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CRITICAL FACETS OF THE STABLE SET POLYTOPE

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Dedicated to the memory of Paul Erdős

A facet of the stable set polytope of a graph G can be viewed as a generalization of the notion of an α-critical graph. We extend several results from the theory of α-critical graphs to facets. The *defect* of a nontrivial, full-dimensional facet $\sum_{v\in V} a(v)x_v \le b$ of the stable set polytope of a graph G is defined by $\delta = \sum_{v\in V} a(v) - 2b$. We prove the upper bound $a(u)+\delta$ for the degree of any node u in a critical facet-graph, and show that $d(u)=2\delta$ can occur only when $\delta=1$. We also give a simple proof of the characterization of critical facet-graphs with defect 2 proved by Sewell [11]. As an application of these techniques we sharpen a result of Surányi [13] by showing that if an α-critical graph has defect δ and contains $\delta+2$ nodes of degree $\delta+1$, then the graph is an odd subdivision of $K_{\delta+2}$.

1. Introduction

Let $G=(V,E)$ be a simple graph on n nodes. Let $\alpha(G)$ denote the maximum size of an independent set of nodes in G. The graph G is called *α-critical* if deleting any edge increases $\alpha(G)$, and (to exclude some trivial complications) G has no isolated node. Since every connected component of an α-critical graph is also α-critical, we often restrict our attention to connected α-critical graphs.

The theory of α-critical graphs was initiated by Erdős and Gallai [4], and contains a variety of interesting structural results (see [8] for a survey). The *defect* $\delta=|V|-2\alpha(G)$ plays the central role in this theory. It was shown in [4]

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that this defect is non-negative, and the only connected α-critical graph with defect 0 is K_2.

Hajnal [5] proved that the degree of any node is at most $\delta+1$, and Surányi [13] proved that equality can hold for at most $\delta+2$ nodes.

Hajnal's theorem implies that the only connected α-critical graphs with defect 1 are odd cycles. Andrásfai [1] proved that every connected α-critical graph with defect 2 is an odd subdivision of K_4 (we replace each edge with a path of odd length). Surányi [13] classified α-critical graphs with $\delta=3$. Lovász [7] proved that α-critical graphs with a fixed defect can be obtained from a finite number of "basic" graphs by odd subdivision.

Chvátal [3] established a rather interesting connection between α-critical graphs and polyhedral combinatorics by showing that if G is a connected α-critical graph, then the inequality $\sum_{v\in V} x_v \le \alpha(G)$ defines a facet of the stable set polytope of G. Thus every facet of the stable set polytope can be viewed as a generalization of the notion of an α-critical graph, and one may ask which results about α-critical graphs extend to facets of the stable set polytope.

The notion of the *defect* can easily be extended to any nontrivial facet $\sum_{v\in V} a(v)x_v \le b$ of the stable set polytope by $\delta = \sum_{v\in V} a(v) - 2b$. It was shown by Lovász and Schrijver [9] that this number is non-negative, and in fact can be characterized in a very natural way as twice the integrality gap of the optimization problem

$$\begin{aligned} \text{maximize} \quad & \sum_v a(v)x_v \\ \text{subject to} \quad & x_v \ge 0, \quad (v \in V(G)) \\ & x_u + x_v \le 1. \quad (uv \in E(G)) \end{aligned}$$

To avoid some trivial complications, we assume throughout that the graph has at least three nodes, it is connected, and every coefficient $a(v)$ is nonzero. Then the facet is different from the (almost trivial) facets defined by edge-constraints, and has positive defect. A graph with a fixed facet will be called a facet-graph.

To obtain more structural results, it is often necessary to restrict ourselves to facet-graphs *critical* with respect to the facet, i.e., to assume that deleting any edge of G the inequality defining the facet does not remain valid any more. We'll state this assumption explicitly wherever needed.

Sewell [11] proved the important fact that $a(v)\le\delta$ for every node. Furthermore, he extended Andrásfai's result by describing all critical facet-graphs with defect at most 2. Recently the authors [6] showed that all critical facets with a given defect can be obtained from a finite number of "basic" facets by odd subdivision.

In this paper we study further properties of α-critical graphs and their extensions to facets. In Theorem 2 we prove the upper bound $a(u)+\delta$ for the degree of any node u in a critical facet-graph, and show that equality can be attained only if $\delta=1$ and G is an odd cycle. Combined with Sewell's result mentioned above, this implies that if $\delta>1$, then every node has degree at most $2\delta-1$. (The example of a wheel shows that this bound is tight for all $\delta > 1$.) The main tool in settling the case of equality is a structural description of critical facet-graphs where the weight of a node is equal to the defect.

Using these methods we give a very simple proof of Sewell's theorem characterizing critical facet-graphs with $\delta = 2$ (Theorem 3). As a further application of these methods, we settle an old question of Surányi by showing in Theorem 6 that among α-critical graphs with defect δ, only the odd subdivisions of $K_{\delta+2}$ contain $\delta+2$ nodes of degree $\delta+1$.

The outline of the paper is the following: In Section 2 we define the stable set polytope and facet-graphs, and prove some preliminary facts. Sections 3, 4, 5 and 6 form the technical core of the paper, where we prove basic properties of the surplus function, we describe three operations that preserve facet-graphs, and study the structure of facet-graphs which have a node with maximum weight or maximum surplus. Sections 7 and 8 contain the main results of the paper.

2. The stable set polytope

Every graph $G=(V,E)$ in this paper will be assumed to be simple, finite, and undirected with node set $V=V(G)$ and edge set $E=E(G)$. The degree of a node $v\in V$ is denoted by $d(v)$, and $N(v)=N_G(v)=\{w:vw\in E(G)\}$ is the set of neighbors of v. A set of nodes $S\subseteq V$ is called *stable* or *independent* if no two nodes of S are joined by an edge.

Given a set of nodes $S\subseteq V$, its *incidence vector* $\chi_S\in\{0,1\}^V$ is defined by

$$\chi_S(u) = \begin{cases} 1 \text{ if } u \in S, \\ 0 \text{ if } u \notin S. \end{cases}$$

The *stable set polytope* of G, denoted by $STAB(G)$, is defined to be the convex hull of the incidence vectors of all stable sets of G:

$$STAB(G) = \text{conv}\{\chi_S : S \text{ is a stable set in } G\}.$$

Since $STAB(G)$ contains the basic unit vectors and the origin ($STAB(G)$ is contained in the unit cube $[0,1]^V$), it is full-dimensional. Hence up to a constant there is a unique system of linear inequalities describing its facets. The facets of $STAB(G)$ can be divided into two classes:

1. **Trivial facets**: $x_v \geq 0$ for all $v \in V$; $x_v \leq 1$, if $v \in V$ is an isolated node in G.
2. **Nontrivial facets**: $\sum_{v\in V} a(v)x_v \leq b$, where all coefficients are nonnegative, and at least two among the $a(v)$'s are positive. If an inequality of the form $x_v + x_w \leq 1$ defines a facet, it will be called an *edge inequality.*

The coefficients $a(v)$ will be referred to as *weights*. Sometimes we will simply write ax instead of $\sum_{v\in V} a(v)x_v$.

In the classification of the facets the inequalities indicate which halfspace the stable set polytope belongs to. To make these inequalities unique we scale them so that their coefficients become integral and their greatest common divisor is 1. The facet defined by the inequality $ax \leq b$ will be sometimes called the facet $ax \leq b$.

The graph G with the weighting a will be called the *weighted graph* (G,a) ($b = \max\{ax : x \in STAB(G)\}$ will be usually omitted from the notation). If $U \subseteq V$ is a stable set in (G,a) with $a(U) = b$, then U will be called a *maximum weight independent* (or *stable*) *set* (we will usually restrict ourselves to one weighting, so the fact that an independent set may be of maximum weight with respect to one weighting, but not with respect to another one, will not cause confusion). The edge $e \in E(G)$ in the weighted graph (G,a) will be called *critical* if the maximum weight of a stable set in $G - e$ is larger than in G. The weighted graph (G,a) will be called *critical* if every edge $e \in E(G)$ is critical. If (G,a) is not critical, we can remove edges from G until it becomes critical. The weighted graph (G,a) will be called a *facet-graph* if $\sum_{v\in V} a(v)x_v \leq b$ defines a facet of $STAB(G)$. The *defect* of this facet is defined by $\delta = \sum_{v\in V} a(v) - 2b = a(V) - 2b$. From now on we assume that every weight is positive ($a > 0$), which can be achieved by deleting the nodes with zero weight.

