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λ_∞, VERTEX ISOPERIMETRY AND CONCENTRATION

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Cheeger-type inequalities are derived relating various vertex isoperimetric constants to a Poincaré-type functional constant, denoted by λ_∞. This approach refines results relating the spectral gap of a graph to the so-called *magnification* of a graph. A concentration result involving λ_∞ is also derived.

1. Introduction

In an important paper, Alon [2] derived a Cheeger–type inequality [8], by bounding from below the second smallest eigenvalue of the Laplacian of a finite undirected graph by a function of a (vertex) isoperimetric constant. More precisely, let $G=(V,E)$ be a finite, undirected, connected graph, and let $\lambda_2(G)$ denote twice (for reasons explained below) the smallest non-zero eigenvalue of the *Laplacian* of G. Recall that the Laplacian of G is the matrix $D(G)-\mathcal{A}(G)$, where $\mathcal{A}(G)$ is a symmetric matrix (indexed by the vertices of G) of order $|V|$ whose i,jth entry is 1 or 0 depending on whether there is an edge or not between the ith and the jth vertex ; and where $D(G)$ is the diagonal matrix whose i,ith element is the degree of the ith vertex. In

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[2] Alon considered the isoperimetric constant h_{out}, given by

$$h_{\text{out}} = \min_{A \subset V} \{|\partial_{\text{out}} A|/|A| : 0 < |A| \le |V|/2\},$$

where for $A \subset V$, $\partial_{\text{out}} A = \{x \in A^C : \exists y \in A, x \sim y\}$; and showed that

$$\lambda_2 \ge \frac{h_{\text{out}}^2}{2 + h_{\text{out}}^2}.$$

(Note that h_{out} was called *magnification* in [2], and was denoted by c.) This was a key result in [2] with useful implications to the so-called *magnifiers* and *expanders* (see [2] for definitions) – special classes of graphs with very many applications in computer science (see [2], [1], [10], [13], [14], [15], [16], ...). In particular, the above result gave an efficient algorithm to generate bounded degree graphs with explicit and efficiently verifiable bounds on h_{out}; the latter aspect is significant in view of the fact that in general determining h_{out} is a computationally hard (NP-hard) problem (see e.g. [5]).

A corollary to one of our main results yields a similar estimate,

$$\lambda_2 \ge \frac{(\sqrt{1 + h_{\text{out}}} - 1)^2}{2}.$$

While the proof in [2] uses basic linear algebra and the max-flow min-cut theorem, the view point here is functional analytic and it allows general probability spaces. In particular the graph can be infinite and the probability measure on the set of vertices can be arbitrary. The special case, normalized counting measure over V, reduces to the framework of [2]. The proof technique is similar to the one used in relating λ_2 to the edge-isoperimetric constant denoted by $i_1(G)$ in [11], and the essential difference is in the choice of the (discrete) gradient. We show here using the same proof technique Cheeger-type inequalities relating λ_2 to isoperimetric constants defined using the notion of *inner* and *symmetric* boundary. Thus an aspect we would like to emphasize here is that one could derive with the same proof, inequalities relating λ_2 to vertex *as well as* edge isoperimetric constants by defining and working with an appropriate discrete gradient. In each case one also needs to derive an appropriate co-area inequality. It is to be noted that we made no attempts to find the best possible (absolute) constants in our theorems, since the main point of this paper is to illustrate the strength of the functional analytic method. For convenience, below we work with finite (undirected and connected) graphs, although our approach easily extends to infinite graphs, and to directed graphs with appropriate minor modifications. Since we are dealing with vertex isoperimetry, with no loss of generality, we may assume that the graphs are simple, i.e., no self-loops nor multiple edges are allowed.

To lower bound λ_2 in terms of h_{out} and other isoperimetric constants, we introduce the Poincaré-type constant λ_∞ (see also [12]), which is such that $\lambda_2/\Delta(G)\le\lambda_\infty\le\lambda_2$, where $\Delta(G)$ is the maximum degree of G. The constant seems to be interesting in its own right and deserves to be explored further. Using this constant we also derive the following concentration result: Let $\rho=\rho_G$ denote the usual graph distance in G. For $A\subset V$, let $(\rho(A,x)\ge\ell)$ be the set of vertices in G which are at distance at least ℓ away from the nearest vertex in A. Then for any A with $\pi(A)\ge 1/2$,

$$\pi(\rho(A,x)\ge\ell)\le c_1 e^{-c_2\sqrt{\lambda_\infty}\ell},$$

where π is the normalized counting measure and where c_1 and c_2 are positive constants. This improves upon a result of Alon and Milman [4] (stated precisely in Section 4 below), which shows concentration wherein the exponent depends on $\lambda_2/\Delta(G)$ in place of our λ_∞.

Finally, note that the above (one-dimensional) concentration inequality can easily be turned into an inequality for the n-dimensional case since (see [12]) if G^n is the Cartesian product of n copies of G, then $\lambda_\infty(G^n)=\lambda_\infty(G)/n$. The concentration result on G^n will then be in terms of the distance ρ which satisfies, $\rho_{G^n}((x_1,\dots,x_n),(y_1,\dots,y_n))=\sum_{i=1}^n\rho_G(x_i,y_i)$, for all $(x_1,\dots,x_n),(y_1,\dots,y_n)\in V^n$, the vertex set of G^n.

