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ORIGINAL ARTICLE

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On the controllability of anomalous diffusions generated by the fractional Laplacian

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Abstract This paper introduces a "spectral observability condition" for a negative self-adjoint operator which is the key to proving the null-controllability of the semigroup that it generates, and to estimating the controllability cost over short times. It applies to the interior controllability of diffusions generated by powers greater than $1/2$ of the Dirichlet Laplacian on manifolds, generalizing the heat flow. The critical fractional order $1/2$ is optimal for a similar boundary controllability problem in dimension one. This is deduced from a subsidiary result of this paper, which draws consequences on the lack of controllability of some one-dimensional output systems from Müntz–Szász theorem on the closed span of sets of power functions.

Keywords Interior controllability · Spectral observability · Control cost · Parabolic equation · Fractional calculus

1 Introduction

In Sect. 2.2 of this paper, an observability condition on the spectral sub-spaces of a negative self-adjoint operator is introduced which ensures *fast controllability*, i.e. the semigroup generated by this operator is null-controllable in arbitrarily small time. In this asymptotic, it also ensures an upper bound for the *controllability cost*,

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i.e. the supremum, over every initial state with norm one, of the norm of the optimal input function which steers it to zero (cf. definitions in Sect 2.1). This *spectral observability condition* is the abstract version of a property proved in [12,14] for the Dirichlet Laplacian Δ on a compact manifold observed on any region.

It applies to the semigroup generated by the fractional Laplacian on manifolds $-(-\Delta)^{\alpha}$ as long as $\alpha > 1/2$. This semigroup is widely used to describe physical systems exhibiting anomalous diffusions (cf. references in Sect. 3.1). Thus new interior null-controllability results for such *fractional diffusions* with non-constant coefficients in any dimension are deduced in Sect. 3.2 (a similar problem with constant coefficients in one dimension and one-dimensional input was recently considered in [16]). In particular, as the control time T tends to 0, the controllability cost grows at most like $C_\beta \exp(c_\beta / T^{\beta})$ where C_β and c_β are positive constants and $\beta > 1/(2\alpha - 1)$ (n.b. a lower bound of the same form with equality $\beta = 1/(2\alpha - 1)$ holds in the case $\alpha = 1$ corresponding to the heat flow). It is proved in Sect. 3.3 that a similar problem in one dimension is not controllable from the boundary for $\alpha \in (0, 1/2]$.

This last result is deduced from a more general remark of independent interest on the lack of controllability of any finite linear combination of eigenfunctions of systems with one-dimensional input, based on the generalized Müntz theorem on the completeness of sets of exponentials.

2 The main result in the abstract setting

After recalling the duality between controllability and observability for parabolic semigroups, this section states the main definition and theorem.

2.1 The abstract setting

Let the generator A be a positive self-adjoint operator with domain $D(A)$ on the Hilbert space $\mathcal{H}$ of states. Let $\mathcal{U}$ be the Hilbert space of inputs. The spaces $\mathcal{H}$ and $\mathcal{U}$ are identified with their duals, and their norms are denoted by $\|\cdot\|$ without subscript.

Let $\mathcal{H}_1$ be the Hilbert space obtained by choosing the graph norm on $D(A)$. Let $\mathcal{H}_{-1}$ be the space dual to $\mathcal{H}_1$. We keep the same notation for the extension of $\{e^{-tA}\}_{t \geq 0}$ to a semigroup on $\mathcal{H}_{-1}$.

Let the observation operator C be bounded from $\mathcal{H}_1$ to $\mathcal{U}$ and let the control operator $B \in \mathcal{L}(\mathcal{U}; \mathcal{H}_{-1})$ be its dual. We make the following equivalent admissibility assumptions on these operators (which generalize $C \in \mathcal{L}(\mathcal{H}; \mathcal{U})$, cf. [25]): for some $T > 0$ (hence for all $T > 0$) there is a positive constant K_T such that

$$\forall v_0 \in D(A), \quad \int_0^T \|C \mathrm{e}^{-tA} v_0\|^2 \mathrm{d}t \leq K_T \|v_0\|^2, \tag{1}$$

$$\forall u \in L^2_{\mathrm{loc}}(\mathbb{R}; \mathcal{U}), \quad \|\int_0^T \mathrm{e}^{-tA} B u(t) \mathrm{d}t\|^2 \leq K_T \int_0^T \|u(t)\|^2 \mathrm{d}t. \tag{2}$$

Therefore the output map $v_0 \mapsto C e^{-tA} v_0$ from $D(A)$ to $L^2([0,T];\mathcal{U})$ has a continuous extension to $\mathcal{H}$, and the differential equation:

$$\dot{\phi} + A\phi = Bu, \quad \phi(0) = \phi_0 \in \mathcal{H}, \qquad u \in L^2_{\mathrm{loc}}(\mathbb{R};\mathcal{U}), \tag{3}$$

has a unique solution $\phi \in C([0,\infty);\mathcal{H})$ defined by the integral formula:

$$\phi(t) = e^{-tA}\phi(0) + \int_0^t e^{(s-t)A} Bu(s)\,ds.$$

Definition 1 *The parabolic control system (3) is said to be null-controllable in time T if for all initial state $\phi_0 \in \mathcal{H}$ there is an input function $u \in L^2_{loc}(\mathbb{R};\mathcal{U})$ such that the solution $\phi \in C([0,\infty);\mathcal{H})$ of (3) satisfies $\phi(T) = 0$.*

By duality (cf. [4]), it is equivalent to the following observability inequality for solutions $v(t) = e^{-tA} v_0$ of the equation without source term: $\dot{v} + Av = 0$.

