeed polynomial curves when M is Euclidean m-space E^m, namely $\mathbb{R}^m$ with metric the Euclidean inner product, it is important to note that they lack many of the properties of Euclidean polynomial curves when $M \neq E^m$. For instance, as noted in [18], a polynomial curve of order $2n - 1$ in the sense of [14] is not necessarily one of order $2(n + 1) - 1$. Therefore, we introduce new terminology for the present paper: we call a solution of (1) a *geodesic of order n*, or simply an *n-geodesic*. This name is justified by noting that Φ_n is a natural nth order generalisation of the functional $\Phi_1 : x \mapsto \int_0^1 \langle x^{(1)}(t), x^{(1)}(t)\rangle \mathrm{d}t$, defined on the set of all C^∞ curves $x : [0, 1] \to M$ satisfying $x(i) = p_i$ for some fixed points $p_i \in M$, $i \in \{0, 1\}$, whose critical points are geodesics, namely solutions of $\nabla_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t) = \mathbf{0}$.

The functional Φ_2 was first introduced in [11,25] (independently), and 2-geodesics, which are usually called *Riemannian cubics*, have since been studied by numerous authors from various points of view. We refer, in particular, to [2–4,6,8,9,13,18,19,21–23,28,34,36]. The trajectory planning problem for rigid body motion remains an important source of motivation, with much of the aforementioned literature focusing on $SO(3)$ or $SE(3)$. Of course, other types of curves can also be used for rigid body motion planning. These include generalisations to manifolds of elastica [17, Chap. 14], splines in tension [15,31,32] and curves constructed using algorithms from the field of computer-aided geometric design [7,12,27,29,30] (see [29] for several more references). There are relatively few cases where mathematical properties of 2-geodesics (Riemannian cubics) are well understood. So it is not surprising that little is known about n-geodesics with $n > 2$. Existing work seems to be limited to a handful of references, notably [14], in which some existence and multiplicity results for solutions of boundary value problems for (1) are established, and [18,36], in which 3-geodesics in $SO(3)$ and $SE(3)$, respectively, are investigated. In the present paper, we generalise several existing results about n-geodesics with $n \in \{2, 3\}$ to arbitrary n.

For most of the paper, we take M to be a Lie group G with bi-invariant semi-Riemannian metric. To put our new results into context, Sect. 2 reviews existing results about n-geodesics in G with $n \in \{2, 3\}$. The existing proofs are rather lengthy, especially for $n = 3$, and give few hints as to how (or even if) the existing results might generalise. By comparison, the proofs of our new results are pleasingly simple, elucidating structure common to all cases $n \geq 2$. In Sect. 3, (1) is reduced to a system of two differential equations: a first-order *linking equation* and an equation of order $2n-1$ in the Lie algebra of G. A Lax pair form of the second equation is found, as is an equivalent equation of order $2n-2$. In Sect. 4, we generalise results of [23] by developing a *duality* theory for n-geodesics, based on the fact that the group inverse of an n-geodesic is also an n-geodesic. When G is *semisimple* (with metric defined by left-translation of the Killing form), our new results allow the linking equation to be solved by quadrature, using methods developed in [24,26]. In Sect. 5, we present the solution in the case $G = SO(3)$. Before focusing on Lie groups, we prove two results about n-geodesics in more general semi-Riemannian manifolds.

Theorem 2 *If $x : I \to M$ is an n-geodesic then the following quantity is constant*:

$$\frac{1}{2}(-1)^{n+1}\langle \nabla_{\mathrm{d}/\mathrm{d}t}^{n-1} x^{(1)}(t), \nabla_{\mathrm{d}/\mathrm{d}t}^{n-1} x^{(1)}(t)\rangle + \sum_{j=2}^{n} (-1)^j \langle \nabla_{\mathrm{d}/\mathrm{d}t}^{2n-j} x^{(1)}(t), \nabla_{\mathrm{d}/\mathrm{d}t}^{j-2} x^{(1)}(t)\rangle.$$

Proof 1 Denoting the above quantity by $b(t)$ and differentiating, most terms in the resulting sums cancel, leaving $b^{(1)}(t) = \langle \nabla_{\mathrm{d}/\mathrm{d}t}^{2n-1} x^{(1)}(t), x^{(1)}(t)\rangle$. So, by (1), and since, for all vector fields X, Y, Z,

$$\langle R(X, Y)Z, Z\rangle = \mathbf{0}, \tag{2}$$

we have $b^{(1)}(t) = -\sum_{j=2}^{n} (-1)^j \langle R(\nabla_{\mathrm{d}/\mathrm{d}t}^{2n-1-j} x^{(1)}(t), \nabla_{\mathrm{d}/\mathrm{d}t}^{j-2} x^{(1)}(t)) x^{(1)}(t), x^{(1)}(t)\rangle = 0$.

Theorem 2 has also been proved independently by Luis Machado, who presented it as part of a talk given at The University of Western Australia in September 2005; the case $n = 2$ was proved in [4,21]. Although these authors only considered the case where $\langle \cdot, \cdot \rangle$ is Riemannian, the proofs are also valid in the general semi-Riemannian case. Our second result holds when M is locally symmetric, i.e. when the covariant differential of the curvature tensor field $(X, Y, Z, W) \mapsto \langle R(X, Y)Z, W\rangle$ is zero. In this case, for all vector fields X, Y, Z, W along a curve $t \mapsto x(t)$ in M,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\langle R(X, Y)Z, W\rangle &= \langle R(\nabla_{\mathrm{d}/\mathrm{d}t} X, Y)Z, W\rangle + \langle R(X, \nabla_{\mathrm{d}/\mathrm{d}t} Y)Z, W\rangle \\ &\quad + \langle R(X, Y)\nabla_{\mathrm{d}/\mathrm{d}t} Z, W\rangle + \langle R(X, Y)Z, \nabla_{\mathrm{d}/\mathrm{d}t} W\rangle. \end{aligned} \tag{3}$$

Note that if M is a Lie group with $\langle \cdot, \cdot \rangle$ bi-invariant then M is locally symmetric (see [20]). Part (i) of the following theorem was proved in [21] (the proof is valid whenever $\langle \cdot, \cdot \rangle$ is semi-Riemannian).