2. λ_∞, inner and outer boundaries

Let $G=(V,E)$ be an undirected, connected graph ($V\neq\emptyset$) equipped with a probability measure π. We write $x\sim y$ to denote that $\{x,y\}\in E$ or $x=y$. For brevity, we often write $\sup_{y\sim x}$ to mean $\sup_{y:y\sim x}$. Let λ_∞ and λ_2 be the optimal constants in the following Poincaré-type inequalities,

$$\lambda_\infty\,\mathrm{var}_\pi(f)\le\int_V \sup_{y:y\sim x}|f(x)-f(y)|^2 d\pi(x),$$

$$\lambda_2\,\mathrm{var}_\pi(f)\le\int_V \sum_{y:y\sim x}|f(x)-f(y)|^2 d\pi(x),$$

where $f:V\longrightarrow\mathbb{R}$ is arbitrary. Note that both λ_2 and λ_∞ depend on G and π. Using the notation of [12], where various discrete gradients are defined, we have $\sup_{y:y\sim x}|f(x)-f(y)|=|\nabla_\infty f(x)|$, while $\sum_{y:y\sim x}|f(x)-f(y)|^2=|\nabla_2 f(x)|^2$.

Clearly, $\lambda_2 \geq \lambda_\infty$ and $\lambda_\infty \geq \lambda_2/\Delta(G)$, where $\Delta(G) = \max_{x\in V}$ degree(x) is the maximum degree of G and so $\lambda_2 > 0$ if and only if $\lambda_\infty > 0$. When π is the normalized counting measure, λ_2 is twice the smallest non-zero eigenvalue of the Laplacian of G. In general, for finite undirected G, λ_2 is also the smallest non-zero eigenvalue of the matrix $D - \mathcal{A}$, where now D and $\mathcal{A}$ depend on π as well. Indeed, let $B(x) = \{y \in V : \{x,y\} \in E, x \neq y\}$, let D be the diagonal matrix with

$$D(x,x) = degree(x) + \frac{\pi(B(x))}{\pi(x)},$$

and let $\mathcal{A}$ be the square matrix, with zeros along the diagonal, and for $x \neq y$ with

$$\mathcal{A}(x,y) = \left(1 + \frac{\pi(y)}{\pi(x)}\right) 1_{\{x,y\}\in E}.$$

Then

$$\sum_x \sum_{y:y\sim x} |f(x) - f(y)|^2 \pi(x) =< f, (D-\mathcal{A})f >_{L^2(\pi)} .$$

Note that D–$\mathcal{A}$ is not necessarily symmetric, but is similar to a symmetric matrix, and hence has real eigenvalues. Indeed, the matrix $\Pi^{1/2}\mathcal{A}\Pi^{-1/2}$, where Π is the diagonal matrix with $\Pi(x,x) = \pi(x)$, is symmetric.

For every set $A \subset V$, let $\partial_{\text{in}} A = \{x \in A : \exists\, y \in A^C, x \sim y\}$ be the *vertex inner boundary* and let $\partial_{\text{out}} A = \{x \in A^C : \exists\, y \in A, x \sim y\}$ be the *vertex outer boundary.* Correspondingly, let

$$h_{\text{in}} = \inf\left\{\frac{\pi(\partial_{\text{in}} A)}{\pi(A)} : 0 < \pi(A) \leq \frac{1}{2}\right\},$$

$$h_{\text{out}} = \inf\left\{\frac{\pi(\partial_{\text{out}} A)}{\pi(A)} : 0 < \pi(A) \leq \frac{1}{2}\right\}.$$

In [11], the above isoperimetric constants are respectively denoted h_∞^+ and h_∞^-.

Since the work of Cheeger, it is known that it is natural to try to understand the Poincaré constants in terms of the isoperimetric constants. Towards this goal, we present:

Theorem 1. *$\lambda_\infty \geq \frac{h_{\text{in}}^2}{4}$ and $\lambda_\infty \geq \frac{\left(\sqrt{1+h_{\text{out}}}-1\right)^2}{2}$.*

First, let

$$Mf(x) = \sup_{y:y\sim x} [f(x) - f(y)] = f(x) - \inf_{y:y\sim x} f(y),$$

and

$$Lf(x) = \sup_{y:y\sim x}[f(y) - f(x)] = \sup_{y:y\sim x} f(y) - f(x).$$

Note that $Mf(x)\geq 0$, and $Lf(x)\geq 0$, for all $x\in V$, since $x\in\{y:y\sim x\}$. (M and L are respectively ∇^+_∞ and ∇^-_∞ in [12]). These two functionals lead to:

Lemma 1 (co-area formulas). *For all $f:V\longrightarrow\mathbb{R}$,*

$$\int_V Mf d\pi = \int_{-\infty}^{+\infty} \pi(\partial_{\text{in}}(f>t))dt,$$

and

$$\int_V Lf d\pi = \int_{-\infty}^{+\infty} \pi(\partial_{\text{out}}(f>t))dt.$$

Proof. Indeed,

$$\int_V f d\pi = \int_0^{+\infty} \pi(f>t)dt - \int_{-\infty}^0 \pi(f<t)dt,$$

and $\inf_{y:y\sim x} f(y)>t$ if and only if for all $y\sim x, f(y)>t$, and so

$$\int_V \inf_{y:y\sim x} f(y)d\pi =$$
$$\int_0^{+\infty} \pi(\{x:\forall\, y\sim x, f(y)>t\})dt - \int_{-\infty}^0 \pi(\{x:\exists\, y\sim x, f(y)<t\})dt.$$

Therefore,

$$\begin{aligned}\int_V Mf d\pi &= \int_V f d\pi - \int_V \inf_{y\sim x} f(y)d\pi(x)\\ &= \int_0^\infty \pi\{x:f(x)>t\}dt - \int_0^\infty \pi\{x:\forall\, y\sim x, f(y)>t\}dt\\ &\quad - \int_{-\infty}^0 \pi\{x:f(x)<t\}dt + \int_{-\infty}^0 \pi\{x:\exists\, y\sim x, f(y)<t\}dt\\ &= \int_0^\infty \pi\{x:f(x)>t, \exists\, y\sim x, f(y)\leq t\}dt\end{aligned}$$

$$+\int_{-\infty}^{0} \pi\{x : f(x) > t, \exists\, y \sim x, f(y) \le t\} dt$$
$$= \int_{-\infty}^{\infty} \pi(\partial_{\text{in}}\{x \in V : f(x) > t\}) dt.$$