Definition 2 *The parabolic semigroup $\{e^{-tA}\}_{t\geq 0}$ is said final-observable through C in time T if there is a positive constant C_T such that:*

$$\forall v_0 \in \mathcal{H}, \quad \|e^{-TA} v_0\| \leq C_T \|C e^{-tA} v_0\|_{L^2(0,T;\mathcal{U})}. \tag{4}$$

The smallest positive constant C_T in (4) is the controllability cost in time T.

By duality, the controllability cost is also the smallest positive constant C_T such that, for all ϕ_0, there is a u as in definition 1 with a norm satisfying: $\|u\|_{L^2(0,T;\mathcal{U})} \leq C_T\|\phi_0\|$.

2.2 The main result

Now we introduce the spectral observability condition of order $\gamma > 0$ for the generator A and observation operator C. This definition is quite natural for dissipative problems as illustrated in Sect. 4: it allows to compare the free dissipation of high modes to the cost of controlling low modes.

Our spectral notations are the following. Given $\gamma > 0$ and $\mu > 1$, applying the functional calculus for self-adjoint operators to the positive operator A^γ and the bounded function on $\mathbb{R}^+$ defined by $\mathbf{1}_{\lambda\leq\mu} = 1$ if $\lambda \leq \mu$ and $\mathbf{1}_{\lambda\leq\mu} = 0$ otherwise yields the spectral projector $\mathbf{1}_{A^\gamma\leq\mu}$. The image of $\mathcal{H}$ under this projection operator is just the *spectral subspace* $\mathbf{1}_{A^\gamma\leq\mu}\mathcal{H}$ of A^γ. N.b. when there are only eigenvalues in the spectrum of A, $\mathbf{1}_{A^\gamma\leq\mu}\mathcal{H}$ is the set of linear combinations of the eigenvectors of A with eigenvalues lower or equal to $\mu^{1/\gamma}$. In short, $\mathbf{1}_{A^\gamma\leq\mu}\mathcal{H}$ can be considered as the space of generalized modes of A^γ lower or equal to μ.

Definition 3 *Let $\gamma > 0$. The observability of low modes of A^γ through C at exponential cost holds if there are positive constants d_1 and d_2 such that:*

$$\forall \mu > 1, \quad \forall v \in \mathbf{1}_{A^\gamma\leq\mu}\mathcal{H}, \qquad \|v\| \leq d_2 e^{d_1\mu}\|Cv\|. \tag{5}$$

The following theorem shows that this is a relevant condition for estimating how violent fast controls are (this problem was solved for $\dim \mathcal{H} < \infty$ in [22]).

Theorem 1 *If Definition 3 holds with* $\gamma \in (0, 1)$ *then the system (3) is null-controllable in any time* $T > 0$ *(cf. Definition 1). Moreover the controllability cost* C_T *(cf. Definition 2) over short times satisfies the upper bound:*

$$\forall \beta > \frac{\gamma}{1-\gamma}, \quad \exists C_1 > 0, \quad \exists C_2 > 0, \quad \forall T \in (0, 1), \qquad C_T \leq C_2 \exp\left(\frac{C_1}{T^\beta}\right)$$

3 Application to the fractional diffusion

This section considers the controllability of the semigroup generated by the fractional Laplacian on a manifold $-(-\Delta)^\alpha$, where Δ denotes the usual Laplacian operator. When the manifold is the whole Euclidean space R^d, $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_d^2$. When the manifold has a boundary, the null Dirichlet condition is always assumed.

3.1 Background of anomalous diffusion models

In recent years, the use of fractional derivatives in dynamical models of physical processes exhibiting anomalously slow or fast diffusion has diffused (cf. the surveys [15,23]). Fractional calculus includes various extensions of the usual derivative from integer to real order. In this paper, we always use the fractional Laplacian, which is not a local operator when the power α is not an integer. Moreover, the model of anomalous diffusion considered here do not include fractional derivatives of any kind, with respect to the time variable (cf. [8,15,23] and references therein).

When the manifold is the whole Euclidean space R^d, the dynamics considered here is the same as the "isotropic space-fractional diffusion equation" in [9], the "strictly space fractional diffusion equation" in [8] and the "Lévy fractional diffusion equation" in [15]. In this case, the fractional powers of the Laplacian are also known as Riesz fractional derivatives [8] or Riesz–Weyl operator [15]. They are easily defined through the Fourier transform $\mathcal{F}$: $\mathcal{F}(-\Delta)^\alpha f(\xi) = |\xi|^{2\alpha}\mathcal{F}f(\xi)$.