Theorem 3 *Suppose M is locally symmetric and let $x : I \to M$ be an n-geodesic. Then*

(i) *if $n = 2$, the following quantity is constant*:

$$\langle \nabla^2_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t), \nabla^2_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t)\rangle + \langle R(\nabla_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t), x^{(1)}(t))x^{(1)}(t), \nabla_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t)\rangle,$$

(ii) *if $n \geq 3$, the following quantity is constant*:

$$\begin{aligned}&\langle \nabla^{2n-2}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t), \nabla^{2n-2}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t)\rangle\\ &\quad + \sum_{j=2}^{n} \langle R(\nabla^{2n-1-j}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t), \nabla^{j-2}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t))\nabla^{j-2}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t), \nabla^{2n-1-j}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t)\rangle\\ &\quad - 2\sum_{i=3}^{n}\sum_{j=i}^{n} (-1)^{i+j} \langle R(\nabla^{2n-1-j}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t), \nabla^{j-2}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t))\nabla^{i-3}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t), \nabla^{2n-i}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t)\rangle.\end{aligned}$$

Proof 2 Part (ii) is proved by differentiating the above quantity with respect to t, using (3), and then using (1), (2) and the fact that $\langle R(X, Y)Z, W\rangle = \langle R(Z, W)X, Y\rangle$ for all X, Y, Z, W. The calculation is straightforward but somewhat tedious, so we omit the details but note that

$$\begin{aligned}\xi^{(1)}(t) &= \sum_{j=3}^{n} \langle R(\nabla^{2n-j}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t), \nabla^{j-3}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t))\nabla^{j-2}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t), \nabla^{2n-j}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t)\rangle\\ &\quad + \sum_{j=3}^{n} \langle R(\nabla^{2n-1-j}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t), \nabla^{j-2}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t))\nabla^{j-2}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t), \nabla^{2n-j}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t)\rangle\\ &\quad - \sum_{j=3}^{n} (-1)^{j} \langle R(\nabla^{2n-1-j}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t), \nabla^{j-2}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t))x^{(1)}(t), \nabla^{2n-2}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t)\rangle,\end{aligned}$$

where $\xi(t) := \sum_{i=3}^{n}\sum_{j=i}^{n}(-1)^{i+j}\langle R(\nabla^{2n-1-j}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t), \nabla^{j-2}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t))\nabla^{i-3}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t),$ $\nabla^{2n-i}_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t)\rangle$.

2 Review of existing results

From now on, take M to be a Lie group G, with identity e, Lie algebra $\mathcal{G} := T_eG$ and Lie bracket $[\cdot,\cdot]$. First we recall some facts about Lie groups and semi-Riemannian geometry; a reference for Lie groups is [33]. For $g \in G$, let $L_g, R_g : G \to G$ be the left and right multiplications by g, i.e. $L_g(h) := gh$ and $R_g(h) := hg$ for all $h \in G$. The semi-Riemannian metric $\langle\cdot,\cdot\rangle$ on G is *bi-invariant* if it is left-invariant and right-invariant, i.e. if, for all $g \in G$, L_g and R_g are both isometries. The derivative $\mathrm{Ad}_g := (dI_g)_e : \mathcal{G} \to \mathcal{G}$ at e of the inner automorphism $I_g := R_{g^{-1}} \circ L_g$ of G is a Lie algebra automorphism. The derivative ad at e of the *adjoint representation* $\mathrm{Ad} : g \mapsto \mathrm{Ad}_g$ of G is given by $\mathrm{ad}_u(v) = [u, v]$ for all $u, v \in \mathcal{G}$. An arbitrary symmetric bilinear form $\langle\cdot,\cdot\rangle_e$ on $\mathcal{G}$ (not necessarily the restriction of $\langle\cdot,\cdot\rangle$ to $\mathcal{G}$) is *ad-invariant* if

$$\langle [u, v], w\rangle_e = \langle [w, u], v\rangle_e \quad \text{for all } u, v, w \in \mathcal{G}. \tag{4}$$

If $\langle\cdot,\cdot\rangle$ is bi-invariant then (4) holds. Conversely, if a non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle_e$ on $\mathcal{G}$ is *ad*-invariant then the left-invariant semi-Riemannian metric $\langle\cdot,\cdot\rangle$ on G defined by *left-translation* of $\langle\cdot,\cdot\rangle_e$, namely by setting $\langle v, w\rangle_g := \langle (dL_{g^{-1}})_g(v), (dL_{g^{-1}})_g(w)\rangle_e$ for all $g \in G$ and all $v, w \in T_gG$, is bi-invariant. Many Lie groups, including $SO(3)$ and $SE(3)$, admit a bi-invariant Riemannian, or at least semi-Riemannian, metric:

Example 1 The symmetric bilinear Killing form $K : \mathcal{G} \times \mathcal{G} \to \mathbb{R}$, defined by $K(u, v) := \text{trace}(ad_u \circ ad_v)$, is *ad*-invariant. If K is non-degenerate then G is called *semisimple*. In this case, left-translation of K defines a bi-invariant semi-Riemannian metric on G.

Example 2 Suppose $G = SO(3)$. Then $\mathcal{G} = so(3)$, the set of all skew-symmetric real 3×3 matrices. Recall that $\mathbb{R}^3$ is a Lie algebra with Lie bracket the cross product $\times$, and that the map $B : \mathbb{R}^3 \to so(3)$ defined by $B(v)w = v \times w$ is a Lie algebra isomorphism. Any inner product $\langle\cdot,\cdot\rangle'$ on $\mathbb{R}^3$ satisfying (4), i.e. $\langle u \times v, w\rangle' = \langle w \times u, v\rangle'$, is a positive multiple of the Euclidean inner product. Therefore, $SO(3)$ admits a bi-invariant Riemannian metric $\langle\cdot,\cdot\rangle$, namely the left-translation of the inner product on $so(3)$ defined by declaring B to be an isometry from E^3 to $so(3)$, and any bi-invariant Riemannian metric on $SO(3)$ is a positive multiple of $\langle\cdot,\cdot\rangle$. (Note that such a Riemannian metric is also defined by left-translation of a negative multiple of the Killing form, which is non-degenerate and negative-definite in the case of $SO(3)$).

Example 3 If $G = SE(3)$ then

$$\mathcal{G} = se(3) := \left\{ \begin{bmatrix} \theta & v \\ \mathbf{0} & 0 \end{bmatrix} : \theta \in so(3) \text{ and } v \in \mathbb{R}^3 \right\}.$$

Suppose constants $\alpha, \beta \in \mathbb{R}$ are chosen such that the symmetric bilinear form on $\mathbb{R}^6$ defined by $(u, v) \mapsto u^T Q(\alpha, \beta) v$, where

$$Q(\alpha, \beta) := \begin{bmatrix} \alpha I & \beta I \\ \beta I & \mathbf{0} \end{bmatrix}$$

and I denotes the 3×3 identity matrix, is non-degenerate. Then, identifying elements of $se(3)$ with 6-dimensional column vectors $[B^{-1}(\theta)^T v^T]^{\mathrm{T}}$, the semi-Riemannian metric on $SE(3)$ defined by left-translation of this form is bi-invariant [35]. One such non-degenerate form is the Klein form, which corresponds to $Q(0, 1)$. The Killing form, corresponding to $Q(1, 0)$, is degenerate.

Example 4 If G is compact then G admits a bi-invariant metric [10, p. 47].

From now on, assume the semi-Riemannian metric $\langle\cdot,\cdot\rangle$ on G is bi-invariant. Let $x : I \to G$ be a C^∞ curve. The *Lie reduction* of a C^∞ vector field u along x is the C^∞ curve $U : I \to \mathcal{G}$ defined by

$$U(t) := (\mathrm{d}L_{x(t)^{-1}})_{x(t)} u(t).$$

The Lie reduction of $x^{(1)}$ is denoted by V, i.e.

$$V(t) := (\mathrm{d}L_{x(t)^{-1}})_{x(t)} x^{(1)}(t). \tag{5}$$

Equivalently, x satisfies the first-order differential equation

$$x^{(1)}(t) = (\mathrm{d}L_{x(t)})_e V(t), \tag{6}$$

which we call the *linking equation*, as in [23]. Allowing for our opposite sign convention for R, [20, Theorem 21.3] (which applies whenever $\langle\cdot,\cdot\rangle$ is semi-Riemannian) gives the following lemma.