Define $c_{uv} = \max\{ax : x \in STAB(G - uv)\} - b$ for any edge $uv \in E$ of G. The quantity c_{uv} will be called the *strength* of the edge uv, and it shows by how much the maximum weight of an independent set increases if we delete the edge uv. Clearly the strength of an edge is positive if and only if the edge is critical. Furthermore, the strength of an edge is at most the minimum of the weights of the two endpoints of the edge.

We need two lemmas that produce many maximum weight stable sets in facet-graphs.

Lemma 1. (a) *For every edge uv in a facet-graph with at least three nodes there is a maximum weight stable set M such that $u, v \notin M$.*

(b) *For every two nodes u and v there is a maximum weight stable set containing exactly one of them.*

Proof. It suffices to note that there must be vertex on the facet defined by $ax \le b$ which is not on the hyperplanes $x_u + x_v = 1$ and $x_u = x_v$, respectively. ∎

Lemma 2. *Let uv be an edge in a facet-graph, and assume that $a(u) = c_{uv}$. Then there is a maximum weight stable set M (not containing u) such that $M \cap N(u) = \{v\}$.*

Remark. The most common situation when this assumption is fulfilled is when uv is a critical edge and $a(u) = 1$.

Proof. There exists a stable set S in $G - uv$ with $a(S) = b + c_{uv} = b + a(u)$. Obviously $u, v \in S$, hence $M = S \setminus \{u\}$ is independent in G with $a(M) = b$. Clearly $M \cap N(u) = \{v\}$. ∎

3. The surplus function

Let (G, a) be a facet-graph. As we mentioned earlier we assume that the defect of the corresponding facet $ax \le b$ is positive (i.e. the facet is nontrivial and not an edge inequality). A useful tool in the study of stable sets is the following set function: For $S \subseteq V$, define $a(S) := \sum_{v \in S} a(v)$, and call the difference $\sigma(X) := a(N(X)) - a(X)$ the *surplus* of the stable set $X \subseteq V$. The empty set has surplus 0; if M is a maximum weight stable set, then its surplus is

$$\sigma(M) = a(V \setminus M) - a(M) = a(V) - 2b = \delta.$$

In this section we summarize some basic properties of this function, generalizing known properties of the surplus function of α-critical graphs (see [8, p. 449]).

Our first lemma about the surplus function is a slight extension of a lemma of Sewell [11].

Lemma 3. *Let M be a maximum weight independent set in V, and let X be any independent set. Then:*

(a) $a(X \setminus M) \le a(N(X) \cap M)$.

(b) $\sigma(M \cap X) \le \sigma(X)$. *If equality holds, then* $a(X \setminus M) = a(N(X) \cap M)$ *and* $N(M \cap X) = N(X) \setminus M$.

(c) *If X is non-empty, then* $\sigma(X) > 0$.

Proof. Let $S_1 = M \cap X$, $S_2 = N(X) \cap M$, $T_1 = X \setminus S_1$, and $T_2 = N(X) \setminus S_2$. Since $(M \setminus S_2) \cup T_1$ is also an independent set, and M is of maximum weight, we get that $a(T_1) \le a(S_2)$, and the first inequality of the lemma is proved. Obviously $N(S_1) \subset T_2$, since $S_1 \cup S_2$ is independent. Hence

$$a(N(S_1)) - a(S_1) \le a(T_2) - a(S_1) \le a(T_2) - a(S_1) + a(S_2) - a(T_1) = \\ = a(T_2 \cup S_2) - a(T_1 \cup S_1) = a(N(X)) - a(X),$$

which proves the second inequality of the lemma. The statements about the case of equality are easily checked.

By Lemma 1(a), the intersection of all maximum weight independent sets is empty, hence a repeated application of the second inequality gives that

$$\sigma(X) \ge 0.$$

Suppose that equality holds here for some non-empty X. Let $u \in N(X)$, and let $\{v_1, \ldots, v_k\} = N(u) \cap X$. By Lemma 1 there are maximum weight stable sets $M_1, \ldots, M_k$ such that $u, v_i \notin M_i$ for all i. Apply (b) repeatedly to get that

$$\sigma(X) \ge \sigma(X \cap M_1 \cap \ldots \cap M_k) \ge 0.$$

Since we have equality here, (b) implies that $u \in N(X \cap M_1 \cap \ldots \cap M_k)$. But this is impossible by the choice of the sets M_i. ∎

From the equality case of part (b) it is easy to show by induction the following:

Lemma 4. *Let X be a stable set, and let $M_1, M_2, \ldots, M_k$ be maximum weight independent sets. Then $\sigma(X \cap M_1 \cap \ldots \cap M_k) \le \sigma(X)$; if equality holds, then $N(X \cap M_1 \cap \ldots \cap M_k) = N(X) \setminus M_1 \setminus \ldots \setminus M_k$.*

The third inequality in Lemma 3 says that the surplus of *every* non-empty independent set is positive. This was sharpened by Mahjoub [10]:

Lemma 5. *Suppose that (G, a) is a facet-graph with at least three nodes. If v and w are joined by an edge, then*

$$a(v) \le a(N(v) \setminus \{w\}).$$

Sewell [11] proved the following lower bound for the surplus of an independent set, which we augment with a necessary condition for equality:

Lemma 6. (a) *Suppose that X is an independent set and $v \in N(X)$. Then $\sigma(X) \ge a(v)$.*

(b) *If equality holds, then for every maximum weight stable set M either* (i) *$v \in M$, or* (ii) *$N(M \cap X) = N(X) \setminus M$, or* (iii) *$M \cap X = \emptyset$ and $N(X) \setminus \{v\} \subseteq M$.*

Proof. Let $u \in X$ be adjacent to v. Consider a maximum weight independent set M not containing u and v. If $N(M \cap X)$ does not contain v, then Lemma 3 gives that $a(X \setminus M) \le a(M \cap N(X))$, and also that

$$a(M \cap X) \le a(N(M \cap X)) \le a(N(X) \setminus M) - a(v), \tag{1}$$

hence $\sigma(X) \ge a(v)$.

If $M \cap X$ contains a neighbor of v, then we can apply inductively the same procedure to the smaller set $M \cap X$ to obtain that its surplus is at least $a(v)$, and then Lemma 3 shows that the same applies to the original set X.

Suppose now that equality holds, and let M be a maximum weight independent set. Suppose that (i) does not hold, i.e., $v \notin M$. If $v \notin N(M \cap X)$, then we have equality in (1), in particular $M \cap X$ has surplus 0; by Lemma 3 this implies that $M \cap X = \emptyset$. But then equality in the second inequality in (1) gives that $N(X) \setminus M = \{v\}$. Thus (iii) holds.

Finally, if $v \in N(M \cap X)$, then by part (a) and by Lemma 3(b),

$$a(v) = \sigma(X) \ge \sigma(X \cap M) \ge a(v),$$

hence $\sigma(X) = \sigma(X \cap M)$. By Lemma 3(b), this implies that $N(X \cap M) = N(X) \setminus M$. Thus (ii) holds. ∎

Applying this lemma to a maximum independent set missing a node v, we get the following important corollary:

Theorem 1. (Sewell [11]) *If (G, a) is a facet-graph with defect $\delta > 0$, then $a(v) \le \delta$ for all $v \in V$.*

The last lemma in this section shows that at least for certain stable sets, the surplus can also be bounded from above, using the defect. We will call a stable set *closed* if it can be obtained as the intersection of maximum weight independent sets. If the set $\{v\}$ is closed, we simply say that the node v is *closed*. The *closure* of a set is the intersection of all maximum weight stable sets containing it (by convention, this is V if no maximum weight stable set contains the set).

Lemma 7. *If S is a closed set of nodes, then $\sigma(S) \le \delta$.*

Proof. The surplus of a single maximum weight independent set is exactly δ. Repeated application of Lemma 3(b) gives the assertion. ∎

This lemma says that the surplus of any closed node is at most δ. It was conjectured that in the case of critical facet-graphs this holds for non-closed nodes as well. This conjecture can be viewed as a weighted generalization of Hajnal's theorem $d(u) \le \delta + 1$ in α-critical graphs (see [5]). However, the conjecture is false, and we will give a counterexample in Section 6. Theorem 2 on the degree is a weaker version of this conjecture.