The fractional Laplacian $-(-\Delta)^\alpha$ with $\alpha \in (0, 1]$ generates the rotationally invariant 2α-stable Lévy process. For a textbook presentation of this stochastic process, we refer the reader to [21], in particular Example 32.7, and for a survey to [1], in particular Example 5 of Lévy process and Example 2 of generator. For $\alpha = 1$, this process is the Brownian motion B_t on R^d, and for $\alpha < 1$, it is subordinated to B_t by a strictly α-stable subordinator T_t, so that it writes B_{T_t}. The convolution kernels of the corresponding semigroups are the rotationally invariant Lévy stable probability distributions, in particular the Gaussian distribution for $\alpha = 1$ and the Cauchy distribution for $\alpha = 1/2$. For $\alpha < 1$, these distributions have "heavy tails", i.e. far away they decrease like a power as opposed to the exponential decrease found in the Gaussian, which accounts for the "superdiffusive" behavior of the semigroup. The more restrictive range $\alpha \in (1/2, 1)$ is the most widely used to model anomalously fast diffusions (cf. [15]), and it turns out that the controllability result Theorem 2 applies to this range of fractional superdiffusions only. Theorem 2 includes the "subdiffusive" range $\alpha > 1$, but it seems that this model has not been considered in the physics literature on anomalously slow diffusion. N.b. the generalized fractional Laplacian operators associated with anisotropic diffusion, also known as the Riesz–Feller derivatives, generate all stable Lévy processes, i.e. including the non-invariant ones also called the skewed ones (cf. [8,9]). These Lévy processes can be approximated by Lévy flights, and references to random walk models of anomalous diffusion can be found in [8,15].

When the manifold is a domain of the Euclidean space R^d, the Markov process generated by the fractional Dirichlet Laplacian $-(-\Delta)^\alpha$ with $\alpha \in (0,1]$ can be obtained by killing the Brownian motion on R^d upon exiting the domain and then subordinating the killed Brownian motion by the subordinator T_t introduced above (cf. [24]). N.b. reversing the order of killing and subordination yields another process which seems to have been investigated earlier and further.

3.2 Interior controllability of some fractional diffusions

Let M be a smooth connected complete n-dimensional Riemannian manifold with metric g and boundary ∂M. When $\partial M \neq \emptyset$, M denotes the interior and $\overline{M} = M \cup \partial M$. Let Δ denote the Dirichlet Laplacian on $L^2(M)$ with domain $D(\Delta) = H_0^1(M) \cap H^2(M)$ (n.b. Δ denotes a negative differential operator with variable coefficients depending on the metric g). Let T be a positive time and let χ_Ω denote the characteristic function of an open subset $\Omega \neq \emptyset$ of $\overline{M}$.

In this application, the state and input space is $\mathcal{H} = \mathcal{U} = L^2(M)$ and the observation operator C is the multiplication by χ_Ω, i.e. it truncates the input function outside the control region Ω. If M is not compact, assume that Ω is the exterior of a compact set K such that $K \cap \overline{\Omega} \cap \partial M = \emptyset$. In this setting, the observability of low modes of $(-\Delta)^{1/2}$ through C at exponential cost holds (cf. Definition 3). When M is compact this is an inequality on sums of eigenfunctions proved as Theorem 3 in [14] and Theorem 14.6 in [12]. This was generalized to non-compact M in [18]. Applying Theorem 1, with $\mathcal{H} = \mathcal{U} = L^2(M)$, $A = (-\Delta)^\alpha$, $\gamma = 1/(2\alpha)$ and $B = C \in \mathcal{L}(\mathcal{H};\mathcal{U})$ yields:

Theorem 2 *For all $\alpha > 1/2$, the fractional diffusion system:*

$$\partial_t \phi + (-\Delta)^\alpha \phi = \chi_\Omega u, \quad \phi(0) = \phi_0 \in L^2(M), \qquad u \in L^2_{loc}(\mathbb{R}; L^2(M)),$$

is null-controllable in any time $T > 0$ (cf. Definition 1). Moreover the controllability cost C_T (cf. Definition 2) over short times satisfies the upper bound:

$$\forall \beta > 1/(2\alpha - 1), \quad \exists C_\beta > 0, \quad \exists c_\beta > 0, \quad \forall T \in (0,1), \qquad C_T \leq C_\beta \exp\left(\frac{c_\beta}{T^\beta}\right).$$

Remark 1 This upper bound for the fast controllability cost in the case $\alpha = 1$ was already stated without proof in [17]. Micu and Zuazua mention indenpendently in [16] that "a careful analysis of the method of proof in [13,14] shows that it works if $\alpha > 1/2$", but no upper bound.