This proves the first statement of the lemma, the second follows in a similar fashion. ∎

Corollary 1. *For all* $f : V \longrightarrow \mathbb{R}$,

$$\int_V Mf\, d\pi \ge h_{\text{in}} \int_V (f - m(f))^+ d\pi,$$

and

$$\int_V Lf\, d\pi \ge h_{\text{out}} \int_V (f - m(f))^+ d\pi,$$

where m *is a median of* f *for* π. *In particular, if* $f \ge 0$, $\pi(f > 0) \le \frac{1}{2}$, *then*

$$\int_V Mf\, d\pi \ge h_{\text{in}} \int_V f\, d\pi,$$

and

$$\int_V Lf\, d\pi \ge h_{\text{out}} \int_V f\, d\pi.$$

Proof. Indeed, by Lemma 1, and since $\pi(f > m(f)) \le \frac{1}{2}$,

$$\int_V Mf\, d\pi = \int_{-\infty}^{+\infty} \pi(\partial_{\text{in}}(f > t)) dt \ge h_{\text{in}} \int_{m(f)}^{\infty} \pi(f > t) dt$$
$$= h_{\text{in}} \int_V (f - m(f))^+ d\pi.$$

This proves the statements with h_{in}, the ones with h_{out} are proved similarly. ∎

Note that the inequalities of Corollary 1 are functional descriptions of h_{in}, since on indicator functions, these are just the definitions of the isoperimetric constants.

Proof of Theorem 1 . We start with the first statement. Note that if $f \geq 0$,

$$\begin{aligned} Mf^2(x) &= \sup_{y\sim x}[f(x)^2 - f(y)^2] \\ &= \sup[f(x)^2 - f(y)^2]1_{\{f(x)\geq f(y)\}} \qquad \text{(since } f \geq 0) \\ &= \sup(f(x) - f(y))(f(x) + f(y))1_{\{f(x)\geq f(y)\}} \\ &\leq \sup(f(x) - f(y))2f(x) \\ &= 2(Mf(x))f(x). \end{aligned}$$

If additionally $\pi(f > 0) \leq 1/2$, by Corollary 1 applied to f^2,

$$\begin{aligned} h_{\text{in}} \int_V f^2 d\pi &\leq \int_V Mf^2 d\pi \\ &\leq 2\int_V fMf d\pi \qquad \text{(from above)} \\ &\leq 2\sqrt{\int_V (Mf)^2 d\pi}\sqrt{\int_V f^2 d\pi} \qquad \text{(using Cauchy– Schwarz).} \end{aligned}$$

Squaring we get,

$$\frac{h_{\text{in}}^2}{4}\int_V f^2 d\pi \leq \int_V (Mf)^2 d\pi.$$

Now consider the general case. In the definition of λ_∞ one may assume that

$$\pi(f > 0) \leq \frac{1}{2}, \qquad \text{and} \qquad \pi(f < 0) \leq \frac{1}{2},$$

that is a median of f is 0. Set $f^+ = \max(f, 0)$, and $f^- = \max(-f, 0)$, so that $\pi(f^+ > 0) \leq 1/2$, $\pi(f^- > 0) \leq 1/2$, and $f^+, f^- \geq 0$. Therefore,

$$\frac{h_{\text{in}}^2}{4}\int_V {f^+}^2 d\pi \leq \int_V (Mf^+)^2 d\pi$$

$$\frac{h_{\text{in}}^2}{4}\int_V {f^-}^2 d\pi \leq \int_V (Mf^-)^2 d\pi.$$

Note that

$$\begin{aligned} Mf^+(x) &= \sup_{y\sim x}[f^+(x) - f^+(y)] \\ &\leq \sup_{y\sim x}[f(x) - f(y)]1_{\{f>0\}} \\ &\leq \sup_{y\sim x}|f(x) - f(y)|1_{\{f(x)>0\}}. \end{aligned}$$

Also,

$$\begin{aligned} Mf^-(x) &= \sup_{y\sim x}[f^-(x) - f^-(y)] \\ &= \sup_{y\sim x}[f^-(x) - f^-(y)]1_{\{f^-(x)>0\}} \\ &\le \sup_{y\sim x}[-f(x) + f(y)]1_{\{f(x)<0\}} \\ &\le \sup_{y\sim x}|f(x) - f(y)|1_{\{f<0\}}. \end{aligned}$$

As a result,

$$\frac{h_{\rm in}^2}{4}\int\limits_{\{f>0\}} f^2 d\pi \le \int\limits_{\{f>0\}} \sup_{y\sim x}|f(x) - f(y)|^2 d\pi,$$

$$\frac{h_{\rm in}^2}{4}\int\limits_{\{f<0\}} f^2 d\pi \le \int\limits_{\{f<0\}} \sup_{y\sim x}|f(x) - f(y)|^2 d\pi.$$

Summing these inequalities, we obtain the result, since $\mathrm{var}_\pi f \le \int\limits_V f^2 d\pi$. This proves the first statement in the theorem.