Lemma 1 *For any C^∞ vector fields u_1, u_2, u_3 along x,*

(i) $(\mathrm{d}L_{x(t)^{-1}})_{x(t)} \nabla_{\mathrm{d}/\mathrm{d}t} u_1(t) = U_1^{(1)}(t) + \frac{1}{2}[V(t), U_1(t)]$,
(ii) $(\mathrm{d}L_{x(t)^{-1}})_{x(t)} R(u_1(t), u_2(t)) u_3(t) = \frac{1}{4}[U_3(t), [U_1(t), U_2(t)]]$,

for all $t \in I$, where U_i is the Lie reduction of u_i, $i \in \{1, 2, 3\}$.

We also need the following result, which is straightforward to prove.

Lemma 2 *For any differentiable curve $W : I \to \mathcal{G}$,*

$$\frac{\mathrm{d}}{\mathrm{d}t} A\mathrm{d}_{x(t)} W(t) = A\mathrm{d}_{x(t)} (a\mathrm{d}_{V(t)} W(t) + W^{(1)}(t))$$

for all $t \in I$.

We now review existing results of about n-geodesics with $n \in \{2, 3\}$, using the existing proofs in order to demonstrate the inherent difficulties of naïve generalisation to arbitrary n. Although most of the results we review were originally proved only in the case where $\langle\cdot,\cdot\rangle$ is Riemannian, the proofs also apply in the general semi-Riemannian case; we make no further mention of this.

2.1 2-Geodesics

By repeatedly using Lemma 1(i), Lie reductions of covariant derivatives of $x^{(1)}$ can be expressed in terms of derivatives of V. In particular, we have, for all $t \in I$,

$$(\mathrm{d}L_{x(t)^{-1}})_{x(t)}\nabla_{\mathrm{d}/\mathrm{d}t}x^{(1)}(t) = V^{(1)}(t), \tag{7}$$

$$(\mathrm{d}L_{x(t)^{-1}})_{x(t)}\nabla^2_{\mathrm{d}/\mathrm{d}t}x^{(1)}(t) = V^{(2)}(t) + \tfrac{1}{2}[V(t), V^{(1)}(t)], \tag{8}$$

$$(\mathrm{d}L_{x(t)^{-1}})_{x(t)}\nabla^3_{\mathrm{d}/\mathrm{d}t}x^{(1)}(t) = V^{(3)}(t)+[V(t), V^{(2)}(t)]+\tfrac{1}{4}[V(t), [V(t), V^{(1)}(t)]]. \tag{9}$$

By (1), x is a 2-geodesic when

$$\nabla^3_{\mathrm{d}/\mathrm{d}t}x^{(1)}(t) + R(\nabla_{\mathrm{d}/\mathrm{d}t}x^{(1)}(t), x^{(1)}(t))x^{(1)}(t) = \mathbf{0}$$

for all $t \in I$. So (7)–(9) and Lemma 1(ii) give the following result, which was first proved in [25] in the case $G = SO(3)$, and then in generality in [8,9].

Proposition 1 (Crouch and Silva Leite [8,9]) *$x : I \to G$ is a 2-geodesic if and only if*

$$V^{(3)}(t) = [V^{(2)}(t), V(t)] \tag{10}$$

for all $t \in I$.

Solutions of (10) are called *Lie quadratics* in [21–23]. We can integrate (10) once to obtain the following result, which was noted in [25] in the case $G = SO(3)$, and in generality in [21].

Corollary 1 (Noakes [21]) *$x : I \to G$ is a 2-geodesic if and only if*

$$V^{(2)}(t) = [V^{(1)}(t), V(t)] + C$$

for some constant $C \in \mathcal{G}$ and all $t \in I$.

For $n = 2$ and $M = G$, the constants of Theorems 2 and 3 can be rewritten using (4), (5), (7), (8) and left-invariance of $\langle \cdot, \cdot \rangle$, giving the following result. An alternative proof, given in [21], is presented below.

Corollary 2 *If $x : I \to G$ is a 2-geodesic then the following quantities are constant*:

(i) $\langle V^{(2)}(t), V^{(2)}(t)\rangle$,
(ii) $\langle V^{(2)}(t), V(t)\rangle - \frac{1}{2}\langle V^{(1)}(t), V^{(1)}(t)\rangle$.

Proof 3 By Proposition 1 and (4), $\frac{d}{\mathrm{d}t}\langle V^{(2)}(t), V^{(2)}(t)\rangle = 2\langle [V^{(2)}(t), V(t)], V^{(2)}(t)\rangle = 0$, proving (i). By Corollary 1 and (4), $\frac{d}{\mathrm{d}t}\langle V^{(1)}(t), V^{(1)}(t)\rangle = 2\langle [V^{(1)}(t), V(t)] + C, V^{(1)}(t)\rangle = \frac{d}{\mathrm{d}t}2\langle C, V(t)\rangle$ and $\langle C, V(t)\rangle = \langle V^{(2)}(t), V(t)\rangle$, proving (ii).

Now define $V^* : I \to \mathcal{G}$ by

$$V^*(t) := -Ad_{x(t)}V(t). \tag{11}$$

If x is a 2-geodesic, V and V^* are said to be *dual* [23] for the following reason.

Theorem 4 (Noakes [23]) *If $x : I \to G$ is a 2-geodesic then so is $y : I \to G$, where $y(t) := x(t)^{-1}$, and the Lie reduction of $y^{(1)}$ is V^*.*

Proof 4 By Corollary 1, (6) and Lemma 2,

$$\begin{aligned} V^{*(1)}(t) &= -\mathrm{Ad}_{x(t)}(\mathrm{ad}_{V(t)}V(t) + V^{(1)}(t)) = -\mathrm{Ad}_{x(t)}V^{(1)}(t),\\ V^{*(2)}(t) &= -\mathrm{Ad}_{x(t)}(\mathrm{ad}_{V(t)}V^{(1)}(t) + V^{(2)}(t)) = -\mathrm{Ad}_{x(t)}C,\\ V^{*(3)}(t) &= -\mathrm{Ad}_{x(t)}(\mathrm{ad}_{V(t)}C) = -\mathrm{Ad}_{x(t)}[V(t), C] = [V^{*(2)}(t), V^*(t)], \end{aligned}$$

for all $t \in I$. So V^* is a solution of (10). By Proposition 1, it remains to show that V^* is the Lie reduction of $y^{(1)}$: since $x(t)y(t)$ is constant, we have $(\mathrm{d}L_{x(t)})_{y(t)}y^{(1)}(t) + (dR_{y(t)})_{x(t)}x^{(1)}(t) = \mathbf{0}$ and thus, by (6), $(\mathrm{d}L_{x(t)})_{y(t)}y^{(1)}(t) = -(dR_{y(t)})_{x(t)} \circ (\mathrm{d}L_{x(t)})_e V(t) = -Ad_{x(t)}V(t) = V^*(t)$.