Micu and Zuazua considered in [16] a similar controllability problem: the space manifold M and the input space $\mathcal{U}$ are one-dimensional, B is the multiplication by a shape function $f \in L^2(M)$ satisfying extra assumptions (instead of χ_Ω). For such types of controls, sometimes called "lumped" controls, harmonic analysis reduces the controllability to the construction of a basis which is bi-orthogonal to the exponential functions with rates equal to the eigenvalues. Although [6] does not concern the fractional Laplacian, it is mainly based on an estimate of infinite products in Lemma 3.1 which only relies on the asymptotic behavior of the eigenvalues in (3.10). Therefore, Theorem 2.1 in [16] deduces from [6], a sufficient condition on the Fourier coefficients of f and ϕ_0 (involving $\alpha > 1/2$ and $T > 0$) ensuring that there is a u steering ϕ_0 to 0 in time T. The main negative result of [16] is referred to in Remark 3 of the next section.

Remark 2 We should comment on the simplest case $\alpha = 1$, i.e. diffusion by the heat flow. The fast null-controllability for any control region Ω has been known for a decade and the fast controllability cost has been investigated, e.g. [7,17]. It allows us to discuss the optimality of the upper bound in Theorem 2. Namely, a lower bound of the same form with equality $\beta = 2/(2-\alpha)$ holds for $\alpha = 1$ (cf. [17]). When M is a bounded domain of $\mathbb{R}^d$ and Δ has constant coefficients, [7] proves that $\limsup_{T\to 0} T \ln C_T < \infty$ for any Ω. For general (M, g), but under some geometric condition on Ω, an explicit geometric upper bound on $\limsup_{T\to 0} T \ln C_T$ is proved in [17].

3.3 Non-controllability of some one-dimensional fractional diffusions

Although there is no result yet for $\alpha \leq 1/2$ in the setting of the previous section, it seems that the controllability in Theorem 2 does not hold for $\alpha \leq 1/2$ since it does not hold for some similar one-dimensional fractional diffusions problems.

Indeed, [16] concerns such a negative result in the setting of "lumped" interior control described in Remark 1. Micu and Zuazua [16] first recall a result of [5] saying that for any $\alpha \leq 1/2$ and $T > 0$ there is an f and a ϕ_0 that cannot be steered to 0 in time T by any u. In Theorem 3.1, they go much further in the analysis of the space of initial states which are not controllable.

The key assumption in [16] compared to the setting of Theorem 2 (even when M is one-dimensional) is that the input space $\mathcal{U}$ is one-dimensional. This allows to make the well-known reduction to some properties of entire functions and exponential sums (cf. e.g. [2,6,17]). Indeed, as pointed out in the Appendix, it is easy to prove that abstract systems with finite-dimensional inputs have a large set of non-controllable initial states as soon as their eigenvalues satisfy a well-known condition on the completeness of sets of exponentials. As an application, the next theorem states a strong non-controllability result for a one-dimensional boundary control system. N.b. although Theorem 3.1 of [16] is a stronger and more difficult result, here the input space is naturally one-dimensional without extra assumption on the structure of the controlled term.

In the next theorem, the manifold is a segment, i.e. $M = (0, L)$. For this result only, we consider the Neumann Laplacian Δ_{N} which acts as Δ but has a different domain: $D(\Delta_{\mathrm{N}}) = \{\phi \in H^2(M) | \phi'(0) = \phi'(L) = 0\}$. Let $A = (-\Delta_{\mathrm{N}})^{\alpha}$ with $\alpha \in (1/4, 1/2]$. Since $\alpha < 3/4$, $D(A)$ with the graph norm is $X_1 = H^{2\alpha}(0, L)$ (without boundary condition) which injects continuously in the space of continuous functions for $\alpha > 1/4$. Therefore, $b : \phi \mapsto \phi(L)$ is continuous on X_1, and thus defines b in the dual X_{-1} of X_1. N.b. if the metric is not Euclidean, then Δ_{N} has variable coefficients so that the eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ and eigenfunctions $\{\phi_n\}_{n\in\mathbb{N}}$ are not explicit. But they satisfy $\phi_n(L) \neq 0$ and $\lambda_n \sim Cn^{2\alpha}$ where C is a positive constant, so that $b_n = \langle b, \phi_n\rangle \neq 0$ and property (ii) of Theorem 4 holds for $2\alpha < 1$. Therefore Theorem 5 implies:

Theorem 3 *Assume b is the boundary control operator and A is the fractional Neumann Laplacian defined above with $\alpha \in (1/4, 1/2]$. For all finite linear combination $x^0 \neq 0$ of the eigenvectors of A and for all $T > 0$, there is no input function $u \in L^2(0, T; \mathbb{C})$ such that the solution $x \in C(0, T; X_{-1})$ of $\dot{x}(t) + Ax(t) = bu(t)$ with initial state $x(0) = x^0$ satisfies $x(T) = 0$.*

4 Proof of the main theorem

This section concerns the proof of Theorem 1. In the first step, from the stationary condition in Definition 3, we deduce the observability of low modes over any positive time in the corresponding dynamics (this is the abstract version of Sect. 4 in [18]). In the second step, using an abstract version of the iterative control strategy introduced by Lebeau and Robbiano in [13] (cf. Sect. 5 in [18]), we prove the full null-controllability in an arbitrarily small time. The main novelty is the last step, in which we estimate the controllability cost as the control time tends to zero.

