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AN ANTI-RAMSEY THEOREM

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Let $t_p(n)$ be the Turán number which gives the maximum size of a graph of order n containing no subgraph isomorphic to K_p.

In 1973, Erdős, Simonovits and Sós [5] proved the existence of an integer $n_0(p)$ such that for every integer $n>n_0$, the minimum number of colours $h_{p+1}(n)$, such that every $h_{p+1}(n)$-colouring of the edges of K_n which uses all the colours produces at least one K_{p+1} all whose edges have different colours, is given by $h_{p+1}(n)=t_p(n)+2$. However, no estimation of $n_0(p)$ was given in [5]. In this paper we prove that $h_{p+1}(n)=t_p(n)+2$ for $3\le p<n$. This formula covers all the relevant values of n and p.

Let Γ be an edge-colouring of the complete graph K_n of order n. A subgraph S of K_n will be called *totally multicoloured* (TMC) if S contains no two edges of the same colour. If Γ uses exactly c colours, Γ will be called a *full c-colouring.*

Let $h_r(n)$ be the minimum number of colours c such that every full c-colouring of the edges of K_n produces at least one TMC copy of K_r.

In [7], Turán proved that the maximum size $t_p(n)$ of a graph of order n which does not contain a copy of K_p is given by $t_p(n)=\frac{(p-2)(n^2-r^2(n,p-1))}{2(p-1)}+\binom{r(n,p-1)}{2}$ where $r(n,p-1)$ is the residue of n mod $p-1$.

In 1973, Erdős, Simonovits and Sós [5, Theorem 4] proved the existence of a number $n_0(p)>p$ such that $h_{p+1}(n)=t_p(n)+2$ for $n>n_0(p)$. The equality $h_3(n)=n$ for $n\ge 3$ was also proved in [5]. The aim of this paper is to prove the following

Theorem 1. *For all integers n and p such that $3\le p<n$, $h_{p+1}(n)=t_p(n)+2$.*

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We remark that this formula covers all the relevant values of n and p. Our proof resembles the minimal degree deletion proof of Turán's Theorem due to Dirac [4].

Related problems have been considered in [1,2,6].

For general concepts, we refer the reader to [3].

In the proof of Theorem 1 we will use the following five lemmas.

Lemma 2. [5] *For all integers n and p such that $n > p \geq 3$, $h_{p+1}(n) \geq t_p(n)+2$.*

Proof. Take a Turán graph of n vertices, with p classes and colour its edges by $t_p(n)$ distinct colours, the remaining pairs by a common colour, different from the previous ones. This is a full $(t_p(n)+1)$-colouring of K_n such that no TMC copy of K_{p+1} is produced. ∎

Let $a(p,n)=\frac{(p-2)(n-1)+r(n-1,p-1)}{p-1}$ for $n>p\geq 3$.

Lemma 3. *Let n and p be positive integers such that $n-2\geq p\geq 3$. Then*

(i) $t_p(n)-t_p(n-1)=a(p,n)$.
(ii) *If $p\geq 4$ then $a(p,n)\geq p$. Further, $a(3,n)\geq 2$.*
(iii) $a(p,n)=(n-1)-\lfloor\frac{n-1}{p-1}\rfloor>-1+\frac{n(p-2)}{p-1}$.

Proof. Observe that $r(n–1,p–1)=p–2$ or $r(n,p–1)–1$ depending on whether $r(n,p–1)=0$ or not. In both cases the proof of (i) is a matter of routine. Since $a(p,n)\geq\frac{(p-2)(p+1)}{p-1}=p-\frac{2}{p-1}$ and $a(p,n)$ is an integer, we get (ii). Finally, $a(p,n)=n-1-\frac{(n-1)-r(n-1,p-1)}{p-1}=n-1-\lfloor\frac{n-1}{p-1}\rfloor>n-1-\frac{n}{p-1}=-1+\frac{n(p-2)}{p-1}$. ∎

Let Γ be a full r-colouring of $E(K_n)$. If $x\in V(K_n)$, then $\nu(x,\Gamma)$ will denote the difference $r-|\Gamma(E(K_n-x))|$, i.e. the difference between r and the number of colours appearing in $E(K_n-x)$.

Suppose that $n-2\geq p\geq 3$ and let $C(p,n)$ be the set of full $(h_{p+1}(n)-1)$-colourings of $E(K_n)$ which have no TMC copy of K_{p+1}.

