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ON THE EXISTENCE OF DISJOINT CYCLES IN A GRAPH

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A simple proof of the following result is given: Suppose G is a graph of order at least $3k$ with $\sigma_2(G) \geq 4k-1$. Then G contains k vertex-disjoint cycles.

1. Introduction

In this note, we only consider finite undirected graphs without loops and multiple edges. For a vertex x of a graph G, the neighborhood of x in G is denoted by $N_G(x)$, and $d_G(x) := |N_G(x)|$ is the degree of x in G. With a slight abuse of notation, for a subgraph H of G and a vertex $x \in V(G) - V(H)$, we also denote $N_H(x) := N_G(x) \cap V(H)$ and $d_H(x) := |N_H(x)|$. The minimum degree of G is denoted by $\delta(G)$. For a noncomplete graph G, let

$$\sigma_2(G) := \min\{d_G(x) + d_G(y) | x \text{ and } y \text{ are nonadjacent vertices of } G\},$$

and $\sigma_2(G) := \infty$ when G is a complete graph. For a subgraph H of G, $G-H$ denotes the subgraph induced by $V(G)-V(H)$, and $|H| := |V(H)|$ is the order of H. K_n denotes a complete graph of order n. For a graph G, mG denotes the union of m copies of G. For graphs G and H, $G+H$ denotes the join of G and H. For other graph-theoretic terminology and notation, we refer the reader to [2].

In [1], Brandt et al. gave the following sufficient conditions to partition a graph into a specified number of vertex-disjoint cycles:

Theorem 1. *Suppose $|G| = n \geq 4k$ and $\sigma_2(G) \geq n$. Then G can be partitioned into k vertex-disjoint cycles, that is, there exist k vertex-disjoint cycles $H_1, \cdots, H_k$ such that $V(G) = \cup_{i=1}^{k} V(H_i)$.* ∎

To prove this theorem, they used the following result:

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Theorem 2. (Justesen [4]) *Suppose* $|G|=n\geq 3k$ *and* $\sigma_2(G)\geq 4k$. *Then* G *contains* k *vertex-disjoint cycles.* ∎

This is a generalization of the following classical result of Corrádi and Hajnal:

Theorem 3. ([3]) *Suppose* $|G|=n\geq 3k$ *and* $\delta(G)\geq 2k$. *Then* G *contains* k *vertex-disjoint cycles.* ∎

Unfortunately, no proofs of Theorem 2 were given in [4]. The purpose of this paper is to give a simple proof of the following extension of Theorem 2.

Theorem 4. *Suppose* $|G|=n\geq 3k$ *and* $\sigma_2(G)\geq 4k-1$. *Then* G *contains* k *vertex-disjoint cycles.* ∎

Since $K_{2k-1}+mK_1$ does not contain k vertex-disjoint cycles, the assumption $\sigma_2(G)\geq 4k-1$ is weakest possible.

In the proof of Theorem 1, the assumption $n\geq 4k$ is only used to apply Theorem 2. By using Theorem 4 instead of Theorem 2, Theorem 1 can be slightly improved.

Theorem 1'. *Suppose* $|G|=n\geq 3k$ *and* $\sigma_2(G)\geq \max\{4k-1,n\}$. *Then* G *can be partitioned into* k *vertex-disjoint cycles.* ∎

In [3], Corrádi and Hajnal proved a stronger result than Theorem 3: Suppose $|G|=kl+t$ with $l\geq 3$, $0\leq t\leq k-1$, and $\delta(G)\geq 2k$. Then G contains vertex-disjoint cycles $C_1,\cdots,C_k$ such that $|C_i|\leq l$ for $1\leq i\leq k-t$ and $|C_i|\leq l+1$ for $k-t<i\leq k$. It would be interesting to decide whether the same conclusion holds if we replace the assumption $\delta(G)\geq 2k$ with $\sigma_2(G)\geq 4k-1$.

2. Proof of Theorem 4

Let G be an edge-maximal counterexample. Since a complete graph of order $\geq 3k$ contains k disjoint cycles, G is not complete. Let x and y be nonadjacent vertices of G, and define $G'=G+xy$, the graph obtained from G by adding the edge xy. Then G' is not a counterexample by the maximality of G, and so G' contains disjoint cycles $C_1,\cdots,C_k$. Without loss of generality, we may assume that $xy\notin\bigcup_{i=1}^{k-1}E(C_i)$, that is, G contains $k-1$ disjoint cycles $C_1.\cdots,C_{k-1}$ such that $\sum_{i=1}^{k-1}|C_i|\leq n-3$. Let H be the subgraph of G induced by $\bigcup_{i=1}^{k-1}V(C_i)$, $M:=G-H$, and P be a longest path in M. Choose $C_1,\cdots,C_{k-1}$ so that

(1) $$|H|=\sum_{i=1}^{k-1}|C_i| \text{ is as small as possible.}$$

(2) Subject to condition (1), $|P|$ is as large as possible.