When $G = SO(3)$, Theorem 4 forms part of a method developed in [23] for solving (6) by quadrature for a 2-geodesic x in terms of a Lie quadratic V; in rare cases, x^{-1} must be found in terms of V^*. Of course, to use such a method in practice, we need to be able to solve (10). On the other hand, the complexity of some of the Lie quadratics in [22] suggests that any kind of integrability result for 2-geodesics is worth having. The method of [23] relies on the fact that V is part of a *Lax pair* (V, Z_2) of curves in $\mathcal{G}$: setting $Z_2 := V^{(2)}$, (10) is equivalent to the *Lax equation*

$$Z_2^{(1)}(t) = [Z_2(t), V(t)]. \tag{12}$$

The method was extended in [26] to allow solution of (6) when V satisfies an (almost) arbitrary equation of the form (12). The papers [23,26] also solve (6) when $G = SO(1,2)$, the group of all real 3×3 matrices that preserve the Lorentz inner product on $\mathbb{R}^3$ and have determinant 1. Noakes [24] has since developed solutions of (6) subject to constraints of the form (12) in all *semisimple* G. In Sect. 3, we show that V satisfies a Lax equation when x is an n-geodesic with $n > 2$, so that (6) can be solved for x (when G is *semisimple*); as illustration, the solution in $G = SO(3)$ is presented in Sect. 5. Lax equations are central to the theory of integrable systems, since if a matrix differential equation can be put in the form (12) then the spectrum of Z_2 is preserved by the flow. They appear in many classical mechanical systems; a well-known example is Euler's equation for geodesics in a Lie group with left-invariant Riemannian metric. More background on Lax equations can be found in, for instance, [1].

2.2 3-Geodesics

By (1), x is a 3-geodesic when

$$\nabla^5_{\mathrm{d}/\mathrm{d}t}x^{(1)}(t)+R(\nabla^3_{\mathrm{d}/\mathrm{d}t}x^{(1)}(t), x^{(1)}(t))x^{(1)}(t)-R(\nabla^2_{\mathrm{d}/\mathrm{d}t}x^{(1)}(t), \nabla_{\mathrm{d}/\mathrm{d}t}x^{(1)}(t))x^{(1)}(t) = \mathbf{0} \tag{13}$$

for all $t \in I$. By Lemma 1(i), with $U_1(t)$ the right-hand side of (9), and the Jacobi identity,

$$\begin{aligned}(\mathrm{d}L_{x(t)^{-1}})_{x(t)}\nabla^4_{\mathrm{d}/\mathrm{d}t}x^{(1)}(t) &= V^{(4)}(t) + \tfrac{3}{2}[V(t), V^{(3)}(t)] + [V^{(1)}(t), V^{(2)}(t)]\\ &\quad + \tfrac{1}{4}[V^{(1)}(t), [V(t), V^{(1)}(t)]] + \tfrac{3}{4}[V(t), [V(t), V^{(2)}(t)]]\\ &\quad + \tfrac{1}{8}[V(t), [V(t), [V(t), V^{(1)}(t)]]]\end{aligned}$$

for all $t \in I$. Similarly,

$$\begin{aligned}(\mathrm{d}L_{x(t)^{-1}})_{x(t)}\nabla^5_{\mathrm{d}/\mathrm{d}t}x^{(1)}(t) &= V^{(5)}(t) + 2[V(t), V^{(4)}(t)] + \tfrac{5}{2}[V^{(1)}(t), V^{(3)}(t)]\\ &\quad + \tfrac{1}{4}[V^{(2)}(t), [V(t), V^{(1)}(t)]] + [V^{(1)}(t), [V(t), V^{(2)}(t)]]\\ &\quad + \tfrac{1}{8}[V^{(1)}(t), [V(t), [V(t), V^{(1)}(t)]]]\\ &\quad + \tfrac{3}{2}[V(t), [V(t), V^{(3)}(t)]]\\ &\quad + \tfrac{5}{4}[V(t), [V^{(1)}(t), V^{(2)}(t)]]\\ &\quad + \tfrac{1}{4}[V(t), [V^{(1)}(t), [V(t), V^{(1)}(t)]]]\\ &\quad + \tfrac{1}{2}[V(t), [V(t), [V(t), V^{(2)}(t)]]]\\ &\quad + \tfrac{1}{16}[V(t), [V(t), [V(t), [V(t), V^{(1)}(t)]]]].\end{aligned}$$

Using this expression together with (7)–(9), Lemma 1(ii), (13) and the Jacobi identity, a lengthy calculation gives the following result. This result was first proved by Žefran et al. [36] in the case $G = SO(3)$, and the proof is essentially the same for any G.

Proposition 2 (Žefran et al. [36]) *$x : I \to G$ is a 3-geodesic if and only if*

$$\begin{aligned}V^{(5)}(t) &= 2[V^{(4)}(t), V(t)] + \tfrac{5}{2}[V^{(3)}(t), V^{(1)}(t)] - \tfrac{5}{4}[[V^{(3)}(t), V(t)], V(t)]\\ &\quad - \tfrac{5}{4}[[V^{(2)}(t), V^{(1)}(t)], V(t)] - \tfrac{5}{4}[[V^{(2)}(t), V(t)], V^{(1)}(t)]\\ &\quad + \tfrac{1}{4}[[[V^{(2)}(t), V(t)], V(t)], V(t)] + \tfrac{1}{2}[[[V^{(1)}(t), V(t)], V^{(1)}(t)], V(t)]\end{aligned} \tag{14}$$

for all $t \in I$.

Note that Žefran et al. [36] actually investigated 3-geodesics in $SE(3)$ with the *left*-invariant (but not bi-invariant) Riemannian metric defined by replacing the matrix $Q(\alpha, \beta)$ in Example 3 by

$$\begin{bmatrix} \alpha I & \mathbf{0} \\ \mathbf{0} & \beta I \end{bmatrix} \quad \text{for some } \alpha, \beta > 0.$$

The projections of these 3-geodesics to $SO(3)$ are 3-geodesics of a bi-invariant Riemannian metric. It was shown in [36] that the Lie reductions of the velocity vector fields of these projections satisfy (14). Next, another lengthy calculation gives the following analogue of Corollary 1, which was proved in the case $G = SO(3)$ by Krakowski [18] (who also gives a proof of the case $G = SO(3)$ of Proposition 2). Again, the proof is essentially the same for any G.

Corollary 3 (Krakowski [18]) *$x : I \to G$ is a 3-geodesic if and only if*

$$V^{(4)}(t) = 2[V^{(3)}(t), V(t)] + \tfrac{1}{2}[V^{(2)}(t), V^{(1)}(t)] - \tfrac{5}{4}[[V^{(2)}(t), V(t)], V(t)] + \tfrac{1}{4}[[[V^{(2)}(t), V(t)], V(t)], V(t)] + C \tag{15}$$

for some constant $C \in \mathcal{G}$ and all $t \in I$.