4.1 From the stationary to the evolution equation

Let dE_λ denote the projection valued measure associated to the self-adjoint operator A^γ by the spectral theorem. Assume that Definition 3 holds. Let $\tau \in (0, 1]$, $\mu \geq 1$ and $v_0 \in \mathbf{1}_{A^\gamma \leq \mu}\mathcal{H}$.

For all $t \in [0, \tau]$, we may apply (5) to $v = \mathrm{e}^{-tA}v_0$ since it is in $\mathbf{1}_{A^\gamma \leq \mu}\mathcal{H}$:

$$d_2^2 \mathrm{e}^{2d_1\mu}\|C\mathrm{e}^{-tA}v_0\|^2 \geq \|\mathrm{e}^{-tA}v_0\|^2 = \int_0^\mu \mathrm{e}^{-2t\lambda^{1/\gamma}}\mathrm{d}(E_\lambda v_0, v_0).$$

First integrating on $[0, \tau]$ with the new variable $s = t/\tau$, then using $\tau \leq 1$ and finally $\int_0^1 \exp(-\alpha t)\mathrm{d}t = (1 - \exp(-\alpha))/\alpha \geq (2\alpha)^{-1}$ for $\alpha \geq \ln 2$ yields:

$$d_2^2 \mathrm{e}^{2d_1\mu}\int_0^\tau \|C\mathrm{e}^{-tA}v_0\|^2\mathrm{d}t \geq \tau \int_0^1 \int_0^\mu \mathrm{e}^{-2\tau s\lambda^{1/\gamma}}\mathrm{d}(E_\lambda v_0, v_0)\,\mathrm{d}s$$
$$\geq \tau \int_0^1 \mathrm{e}^{-2s\mu^{1/\gamma}}\mathrm{d}s \int_0^\mu \mathrm{d}(E_\lambda v_0, v_0) \geq \frac{\tau}{4\mu^{1/\gamma}}\|v_0\|^2.$$

Therefore, for any $D_1 > d_1$, there is a $D_2 > 0$ such that *low modes fast observability for* e^{-tA} *at exponential cost* holds: $\exists D_1 > 0, \exists D_2 > 0, \forall \mu > 1$,

$$\forall \tau \in (0, 1], \quad \forall v_0 \in \mathbf{1}_{A^\gamma \leq \mu}\mathcal{H}, \qquad \|\mathrm{e}^{-\tau A}v_0\| \leq \frac{D_2}{\sqrt{\tau}}\mathrm{e}^{D_1\mu}\|C\mathrm{e}^{-tA}v_0\|_{L^2(0,\tau;\mathcal{U})}. \tag{6}$$

By duality (cf. [4]), this is equivalent to the following null-controllability: for all $\tau \in (0,\ 1]$ and $\mu \geq 1$, there is a bounded operator $S_\mu^\tau : \mathcal{H} \to L^2(0, \tau; \mathcal{U})$ such that for all $\phi_0 \in \mathbf{1}_{A^\gamma \leq \mu} \mathcal{H}$, the solution $\phi \in C([0, \infty), \mathcal{H})$ of (3) with control function $u = S_\mu^\tau \phi_0$ satisfies $\mathbf{1}_{A^\gamma \leq \mu} \phi(\tau) = 0$, and $\|S_\mu^\tau\| \leq \left(D_2/\sqrt{\tau}\right) \mathrm{e}^{D_1 \mu}$. (This is the cost estimate.)

4.2 From low modes to full controllability

From now on, we need to assume that γ in Definition 3 is lower than 1. We introduce a dyadic scale of modes $\mu_k = 2^k$ ($k \in \mathbb{N}$) and a sequence of time intervals $\tau_k = \sigma_\delta T / \mu_k^\delta$ where $\delta \in (0, \gamma^{-1} - 1)$ and $\sigma_\delta = (2 \sum_{k \in \mathbb{N}} 2^{-k\delta})^{-1} > 0$, so that the sequence of times defined recursively by $T_0 = 0$ and $T_{k+1} = T_k + 2\tau_k$ converges to T. The strategy of Lebeau and Robbiano in [13] is to steer the initial state ϕ_0 to 0, through the sequence of states $\phi_k = \phi(T_k) \in \mathbf{1}_{A^\gamma > \mu_{k-1}} \mathcal{H}$ composed of ever higher modes, by applying recursively the input function $u_k = S_{\mu_k}^{\tau_k} \phi_k$ to ϕ_k during a time τ_k and no input during a time τ_k. This strategy is successful if ϕ_k tends to zero and the full input function $u(t) = \sum_k \mathbf{1}_{0 \leq t - T_k \leq \tau_k} u_k(t - T_k)$ is in $L^2(0, T; \mathcal{U})$.