Claim 1. *For any $x \in V(M)$ and for any i, $1 \le i \le k-1$, $d_{C_i}(x) \le 3$. Furthermore, $d_{C_i}(x) = 3$ implies $|C_i| = 3$.*

Proof. This is easily seen by the extremality condition (1). ∎

Claim 2. *Suppose $x, y \in V(M)$ and $d_{C_i}(x) + d_{C_i}(y) \ge 5$. Then C_i is a triangle, and there exists $z \in N_{C_i}(x)$ such that $(V(C_i) - \{z\}) \cup \{y\}$ induces a triangle.*

Proof. Since $d_{C_i}(x) \ge 3$ or $d_{C_i}(y) \ge 3$, C_i is a triangle by Claim 1. If $d_{C_i}(y) = 3$, any $z \in N_{C_i}(x)$ satisfies the conclusion. If $d_{C_i}(y) = 2$, the unique vertex $z \in V(C_i) - N_{C_i}(y)$ satisfies the conclusion. ∎

Claim 3. *Suppose x and $y \in V(M)$ are nonadjacent, and $d_M(x) + d_M(y) \le 2$. Then $d_{C_i}(x) + d_{C_i}(y) \ge 5$ for some i, $1 \le i \le k-1$.*

Proof. Since x and y are nonadjacent,

$$d_H(x) + d_H(y) \ge \sigma_2(G) - (d_M(x) + d_M(y)) \ge 4k - 3 > 4(k-1).$$

Hence $d_{C_i}(x) + d_{C_i}(y) \ge 5$ for some i, $1 \le i \le k-1$. ∎

Let $P = (x_1, x_2, \cdots, x_t)$.

Claim 4. *$V(P) = V(M)$, that is, P is a Hamilton path of M.*

Proof. Suppose $V(P) \ne V(M)$. Since M is a forest, there exists a vertex $y \in V(M) - V(P)$ such that $d_M(y) \le 1$. Since P is a longest path in M, x_t and y are nonadjacent. By Claim 3, $d_{C_i}(x_t) + d_{C_i}(y) \ge 5$ for some i, $1 \le i \le k-1$. By Claim 2, C_i is a triangle, and there exists $z \in N_{C_i}(x_t)$ such that $(V(C_i) - \{z\}) \cup \{y\}$ induces a triangle. This contradicts the extremality condition (2). ∎

Since $d_M(x_1) = d_M(x_t) = 1$, $d_{C_i}(x_1) + d_{C_i}(x_t) \ge 5$ for some i, $1 \le i \le k-1$, by Claim 3. Without loss of generality, we may assume that $i = k-1$. By Claim 2, C_{k-1} is a triangle. Let $V(C_{k-1}) = \{x_0, u, u'\}$. We may assume that $d_{C_{k-1}}(x_1) = 3$ and $\{u, u'\} \subseteq N_{C_{k-1}}(x_t)$. Let M_1 be the subgraph induced by $V(M) \cup V(C_{k-1})$, and $H_1 := G - M_1 = H - C_{k-1}$.

Claim 5. *$N_{M_1}(x_0) \subseteq \{u, u', x_1, x_t\}$ and $N_{M_1}(x_2) = \{x_1, x_3\}$.*

Proof. If x_0 is adjacent to some x_i, $2 \le i \le t-1$, then $(x_0, x_1, \cdots, x_i, x_0)$ and (u, u', x_t, u) are disjoint cycles in M_1 (see Figure 1). Hence $N_{M_1}(x_0) \subseteq \{u, u', x_1, x_t\}$. If $N_M(x_2) \ne \{x_1, x_3\}$, M contains a cycle. If x_2 and u are adjacent, $(u, x_2, \cdots, x_t, u)$ and (u', x_0, x_1, u') are disjoint cycles in M_1. Hence x_2 and u are nonadjacent. Similarly, x_2 and u' are nonadjacent. Hence $N_{M_1}(x_2) = \{x_1, x_3\}$. ∎