When x is an n-geodesic with $n > 3$, (1) and Lemma 1 again give a differential equation for V, as in Propositions 1 and 2. However, as illustrated by (14), this equation becomes very complicated as n increases, since the Lie reduction of $\nabla^k_{\mathrm{d}/\mathrm{d}t} x^{(1)}$ becomes very complicated as k increases. This makes it very difficult to prove further results, especially ones which are valid for arbitrary n. Even for $n = 3$, the fact that (14) can be integrated to give (15) is far from obvious. So, although it seems plausible that an analogue of Corollary 1 holds for arbitrary n, it is unclear how this can be proved. The following two sections show how these difficulties can be avoided, allowing not only Corollary 1, but all results of Sect. 2.1, to be generalised.

3 Lax equations and constants of motion

Let $x : I \to G$ be a C^∞ curve and $n \geq 2$ a positive integer. Again define $V : I \to \mathcal{G}$ by (5). Similarly, for each integer $k \geq 1$, define $V_k : I \to \mathcal{G}$ by

$$V_k(t) := (\mathrm{d}L_{x(t)^{-1}})_{x(t)} \nabla^k_{\mathrm{d}/\mathrm{d}t} x^{(1)}(t). \tag{16}$$

Set $V_0 := V$, so that (16) also holds for $k = 0$. We state and prove our new results in terms of the V_k, not in terms of derivatives of V as in Sect. 2. This approach has many advantages, the first of which is that, when x is an n-geodesic, the $(2n-1)$th order differential equation for V can be written down immediately in terms of the V_k. Indeed, defining $Y_n : I \to \mathcal{G}$ by

$$Y_n(t) := \frac{1}{2}\sum_{j=2}^{n}(-1)^j [V_{2n-1-j}(t), V_{j-2}(t)],$$

it is clear from (1) and Lemma 1(ii) that Proposition 1 generalises as follows.

Proposition 3 *$x : I \to G$ is an n-geodesic if and only if*

$$V_{2n-1}(t) = \frac{1}{2}[Y_n(t), V(t)] \tag{17}$$

for all $t \in I$.

The simple form (17) of the otherwise complicated $(2n-1)$th order differential equation for V allows the remaining results of Sect. 2.1 to be generalised in a fairly straightforward manner. First note the following immediate consequence of Lemma 1(i) (which could be used to write (17) in terms of derivatives of V).

Lemma 3 *For all $k \geq 1$, $V_k(t) = V_{k-1}^{(1)}(t) + \frac{1}{2}[V(t), V_{k-1}(t)]$ for all $t \in I$.*

Define $Z_n : I \to \mathcal{G}$ by

$$Z_n(t) := V_{2n-2}(t) + Y_n(t).$$

Then (12) and Corollaries 1 and 3 generalise as follows.

Theorem 5 *The following statements are equivalent*:

(i) *$x : I \to G$ is an n-geodesic.*
(ii) *The Lie reduction $V : I \to \mathcal{G}$ of $x^{(1)}$ satisfies, for some constant $C \in \mathcal{G}$ and all $t \in I$,*

$$V_{2n-2}(t) = Y_n(t) + C.$$

(iii) *The Lie reduction $V : I \to \mathcal{G}$ of $x^{(1)}$ satisfies, for all $t \in I$, the Lax equation*

$$Z_n^{(1)}(t) = [Z_n(t), V(t)].$$

Proof 5 We first compute $Y_n^{(1)}(t)$. By Lemma 3 and the Jacobi identity,

$$Y_n^{(1)}(t) = \frac{1}{2}[Y_n(t), V(t)] + \frac{1}{2}\sum_{j=2}^{n}(-1)^j[V_{2n-j}(t), V_{j-2}(t)]$$
$$+\frac{1}{2}\sum_{j=2}^{n}(-1)^j[V_{2n-1-j}(t), V_{j-1}(t)]$$

for all $t \in I$. Most terms in the sums cancel, leaving

$$Y_n^{(1)}(t) = \frac{1}{2}[Y_n(t), V(t)] + \frac{1}{2}[V_{2n-2}(t), V(t)]. \tag{18}$$

We now prove the equivalence of (i) and (ii). First suppose (i) holds. Then, by Proposition 3, (18) and Lemma 3, $Y_n^{(1)}(t) = V_{2n-2}^{(1)}(t)$. So (ii) holds. Conversely, if (ii) holds then $V_{2n-2}^{(1)}(t) = Y_n^{(1)}(t)$. So, by (18) and Lemma 3, $V_{2n-1}(t) = \frac{1}{2}[Y_n(t), V(t)]$. Thus (i) holds, by Proposition 3. It now suffices to prove the equivalence of (ii) and (iii). First suppose (ii) holds. Then $Z_n^{(1)}(t) = 2Y_n^{(1)}(t)$. On the other hand, $[Z_n(t), V(t)] = 2Y_n^{(1)}(t)$, by (18). So (iii) holds. Conversely, if (iii) holds then $V_{2n-2}^{(1)}(t) + Y_n^{(1)}(t) = [Z_n(t), V(t)] = 2Y_n^{(1)}(t)$, which implies (ii).

For $M = G$, the constant of Theorem 3 can be rewritten using (4), Lemma 1(ii) and left-invariance of $\langle\cdot,\cdot\rangle$ to give part (i) of the following corollary (which we also prove in a different way), which generalises Corollary 2(i). For part (ii), note that, by Ado's Theorem, the finite-dimensional Lie algebra $\mathcal{G}$ can be identified with a matrix Lie algebra (see [16] for a proof).

Corollary 4 *If $x : I \to G$ is an n-geodesic then*

(i) $\langle Z_n(t), Z_n(t)\rangle$ *is constant,*
(ii) *the spectrum of $Z_n(t)$ is independent of t.*

Proof 6 By Theorem 5, the Lax equation $Z_n^{(1)}(t) = [Z_n(t), V(t)]$ holds. So (4) implies (i). It is well known that (ii) holds for an arbitrary Lax equation; the proof can be found in [1].

The constant of Theorem 2 can be rewritten as follows, generalising Corollary 2(ii).

Corollary 5 *If $x : I \to G$ is an n-geodesic then the following quantity is constant*:

$$\frac{1}{2}(-1)^{n+1}\langle V_{n-1}(t), V_{n-1}(t)\rangle + \sum_{j=2}^{n}(-1)^j\langle V_{2n-j}(t), V_{j-2}(t)\rangle.$$

The above results reveal properties common to n-geodesics for all $n \geq 2$, with differential equations and constants of motion given in very simple forms. In particular, they not only allow the results of Sect. 2.2 to be readily recovered, but also reveal new properties of 3-geodesics:

Example 5 By Proposition 3, the fifth order differential equation (14) reads $V_5(t) = [Y_3(t), V(t)]$, with $Y_3(t) = \frac{1}{2}([V_3(t), V(t)] - [V_2(t), V_1(t)])$. By Theorem 5, it is equivalent to the fourth order equation $V_4(t) = Y_3(t) + C$. So we recover Corollary 3. Theorem 5 also reveals the Lax pair form of (14), namely $Z_3^{(1)}(t) = [Z_3(t), V(t)]$, with $Z_3(t) = V_3(t) - Y_3(t)$. In particular, $\langle Z_3(t), Z_3(t)\rangle$ is constant if V is a solution of (14), by Corollary 4(i). By Corollary 5, $\frac{1}{2}\langle V_2(t), V_2(t)\rangle + \langle V_4(t), V(t)\rangle - \langle V_3(t), V_1(t)\rangle$ is also constant.