A subset W of $V(K_n-x)$ will be called a *selective* (x,Γ)*-set* provided $|W|=\nu(x,\Gamma)$ and all the xW-edges have different colours which do not appear in K_n-x. If Z is an induced subgraph of K_n, then Γ_Z will denote the (full) colouring of $E(Z)$ induced by Γ. Clearly, for each $x\in V(K_n), h_{p+1}(n)-1=|\Gamma(E(K_n-x))|+\nu(x,\Gamma)\leq h_{p+1}(n-1)-1+\nu(x,\Gamma)$ and so we get the following

Lemma 4. *Let $\Gamma\in C(p,n)$.*

(i) *If $V(Z)$ is a selective (x,Γ)-set then $\nu(\zeta,\Gamma)\leq\nu(\zeta,\Gamma_Z)+n-\nu(x,\Gamma)$ for every $\zeta\in V(Z)$.*

(ii) *If* $n-2 \geq p \geq 3$ *and* x_0 *is a value of* x *which minimizes* $\nu(x,\Gamma)$*, then* $\nu(x_0,\Gamma) \geq h_{p+1}(n) - h_{p+1}(n-1)$. ∎

Let $\nu^*(p,n) = \max_{\Gamma \in C(p,n)} \min_{x \in V(K_n)} \nu(x,\Gamma)$.

Lemma 5. *Let* p *be an integer,* $p \geq 3$*. Then we have*

(i) $h_{p+1}(p+1) = t_p(p+1) + 2$.
(ii) *If for every* $n > p+1, \nu^*(p,n) \leq a(p,n)$ *then* $h_{p+1}(n) = t_p(n) + 2$ *for every* $n > p+1$.

Proof. Observing that $r(p+1,p-1) = 0$ or 2 depending on whether $p=3$ or $p>3$, it is easy to see that $t_p(p+1) = \binom{p+1}{2} - 2$ and since $h_{p+1}(p+1) = \binom{p+1}{2}$, we get (i).

Suppose now that $\nu^*(p,n) \leq a(p,n)$ for every $n > p+1$. By Lemma 3.(i) and Lemma 4.(ii), $t_p(n) - t_p(n-1) = a(p,n) \geq h_{p+1}(n) - h_{p+1}(n-1)$ for every $n > p+1$. Thus $h_{p+1}(n) - t_p(n) \leq h_{p+1}(n-1) - t_p(n-1) \leq h_{p+1}(p+1) - t_p(p+1) = 2$. Applying Lemma 2, the proof ends. ∎

Lemma 6. *Suppose that* $\Gamma \in C(p,n), n-2 \geq p \geq 4, \nu(x_0,\Gamma) = \min\{\nu(x,\Gamma) : x \in V(K_n)\}$ *and* Z *is the subgraph of* K_n *induced by a given selective* (x_0,Γ)*-set. If* $\nu(x_0,\Gamma) \geq a(p,n)+1$ *then* Z *has a vertex* ζ *such that* $\nu(\zeta,\Gamma_Z) \leq \frac{2h_p(\nu(x_0,\Gamma))-4}{\nu(x_0,\Gamma)}$.

Proof. By Lemma 3.(ii), $\nu(x_0,\Gamma) \geq p+1$. Since Z contains no TMC copy of K_p, it follows that the number of colours appearing in Z is at most $h_p(\nu(x_0,\Gamma)) - 1$. For each $z_j \in V(Z)$ let W_j be a selective (z_j,Γ_Z)-set. Define the digraph $\vec{Z}_0$ on the vertex set $V(Z)$ by $A(\vec{Z}_0) = \{z_j z : z_j \in V(Z) \& z \in W_j\}$ and let Z_0 be its underlying graph. Notice that if two arcs of $\vec{Z}_0$ receive the same colour (considered as edges of Z), then they are opposite one to the other and in such a case no other edge of Z can receive that colour. Let $\gamma = |\{zz' \in E(Z_0) : zz', z'z \in A(\vec{Z}_0)\}|$ and $\phi = |\Gamma_Z(E(Z))| - |\Gamma_Z(E(Z_0))|$. Clearly, $\sum_j |W_j| = |A(\vec{Z}_0)| = |\Gamma_Z(E(Z_0))| + \gamma = |\Gamma_Z(E(Z))| + \gamma - \phi$.

We will show now that $\sum_j |W_j| \leq 2h_p(\nu(x_0,\Gamma)) - 4$. Since $|\Gamma_Z(E(Z))| \leq h_p(\nu(x_0,\Gamma)) - 1$, we only have to prove (1) $\gamma - \phi \leq h_p(\nu(x_0,\Gamma)) - 3$. If $\phi \geq 1$, we have $\gamma \leq |\Gamma_Z(E(Z_0))| \leq |\Gamma_Z(E(Z))| - \phi \leq h_p(\nu(x_0,\Gamma)) - 2$ and therefore (1) holds. Assume $\phi = 0$ and suppose $\gamma \geq h_p(\nu(x_0,\Gamma)) - 2$. Then, there exist at least $h_p(\nu(x_0,\Gamma)) - 2$ edges in $E(Z)$ whose corresponding chromatic classes are singular and so the remaining edges of Z (which really do exist for otherwise Z would be a TMC complete subgraph of order at least $p+1$) are coloured with a single new colour. Since $\phi = 0$, all these last edges are adjacent to one vertex v_t. Then, $Z-\{v_t\}$ is a TMC complete subgraph of order at least p, which is impossible. So, (1) holds, and therefore, $\sum_j |W_j| \leq 2h_p(\nu(x_0,\Gamma))-4$ which implies that for some W_q, $|W_q| \leq \frac{2h_p(\nu(x_0,\Gamma))-4}{|V(Z)|}$ and the proof ends. ∎

Proof of Theorem 1. Let $\mathcal{Q}(p)$ be the property that for every n such that $n-2 \geq p \geq 3$, $\nu^*(p,n) \leq a(p,n)$. By Lemma 5, we only have to prove that (1) For every $p \geq 3$, $\mathcal{Q}(p)$ holds.