4. Nodes with maximum weight

In this section, we fix a facet-graph (G,a) and a node u with $a(u) = \delta > 0$. We show that such graphs have a rather strict structure.

Lemmas 6 and 7 imply the following:

Lemma 8. *Let S be a closed stable set containing a neighbor of u. Then the surplus of S is exactly δ.*

Lemma 9. *Let S be a non-empty set that is the intersection of maximum weight stable sets not containing u. Then $u \in N(S)$ and so $\sigma(S) = \delta$.*

Proof. Let $S = M_1 \cap \ldots \cap M_r$, where the M_i are maximum weight stable sets not containing u. We prove by induction that $u \in N(M_1 \cap \ldots \cap M_i)$ for $i = 1, \ldots, r$. This is true for $i = 1$. Suppose that we know that $u \in N(M_1 \cap \ldots \cap M_i)$, $i < r$. Then $\sigma(M_1 \cap \ldots \cap M_i) = \delta$ by Lemma 8. Thus by Lemma 6(b), every maximum weight stable set M satisfies either (i) $u \in M$, or (ii) $N(M \cap (M_1 \cap \ldots \cap M_i)) = N(M_1 \cap \ldots \cap M_i) \setminus M$, or (iii) $M \cap (M_1 \cap \ldots \cap M_i) = \emptyset$ and $M \supseteq N(M_1 \cap \ldots \cap M_i) \setminus \{u\}$. The first and third possibilities are trivially ruled out if $M = M_{i+1}$, and the second completes the induction. ∎

Lemma 10. *For every node $v \ne u$ there is a maximum weight stable set not containing u and v.*

Proof. Let S be the intersection of all maximum weight independent sets avoiding u. Suppose that $v \in S$. Then Lemma 9 implies that $u \in N(S)$. Let $w \in N(u) \cap S$. By Lemma 1 there exists a maximum weight independent set missing u and w, contradicting $w \in S$. ∎

Let u be a node of the facet-graph (G,a). We say that an edge $vw \in E(G)$ is *co-covered with respect to u* if every maximum weight independent set containing u contains either v or w (clearly every edge incident to u is co-covered with respect to u). Similarly, the edge vw is *anticovered* with respect to u if every maximum weight independent set *not* containing u contains either v or w. By Lemma 1, no edge incident to u is anticovered, and no edge can be of both types.

In this section anticovered edges (with respect to the maximum weight node u) will play an important role.

Lemma 11. *Anticovered edges cover all nodes in G except u.*

Proof. Let $v \in V \setminus \{u\}$, and let $M_1, \dots, M_t$ be all maximum weight independent sets disjoint from $\{u,v\}$ (by Lemma 10, there is at least one such set). Let $S = M_1 \cap \dots \cap M_t$. We claim that $v \in N(S)$.

We prove by induction on i that $u, v \in N(M_1 \cap \dots \cap M_i)$. This is clear if $i = 1$. If this is true for i, then $\sigma(M_1 \cap \dots \cap M_i) = \delta$ by Lemma 8. Hence by Lemma 6(b), every maximum weight independent set satisfies one of the following alternatives: (i) $u \in M$, or (ii) $N(M \cap (M_1 \cap \dots \cap M_i)) = N(M_1 \cap \dots \cap M_i) \setminus M$, or (iii) $M \cap (M_1 \cap \dots \cap M_i) = \emptyset$ and $M \supseteq N(M_1 \cap \dots \cap M_i) \setminus \{u\}$. For $M = M_{i+1}$, (i) is trivially ruled out since $u \notin M_{i+1}$, and so is (iii) since $v \notin M_{i+1}$. Thus (ii) holds, which completes the induction.

Now vw is anticovered for any $w \in N(v) \cap S$. ∎

As a consequence, we obtain two analogues of Lemma 10.

Lemma 12. *For every node $v \neq u$ there is a maximum weight stable set containing v but not u.*

Proof. Let vw be an anticovered edge. We know that there exists a maximum weight stable set M avoiding both u and w. Then, clearly, M contains v. ∎

Lemma 13. *The node u is closed; in other words, for every node $v \neq u$ there is a maximum weight stable set containing u but not v.*

Proof. Let $v \neq u$ and let vw be an anticovered edge. By Lemma 1, there is a maximum weight independent set M avoiding both v and w. Since vw is anticovered, we must have $u \in M$. ∎

The next lemma describes main structural properties of anticovered edges. We need a definition: a *perfect a-matching* in a graph $H = (W, F)$ ($a \in \mathbb{Z}_+^W$) is a multiset P of edges such that each node v is contained in exactly $a(v)$ members of P.

Lemma 14. *Let $H = (V \setminus \{u\}, E')$ be the graph formed by anticovered edges. Then H is bipartite, and has a perfect a-matching.*

Remark. The last assertion implies that the inequality $\sum_{v \neq u} a(v) x_v \leq b$ can be obtained as the sum of edge inequalities ($x_v + x_w \leq 1$) corresponding to anticovered edges.

Proof. Let M be any maximum weight independent set avoiding u. Then M contains exactly one node of each edge of H. Hence H is bipartite.

For $v \neq u$, let $S(v)$ be the intersection of all maximum weight stable sets missing both u and v. Clearly $N_H(v) = S(v) \cap N(v)$, and Lemma 11 implies that this set is non-empty.

Let the connected components of H be $H_1, \dots, H_k$, and let (C_i, D_i) be the bipartition of H_i. We may assume that $M = \cup_{i=1}^k C_i$. Let $N = \cup_{i=1}^k D_i$. Clearly every maximum weight independent set not containing u contains exactly one of C_i and D_i for all i. This implies that for $v \in D_i$, $C_i \subseteq S(v) \subseteq M = \cup_{i=1}^k C_i$, and $S(v)$ is the union of certain C_j's. Furthermore, $S(v)$ is the same for any node $v \in D_i$, and we may denote it by S_i. Let $T_i = \cup\{D_j : C_j \subseteq S_i\}$.

Note that if $C_j \subseteq S_i$, then $S_j \subseteq S_i$. Indeed, every maximum weight stable set missing u and D_i also misses D_j (since it contains C_j), and hence S_j is the intersection of a larger family of sets. It follows that we can introduce a pre-order among the sets $C_1, \dots, C_k$: let $C_i \to C_j$ if $S_j \subseteq S_i$; equivalently, if every maximum weight independent set not containing u but containing C_i also contains C_j. Let $\overline{C_i} = \cup\{C_j : S_i = S_j\}$ and $\overline{D_i} = \cup\{D_j : S_i = S_j\}$. Clearly $\overline{C_i} \subseteq S_i$, the sets $\overline{C_i}$ are either equal or disjoint, and S_i is a union of all sets $\overline{C_j}$ for which $S_j \subseteq S_i$.

We show that $N(S_i) = T_i \cup \{u\}$. We have $u \in N(S_i)$ by Lemma 9. Furthermore, let $vw \in E$, $v \in S_i$. We may assume that $v \in C_i$. Let $w \in D_j$. Then every maximum weight stable set missing u and D_i must contain v, so it must miss w, so it must contain C_j. Thus it follows that $S_j \subseteq S_i$, and so $D_j \subseteq T_j \subseteq T_i$.

Furthermore, if $vw \notin E(H)$, then there is a maximum weight stable set missing u, v, and w. This implies that $C_i \not\subseteq S_j$ and so S_j is a proper subset of S_i. Thus all edges spanned by $\overline{C_i} \cup \overline{D_i}$ are edges of H. In particular, since $\overline{D_i}$ is trivially independent in H, $\overline{D_i}$ is independent.

We have $\sigma(S_i) = \delta$ by Lemma 8 (since $u \in N(S_i)$ and $a(u) = \delta$). But

$$a(N(S_i)) - a(S_i) = a(u) + a(T_i) - a(S_i) = \delta + a(T_i) - a(S_i),$$

and hence $a(S_i) = a(T_i)$.

Now we can prove that $a(\overline{C_i}) = a(\overline{D_i})$ for all i by induction on $|S_i|$. If $\overline{C_i} = S_i$, then $\overline{D_i} = T_i$, and we already know that $a(S_i) = a(T_i)$. If S_i is strictly larger than $\overline{C_i}$, say $S_i = \overline{C_i} \cup (\cup_{j \in J} \overline{C_j})$, then $|S_j| < |S_i|$ for $j \in J$, and hence by the induction hypothesis, we have $a(\overline{C_j}) = a(\overline{D_j})$ for all $j \in J$. Hence

$$a(\overline{C_i}) = a(S_i) - \sum_{j \in J} a(\overline{C_j}) = a(T_i) - \sum_{j \in J} a(\overline{D_j}) = a(\overline{D_i}).$$

Next we show that

$$a(D) \leq a(N_H(D)) \tag{2}$$

for each set $D \subseteq N$. Clearly it is enough to show this for the connected components of H, so without loss of generality we may suppose that $D \subseteq D_i$.