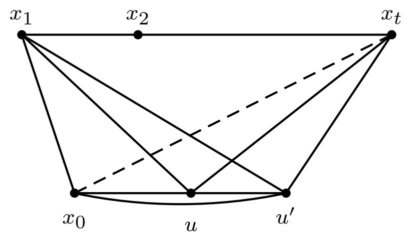

Fig. 1

Claim 6. *For any* i, $1 \le i \le k-2$,

$$2(d_{C_i}(x_0) + d_{C_i}(x_2)) + d_{C_i}(x_1) + d_{C_i}(x_t) \le 12.$$

Proof. Suppose not. Then $d_{C_i}(x_0)+d_{C_i}(x_2) \ge 5$ or $d_{C_i}(x_1)+d_{C_i}(x_t) \ge 5$. By Claim 1, C_i is a triangle. Suppose $d_{C_i}(x_0)=d_{C_i}(x_2)=3$. If $d_{C_i}(x_1)>0$, there are three disjoint cycles in the subgraph induced by $V(M_1) \cup V(C_i)$ as shown in Figure 2. If $d_{C_i}(x_t)>0$, there are three disjoint cycles as in Figure 3. Hence $d_{C_i}(x_1)=d_{C_i}(x_t)=0$. Next, suppose $d_{C_i}(x_0)=3$ and $d_{C_i}(x_2)=2$. If $N_{C_i}(x_1) \cap (V(C_i)-N_{C_i}(x_2)) \ne \emptyset$, there are three disjoint cycles as in Figure 2. If $N_{C_i}(x_1) \cap N_{C_i}(x_2) \ne \emptyset$, there are three disjoint cycles as in Figure 4. If $N_{C_i}(x_t) \cap N_{C_i}(x_2) \ne \emptyset$, there are three disjoint cycles as in Figure 3. Hence $d_{C_i}(x_1)=0$ and $d_{C_i}(x_t) \le 1$. Similarly, if $d_{C_i}(x_0)=2$ and $d_{C_i}(x_2)=3$, then $d_{C_i}(x_1)=0$ and $d_{C_i}(x_t) \le 2$. If $d_{C_i}(x_1)+d_{C_i}(x_t) \ge 5$, then $d_{C_i}(x_2)=0$ similarly to Claim 5. In all these cases,

$$2(d_{C_i}(x_0) + d_{C_i}(x_2)) + d_{C_i}(x_1) + d_{C_i}(x_t) \le 12.$$ ∎

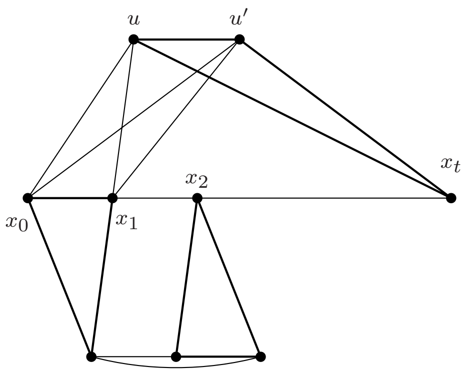

Fig. 2

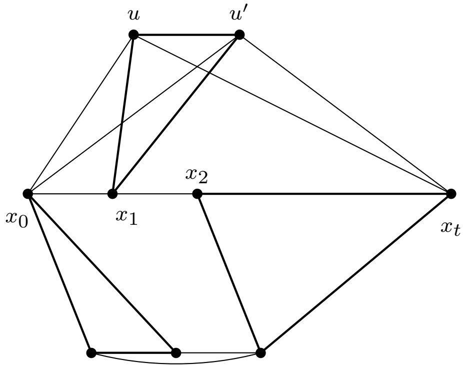

Fig. 3

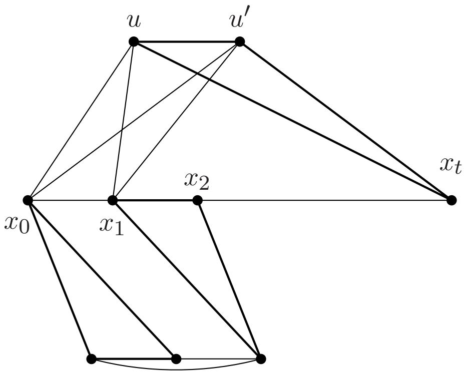

Fig. 4

By Claim 5 and Claim 6,

$$\begin{aligned}
&2(d_G(x_0) + d_G(x_2)) + d_G(x_1) + d_G(x_t) \\
&\qquad \le 12(k-2) + 2(4+2) + 4 + 4 \\
&\qquad = 12k - 4 \\
&\qquad < 3\sigma_2(G).
\end{aligned}$$

This contradicts the assumption. ∎

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