Introducing the notations

$$\varepsilon_k = \|\phi_k\|, \quad C_k = D_2 \frac{\mathrm{e}^{D_1 \mu_k}}{\sqrt{\tau_k}} \quad \text{and} \quad \rho_k = \left(\frac{C_{k+1}\varepsilon_{k+1}}{C_k \varepsilon_k}\right)^2, \tag{7}$$

the cost estimate of the previous step writes $\|S_{\mu_k}^{\tau_k}\| \leq C_k$ and implies:

$$\|u\|^2_{L^2(0,T;\mathcal{U})} = \sum_{k \in \mathbb{N}} \|u_k\|^2_{L^2(0,\tau_k;\mathcal{U})} \leq \sum_{k \in \mathbb{N}} C_k^2 \varepsilon_k^2. \tag{8}$$

It only remains to check that the last series converges (this implies $\lim_k \varepsilon_k = 0$). This shall be achieved by comparing it to a geometric series, i.e. by proving that there is a $\rho \in (0, 1)$ such that $\rho_k \leq \rho$ for all k large enough.

The integral formula for $\phi(T_k + \tau_k)$ in terms of $\phi(T_k)$ and Bu_k implies, using the admissibility assumption (2) for B (over the time $1 \geq T \geq \tau_k$) and the contractivity inequality $\|\mathrm{e}^{-tA}\| \leq 1$ due to the positivity of A: $\|\phi(T_k + \tau_k)\|^2 \leq 2\|\mathrm{e}^{-\tau_k A}\phi(T_k)\|^2 + 2K_1\|u\|^2_{L^2(0,\tau_k;\mathcal{U})} \leq 2(1 + K_1 C_k^2)\varepsilon_k^2$. Since $\mathbf{1}_{A^\gamma \leq \mu_k} \phi(T_k + \tau_k) = 0$ implies $\varepsilon_{k+1} \leq \mathrm{e}^{-\tau_k \mu_k^{1/\gamma}} \|\phi(T_k + \tau_k)\|$, we deduce: $\varepsilon_{k+1}^2 \leq 2\mathrm{e}^{-2\tau_k \mu_k^{1/\gamma}}(1 + K_1 C_k^2)\varepsilon_k^2$. Since $C_{k+1}/C_k = 2^{\delta/2}\mathrm{e}^{D_1 \mu_k}$, we deduce that, for any $D_3 > 4D_1$, there is a $D_4 > 0$ such that:

$$\rho_k \leq 2^{1+\delta}\left(\mathrm{e}^{-2D_1\mu_k} + \frac{K_1 D_2^2}{\tau_k}\right)\mathrm{e}^{4D_1\mu_k - 2\tau_k \mu_k^{1/\gamma}} \leq \frac{D_4}{T}\mathrm{e}^{D_3 \mu_k - 2\sigma_\delta T \mu_k^{\gamma^{-1}-\delta}}. \tag{9}$$

Since $\gamma^{-1} - \delta > 1$, this implies: $\forall \rho \in (0, 1), \exists N \in \mathbb{N}, k \geq N \Rightarrow \rho_k \leq \rho$. As explained after (8), this completes the proof of the first assertion of Theorem 1.

4.3 Estimate of the controllability cost over short times

We keep the notations in (7). Since $l \le \mu_l$, $\sum_{0\le k\le l-1} \mu_k \le \mu_l$ and

$$\sum_{0\le k\le l-1} \mu_k^{\gamma^{-1}-\delta} = \frac{2^{(\gamma^{-1}-\delta)l}-1}{2^{(\gamma^{-1}-\delta)}-1} \ge \mu_l^{\gamma^{-1}-\delta}\frac{1-1/2}{2^{(\gamma^{-1}-\delta)}} = \frac{\mu_{l-1}^{\gamma^{-1}-\delta}}{2},$$

(9) implies $\prod_{0\le k\le l-1} \rho_k \;\le\; \exp\left((D_3+\ln(D_4/T))\,\mu_l - \sigma_\delta T \mu_{l-1}^{\gamma^{-1}-\delta}\right)$. Hence, setting $q = 2^{\gamma^{-1}-\delta} \in (2,\ 2^{\gamma^{-1}})$ and $T' = \sigma_\delta T/q$:

$$\forall l \ge 1, \quad \prod_{0\le k\le l-1} \rho_k \le \exp\left(D_{T'}2^l - T'q^l\right) \text{ with } \quad D_{T'} \underset{T'\to 0}{\sim} \ln\left(\frac{1}{T'}\right).$$

Using (8) and setting $D_5 = D_2^2 \mathrm{e}^{2D_1}/q$, we deduce the cost estimate:

$$C_T^2 \le C_0^2 \left(1+\sum_{l\ge 1}\prod_{0\le k\le l-1}\rho_k\right) \le \frac{D_5}{T'}\left(1+\sum_{k\ge 1}\exp\left(D_{T'}2^k - T'q^k\right)\right). \quad (10)$$

To estimate the last sum, we shall use the simple estimate:

$$\forall t>0, \quad f(t) := \sum_{k\ge 1}\exp\left(-tq^k\right) \le \sum_{k\ge 1}\exp(-tk) = \frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}} \le \frac{1}{t}. \quad (11)$$