In the following section, Theorem 4 is generalised to n-geodesics with $n > 2$. The result is used, together with the Lax pair form of (17) given in Theorem 5, in Sect. 5 to solve (6) by quadrature for an n-geodesic x in terms of a solution V of (17) in the case $G = SO(3)$.

4 Duality

With all definitions as in Sect. 3, now define $y : I \to G$ by $y(t) := x(t)^{-1}$. The theory presented here relies on the following result.

Lemma 4 *For all $k \geq 0$ and all $t \in I$,*

$$(\mathrm{d}L_{y(t)^{-1}})_{y(t)} \nabla^k_{\mathrm{d}/\mathrm{d}t} y^{(1)}(t) = -Ad_{x(t)} V_k(t). \tag{19}$$

Proof 7 Since $x(t)y(t)$ is constant, $(\mathrm{d}L_{x(t)})_{y(t)} y^{(1)}(t) + (\mathrm{d}R_{y(t)})_{x(t)} x^{(1)}(t) = \mathbf{0}$ for all $t \in I$. So, by (6), (19) holds for $k = 0$:

$$(\mathrm{d}L_{x(t)})_{y(t)} y^{(1)}(t) = -(\mathrm{d}R_{y(t)})_{x(t)} \circ (\mathrm{d}L_{x(t)})_e V(t) = -Ad_{x(t)} V(t). \tag{20}$$

The proof is completed by induction on k. Suppose (19) holds (for some k) and set

$$U(t) := (\mathrm{d}L_{y(t)^{-1}})_{y(t)} \nabla^{k+1}_{\mathrm{d}/\mathrm{d}t} y^{(1)}(t).$$

By (19), Lemma 1(i) and (20), $U(t) = \frac{\mathrm{d}}{\mathrm{d}t}(-Ad_{x(t)} V_k(t)) + \frac{1}{2}[-Ad_{x(t)} V(t), -Ad_{x(t)} V_k(t)]$. So

$$\begin{aligned} U(t) &= -Ad_{x(t)} V_k^{(1)}(t) - \frac{1}{2}[-Ad_{x(t)} V(t), -Ad_{x(t)} V_k(t)] \\ &= -Ad_{x(t)} \left(V_k^{(1)}(t) + \frac{1}{2}[V(t), V_k(t)] \right), \end{aligned}$$

by Lemma 2. Thus, by Lemma 3, $U(t) = -Ad_{x(t)} V_{k+1}(t)$, as required.

As in Sect. 2.1, define $V^* : I \to \mathcal{G}$ by (11). For each $k \geq 1$, define $V_k^* : I \to \mathcal{G}$ by

$$V_k^*(t) := -Ad_{x(t)} V_k(t). \tag{21}$$

Set $V_0^* := V^*$, so that (21) holds for $k = 0$. By Lemma 4,

$$V_k^*(t) = (\mathrm{d}L_{y(t)^{-1}})_{y(t)} \nabla^k_{\mathrm{d}/\mathrm{d}t} y^{(1)}(t) \tag{22}$$

for all $k \geq 0$ and all $t \in I$. Define $Y_n^* : I \to \mathcal{G}$ by

$$Y_n^*(t) := \frac{1}{2} \sum_{j=2}^{n} (-1)^j [V_{2n-1-j}^*(t), V_{j-2}^*(t)].$$

The *duality* theory of 2-geodesics (Theorem 4) generalises as follows.

Theorem 6 *If $x : I \to G$ is an n-geodesic then so is $y : I \to G$, where $y(t) := x(t)^{-1}$, and the Lie reduction of $y^{(1)}$ is V^*.*

Proof 8 By Proposition 3 and (22), it remains to show that $V_{2n-1}^*(t) = \frac{1}{2}[Y_n^*(t), V^*(t)]$ for all $t \in I$. By (21), and since $Ad_{x(t)}$ is a Lie algebra automorphism, $Ad_{x(t)} Y_n(t) = Y_n^*(t)$. So, by Proposition 3, $-Ad_{x(t)} V_{2n-1}(t) = -Ad_{x(t)} \frac{1}{2}[Y_n(t), V(t)] = \frac{1}{2}[Y_n^*(t), V^*(t)]$.

Define $Z_n^* : I \to \mathcal{G}$ by

$$Z_n^*(t) := V_{2n-2}^*(t) + Y_n^*(t).$$

Theorems 5 and 6 and Corollary 4 give the following result. (For statement (iii), recall again that $\mathcal{G}$ can be identified with a matrix Lie algebra.)

Corollary 6 *If $x : I \to G$ is an n-geodesic then*

(i) $Z_n^{*(1)}(t) = [Z_n^*(t), V^*(t)]$ *for all* $t \in I$,
(ii) $\langle Z_n^*(t), Z_n^*(t)\rangle$ *is constant,*
(iii) *the spectrum of* $Z_n^*(t)$ *is independent of* t,
(iv) $V_{2n-2}^*(t) = Y_n^*(t) + C^*$ *for some constant* $C^* \in \mathcal{G}$ *and all* $t \in I$.

By bi-invariance of $\langle\cdot,\cdot\rangle$, the constant of Corollary 5 is unchanged if the V_k are replaced by V_k^*. The following results, which hold in all G, are used in the next section, where we take $G = SO(3)$.

Corollary 7 *If $x : I \to G$ is an n-geodesic then*

(i) $Z_n(t) = -Ad_{x(t)^{-1}}C^*$ *for all* $t \in I$, *and if* $Z_n(t)$ *is ever* $\mathbf{0}$ *then* $Z_n(t) = \mathbf{0}$ *for all* $t \in I$,
(ii) $Z_n^*(t) = -Ad_{x(t)}C$ *for all* $t \in I$, *and if* $Z_n^*(t)$ *is ever* $\mathbf{0}$ *then* $Z_n^*(t) = \mathbf{0}$ *for all* $t \in I$.

Proof 9 By definition of V_k^* and Z_n^*, and by Theorem 5,

$$Z_n^*(t) = -\mathrm{Ad}_{x(t)}V_{2n-2}(t) + \mathrm{Ad}_{x(t)}Y_n(t) = -\mathrm{Ad}_{x(t)}C \tag{23}$$

for all $t \in I$, where C is constant. Therefore, if $Z_n^*(t_*) = \mathbf{0}$ for some $t_* \in I$ then $C = \mathbf{0}$, and thus $Z_n^*(t)$ is identically $\mathbf{0}$, completing the proof of (ii). Similarly, (i) follows from Corollary 6(iv).