Let L be a maximum weight independent set not containing u but containing D_i (such a set exists by Lemma 12). The set $L' = (L \setminus T_i) \cup (S_i \setminus L)$ is independent. Indeed, assume that xy is an edge spanned by L', $x \in M$, $y \in N$. Then trivially $x \in S_i$ and hence $y \in T_i$, which is impossible since L' is disjoint from T_i.

We claim that $L'' = (L' \setminus N_H(D)) \cup D$ is also independent. Suppose that it spans an edge xy, where the only case we have to worry about is when $x \in D$ and $y \in L' \setminus N_H(D)$. Since L is stable, and $x \in L$, we must have $y \notin L$, so $y \in L'$ implies that $y \in S_i$. Since $xy \notin E(H)$, this implies that $x \in T_j$ for some j with $S_j \subset S_i$; but this contradicts $x \in D \subseteq D_i$.

Now we just have to count nodes. L contains exactly one of $\overline{C_j}$ and $\overline{D_j}$ for all j, hence $a(L) = a(L')$, i.e. L' is also of maximum weight. Thus $a(L') \geq a(L'') = a(L') + a(D) - a(N_H(D))$, which implies (2).

The second statement of the lemma follows by the Kőnig–Egerváry theorem.

Applying inequality (2) to each D_j in the set $\overline{D_i}$ we easily obtain that $a(\overline{D_i}) = a(\overline{C_i})$ implies $a(D_j) = a(C_j)$ for all j, and the lemma is proved. ∎

Remark. We could derive a few more properties of the graph H from the arguments above. In particular, we may note that if D is a non-empty proper subset of D_i, then strict inequality holds in (2). This implies that every edge of H is contained in a perfect a-matching.

We conclude this section with some consequences of this structure theorem.

Lemma 15. (a) *All critical edges in $(G-u, a)$ are anticovered in G, and each node v of $G-u$ is incident to at most $a(v)$ such edges.*

(b) *Suppose that (G, a) is a critical facet-graph. Then every node v in $(G-u-N(u), a)$ is incident to at most $a(v)$ noncritical edges.*

Proof. Lemma 14 implies that there is a perfect a-matching P in $G-u$ using only anticovered edges. Every edge of $G-u$ not in P must be noncritical (since the inequality $\sum_{v \in V \setminus \{u\}} a(v) x_v \leq b$ can be derived using only edge-constraints for edges in P). Hence every edge critical in $G-u$ is anticovered and belongs to P, which proves (a).

For (b), it suffices to notice that if (G, a) is critical, then every noncritical edge of $(G-u-N(u), a)$ is critical in $(G-u, a)$. ∎

Remark. Part (b) of this last lemma implies that we can delete at most $a(v)$ edges incident to any node v in $G-u-N(u)$ to obtain a critical weighted graph (it won't necessarily become a facet). This generalizes Lemma 3.7 in [13].

Lemma 16. *Let (G,a) be a facet-graph with defect $\delta > 1$. Suppose that $a(u)=a(v)=\delta$. Then there is a maximum weight stable containing both u and v. In particular, u and v are not adjacent.*

Proof. Suppose that no maximum weight stable set contains both u and v. By Lemma 11, anticovered edges with respect to u cover all nodes but u, and similarly, anticovered edges with respect to v cover all nodes but v. Start from u, and build a path whose edges are anticovered with respect to v and u alternatingly. Since every node different from u and v is incident to anticovered edges both with respect to u and with respect to v, we must obtain a cycle or reach v.

Suppose that we get a cycle C. If C is even, then it contains anticovered edges with respect to u and v alternatingly. Hence a maximum weight independent set M not containing u must intersect every second edge of C, hence $M\cap C=|C|/2$. Similarly if M does not contain v, it must intersect every second edge of C, thus again $M\cap C=|C|/2$. Since no maximum weight stable set contains both u and v, $M\cap C=|C|/2$ holds for all maximum weight independent sets. Thus the facet we consider is contained in the hyperplane $\sum_{i\in C} x_i=|C|/2$, which is a contradiction.

If we get on odd cycle C with length $2n+1$, then the two kinds of anticovered edges still alternate except at one node, where two edges anticovered with respect to (without loss of generality) u meet. We claim that for any maximum weight independent set M we have $M\cap C=(|C|-1)/2$. Clearly no stable set can contain more than n nodes of an odd cycle of length $2n+1$. Hence it is enough to show that each maximum weight independent set contains at least n nodes of C. If M does not contain u, then M intersects every anticovered edge with respect to u. Since C contains n disjoint anticovered edges with respect to u, we get $|M\cap C|\geq n$. Similar argument holds if $v\notin M$. Since $u,v\in M$ cannot occur, this proves that every maximum weight independent set contains exactly n nodes of C, again a contradiction.

If we arrive to v on a path P, then clearly the last edge was anticovered with respect to u, and similarly as before, P is an alternating path of even length. The same argument shows that every maximum weight independent set must contain exactly $|P|/2$ nodes of the path P. This shows that this case is also impossible, except if the graph itself is an odd cycle, but then $\delta=1$. ∎

As a further application of anticovered edges, we derive another class of closed nodes:

Lemma 17. *Let* (G,a) *be a critical facet-graph with defect* $\delta > 0$, *and* $uv \in E(G)$. *Assume* $a(u) = \delta$ *and* $a(v) = 1$. *Then* v *is of degree* 2. *If the other neighbor of* v *has weight 1, then* v *is closed.*

It may be true that the first assumption implies the second, i.e., if $a(u) = \delta$ and its neighbor v has weight 1, then v is closed. This then would imply by Lemma 7 that the weight of the other neighbor of v is 1.

Proof. Since $a(u) = \delta$, we have edges anticovered with respect to u incident to all nodes but u. Let vw be such an edge (clearly $w \neq u$). Suppose that there is a third edge vz incident to v $(z \neq u, w)$. Lemma 2 implies that there is a maximum weight independent set not containing u, v, and w, contradicting the assumption that the edge vw is anticovered. Hence v has degree 2.

Now suppose that $a(w) = 1$. Let S be the closure of v. First we show that no other neighbor z of u is in S. By Lemma 1, there is a maximum weight independent set M containing neither u nor z. If $v \in M$, then $z \notin S$ by the definition of closure. If $v \notin M$, then w, the other neighbor of v, must be in M, and $a(v) = a(w)$ implies that $M' = (M \cup \{v\}) \setminus \{w\}$ is a maximum weight independent set as well. Thus $S \subseteq M'$, and hence $z \notin S$.

Finally, let M be a maximum weight independent set not containing u and v. Then clearly $w \in M$. Since $S \cap N(u) = \{v\}$, Lemma 6 implies that $M \cap S = \emptyset$ and $M \cap N(S) = N(S) \setminus \{u\}$. Since $(M \cup \{v\}) \setminus \{w\}$ is also a maximum weight independent set, we obtain that $\{v\} \subseteq S = S \cap ((M \cup \{v\}) \setminus \{w\}) \subseteq \{v\}$, finishing the proof. ∎

5. Operations on critical facets

In this section we examine several operations to obtain new facet-graphs. The first one is a weighted generalization of the odd subdivision operation for α-critical graphs, the second one is a generalization of a similar operation introduced by Barahona and Mahjoub [2], while the last one is a generalization of an operation introduced in [6].

Subdivision of a facet-graph (G,a)**.** This operation was introduced by Wolsey [14], and is defined as follows: Let uv be a critical edge. Introduce now two new nodes u' and v' (of degree 2) on the edge uv and give them weights c_{uv} (the strength of uv). Let (G',a') be the resulting weighted graph (see Figure 1.). When this subdivision operation is repeated a finite number of times, the final weighted graph will be called an *odd subdivision* of the original facet-graph.