Let $\varepsilon \in (0,1)$ and $h_\varepsilon(x) = D_{T'}2^x - \varepsilon T' q^x$. The maximum of the function h_ε on $\mathbb{R}$ is obtained at a point x_ε which satisfies, since $D_{T'} \underset{T'\to 0}{\sim} \ln(1/T')$:

$$x_\varepsilon = \ln\frac{\left(D_{T'}\ln 2/\varepsilon T'\ln q\right)}{\ln(q/2)} \underset{T'\to 0}{\sim} \frac{\ln(1/T')}{\ln(q/2)} = \frac{1+\beta_q}{\ln q}\ln\left(\frac{1}{T'}\right),$$

where $\beta_q = [(\ln q/\ln 2)-1]^{-1}$. Therefore, $\forall \beta > \beta_q, \exists T_\beta > 0, \forall T' \in (0, T_\beta)$:

$$x_\varepsilon \ln q \le (1+\beta)\ln(1/T'),$$

hence

$$h_\varepsilon(x_\varepsilon) = \frac{\varepsilon T'}{\beta_q} q^{x_\varepsilon} \le \frac{\varepsilon}{\beta_q T'^\beta}.$$

Applying $h_1(x) \le h_\varepsilon(x_\varepsilon) - (1-\varepsilon)T'q^x$ to $x = k$ for $k \ge 1$ and (11) yields:

$$\sum_{k\ge 1}\exp\left(D_{T'}2^k - T'q^k\right) \le \mathrm{e}^{h_\varepsilon(x_\varepsilon)} f((1-\varepsilon)T') \le \exp\left(\frac{\varepsilon}{\beta_q T'^\beta}\right)\frac{1}{(1-\varepsilon)T'}.$$

Plugging this in (10) yields the cost estimate: $\forall \beta > \beta_q, \exists D_6 > 0, \exists D_7 > 0, \forall T' > 0$, $C_T^2 \le D_6 \exp\left(D_7/T'^\beta\right)$. Since $T' = \sigma_\delta T/q$ and β_q decreases to $\gamma/(1-\gamma)$ as δ

decreases to 0 (q increases to $2^{\gamma^{-1}}$), this completes the proof of the second assertion of Theorem 1.

Appendix: Lack of controllability based on Müntz theorem

This appendix concerns control systems having a Riesz basis of eigenvectors and a one-dimensional input space. It is well-known that their exact, null- and approximate controllability are related to properties of sets of exponentials (cf. [2]). Such systems were recently considered in [10,11,19]. In particular, a necessary and sufficient condition for null-controllability in terms of the eigenvalues is given in [10]. This condition is enough to prove that null-controllability does not hold in Theorem 3. This appendix concerns a much stronger property which has not drawn much attention yet: finite linear combination of the eigenvectors are initial state ones which cannot be steered to zero by any input function. Theorem 5 gives a sufficient condition in terms eigenvalues which is applied in Theorem 3.

The generalized Müntz theorem referred to in the title of this appendix is the following Theorem 7 of [20] (the original Müntz–Szász theorem concerned the approximation by power functions $x \mapsto x^{\lambda_n}$, with positive exponents λ_n, instead of exponentials; we refer to [3] for more results and references):

Theorem 4 *Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence of distinct non-zero complex numbers and let $\{e_n\}_{n\in\mathbb{N}}$ be the corresponding sequence of exponential functions defined by $e_n(t) = \exp(\lambda_n t)$.*
If $\{\lambda_n\}_{n\in\mathbb{N}}$ satisfies one of these properties:

(i) $\exists\varepsilon > 0, \displaystyle\sum_n \left(1/|\lambda_n|^{1+\varepsilon}\right) = \infty,$

(ii) $\displaystyle\sum_n \left|\mathrm{Re}(1/\lambda_n)\right| = \infty,$

(iii) $\{|\lambda_n|\}_{n\in\mathbb{N}}$ *increases and there exists a sequence* $\{\theta_n\}_{n\in\mathbb{N}}$ *of non-negative real numbers such that* $\displaystyle\sum_n \left(1/n^{\theta_n}\right) < \infty$, *and* $\displaystyle\sum_n \left(1/|\lambda_n|^{\theta_n}\right) = \infty,$

then, for all $T > 0$, $\{e_n\}_{n\in\mathbb{N}}$ is complete in $L^2(0, T; \mathbb{C})$, i.e. any function of $L^2(0, T; \mathbb{C})$ is an infinite linear combinations of these exponential functions converging in the norm of this space.

On a Hilbert space $\mathcal{X}$, we consider the system described by the following differential equation for $t \geq 0$:

$$\dot{x}(t) + Ax(t) = bu(t), x(0) = x^0 \in \mathcal{X}, \quad u \in L^2_{\mathrm{loc}}(\mathbb{R}; \mathbb{C}) . \tag{12}$$