The C^∞ curve $x : I \to G$ is a geodesic if and only if $V_1(t)$ is identically $\mathbf{0}$. Moreover, if $V_1(t)$ is identically $\mathbf{0}$ then so are $V_k(t)$ and $V_k^*(t)$ for all $k \geq 1$, by Lemma 3 and (21). In this case, $Z_n(t) = Z_n^*(t) = \mathbf{0}$ for all $t \in I$, by definition. So x is a geodesic if and only if $Z_n(t)$, $Z_n^*(t)$ and $V_1(t)$ are all identically $\mathbf{0}$. For convenience, we call an n-geodesic *trivial* if it is a geodesic.

Corollary 8 *If $x : I \to G$ is an n-geodesic with $Z_n(t) = Z_n^*(t) = \mathbf{0}$ for all $t \in I$ then*

(i) $V_{2n-2}(t) = \mathbf{0}$ *for all* $t \in I$,
(ii) *either x is trivial or there exists* $k \in \{0, \ldots, 2n-5\}$ *such that, for all* $t \in I$,

$$V_{2n-3-k}^{(1)}(t) = \frac{1}{2}[V_{2n-3-k}(t), V(t)] \quad \textit{and} \quad V_{2n-3-k}(t) \neq \mathbf{0}. \tag{24}$$

Proof 10 Since $Z_n^*(t)$ is identically $\mathbf{0}$, (23) gives $C = \mathbf{0}$. Therefore, and by Theorem 5(ii), we have (i): $2V_{2n-2}(t) = Z_n(t) + C = \mathbf{0}$. By (i) and Lemma 3, $V_{2n-3}^{(1)}(t) = \frac{1}{2}[V_{2n-3}(t), V(t)]$. So if (24) does not hold for $k = 0$ then $V_{2n-3}(t_*) = \mathbf{0}$ for some $t_* \in I$. Then $\nabla_{\mathrm{d}/\mathrm{d}t}^{2n-3} x^{(1)}(t_*) = \mathbf{0}$, by (16). Since $V_{2n-2}(t)$ is identically $\mathbf{0}$, we have $\nabla_{\mathrm{d}/\mathrm{d}t}^{2n-2} x^{(1)}(t) = \nabla_{\mathrm{d}/\mathrm{d}t}(\nabla_{\mathrm{d}/\mathrm{d}t}^{2n-3} x^{(1)})(t) = \mathbf{0}$ for all $t \in I$, i.e. $\nabla_{\mathrm{d}/\mathrm{d}t}^{2n-3} x^{(1)}(t)$ is the parallel translation of $\mathbf{0} \in T_{x(t_*)}G$ along x to $T_{x(t)}G$ and is thus identically $\mathbf{0}$. So $V_{2n-3}(t) = \mathbf{0}$ for all $t \in I$, by (16). If $n = 2$ then x is trivial. Otherwise, $V_{2n-4}^{(1)}(t) = \frac{1}{2}[V_{2n-4}(t), V(t)]$, by Lemma 3. Reasoning as for $V_{2n-3}(t)$, if (24) does not hold for $k = 1$ then $V_{2n-4}(t)$ is identically $\mathbf{0}$. Then $V_{2n-5}^{(1)}(t) = \frac{1}{2}[V_{2n-5}(t), V(t)]$, and so on.

5 $G = SO(3)$: solution of the linking equation

When G is *semisimple*, the Lax pair form of (17) given in Theorem 5 allows us to solve the linking equation (6) by quadrature for an n-geodesic x in terms of a solution V of (17) using methods developed in [24]. Comparing Corollary 7(i) with [24, Theorem 2.1], we see that (6) can be solved provided the constant C^* lies in some (specified) open dense subset of $\mathcal{G}$. Rather than discussing the general methods of [24], we shall instead solve (6) in the case $G = SO(3)$ using a different method, developed in the earlier paper [26]. So let $x : I \to SO(3)$ be an arbitrary C^∞ curve and define $V : I \to so(3)$ by (5). Since $SO(3)$ is a matrix group, (6) reads

$$x^{(1)}(t) = x(t)V(t).$$

As noted in Example 2, $SO(3)$ admits a bi-invariant Riemannian metric $\langle \cdot, \cdot \rangle$ that is unique up to a positive multiple. Assume (without loss of generality) that $\langle \cdot, \cdot \rangle$ is chosen (scaled) such that the Lie algebra isomorphism B is an isometry from E^3 to $so(3)$, and let $\langle \cdot, \cdot \rangle'$ denote the Euclidean inner product on E^3. First, note that if $Z : I \to so(3)$ is a curve satisfying, for all $t \in I$,

$$Z^{(1)}(t) = [Z(t), V(t)] \quad \text{and} \quad Z(t) \neq \mathbf{0} \tag{25}$$

then, by (4) and since B is an isometry, $\langle Z(t), Z(t) \rangle = \langle B^{-1}(Z(t)), B^{-1}(Z(t)) \rangle' = c$ for some positive constant c and all $t \in I$. Note also that for any C^∞ map $W : I \to S^2$, where $S^2 \subset E^3$ is the unit 2-sphere, there exists a C^∞ map $W_1 : I \to S^2$ with $\langle W(t), W_1(t) \rangle'$ identically 0. To see this, note that $\{(t, v) \in I \times S^2 : \langle W(t), v \rangle' = 0\}$ is a C^∞ fibre bundle over I with fibre the unit circle. Since I is contractible, the bundle is trivial and W_1 can be defined using any cross-section. Bearing these observations in mind, we now state [26, Theorem 1].

Theorem 7 (Noakes and Popiel [26]) *Suppose there exists a curve $Z : I \to so(3)$ satisfying (25) for all $t \in I$ and let c denote the positive constant $\langle Z(t), Z(t) \rangle = \langle B^{-1}(Z(t)), B^{-1}(Z(t)) \rangle'$. Define $W_3 : I \to S^2$ by $W_3(t) := \frac{1}{\sqrt{c}} B^{-1}(Z(t))$ and let*

$W_1 : I \to S^2$ be a C^∞ map satisfying $\langle W_3(t), W_1(t)\rangle' = 0$ for all $t \in I$. Define $W_2 : I \to S^2$ by $W_2(t) := W_3(t) \times W_1(t)$, set

$$\theta(t) := \int_{t_0}^{t} \langle W_1(\zeta), \dot{W}_2(\zeta) + [B^{-1}(V(\zeta)), W_2(\zeta)]\rangle' \mathrm{d}\zeta,$$
$$U_1(t) := W_1(t)\cos(\theta(t)) + W_2(t)\sin(\theta(t)),$$
$$U_2(t) := W_2(t)\cos(\theta(t)) - W_1(t)\sin(\theta(t)),$$

for some fixed $t_0 \in I$, and all $t \in I$, and define $U : I \to SO(3)$ by $U(t) := [U_1(t)U_2(t)W_3(t)]$. Then $x(t) = x(t_0)U(t_0)U(t)^T$ for all $t \in I$.