The proof of (1) will be done by induction on p. Let $p=3$ and suppose that $\mathcal{Q}(3)$ does not hold. So for some n there exists $\Gamma \in C(3,n)$ such that (2) $\nu(x_0,\Gamma) \geq 1+a(3,n)$, where x_0 is a value of x which minimizes $\nu(x,\Gamma)$. By Lemma 3.(ii) we have (3) $\nu(x_0,\Gamma) \geq 3$. Let $\{z_1,z_2,\ldots,z_k\}$ be a selective (x_0,Γ)-set, so $k=\nu(x_0,\Gamma)$. Let Z be the (complete) subgraph of K_n induced by $\{z_1,z_2,\ldots,z_k\}$. Suppose that $\nu(z_j,\Gamma_Z) \geq 2$ and let z and z' be two different vertices in a selective (z_j,Γ_Z)-set; clearly $\{x_0,z_j,z,z'\}$ induces a TMC copy of K_4, which is impossible. So $\nu(z_j,\Gamma_Z) \leq 1$. From Lemma 4.(i), we have $\nu(z_j,\Gamma) \leq 2+(n-1-\nu(x_0,\Gamma)) = n-\nu(x_0,\Gamma)+1$.

We will prove now that we have (4) For some j, $\nu(z_j,\Gamma) \leq n-\nu(x_0,\Gamma)$. Suppose that our assertion is false. Then, (5) $\nu(z_j,\Gamma) = n-\nu(x_0,\Gamma)+1$ for every j and this implies that all the $Z(K_n-Z)$-edges, together with the edges of some (perfect) matching of Z, must have different colours. Since $\nu(x_0,\Gamma) \geq 3$, the matching contains at least two edges. Moreover, we have (6) $\nu(x_0,\Gamma) \leq n-2$ since $\nu(x_0,\Gamma) \leq \nu(z_j,\Gamma) = n-\nu(x_0,\Gamma)+1 \leq n-2$. Taking z_i, z_j and $w \in V(K_n-Z)$, $w \neq x_0$ such that the colours of x_0z_i, x_0z_j and x_0w are all different and z_iz_j belongs to the matching, we obtain the TMC copy of K_4 induced by $\{x_0,z_i,z_j,w\}$. Then (4) holds and we obtain $\nu(x_0,\Gamma) \leq \lfloor \frac{n}{2} \rfloor$ which implies $\mathcal{Q}(3)$. This yields a contradiction.

Assume now that $\mathcal{Q}(k)$ holds for every $k<p$. By Lemma 5.(ii) we have $h_{k+1}(n) = t_k(n)+2$. Suppose that $\mathcal{Q}(p)$ does not hold. So for some $n \geq p+2$ there exists $\Gamma \in C(p,n)$ such that (7) $\nu(x_0,\Gamma) \geq 1+a(p,n)$. By Lemma 3.(ii) we have (8) $\nu(x_0,\Gamma) > p$. Let Z be as above. Since Z contains no TMC copy of K_p it follows that the number of colours appearing in Z is at most $h_p(\nu(x_0,\Gamma))-1$. By Lemma 6 we have (9) $\nu(\zeta,\Gamma_Z) \leq \frac{2h_p(\nu(x_0,\Gamma))-4}{\nu(x_0,\Gamma)}$ for some $\zeta \in V(Z)$. Now, by the induction hypothesis $h_p(\nu(x_0,\Gamma)) = t_{p-1}(\nu(x_0,\Gamma))+2$, and then, after some easy calculations, we obtain, (10) $\nu(\zeta,\Gamma_Z) \leq \nu(x_0,\Gamma)\frac{p-3}{p-2}$.

Applying Lemma 4.(i), it follows that $\frac{n(p-2)}{p-1} \geq \nu(x_0,\Gamma)$ and from Lemma 3.(iii) we obtain $a(p,n)+1 > \nu(x_0,\Gamma)$ which implies $\mathcal{Q}(p)$. This yields a contradiction. ∎

Final comments. We remark that if $n-2 \geq p \geq 3$ then for every $\Gamma \in C(p,n)$, $\min\{\nu(x,\Gamma) : x \in V(K_n)\} = \nu^*(p,n)$. This follows from the inequalities $h_{p+1}(n) - h_{p+1}(n-1) \leq \min\{\nu(x,\Gamma) : x \in V(K_n)\} \leq \nu^*(p,n) \leq a(p,n)$ included in Lemma 4.(ii) and in the proof of Theorem 1, and from the fact that $h_{p+1}(n) - h_{p+1}(n-1) = a(p,n)$, because of Theorem 1 and Lemma 3.(i).

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