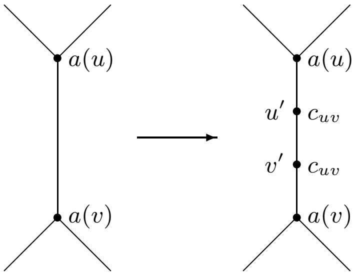

Fig. 1. Subdivision of a facet-graph

The following lemma shows that the odd subdivision of a facet-graph is a generalization of the odd subdivision of α-critical graphs:

Lemma 18. (Wolsey [14]) *An odd subdivision (G',a) of a critical facet-graph (G,a) on at least three nodes is again a critical facet-graph with the same defect. The three new edges have strengths c_{uv}.*

The converse of this lemma is not always true (one counterexample is the third graph in Figure 5., page 82); however, the following partial result holds, proved by Barahona and Mahjoub [2]:

Lemma 19. *Let (G,a) be a facet-graph. If $d(u)=d(v)=d(w)=2$, and $uv,vw,wz \in E(G)$, then $(G',a)=(G-\{v,w\}+uz,a)$ is also a facet-graph with the same defect. If $a(u)=a(v)$, then the same statement holds even without the assumption that $d(u)=2$.*

Subdivision of a star. Let (G,a) be a facet-graph, $u \in V(G)$, $d(u)=k$, and suppose that the neighbors of u are $V_u=\{v_1,\dots,v_k\}$. Introduce a new node v_i' on each edge uv_i, $i=1,2,\dots,k$, and denote the resulting graph by G'. Let $C_u=\sum_{v\in V_u} c_{uv}$. Define the new weighting a' as follows:

$$a'(v)=\begin{cases} C_u-a(u) & \text{if } v=u,\\ c_{uv_i} & \text{if } v=v_i',\ i=1,2,\dots,k,\\ a(v) & \text{otherwise.}\end{cases}$$

This operation is illustrated on Figure 2. Note that the new weights c_{uv_i} are positive whenever the edges uv_i are critical, but it is not obvious that the new weight of u is also positive. However, Lemma 5 guarantees that this is always the case.

This subdivision of a star operation is a generalization of the subdivision of a star operation introduced by Barahona and Mahjoub in [2].

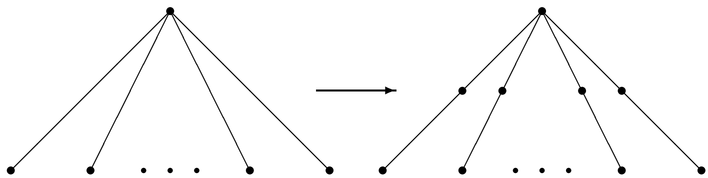

Fig. 2. Subdivision of a star

Lemma 20. *If (G,a) is a critical facet-graph with defect $\delta > 0$, then (G',a') is also a critical facet-graph with the same defect.*

Proof. First consider the special case when $c_{uv_i} = a(v_i)$ for all neighbors v_i of u in G (then $C_u = a(V_u)$). We prove that the inequality $\sum_{v \in V'} a'(v)x_v \leq b + C_u - a(u)$ defines a facet of $STAB(G')$.

First we prove that it is valid. Let M be a maximum weight independent set in G'. If $u \in M$, then $M \setminus \{u\}$ is stable in G, hence

$$\begin{aligned} a'(M) &= a'(M \setminus \{u\}) + a'(u) \\ &= a(M \setminus \{u\}) + C_u - a(u) \leq b + C_u - a(u). \end{aligned}$$

If $u \notin M$, then let $M' = M \setminus (N_{G'}(u) \cup V_u)$. Since $M' \cup \{u\}$ is stable in G, we get that

$$\begin{aligned} a'(M) &= a'(M') + a'(M \cap (N_{G'}(u) \cup V_u)) \\ &\leq (a(M' \cup \{u\}) - a(u)) + a(V_u) \leq b - a(u) + C_u. \end{aligned}$$

Next we prove that this inequality gives rise to a facet. For this we need $|V(G')|$ linearly independent, maximum weight independent sets in G'. Since (G,a) is a facet-graph, we have $|V(G)|$ such sets in G, and we can "lift" each of them to a similar set in G' by adding u (if u was not in the set) or replacing u by $N_{G'}(u) = \{v'_1, \ldots, v'_k\}$ (if u was in the set). Clearly these maximum weight independent sets in G' remain linearly independent. G' has $d(u)$ more nodes than G, so we need $d(u)$ more such sets. Since the edge uv_i is critical with strength c_{uv_i}, we have a stable set M_i in $G - uv_i$ with $a(M_i) = b + c_{uv_i}$, and clearly $u, v_i \in M_i$ for all $i = 1, 2, \ldots, k$. Then $M'_i = (M_i \cup N_{G'}(u)) \setminus \{u, v'_i\}$ is a maximum weight independent set in G', since

$$\begin{aligned} a'(M'_i) &= a(M_i \setminus u) + a'(N_{G'}(u)) - a'(v'_i) \\ &= b + c_{uv_i} - a(u) + C_u - c_{uv_i} \\ &= b + C_u - a(u). \end{aligned}$$

That these new maximum weight independent sets remain linearly independent follows from the fact that all previous such sets contained either all nodes of $N_{G'}(u)$ or none of them, while the new sets contain exactly $|N_{G'}(u)|-1$ of them ($v_i' \notin M_i'$). The criticality of the edges is obvious, and it is easy to see that the defect of (G',a') is the same as that of (G,a). (Remark: This proof also works if some of the edges incident to u are not critical, because then the new node has weight 0.)

Now we turn to the general case. For all neighbors v_i of u apply the odd subdivision operation for the edge uv_i. (It is enough to apply this operation for those neighbors v whose weight is not equal to the strength of the edge uv.) This way we obtain the critical facet-graph (G'',a'') with the same defect, where now $c''_{uv} = a''(v)$ for all neighbors v of u, hence by the previous argument, if we apply the subdivision of a star operation on u, we get a critical facet-graph (G''',a'''). Then by Lemma 19 we can delete all nodes introduced in the first step to get the critical facet-graph (G',a') with the same defect. ∎

Splitting of a node. This is a slight generalization of the similar operation defined in [6]. Let (G,a) be a facet-graph, and choose a subset $S \subseteq N(v)$ of the neighbors of the node v with the property that

$$b < \max\{ax : x \in STAB(G - E(v,S))\} < b + a(v), \tag{3}$$

where $E(v,S)$ denotes the set of edges between v and S. (The easiest application of this operation is when S contains a single node w, and vw is a critical edge with $c_{vw} < a(v)$.) Denote the maximum occurring in (3) by $b_{v,S}$, and define $c_{v,S} := b_{v,S} - b$. Next,

1. Introduce a new node, v', and join it to every neighbor of v except those in S. Let the resulting graph be G'.
2. Define the new weighting a' by

$$a'(u) = \begin{cases} c_{v,S} & \text{if } u = v, \\ a(v) - c_{v,S} & \text{if } u = v', \\ a(u) & \text{otherwise.} \end{cases}$$

This definition makes sense, because $0 < c_{v,S} < a(v)$. The operation is demonstrated on Figure 3.

The following lemma shows that using this operation we can obtain new facet-graphs:

Lemma 21. *If (G,a) is a facet-graph, and $0 < c_{v,S} < a(v)$ for a node $v \in V$ and a subset $S \subset N(v)$, then (G',a') is also a facet-graph with the same defect.*

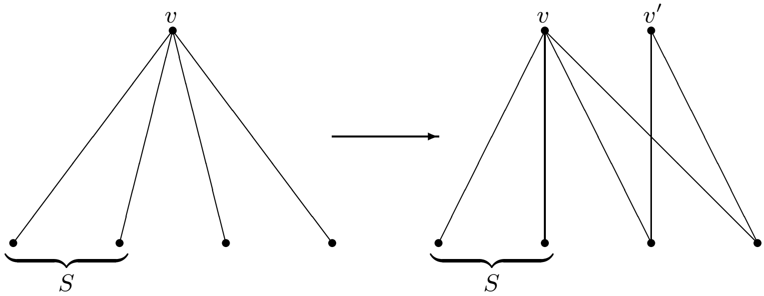

Fig. 3. Splitting of the node v

Proof. The proof is essentially the same as the proof of a similar statement in [6], we include it for completeness.