For the second, note that for $f \ge 0$,

$$\begin{aligned} Lf^2(x) &= \sup_{y\sim x}(f(y)^2 - f(x)^2) \\ &= \sup_{y\sim x}(f(y) - f(x))(f(y) + f(x))1_{\{f(y)\ge f(x)\}} \\ &= \sup_{y\sim x}((f(y) - f(x))^2 + (f(y) - f(x))2f(x))1_{(f(y)\ge f(x))} \\ &\le \sup_{y\sim x}(f(y) - f(x))^2 1_{(f(y)\ge f(x))} + 2f(x)\sup_{y\sim x}(f(y) - f(x))1_{\{f(y)\ge f(x)\}}. \end{aligned}$$

Then, by Corollary 1,

$$\begin{aligned} h_{\rm out}\int\limits_V f^2 d\pi \le &\int\limits_V \sup_{y\sim x}[(f(y) - f(x))^2 1_{(f(y)>f(x))}]d\pi(x) \\ &+2\int\limits_V f(x)\sup_{y\sim x}[(f(y) - f(x))1_{\{(f(y)\ge f(x))\}}]d\pi(x) \end{aligned}$$

If we set

$$A^2 = \int\limits_V f^2 d\pi, \qquad B^2 = \int\limits_V \sup[f(y) - f(x)]^2 1_{\{f(y)\ge f(x)\}}d\pi(x),$$

we get,

$$h_{\text{out}} A^2 \le B^2 + 2AB,$$

which is equivalent to

$$\left(\sqrt{1+h_{\text{out}}} - 1\right)^2 A^2 \le B^2.$$

That is

$$\left(\sqrt{1+h_{\text{out}}} - 1\right)^2 \int_V f^2 d\pi \le \int_V \sup_{y\sim x}[(f(y)-f(x))^2 1_{(f(y)\ge f(x))}] d\pi(x). \tag{1}$$

In general, assuming again that the median of f is 0, let f^+ and f^- be as before. Using (1) with f^+ and f^-, we get

$$\left(\sqrt{1+h_{\text{out}}} - 1\right)^2 \int_V (f^+)^2 d\pi \le \int_V \sup_{y\sim x}[(f^+(y)-f^+(x))^2 1_{(f^+(y)\ge f^+(x))}] d\pi(x),$$

and

$$\left(\sqrt{1+h_{\text{out}}} - 1\right)^2 \int_V (f^-)^2 d\pi \le \int_V \sup_{y\sim x}[(f^-(y)-f^-(x))^2 1_{(f^-(y)\ge f^-(x))}] d\pi(x).$$

But, $\int_V (f^+)^2 d\pi + \int_V (f^-)^2 d\pi = \int_V f^2 d\pi$. Moreover,

$$\sup_{y\sim x}[(f^+(y)-f^+(x))^2 1_{(f^+(y)\ge f^+(x))}] + \sup_{y\sim x}[(f^-(y)-f^-(x))^2 1_{(f^-(y)\ge f^-(x))}] \le 2\sup_{y\sim x}|f(y)-f(x)|^2,$$

yielding, for all f with $m(f)=0$,

$$\frac{\left(\sqrt{1+h_{\text{out}}} - 1\right)^2}{2} \int_V f^2 d\pi \le \int_V \sup_{y\sim x}|f(y)-f(x)|^2 d\pi(x).$$

∎

3. λ_∞ and symmetric boundary

For every set $A \subset V$, let the *symmetric vertex boundary* be defined via

$$\begin{aligned}\partial A &= \partial_{\text{in}} A \cup \partial_{\text{out}} A \\ &= \{x \in A : \exists\, y \notin A, x \sim y\} \cup \{x \notin A : \exists\, y \in A, x \sim y\}.\end{aligned}$$

Clearly, $\partial A = \partial A^c$, for all $A \subset V$. In addition let

$$\begin{aligned}h\ (= h_{\text{vertex}}) &= \inf\left\{\frac{\pi(\partial A)}{\pi(A)} : 0 < \pi(A) \le \frac{1}{2}\right\} \\ &= \inf\left\{\frac{\pi(\partial A)}{\min(\pi(A), \pi(A^c))} : 0 < \pi(A) < 1\right\}.\end{aligned}$$

Theorem 2. $2h \ge \lambda_\infty \ge \dfrac{\left(\sqrt{h+1}-1\right)^2}{4}$.

Proof. First the easy inequality: $2h \ge \lambda_\infty$. For $A \subset V$, let $f = 1_A$. Then $\mathrm{var}_\pi 1_A = \pi(A)\pi(A^c)$ and

$$\int_V \sup_{y:y\sim x} (1_A(x) - 1_A(y))^2 d\pi(x) = \pi(\partial_{\text{in}} A) + \pi(\partial_{\text{out}} A) = \pi(\partial A),$$

noting that $\partial_{\text{in}} A \cap \partial_{\text{out}} A = \emptyset$, and $\partial_{\text{in}} A \cup \partial_{\text{out}} A = \partial A$. Thus

$$\lambda_\infty = \inf_f \frac{\int_V \sup_{y:y\sim x} (f(x) - f(y))^2 d\pi}{\mathrm{var}_\pi f} \le \inf_{A:\pi(A)\le\frac{1}{2}} \frac{\pi(\partial A)}{\pi(A)\pi(A^c)} \le 2h.$$

Now the nontrivial inequality: $\lambda_\infty \ge \dfrac{\left(\sqrt{h+1}-1\right)^2}{4}$. Recall that by Lemma 1, we have

$$\int_V \sup_{y:y\sim x} (f(x) - f(y)) d\pi = \int_{-\infty}^{\infty} \pi(\partial_{\text{in}}(f > t)) dt,$$

$$\int_V \sup_{y:y\sim x} (f(y) - f(x)) d\pi = \int_{-\infty}^{\infty} \pi(\partial_{\text{out}}(f > t)) dt.$$

Using the above two equations, we get that

$$2\int_V \sup_{y:y\sim x} |f(x) - f(x)| d\pi \geq \int_{-\infty}^{\infty} \pi(\partial(f > t))dt$$

$$\geq h \int_{-\infty}^{\infty} \min(\pi(f > t), \pi(f \leq t))dt. \tag{2}$$

But, recall that for f with $m(f)=0$, $\int_{-\infty}^{\infty} \min(\pi(f>t),\pi(f\leq t))dt = E_\pi|f|$.