Now suppose $x : I \to SO(3)$ is a non-trivial n-geodesic, where $n \geq 2$. In order to apply Theorem 7, we assume that we know how to solve (17). First suppose $Z_n(t) \neq \mathbf{0}$ for all $t \in I$. Then, by Theorem 5, x is found by taking $Z = Z_n$ in Theorem 7. Now suppose $Z_n(t_*) = \mathbf{0}$ for some $t_* \in I$. Then $Z_n(t) = \mathbf{0}$ for all $t \in I$, by Corollary 7(i). In this case, first suppose $Z_n^*(t) \neq \mathbf{0}$ for all $t \in I$. By Theorem 6, the Lie reduction V^* of $x^{-1} : I \to SO(3)$ is also a solution of (17). So, assuming we know how to solve (17), we can find V^*. Then, by Corollary 6(i), we can replace V by V^* and take $Z = Z_n^*$ in Theorem 7 to find x^{-1}. Now suppose $Z_n^*(t_*) = \mathbf{0}$ for some $t_* \in I$. Then $Z_n^*(t) = \mathbf{0}$ for all $t \in I$, by Corollary 7(ii). In this case, since x is non-trivial, Corollary 8(ii) guarantees that (24) holds for some $k \in \{0, \ldots, 2n-5\}$ and all $t \in I$. So we can take $Z = V_{2n-3-k}$ in Theorem 7 to solve for the curve $\tilde{x} : I \to G$ satisfying $x(t) = \tilde{x}(2t)$.

Of course, the problem of solving (17) is considerably more difficult. Although solutions can be written down in terms of the exponential map when G is abelian [5, Theorem 3.6], it seems unlikely that quadrature (or closed form) solutions can be found in more interesting cases. Noakes [21,22] has investigated symmetries and asymptotics of solutions of (17) with $n = 2$, i.e. (10), with particular attention given to the case $G = SO(3)$. The extreme complexity of these curves is particularly evident in [22]. Belta and Kumar [2,3] and Žefran and Kumar [34] have investigated 2-geodesics of the left-invariant Riemannian metric on $SE(3)$ discussed after Proposition 2. The projections to $SO(3)$ of these curves satisfy (6) and (10); numerical solutions of this system are computed in [2,3,34]. Žefran et al. [36] have investigated 3-geodesics of the aforementioned Riemannian metric on $SE(3)$, the projections to $SO(3)$ of which satisfy (6) and (14) (as mentioned earlier), and computed numerical solutions of this system. In [34], Žefran and Kumar have also numerically computed 2-geodesics of the bi-invariant semi-Riemannian metric on $SE(3)$ corresponding to the matrix $Q(0, 1)$ in Example 3. It seems little is known about solutions of (17) in any cases other than $n = 2$ and $n = 3$.

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References

1. Babelon O, Bernard D, Talon M (2003) Introduction to classical integrable systems. Cambridge University Press, Cambridge
2. Belta C, Kumar V (2002) On the computation of rigid body motion. Electron J Comput Kinemat 1
3. Belta C, Kumar V (2002) Euclidean metrics for motion generation on $SE(3)$. J Mech Eng Sci C 216:47–60
4. Camarinha M (1996) The geometry of cubic polynomials on Riemannian manifolds. PhD Thesis, University of Coimbra, Portugal
5. Camarinha M, Silva Leite F, Crouch P (1995) Splines of class C^k on non-Euclidean spaces. IMA J Math Control Inform 12:399–410
6. Camarinha M, Silva Leite F, Crouch P (2001) On the geometry of Riemannian cubic polynomials. Diff Geom Appl 15:107–135
7. Crouch P, Kun G, Silva Leite F (1999) The de Casteljau algorithm on Lie groups and spheres. J Dyn Control Syst 5:397–429
8. Crouch P, Silva Leite F (1991) Geometry and the dynamic interpolation problem. Proc Amer Control Conf, Boston, pp 1131–1136
9. Crouch P, Silva Leite F (1995) The dynamic interpolation problem: on Riemannian manifolds, Lie groups and symmetric spaces. J Dyn Control Syst 1:177–202
10. do Carmo M (1992) Riemannian geometry. Birkhäuser, Boston
11. Gabriel S, Kajiya J (1985) Spline interpolation in curved space. State of the art in image synthesis, SIGGRAPH '85 course notes. ACM Press, New York, pp 1–14
12. Ge Q, Ravani B (1994) Geometric construction of Bézier motions. Trans ASME J Mech Des 116:749–755
13. Giambo R, Giannoni F, Piccione P (2002) An analytical theory for Riemannian cubic polynomials. IMA J Math Control Inform 19:445–460
14. Giambo R, Giannoni F, Piccione P (2004) Optimal control on Riemannian manifolds by interpolation. Math Control Signals Syst 16:278–296
15. Hussein I, Bloch A (2004) Dynamic interpolation on Riemannian manifolds: an application to interferometric imaging. Proc Amer Control Conf, Boston, pp 413–418
16. Jacobson N (1962) Lie algebras. Interscience, New York
17. Jurdjevic V (1997) Geometric control theory. Cambridge University Press, Cambridge
18. Krakowski K (2002) Geometrical methods of inference. PhD Thesis, University of Western Australia
19. Krakowski K (2005) Envelopes of splines in the projective plane. IMA J Math Control Inform 22:171–180
20. Milnor J (1963) Morse theory. Princeton University Press, Princeton
21. Noakes L (2003) Null cubics and Lie quadratics. J Math Phys 44:1436–1448
22. Noakes L (2004) Non-null Lie quadratics in E^3. J Math Phys 45:4334–4351
23. Noakes L (2006) Duality and Riemannian cubics. Adv Comput Math 25:195–209
24. Noakes L (2006) Lax constraints in semisimple Lie groups. Q J Math 57:527–538
25. Noakes L, Heinzinger G, Paden B (1989) Cubic splines on curved spaces. IMA J Math Control Inform 6:465–473
26. Noakes L, Popiel T (2005) Quadratures and cubics in $SO(3)$ and $SO(1,2)$. IMA J Math Control Inform 23:463–473
27. Park F, Ravani B (1995) Bézier curves on Riemannian manifolds and Lie groups with kinematic applications. Trans ASME J Mech Des 117:36–40
28. Park F, Ravani B (1997) Smooth invariant interpolation of rotations. ACM Trans Graph 16:277–295
29. Popiel T, Noakes L (2006) C^2 spherical Bézier splines. Comput Aided Geom Des 23:261–275
30. Shoemake K (1985) Animating rotations with quaternion curves. ACM SIGGRAPH Comput Graph 19:245–254
31. Silva Leite F, Camarinha M, Crouch P (1995) Two higher order variational problems on Riemannian manifolds and the interpolation problem. Proceedings of the 3rd European control conference, Rome, pp 3269–3274

32. Silva Leite F, Camarinha M, Crouch P (2000) Elastic curves as solutions of sub-Riemannian control problems. Math Control Signals Syst 13:140–155
33. Varadarajan V (1984) Lie groups, Lie algebras, and their representations. Springer, New York
34. Žefran M, Kumar V (1998) Interpolation schemes for rigid body motions. Comput Aided Des 30:179–189
35. Žefran M, Kumar V, Croke C (1996) Choice of Riemannian metrics for rigid body kinematics. Proc ASME 24th Biennial Mechanisms Conf, Irvine, California
36. Žefran M, Kumar V, Croke C (1998) On the generation of smooth three-dimensional rigid body motions. IEEE Trans Robot Autom 14:576–589