First we prove that $\sum_{u\in V(G')} a'(u)x_u \le b$ is valid for $STAB(G')$. Take a maximum weight independent set M' in (G',a'). If v is in M', then $M'\cup\{v'\}$ is also stable in (G',a'), hence $M=M'\setminus\{v'\}$ is a stable set in G, implying $a'(M')\le a'(M'\cup\{v'\})=a(M)\le b$. If neither v nor v' is in M', then $M=M'$ is stable in G, hence $a'(M')=a(M)\le b$. Finally, if v' is in M', but v is not, then $M=(M'\setminus\{v'\})\cup\{v\}$ is stable in $G-E(v,S)$, hence $a(M)\le b_{v,S}$, thus $a'(M')=a(M)-a(v)+a'(v')\le b_{v,S}-a(v)+(a(v)-c_{v,S})=b$.

Next we show that $a'x\le b$ gives rise to a facet. For this there should exist $|V(G')|$ linearly independent maximum weight stable sets in G'. Since $ax\le b$ defines a facet of $STAB(G)$, there are $|V(G)|$ such sets in G. Now note that each such set can be "lifted" to a similar set in G' by adding v' if v is in the set. Since the number of nodes increased by 1 during the splitting of v, we only have to find one more such set. The definition of $b_{v,S}$ implies that there exists a stable set M in $G-E(v,S)$ with $v\in M$ and $a(M)=b_{v,S}$. But then $M'=(M\setminus\{v\})\cup\{v'\}$ is stable in G' with $a'(M')=b$, thus M' is also a maximum weight stable set, which is linearly independent from the lifted ones, because M' contains exactly one of v,v', while the others contain both v and v' or neither of them. Since the defect of the facet clearly has not changed, this completes the proof. ∎

Note that while the previous two operations preserved criticality of the facet-graph, edges incident to v or v' may become noncritical after the splitting of the node v, so one may need to remove some noncritical edges to obtain a critical facet-graph again.

6. Nodes with maximum surplus

In this section we study closed nodes which are "extremal" in the sense that they have surplus δ, and prove analogues of the results in section 4. Several of these results could be obtained by reduction using the subdivision of a star operation, at least for critical facet-graphs.

Lemma 22. *Let (G,a) be a facet-graph with defect $\delta > 0$. Suppose that $u \in V(G)$ is a closed node with surplus δ. Then the edges co-covered with respect to u cover all nodes in G.*

Proof. Lemma 7 implies that every stable X set containing u has surplus at least δ. If X is also closed, then 4 implies that it has surplus exactly δ. Choose any node $v \neq u$, and let $M_1, \dots, M_k$ be those maximum weight independent sets that contain u but not v (such a set exists, because u is closed). Then Lemma 4 implies that

$$N(M_1 \cap \ldots \cap M_k) = V \setminus (M_1 \cup \ldots \cup M_k). \tag{4}$$

Since v is contained in the set in the right-hand side of (4), there is a (not necessarily unique) node $v' \in M_1 \cap \ldots \cap M_k$ such that vv' is an edge in G. Clearly this edge is co-covered with respect to u, which proves the lemma. ∎

As we previously mentioned, it is possible that a node in a critical facet-graph has surplus greater than δ, and we will give an example shortly. However, we have the following weaker result:

Lemma 23. *Let (G,a) be a facet-graph with defect δ. Let $u \in V$ and assume that for every neighbor v of u we have $c_{uv} = a(v)$. Then the surplus of u is at most δ, and if equality holds, then u is closed.*

Remark. The condition in the lemma is fulfilled if (G,a) is a critical facet-graph and all neighbors of u have weight 1. It follows that in this case $d(u) \le a(u) + \delta$.

Proof. Let S be the closure of u, and let $X = S \setminus \{u\}$. Let $uv \in E$. Then by Lemma 2 there exists a maximum weight independent set M with $M \cap N(v) = \{u\}$. By the definition of S, we have $M \supseteq S$, and hence $N(v)$ and X are disjoint. Applying this for every edge incident to u we get $N(X) \cap N(u) = \emptyset$, so the surplus of S is the sum of the surpluses of u and X. Lemma 7 implies that the surplus of S is at most δ. Hence the surplus of u is at most δ, and if equality holds, then the surplus of X must be 0. But then Lemma 3 implies that $X = \emptyset$. ∎

One way to obtain a graph with a closed node of surplus δ is the following: Suppose that (G,a) is a critical facet-graph with facet-defining inequality $ax \le b$. Add a new node v to G and join it to every node of G to get the graph $G+v$. Set $a(v)=b$, then it is easy to check that $(G+v,a)$ is again a critical facet-graph, and the surplus of v is $a(V(G))-a(v)=a(V)-b$, which is exactly the defect of $(G+v,a)$. (This method to generate new facet-graphs is a special case of "lifting" a facet of $STAB(G)$ to that of $STAB(G+v)$.)

There are other examples for critical facet-graphs with closed nodes of surplus δ. One example is shown in Figure 4., where the nodes u and v have weight 2, every other weight is 1, and the surplus of u is the defect $\delta=3$.

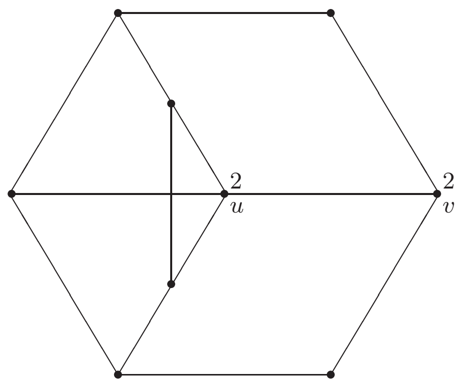

Fig. 4. A node u with surplus δ

Using the first example and the splitting of a node operation we can now obtain an example for a node of surplus bigger than δ (of course, not closed) as follows.

Let the graph G_1 be the disjoint union of C_5 and C_7 adding every edge between the nodes of the two odd cycles. It is easy to see that the inequality

$$\sum_{i=1}^{5} 3x_i + \sum_{i=6}^{12} 2x_i \leq 6.$$

defines a facet of $STAB(G_1)$ (nodes 1–5 correspond to the pentagon, nodes 6–12 correspond to the 7-cycle). Remove the noncritical edge between nodes 1 and 6, then add a new node (node 13) and join it to every other node to get the graph G_2 with facet-defining inequality

$$\sum_{i=1}^{5} 3x_i + \sum_{i=6}^{12} 2x_i + 6x_{13} \leq 6.$$

By our previous observation node 13 has the same surplus as the defect of this facet ($\delta=\sigma(\{13\})=23$ in this example). Finally apply the splitting of a node operation with $v = 13$ and $S = \{1,6\}$. Clearly $b_{v,S} = 11$, $c_{v,S} = 5$, hence the new node (node 14) will have weight 1, and the weight of node 13 decreases to 5. Hence the final graph, G_3, has the facet

$$\sum_{i=1}^{5} 3x_i + \sum_{i=6}^{12} 2x_i + 5x_{13} + x_{14} \leq 6, \tag{5}$$

and the surplus of node 13 has increased to 24, hence it became larger than the defect, which remained the same. Although the facet (5) is not critical, notice that all edges incident to node 13 are, hence after deleting noncritical edges we can obtain a critical facet-graph where node 13 has still larger surplus than δ.

The subdivision of a star operation and Lemma 14 imply a similar structure theorem for co-covered edges:

Lemma 24. *Assume that in a facet-graph (G,a) a node u has surplus δ and $c_{uv} = a(v)$ for all neighbors v of u. Then the graph H formed by co-covered edges not incident to u is bipartite on node set $V \setminus \{u\} \setminus N(u)$, and has a perfect a-matching.*

Remark. It follows again that the inequality

$$\sum_{v \notin \{u\} \cup N(u)} a(v)x_v \leq b - a(u)$$

can be obtained as a sum of edge inequalities corresponding to co-covered edges.

7. The maximum degree of a critical facet-graph

The following theorem gives an upper bound on the degree of any node in a critical facet-graph:

Theorem 2. *If (G,a) is a critical facet-graph with defect δ, then $d(u) \leq a(u) + \delta \leq 2\delta$ for any node $u \in V$, and also $d(u) \leq 2\delta - 1$ when $\delta > 1$.*

Remark. This theorem shows that $d(u) = 2\delta$ can occur only when $\delta = 1$, in which case G is an odd cycle. The wheel with $2\delta - 1$ spokes (see Figure 6.), with weight $\delta - 1$ in the middle and weights 1 along the rim, shows that the bound $2\delta - 1$ is tight for all $\delta \geq 2$.