We assume that $-A$ is the infinitesimal generator of a C_0-semigroup $\{\mathrm{e}^{-tA}\}_{t\geq 0}$ on $\mathcal{X}$, which has a sequence of normalized eigenvectors $\{\phi_n\}_{n\in\mathbb{N}}$ forming a Riesz basis of $\mathcal{X}$, with associated eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$, that is, $A\phi_n = \lambda_n\phi_n$. We denote by $\mathcal{X}_1$ the Hilbert space obtained by choosing the graph norm on the domain $D(A)$ of the unbounded operator A on $\mathcal{X}$, by $\mathcal{X}_{-1}$ the space dual to $\mathcal{X}_1$, and we keep the same notation for the extension of $\{\mathrm{e}^{-tA}\}_{t\geq 0}$ to a semigroup on $\mathcal{X}_{-1}$. We also assume that the "control vector" b is in $\mathcal{X}_{-1}$ so that the solution $x \in C(0, T; \mathcal{X}_{-1})$

of (12) is defined for $T \geq 0$ by the integral formula:

$$x(T) = \mathrm{e}^{-TA}x^0 + \int_0^T \mathrm{e}^{-(T-t)A}bu(t)\mathrm{d}t. \tag{13}$$

There is a sequence of eigenvectors $\{\psi_n\}_{n\in\mathbb{N}}$ of A^* forming a Riesz basis of $\mathcal{X}$, with associated eigenvalues $\{\bar{\lambda}_n\}_{n\in\mathbb{N}}$, which is bi-orthogonal to $\{\phi_n\}_{n\in\mathbb{N}}$, i.e. $\langle\phi_n, \psi_n\rangle = 1$ and $\langle\phi_n, \psi_m\rangle = 0$ if $m \neq n$. We introduce the coefficients $b_n = \langle b, \psi_n\rangle$ in the expansion $b = \sum_{n\in\mathbb{N}} b_n\phi_n$.

Theorem 5 *Assume that $b_n \neq 0$ for all n larger than some integer N_b. If the set of distinct non-zero eigenvalues of A satisfies one of the properties stated in Theorem 4, then, for all non-zero initial state x^0 which is a finite linear combination of the eigenvectors $\{\phi_n\}_{n\in\mathbb{N}}$ and for all $T > 0$, there is no input function $u \in L^2(0, T; \mathbb{C})$ such that the solution $x \in C(0, T; \mathcal{X}_{-1})$ of (12) satisfies $x(T) = 0$.*

Proof Introducing the coefficients $x_n(t) = \langle x(t), \psi_n\rangle$, (13) writes $x_n(T) = \mathrm{e}^{-\lambda_n T}x_n^0 + \int_0^T \mathrm{e}^{-\lambda_n(T-t)}b_n u(t)\mathrm{d}t$. With the notation $e_n(t) = \exp(\lambda_n t)$, $x(T) = 0$ writes:

$$\forall n \in \mathbb{N}, \quad -x_n^0 = b_n \int_0^T e_n(t)u(t)\mathrm{d}t. \tag{14}$$

We make the assumptions on $\{b_n\}_{n\in\mathbb{N}}$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ of the theorem. Arguing by contradiction, we also assume that there are $T > 0$, $x^0 \neq 0$ which is a finite linear combination of the $\{\phi_n\}_{n\in\mathbb{N}}$, and $u \in L^2(0, T; \mathbb{C})$ such that (14) holds. Let x_N^0 be the non-zero coefficient of x^0 with the greatest index, i.e. $x_N^0 \neq 0$ and $x_n^0 = 0$ for $n > N$. Let $M = \max\{N_b, N\}$. For all $n > M$: on the one hand, $M \geq N_b$ implies $b_n \neq 0$; on the other hand, $M \geq N$ implies $x_n^0 = 0$; so that (14) implies $\int_0^T e_n(t)u(t)\mathrm{d}t = 0$. The set of distinct non-zero values of $\{\lambda_n\}_{n>M}$ also satisfies the same property stated in Theorem 4 as $\{\lambda_n\}_{n\in\mathbb{N}}$, so that the corresponding subset of $\{e_n\}_{n>M}$ is complete in $L^2(0, T; \mathbb{C})$. In particular, $e_N = \sum_{n>M} c_n e_n$ for some coefficients $\{c_n\}_{n>M} \in l^2(\mathbb{C})$. Plugging this expansion in (14) with $n = N$ yields the contradiction: $0 \neq -x_N^0 = b_N \sum_{n>M} c_n \int_0^T e_n(t)u(t)\mathrm{d}t = 0$. □

Remark 3 This abstract theorem applies directly to the context of Theorem 3.1 in [16], since (2.10) in [16] corresponds to the hypothesis $b_n \neq 0$ for all n. In an explicit setting where $\lambda_n = n^{2\alpha}$ with $\alpha \in (0, 1/2]$, Micu and Zuazua [16] describe a much larger set of initial data which cannot be steered to zero.

Remark 4 The following weaker result, in the setting of finite-dimensional input space (instead of one-dimensional) but of eigenvectors forming a Hilbert basis (instead of Riesz basis) and of eigenvalues with positive real parts, can be deduced from [2] by combining Theorem III.3.3(d) with Theorem II.2.4 as in the proof of Theorem IV.1.3(c): if the eigenvalues violate the Blaschke condition $\sum_n \operatorname{Re}\lambda_n(1+|\lambda_n|^2)^{-1} < \infty$, then, for all $T > 0$, there is an initial state equal to some eigenvector ϕ_n which cannot be steered to zero in time T by any input function (n.b. when $|\lambda_n| \to \infty$, the violation of the Blaschke condition here is equivalent to the property (ii) in Theorem 4).

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