Thus, for $f\geq 0$ with $m(f)=0$, (2) implies that

$$2E_\pi \sup |f(x) - f(y)| \geq hE_\pi f. \tag{3}$$

(Here and for the rest of the proof, for convenience, we write sup to mean $\sup_{y:y\sim x}$.) Moreover, for all f with $m(f) = 0$, let us write as before $f = f^+ - f^-$. Now applying (3) to $(f^+)^2$,

$$\begin{aligned} hE_\pi {f^+}^2 &\leq 2E_\pi \sup |f^+(x) - f^+(y)|(f^+(x) + f^+(y)) \\ &= 2E_\pi \sup[|f^+(x)-f^+(y)|(f^+(y)-f^+(x))+2|f^+(x)-f^+(y)|f^+(x)] \\ &\leq 2E_\pi \sup |f^+(x) - f^+(y)|^2 + 4E_\pi \sup |f^+(x) - f^+(y)|f^+(x) \\ &\leq 2E_\pi \sup |f(x) - f(y)|^2 + 4E_\pi \sup |f(x) - f(y)|f^+(x), \end{aligned} \tag{4}$$

since $|f^+(x)-f^+(y)|\leq|f(x)-f(y)|$, for all x,y. Similarly, applying (3) to $(f^-)^2$ we get

$$hE_\pi {f^-}^2 \leq 2E_\pi \sup |f(x) - f(y)|^2 + 4E_\pi \sup |f(x) - f(y)|f^-(x). \tag{5}$$

Summing (4) and (5),

$$hE_\pi f^2 \leq hE_\pi {f^+}^2 + hE_\pi {f^-}^2 \leq 4E_\pi|\nabla_\infty f|^2 + 4E_\pi|\nabla_\infty f||f|,$$

where $|\nabla_\infty f(x)|=\sup_{y:y\sim x}|f(x)-f(y)|$. Setting $E_\pi f^2=A^2$, $E_\pi|\nabla_\infty f|^2=B^2$, the above yields $hA^2\leq 4B^2+4AB$. Rewriting,

$$(h+1)A^2 \leq (A+2B)^2,$$

which is equivalent to

$$\frac{B^2}{A^2} \geq \frac{(\sqrt{h+1}-1)^2}{4}.$$

We conclude with

$$\lambda_\infty = \inf_f \left\{ \frac{E_\pi |\nabla_\infty f|^2}{\mathrm{var}_\pi\, f} \right\} \geq \frac{(\sqrt{h+1}-1)^2}{4}.$$

Here we used the fact that $\mathrm{var}_\pi\, f \leq E_\pi(f - m(f))^2$, for all f. ∎

Remark 1. The definitions of $h_{\mathrm{in}}, h_{\mathrm{out}}$ and h imply that

$$h_{\mathrm{in}} \leq 1, \;\; h_{\mathrm{out}} \leq \inf_{0<\pi(A)\leq 1/2} \frac{1-\pi(A)}{\pi(A)}, \quad \text{and } h \leq \inf_{\pi(A)\leq 1/2} \frac{1}{\pi(A)}.$$

In particular, if π is the normalized counting measure, h_{out} is uniformly bounded by 3/2; indeed, in this case it is easy to see that the complete graph is an extremal example, and for the complete graph on $2n+1$ vertices, $h_{\mathrm{out}} \leq (n+1)/n$, which is at most 3/2 (similarly, $h \leq 3$). This uniform bound on h_{out} also shows that Alon's bound implies $\lambda_2 \geq 4h_{\mathrm{out}}^2/17$, slightly better than our bound. However, as the next remark shows, for non uniform measures, our results provide better information.

Remark 2. Consider the two-point space $\{0,1\}$ with $\pi(1) = p < 1/2$. Let $q = 1-p$. It is easy to see that $\lambda_\infty = 1/(pq)$, $h_{\mathrm{out}} = q/p$, $h_{\mathrm{in}} = 1$, and that $h = 1/p$. Since $1/2 \leq q < 1$, this shows that the behavior of λ_∞ is captured accurately by h_{out} or h, up to a constant. In general, the lower bound in Theorem 1 (respectively in Theorem 2) behaves like $h_{\mathrm{out}}^2/8$, for h_{out} small, and like $h_{\mathrm{out}}/2$ for h_{out} large (respectively like $h^2/16$ and $h/4$).

3.1. Some examples

In all of the following examples, π is the normalized counting measure.

Example 1. Let $G = Q_n$ be the n-dimensional (discrete) cube. Note that $\lambda_2(Q_n) = 4$, since (as is well known) the second smallest eigenvalue of the Laplacian of Q_n is 2. It is easy to check that $\lambda_\infty = 4/n$ (for example, using the tensorization of the variance). This shows, in fact, that the bound $\lambda_\infty \geq \lambda_2/\Delta(G)$ is tight. This example also shows that the theorems in the previous sections are all in general tight (up to absolute constants) since h_{in}, h_{out}, and h are all $\Theta(1/\sqrt{n})$.

Example 2. Let G be the so-called *bar-bell* graph on $|V| = n := 6k-1$ vertices, for $k \geq 1$: start with a path on $2k+1$ vertices, labeled as $v_{-k}, v_{-k+1}, \ldots, v_{-1}, v_0, v_1, \ldots, v_k$, from left to right (say). Attach a clique of size $2k$ at either end of this path, using $2k-1$ new vertices (for each clique) and the end vertex of the path. Denote by L (and similarly by R), the set of vertices of the clique attached to v_{-k} (and similarly to v_k). Then $h_{\text{in}} = h_{\text{out}} = 1/(3k-1) = h/2$, an extremal set being L together with the left half of the path. So $\lambda_\infty \geq 1/(3k-1)^2$. By a suitable choice of f, it can be shown that $\lambda_2 \leq c/n^2$, for $c > 0$. (For example, let $f(v_i) = i$ for the vertices on the path, let $f(x) = -k$ for $x \in L$ and let $f(y) = k$ for $y \in R$.) This yields $\lambda_2 = \lambda_\infty = \Theta(1/n^2)$, and shows that the bound $\lambda_2 \geq \lambda_\infty$ is tight up to an absolute constant.