Proof. Let (G,a) be a critical facet-graph. Apply the subdivision operation for all edges incident to u to get the critical facet-graph (G',a') with the same defect. Clearly the degree of u has not changed, so it is enough to prove the theorem for this new facet-graph. To simplify notation, denote this new facet-graph by (G,a).

Let S be the intersection of all maximum weight independent sets containing u, and let $S' = S \setminus \{u\}$. We claim that $N(S')$ and $N(u)$ are disjoint. Suppose that $v \in N(u)$. Then $c_{uv} = a(v)$ because of the subdivision, hence by Lemma 2, there is a maximum weight independent set M in G containing u but no other neighbor of v. Clearly $S \subseteq M$, and hence $v \notin N(S')$.

Now since $N(S')$ and $N(u)$ are disjoint, the surplus of S is the sum of the surpluses of u and S'. Since S is closed, Lemma 7 implies that the surplus of S is at most δ, and then using Lemma 3 we get that

$$\begin{aligned} d(u) &\le a(N(u)) = a(N(S)) - a(N(S')) \\ &= a(N(S)) - a(S) + a(S) - a(N(S')) \\ &\le \delta + a(u) - (a(N(S')) - a(S')) \\ &\le \delta + a(u). \end{aligned}$$

Now Theorem 1 shows that $a(u) \le \delta$, hence in particular $d(u) \le 2\delta$.

It follows now that if the weight of the node u is less then δ, then its degree is less then 2δ, so it is enough to prove the remaining part of the theorem for $a(u) = \delta$. Now Lemma 13 implies that u is closed, and since its surplus is δ, co-covered edges (with respect to u) cover all nodes in G by Lemma 22. Since $a(u) = \delta$, Lemma 11 implies that anticovered edges (with respect to u) cover all nodes except u.

Clearly every edge incident to u is co-covered. Now start from u and build an alternating path of co-covered and anticovered edges starting with a co-covered edge. Since every node different from u is incident to at least one co-covered and at least one anticovered edge, sooner or later we obtain a cycle C. Using a similar argument as in Lemma 16 we get that either $|M \cap C| = |C|/2$ or $|M \cap C| = (|C|-1)/2$ holds for all maximum weight independent sets depending on the parity of the length of C, which is a contradiction, except in the second case if the graph itself is an odd cycle and $\delta = 1$. This completes the proof. ∎

Now we are able to give a very simple proof of the following characterization of facet-graphs with defect 2:

Theorem 3. (Sewell [11]) *If (G,a) is a critical facet-graph with defect 2, then G is an odd subdivision of one of the five graphs shown in Figure 5. (only weights different from 1 are indicated).*

Proof. Theorem 1 shows that every weight is 1 or 2. If some weights are 2, then Lemma 16 implies that these nodes are nonadjacent, hence a node with

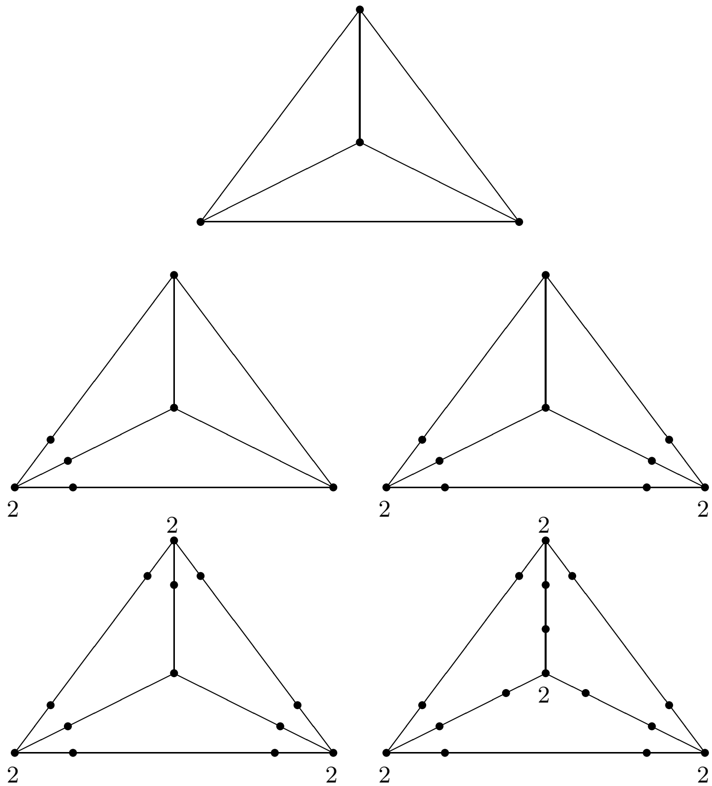

Fig. 5. Basic facet-graphs with defect 2

weight 2 has only neighbors with weight 1. From Lemma 13 we get that if $a(u)=2$, then u is closed, and then Lemma 3 and Theorem 2 together with the fact that every neighbor of u has weight 1 give that

$$1 \le a(N(u)) - a(u) = d(u) - a(u) \le 3 - 2 = 1,$$

hence the degree of u must be 3. Thus applying the subdivision of a star operation to all nodes of weight 2 we get an α-critical graph with defect 2, since the new weight of u will be $a(N(u)) - a(u) = 1$. Then by Andrásfai's theorem (see [1]) this graph must be an odd subdivision of K_4, hence the original graph can be obtained from an odd subdivision of K_4 by applying the subdivision of a star operation to some nodes of degree 3, which completes the proof. ∎

If we allow the subdivision of a star operation, we can state this theorem in a much nicer form, which parallels the description of critical facet-graphs with defect 1:

Theorem 4. *(G, a) is a critical facet-graph with defect 2 if and only if G can be obtained from K_4 using the odd subdivision and the subdivision of a star operations.*

Sewell and Trotter [12] showed that every α-critical graph with defect at least 2 contains an odd subdivision of K_4. It is an interesting open problem whether a facet-graph of defect at least 2 contains an odd subdivision of one of the basic facet-graphs with defect 2 shown on Figure 5.

We have seen that $d(u) = 2\delta$ can occur only if $\delta = 1$ in the odd cycle. Consider now the case $d(u) = 2\delta - 1$ for $\delta > 1$. From Theorem 2 we can see that there are two possible ways to achieve this: $a(u) = \delta$ or $a(u) = \delta - 1$. As noted, the wheel with $2\delta - 1$ spokes and weight $\delta - 1$ in the middle attains equality. Applying the subdivision of a star operation to u (Lemma 20), we get another example where the middle node has weight δ. The next theorem says that under the hypothesis that all other nodes have weight 1, these are the only examples:

Theorem 5. *Let (G, a) be a facet-graph with defect $\delta > 1$. Suppose that $d(u) = 2\delta - 1$, and that every weight is 1 except possibly $a(u)$. Then G is either an odd subdivision of a wheel of size $2\delta - 1$, or an odd subdivision of the graph obtained from this wheel by applying the subdivision of a star operation to the center node.*

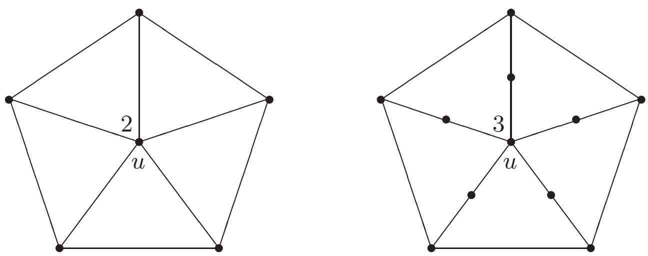

Fig. 6. Critical facet-graphs with $d(u) = 2\delta - 1$

Remark. The case $\delta = 3$ is illustrated on Figure 6. (only weights different from 1 are shown). The last three graphs on Figure 5. show that there are other graphs with $d(u) = 2\delta - 1$ as well. We do not know if there are examples which cannot be obtained from the wheel using the odd subdivision and the subdivision of a star operations.

Proof. By Theorem 2 we must have $a(u) = \delta$ or $a(u) = \delta - 1$. Suppose first that $a(u) = \delta$. Apply the inverse of the subdivision operation while possible to get a graph G where (by Lemma 19) nodes of degree 2 are not adjacent. Then by Lemma 11 anticovered edges with respect to u cover all nodes but u. Since every weight other than $a(u)$ is 1, and the defect is δ, the weight of a maximum weight independent set is $b = (a(V) - \delta)/2 = (|V| - 1)/2$. Delete the node u and its neighbors $N(u)$ from G (we omit altogether 2δ nodes). The remaining weighted graph is usually not critical, so remove noncritical edges until we get a critical facet-graph G'. Since every weight is 1, this is an α-critical graph (possibly disconnected). If M is a maximum weight independent set in G', then $M \cup \{u\}$ is a maximum weight independent set in (G, a), hence $|M| = \frac{|V|-1}{2} - \delta$. Thus the defect of G' is

$$a(V') - 2a(M) = |V'| - 2|M| = (|V| - 2\delta) - 2\left(\frac{|V| - 1}{2} - \delta\right) = 1.$$

Notice that G' cannot contain an isolated node, because Lemma 13 implies that u is closed, hence for any other node v there is a maximum weight independent set containing u but not v, and if v became isolated, it could be added to this maximum weight independent set. Hence every connected component of G' has nonnegative defect, and since the defect is additive for disjoint union of α-critical graphs, we get that exactly one component has defect 1, the others have defect 0. Hence one component is an odd cycle, the rest are independent edges.

Lemma 15 implies that all critical edges of $G - u$ form a matching (and hence so do the noncritical edges of $G - u - N(u)$). Thus if a connected component of G' is an edge (with defect 0), then both of its endpoints must be of degree 2 in G (in G every degree is at least 2). Since every weight is 1 except $a(u)$, Lemma 19 gives that we can apply the inverse of the subdivision operation to this edge. We assumed, however, that this is not possible, hence G' must have only one connected component, an odd cycle. If there is a chord in this odd cycle in G, then this chord divides the original cycle into an odd and an even cycle (the chord belongs to both), and then the edges of the even cycle adjacent to this chord are also noncritical as well as the chord (because instead of omitting the chord we could omit every second edge of the even cycle to get an α-critical graph G'), contradicting Lemma 15.

Hence $G - u - N(u)$ is a (chordless) odd cycle C. Now Lemma 17 shows that every neighbor of u is of degree 2, hence it is joined to exactly one node of C (if two neighbors of u were joined, then u would be a cut node). It follows that every edge between $N(u)$ and C is critical in $G - u$ and hence anticovered. It also follows that these edges go to different nodes of C. Since every node but u is incident to at least one anticovered edge, if a node of the odd cycle is not joined to any neighbor of u, then its neighbor along the anticovered edge is not joined to any neighbor of u either, hence we could apply the inverse of the subdivision operation to this edge. Thus every node of the odd cycle is joined to exactly one neighbor of u, hence G is the graph obtained from the wheel by applying the subdivision of a star operation to the central node.

Now consider the second case, when $a(u)=\delta-1$ and $d(u)=2\delta-1$. Apply the subdivision of a star operation to u. Since every neighbor of u has weight 1, in the resulting graph $a'(u)=d(u)-a(u)=\delta$, and every other weight remains 1. Since the degree of u has not changed, from the first part of the proof we get that our graph is an odd subdivision of a graph obtained from a wheel by applying the subdivision of a star operation to the center node. Hence the original graph was an odd subdivision of the wheel, finishing the proof. ∎

8. An application to α-critical graphs

Here we use the methods developed in this paper to give an answer to a question posed by Surányi in [13]:

Theorem 6. *If G is a connected α-critical graph with defect $\delta>1$ and it has at least $\delta+2$ nodes of degree $\delta+1$, then G is an odd subdivision of $K_{\delta+2}$.*

Proof. Surányi [13] already proved that an α-critical graph with defect $\delta>1$ has at most $\delta+2$ nodes of degree $\delta+1$ (this is the maximum degree by Theorem 2), which is clearly achieved in any odd subdivision of $K_{\delta+2}$.

So it is enough to show that no other graphs can have $\delta+2$ nodes of degree $\delta+1$. The proof goes by induction. The case $\delta=2$ was proved by Andrásfai (see [1]) and also follows from Theorem 3. Now suppose that the statement is true for $\delta-1$, and show it for $\delta>2$. Let u be a node of degree $\delta+1$ in G. Since G is α-critical and connected, $(G,\mathbf{1})$ is a critical facet-graph by Chvátal's theorem. We may assume that the inverse of the subdivision operation cannot be applied to G, hence by Lemma 19 no two nodes of degree 2 are adjacent.

Apply the subdivision of a star operation to u. Since the surplus of u is $(\delta+1)-1=\delta$, the new weight of u is δ, hence by Lemma 11 edges of type 2 with respect to u cover all nodes but u. Delete u and its neighbors (these were introduced in the subdivision of a star operation) as in the proof of Theorem 5, then delete noncritical edges to get an α-critical graph G' in

$G-u$. Since by Lemma 15 every degree decreased by at most one, G' has at least $\delta+1$ nodes of degree δ. We claim that G' is connected. Similarly as in Theorem 5, G' cannot have an isolated node or an edge as a connected component, hence every connected component of G' has positive defect. Since G' has a node of degree δ, the component containing that node must have defect at least $\delta-1$ by Theorem 2, and the additivity of the defect implies that G' cannot have more connected components (the defect of G' is clearly $\delta-1$, since there is a maximum weight independent set in G not containing u).

Hence by the induction hypothesis G' must be an odd subdivision of $K_{\delta+1}$, and since we already have at least $\delta+1$ nodes of degree δ, these nodes must coincide with the degree δ nodes of the odd subdivision of $K_{\delta+1}$.

Since the degree of each node of degree δ in G' was originally $\delta+1$ in G, one of the edges incident to this node is either joined to u or noncritical in $G-u$. If every such node is joined to u in G, then G contains an odd subdivision of $K_{\delta+2}$, and its criticality implies that G itself is an odd subdivision of $K_{\delta+2}$.

Now suppose that a node v of degree δ in G' is not joined to u in G. Then it must be joined to some other node w in $G-u$. There are two possibilities for this to happen:

(1) w is of degree δ in G'. Since G' is an odd subdivision of $K_{\delta+1}$, there is a path P from v to w in G' containing only nodes of degree 2 (except v and w). Then replacing the path P with the edge vw we get another odd subdivision of $K_{\delta+1}$ in $G-u$, hence this odd subdivision and every second edge of P also form an α-critical subgraph of $G-u$, hence the first (and every odd) edge of P is noncritical in $G-u$. But then this first edge and vw are both noncritical, contradicting Lemma 15.

(2) w is not of degree δ in G'. Since G' is an odd subdivision of $K_{\delta+1}$, the degree of w must be 2. Suppose that w is on the path P joining x and y, where x and y are nodes of degree δ in G', and every other node of P is of degree 2. The number of edges of P is odd (G' is an odd subdivision). We may suppose that the number of edges in the subpath of P joining w and x is even (otherwise exchange the role of x and y). First suppose that $v\neq x,y$. We claim that the first edge on the path joining v and x is also noncritical, and then we get a similar contradiction as in case (1). Let the path joining v and x be $R=vv_1v_2\dots v_{2k}x$, and the path joining x and w be $xw_1w_2\dots w_{2m+1}w$. If vv_1 is critical in $G'+vw$, then there is a larger stable set M in $G'+vw-vv_1$ than the maximum stable set in G'. Clearly $v,v_1\in M$, hence we must also have $v_3,v_5,\dots,v_{2k-1},x\in M$, otherwise we could get a stable set M' in $G'+vw$ with $|M|=|M'|$ by replacing the nodes of $M\cap R$ by $v,v_2,v_4,\dots,v_{2k}$. Now $v,x\in M$ implies $w,w_1\notin M$, hence M contains

exactly m nodes from $w_2, w_3, \dots, w_{2m+1}$. Hence we can replace these nodes by $w_3, w_5, \dots, w_{2m+1}$, x by w_1, and $v_1, v_3, \dots, v_{2k-1}$ by $v_2, v_4, \dots, v_{2k}$ to obtain a stable set M' in $G'+vw$ with $|M'|=|M|$, contradiction. Finally, the cases $v=x$ or $v=y$ lead to a similar contradiction (the details are omitted), and the theorem is proved. ∎

Finally, we conjecture the following weighted generalization of Theorem 6:

Conjecture 1. A critical facet-graph with defect $\delta > 1$ can have at most $\delta+2$ nodes of weight δ. If equality holds, then the graph is obtained from $K_{\delta+2}$ by the subdivision of a star operation applied to each node, and then the odd subdivision operation applied to any set of edges.

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