Example 3. Let G be the so-called *dumbbell* graph on $n = 2k$ vertices – two cliques of size k joined by an edge (between two arbitrarily chosen vertices of the cliques). This example shows that λ_∞ and λ_2 can be the same (up to a constant) as h_{in}, h_{out}, and h. All the quantities are $\Theta(1/n)$. (This is an example where the degrees can become unbounded.)

Example 4. Let $G = K_n$ be the complete graph on n vertices; and let n be even for convenience. Then $h_{\text{in}} = h_{\text{out}} = 1$, and $h = 2$. This shows that the inequality $h \geq h_{\text{in}} + h_{\text{out}}$ can be tight. Also, $\lambda_2 = 2n$, and $\lambda_\infty = 4$, showing that λ_2 can become unbounded, while (as remarked in the next section) λ_∞ cannot be.

4. λ_∞ and concentration for graphs

Let $G = (V, E)$ be a finite connected graph with $|V| \geq 2$, and for simplicity, let E be symmetric. Let π be a probability measure on V. As before, for $x, y \in V$, $x \sim y$ means that either $\{x, y\} \in E$ or $x = y$.

When π is the counting measure, Alon and Milman proved the following (one dimensional) concentration result (Theorem 2.6 of [4]), which also gives an (n-dimensional) concentration result for the Cartesian product of n copies of G. Let (G, π) satisfy a Poincaré inequality with constant λ_2, and let $\Delta = \Delta(G)$ be the maximum degree of G. For A and B disjoint subsets of V, let $\rho(A, B)$ be the graph distance between A and B, and further assume that $\rho(A, B) > \ell \geq 1$. Then

$$\pi(B) \leq (1 - \pi(A)) \exp\left(-\sqrt{(\lambda_2/4\Delta)}\,\ell \log(1 + 2\pi(A))\right). \tag{6}$$

We report here a qualitative improvement, by being able to replace λ_2/Δ with λ_∞ (see Corollary 3 for the precise formulation and also [12]). Once again, in our case π is an arbitrary probability measure. As before, let λ_∞ be the optimal constant in,

$$\lambda_\infty \operatorname{var}_\pi(f) \leq E_\pi |\nabla_\infty f|^2, \tag{7}$$

where $f : V \longrightarrow \mathbb{R}$ is arbitrary, and

$$|\nabla_\infty f(x)| = \sup_{y:y\sim x} |f(x) - f(y)|.$$

Remark 3. Applying (7) to $f = 1_A$, gives $\lambda_\infty \le \frac{1}{\pi(A)(1-\pi(A))}$. In particular, if for some $A \subset V$, $\pi(A) = 1/2$, then $\lambda_\infty \le 4$. This is the case when π is the normalized counting measure and $|V| = 2n$ is even. If $|V| = 2n+1$, $n = 1, 2, 3\ldots$, then the maximum of $\pi(A)(1-\pi(A))$ is equal to $\frac{n(n+1)}{(2n+1)^2}$, so

$$\lambda_\infty \le \frac{(2n+1)^2}{n(n+1)}. \tag{8}$$

The sequence on the right in (8) tends to 4 and is maximal for $n = 1$. Thus, for all n,

$$\lambda_\infty \le \frac{9}{2}. \tag{9}$$

Therefore, this estimate holds for all finite graphs with the normalized counting measure.

We will now deduce from the Poincaré inequality (7), a concentration inequality. By definition, a function $f : V \longrightarrow \mathbb{R}$ is Lipschitz, if

$$x \sim y \Rightarrow |f(x) - f(y)| \le 1.$$

Theorem 3. *Let $f : V \longrightarrow \mathbb{R}$ be Lipschitz with $E_\pi f = 0$, then for all $t \in \mathbb{R}$ such that*

$$|t| \le \frac{1}{4\sqrt{1+\lambda_\infty}}, \tag{10}$$

we have

$$E_\pi e^{2\sqrt{\lambda_\infty} tf} \le 4. \tag{11}$$

First a preliminary inequality:

Lemma 2. *Let $a, b \in \mathbb{R}$ be such that $|a-b| \le c$, then*

$$|sh(a) - sh(b)|^2 \le e^{2c} ch^2(a)(a-b)^2 \le c^2 e^{2c} ch^2(a), \tag{12}$$

where $sh(a) = \frac{e^a - e^{-a}}{2}$, $ch(a) = \frac{e^a + e^{-a}}{2}$.

Proof. There exists a middle point a' between a and b such that

$$\frac{sh(a)-sh(b)}{a-b}=ch(a').$$

Moreover, since $a-c\leq a'\leq a+c$, then $ch(a')\leq \max(ch(a+c),ch(a-c))$ and thus

$$ch(a+c)=\frac{e^{a+c}+e^{-a-c}}{2}\leq e^c\frac{e^a+e^{-a}}{2}=e^c ch(a),$$

$$ch(a-c)=\frac{e^{a-c}+e^{-a+c}}{2}\leq e^c\frac{e^a+e^{-a}}{2}=e^c ch(a). \qquad \blacksquare$$

Proof of Theorem 3. Let us tensorize (7): for every $g:V\times V\longrightarrow\mathbb{R}$,

$$\mathrm{var}_{x,y}(g)\leq E_{x,y}[\mathrm{var}_x(g)+\mathrm{var}_y(g)],$$

where $E_{x,y}$ and $\mathrm{var}_{x,y}$ denote the expectation and variance with respect to the measure $\pi\times\pi$, and var_x (resp. var_y) denotes the variance with respect to the x (resp. y) coordinate when the other is fixed. Thus, from (7),

$$\lambda_\infty\,\mathrm{var}_{x,y}(g)\leq E_{x,y}\left[\sup_{x'\sim x}|g(x,y)-g(x',y)|^2+\sup_{y'\sim y}|g(x,y)-g(x,y')|^2\right]. \tag{13}$$

Then we will apply (13) to the function

$$g(x,y)=sh\left(\sqrt{\lambda_\infty}t(f(x)-f(y))\right),\qquad x,y\in V;\quad t\geq 0.$$

This function is symmetrically distributed around 0, thus $E_{x,y}g=0$, and so

$$\mathrm{var}_{x,y}(g)=E_{x,y}g^2=E_{x,y}sh^2\left(\sqrt{\lambda_\infty}t(f(x)-f(y)).\right) \tag{14}$$

Let f be Lipschitz on V, with $E_\pi f=0$. Using Lemma 2 with

$$\begin{aligned} a&=\sqrt{\lambda_\infty}t(f(x)-f(y)),\\ b&=\sqrt{\lambda_\infty}t(f(x')-f(y)),\qquad \text{where } x'\sim x\\ c&=\sqrt{\lambda_\infty}t,\end{aligned}$$

we can conclude that

$$\begin{aligned}|g(x,y)-g(x',y)|^2&=|sh(a)-sh(b)|^2\leq c^2e^{2c}ch^2(a)\\ &=\lambda_\infty t^2e^{2\sqrt{\lambda_\infty}t}ch^2\left(\sqrt{\lambda_\infty}t(f(x)-f(y))\right).\end{aligned} \tag{15}$$

The same estimate holds true for $|g(x,y)-g(x,y')|^2$. Combining (13)-(14)-(15) and noting that $sh^2 = ch^2 - 1$, we get

$$\begin{aligned}E_{x,y} ch^2 \left(\sqrt{\lambda_\infty} t(f(x) - f(y))\right) &- 1 \\ &\le 2t^2 e^{2\sqrt{\lambda_\infty} t} E_{x,y} ch^2 \left(\sqrt{\lambda_\infty} t(f(x) - f(y))\right),\end{aligned}$$

that is,

$$E_{x,y} ch^2 \left(\sqrt{\lambda_\infty} t(f(x) - f(y))\right) \le \frac{1}{1 - 2t^2 e^{2\sqrt{\lambda_\infty} t}}, \tag{16}$$

provided

$$1 - 2t^2 e^{2\sqrt{\lambda_\infty} t} > 0. \tag{17}$$

The function $u = \sqrt{\lambda_\infty} t(f(x) - f(y))$ is also symmetrically distributed, hence $E_{x,y} e^{-2u} = E_{x,y} e^{2u}$, and therefore,

$$E_{x,y} ch^2 u = E_{x,y} \left(\frac{e^u + e^{-u}}{2}\right)^2 = E_{x,y} \frac{e^{2u} + e^{-2u} + 2}{4} = \frac{1}{2}\left(E_{x,y} e^{2u} + 1\right).$$

Thus, from (16) under (17),

$$E_{x,y} e^{2\sqrt{\lambda_\infty} t(f(x)-f(y))} + 1 \le \frac{2}{1 - 2t^2 e^{2\sqrt{\lambda_\infty} t}}. \tag{18}$$

But

$$\begin{aligned}E_{x,y} e^{2\sqrt{\lambda_\infty} t(f(x)-f(y))} &= E_x e^{2\sqrt{\lambda_\infty} tf(x)} E_y e^{-2\sqrt{\lambda_\infty} tf(y)} \\ &\ge \left(E_x e^{2\sqrt{\lambda_\infty} tf(x)}\right) e^{-2\sqrt{\lambda_\infty} t E_y f(y)} \\ &= E_\pi e^{2\sqrt{\lambda_\infty} tf},\end{aligned}$$

where we used Jensen's inequality, and the fact that $E_\pi f = 0$. Thus, from (18),

$$E_\pi e^{2\sqrt{\lambda_\infty} tf} \le \frac{1 + 2t^2 e^{2\sqrt{\lambda_\infty} t}}{1 - 2t^2 e^{2\sqrt{\lambda_\infty} t}} \le \frac{2}{1 - 2t^2 e^{2\sqrt{\lambda_\infty} t}}, \tag{19}$$

for all $t \ge 0$ satisfying (17). For small $t \ge 0$, of course,

$$1 - 2t^2 e^{2\sqrt{\lambda_\infty} t} \ge \frac{1}{2}. \tag{20}$$

This is equivalent to

$$e^{-2\sqrt{\lambda_\infty} t} \ge 4t^2. \tag{21}$$

Using $e^{-x} \geq 1-x$, for all $x \geq 0$, (21) will follow from

$$1 - 2\sqrt{\lambda_\infty}t \geq 4t^2,$$

which is solved as

$$t \leq \frac{1}{2\left(\sqrt{\lambda_\infty} + \sqrt{1+\lambda_\infty}\right)},$$

which, in turn, will follow from

$$t \leq \frac{1}{4\sqrt{1+\lambda_\infty}}.$$

Theorem 3 is proved. ∎

In the case of the normalized counting measure, we have by (9) that $\lambda_\infty \leq 9/2$, which we can use to get slightly better constants. Proceeding as before, we arrive at the first inequality in (19),

$$E_\pi e^{2\sqrt{\lambda_\infty}tf} \leq \frac{1 + 2t^2 e^{2\sqrt{\lambda_\infty}t}}{1 - 2t^2 e^{2\sqrt{\lambda_\infty}t}}, \tag{22}$$

for all $t \geq 0$ satisfying (17). Using $\lambda_\infty \leq 9/2$ and choosing $t = 1/4$, and after some computation, leads to:

Corollary 2. *With respect to the normalized counting measure, for all $f : V \longrightarrow \mathbb{R}$ Lipschitz with $E_\pi f = 0$,*

$$E_\pi e^{\frac{1}{2}\sqrt{\lambda_\infty}f} \leq \frac{9}{4}. \tag{23}$$

Let us now describe how Corollary 2 leads to concentration with λ_∞. By Chebyshev's inequality, for all $\ell > 0$,

$$\pi(f \geq \ell) \leq \frac{9}{4} e^{-\frac{1}{2}\sqrt{\lambda_\infty}\ell}. \tag{24}$$

For any set $A \subset V$, let $f(x) = \rho(A,x) - E_\pi \rho(A,x), x \in V$, then (24) gives

$$\pi(f \geq \ell) = \pi(\rho(A,x) \geq \ell + E_\pi\rho(A,x)) \leq \frac{9}{4} e^{-\frac{1}{2}\sqrt{\lambda_\infty}\ell},$$

or, for all $\ell \geq E_\pi \rho(A,x)$,

$$\pi(\rho(A,x) \geq \ell) \leq \frac{9}{4} e^{-\frac{1}{2}\sqrt{\lambda_\infty}\ell} e^{\frac{1}{2}\sqrt{\lambda_\infty}E_\pi\rho(A,x)}. \tag{25}$$

Before concluding, we state:

Remark 4. It follows from (24) that

$$\pi\left(f \geq \frac{2\log(9/2)}{\sqrt{\lambda_\infty}}\right) \leq \frac{1}{2};$$

therefore, for a median, $m(f)$, of f we have

$$m(f) \leq \frac{2\log(9/2)}{\sqrt{\lambda_\infty}} \leq \frac{4}{\sqrt{\lambda_\infty}}.$$

Lemma 3. *Let the random variable $\rho \geq 0$ be such that $\mathrm{var}_\pi(\rho) \leq \sigma^2$, $\pi(\rho=0) \geq p$, then $\sqrt{p}E_\pi\rho \leq \sqrt{q}\sigma$, where $q=1-p$.*

Proof. For completeness we include the proof, which is trivial.

$$\begin{aligned}(E_\pi\rho)^2 &\leq \left(\int_{\rho>0} \rho^2(x)d\pi(x)\right)(1-p)\\ &= \left(\mathrm{var}_\pi(\rho) + (E_\pi\rho)^2\right)(1-p). \blacksquare\end{aligned}$$

To finish, we apply Lemma 3 to $\rho=\rho(A,x)$, where $A \subset V$ is any subset of measure $\pi(A) \geq \frac{1}{2}$. Since ρ is a Lipschitz function, it follows from the Poincaré inequality (7) that $\mathrm{var}_\pi(\rho) \leq \sigma^2 = \frac{1}{\lambda_\infty}$. Thus

$$E_\pi\rho(A,x) \leq \frac{1}{\sqrt{\lambda_\infty}},$$

and according to (25)

$$\pi(\rho(A,x) \geq \ell) \leq \frac{9}{4}e^{\frac{1}{2}}e^{-\frac{1}{2}\sqrt{\lambda_\infty}\ell}, \tag{26}$$

which holds for all $\ell \geq E_\pi\rho(A,x)$, and in particular, for $\ell \geq \frac{1}{\sqrt{\lambda_\infty}}$. But for $\ell \in \left(0, \frac{1}{\sqrt{\lambda_\infty}}\right]$, the inequality (26) also holds, since $\frac{9}{4} > \frac{1}{2}$. We have derived:

Corollary 3. *For every set $A \subset V$ of normalized counting measure $\pi(A) \geq \frac{1}{2}$, for all integer $\ell > 0$,*

$$1 - \pi(A^{\ell-1}) \leq \frac{9}{4}e^{\frac{1}{2}}e^{-\frac{1}{2}\sqrt{\lambda_\infty}\ell}.$$

Concluding Remarks

• Let G^n, be the Cartesian product of n copies of G. Then, as observed in [12], it is easy to show that $\lambda_\infty(G^n) = \lambda_\infty(G)/n$. Hence the above concentration results can automatically be translated into concentration results on G^n, wherein the graph distance satisfies, $\rho_{G^n}((x_1,\dots,x_n),(y_1,\dots,y_n)) = \sum_{i=1}^n \rho_G(x_i,y_i)$. In this context, for large distances, $\rho \gg n^{1/2}$, a recent result of [3] provides an asymptotically tight estimate using the so-called *spread constant* of G. Recall (see [6], [3]) that the spread constant, $c(G)$, is the maximal variance of f, over all Lipschitz functions (with respect to the graph distance) f defined on V. In the definition of λ_∞, restricting ourselves to Lipschitz f – namely, that $|f(x)-f(y)| \le 1$, whenever $\{x,y\} \in E$ – we see that $\lambda_\infty(G) \le 1/c(G)$. However, an example, such as the dumbbell graph, shows that λ_∞ can be much smaller than $1/c(G)$. Indeed, consider a dumbbell graph on an even number, n, of vertices and with the uniform measure on the vertices. It can be described as two cliques of size $n/2$ joined by an edge. Let the end points of the edge be x and y. Then the variance of any Lipschitz function on this graph is bounded from above by an absolute constant. For, the diameter of this graph is 3, and so any Lipschitz function can be restricted to an interval of width at most 3. However, the choice of $f=1$ on the clique containing x and $f=0$ on the other clique shows that $\lambda_\infty \le 4/n$.

• The Poincaré constant λ_2, having an alternative characterization as an eigenvalue of a matrix, is computable in polynomial time in the size of the graph. On the other hand, the complexity of computing λ_∞ is an interesting open problem. Efficient computation of λ_∞ would have interesting applications, particularly in the spirit of Alon's work, by way of providing an efficient algorithm to check a randomly generated graph for *magnification* and *expansion* properties (see [2]).

• Proceeding as in [7] (or as in the discrete analog, [12]), it is easy to derive inequalities relating the diameter (with respect to the graph distance) of G to $\lambda_\infty(G)$, and also to derive concentration for Lipschitz functions and diameter bounds using the corresponding log-Sobolev constant, obtained by replacing the variance by the entropy in the definition of λ_∞.

Note added in proof. It has been pointed out to us by Santosh Vempala that λ_∞ can indeed be computed efficiently, since its definition can be rewritten so that its computation amounts to minimizing a linear function subject to convex constraints.

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