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ORIGINAL PAPER

Counting Hamilton Cycles in Dirac Hypergraphs

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Abstract

For $0 \leq \ell < k$, a Hamilton ℓ-cycle in a k-uniform hypergraph H is a cyclic ordering of the vertices of H in which the edges are segments of length k and every two consecutive edges overlap in exactly ℓ vertices. We show that for all $0 \leq \ell < k-1$, every k-graph with minimum co-degree δn with $\delta > 1/2$ has (asymptotically and up to a subexponential factor) at least as many Hamilton ℓ-cycles as a typical random k-graph with edge-probability δ. This significantly improves a recent result of Glock, Gould, Joos, Kühn and Osthus, and verifies a conjecture of Ferber, Krivelevich and Sudakov for all values $0 \leq \ell < k-1$.

Keywords Hamilton cycles · Hypergraphs · Minimum degree · Counting problems

Mathematics Subject Classification 05C65 · 05C38 · 05C45 · 05C80 · 05C07

1 Introduction

A classical theorem of Dirac [3] states that any graph on $n \geq 3$ vertices with minimum degree at least $n/2$ is Hamiltonian. We call graphs that meet this minimum degree requirement *Dirac graphs*.

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The complete bipartite graph $K_{n,n+1}$ is an extremal example for the tightness of this minimum degree condition. Moreover, since adding one edge to it already creates many Hamilton cycles, this suggests that Dirac graphs might contain not only one, but many Hamilton cycles. This leads to the question of *how many* Hamilton cycles are contained in a Dirac graph. In a seminal paper by Sárközy et al. [20] it was proven that n-vertex Dirac graphs contain at least $c^n n!$ many distinct Hamilton cycles for some small positive constant c. As this is clearly the correct order of magnitude, one could further ask for the correct value of the constant c.

Toward an intelligent guess for the value of c, consider a binomial random graph $G_{n,p}$ on n vertices wherein each edge appears independently with probability p. It is easy to show (for example by Chernoff bounds) that with high probability (i.e., with probability tending to 1 as $n \to \infty$), its minimum degree is $(1 - o(1))np$, and that the expected number of Hamilton cycles is

$$\frac{1}{2}(n-1)!p^n = (1 - o(1))^n n!p^n. \tag{1.1}$$

(It is not so easy to show concentration though! See Janson [9].) This hints that we might take $c \approx \delta(G)/n$, where $\delta(G)$ denotes the the minimum degree in G. Indeed, Cuckler and Kahn [2] impressively showed that $c \approx \delta(G)/n$ is the correct constant, thereby closing the case completely.

It is natural to extend Cuckler and Kahn's result to the hypergraph setting. First, let us introduce a notion of cycle in hypergraphs. For positive integers $0 \le \ell < k$, we define a (k, ℓ)-cycle to be a k-uniform hypergraph (or a "k-graph" for short) whose vertices may be ordered cyclically such that its edges are segments of length k and every two consecutive edges overlap in exactly ℓ vertices. A (k, ℓ)-cycle which contains all the vertices of a given k-graph is called a *Hamilton ℓ-cycle*. We say that a k-graph is *ℓ-Hamiltonian* if it contains a Hamilton ℓ-cycle. When $\ell = k - 1$ we often refer to an ℓ-cycle as a *tight cycle*, and we say that a k-graph is *tight Hamiltonian* or contains a *Hamilton tight cycle*, accordingly. Note that in order for a k-graph on n vertices to be ℓ-Hamiltonian, it is necessary that n be divisible by $k - \ell$. In light of Dirac's theorem, we also consider the more general notion of degrees in hypergraphs. We say that the *co-degree* of a $(k - 1)$-set X in a k-graph H is the number of edges in H that contain X.

There has been much work on analogues of Dirac's theorem in the hypergraph setting. Initial results were due to Katona and Kierstead [11]. Later it was shown that the necessary minimum co-degree for a k-graph H to be ℓ-Hamiltonian is $\delta_{k-1}(H) \approx n/2$, for $\ell = k - 1$ [18, 19], and more generally for ℓ satisfying $(k - \ell) \mid k$ [14]. For values of ℓ satisfying $(k - \ell) \nmid k$, it was proven in [13] that the necessary minimum co-degree for ℓ-Hamiltonicity is $\delta_{k-1}(H) \approx \frac{n}{\lceil k/(k-\ell) \rceil (k-\ell)}$. For more details about (many) other results regarding the minimum co-degree of a hypergraph and ℓ-Hamiltonicity, we refer the reader to the excellent surveys by Rödl and Ruciński [17] and by Kühn and Osthus [12].

In light of these results, and since we consider Hamilton ℓ-cycles for various values of ℓ, we say that H is *δ-Dirac* if $\delta_{k-1}(H) \ge \delta n$ for some constant $\delta > 1/2$. For the sake of simplicity, we have not tried to optimize the value of δ, so we always assume that $\delta > 1/2$ is a constant. However, with some more careful calculations and slight adjustments to the parameters involved, we can modify all our proofs so that they hold for any, say, $\delta \geq 1/2 + \omega\left((\log^3 n/n)^{1/5}\right)$.

A natural guess for the correct lower bound on the number of Hamilton ℓ-cycles in a δ-Dirac graph is the expected number of Hamilton ℓ-cycles in a random hypergraph with edge density δ. That is, we hope to obtain a lower bound of the form

$$(1 - o(1))^n \cdot \Psi_k(n, \ell) \cdot \delta^{\frac{n}{k-\ell}}, \tag{1.2}$$

where $\Psi_k(n, \ell)$ is the number of Hamilton ℓ-cycles in the complete k-graph on n vertices. Ferber et al. [6] realized this hope in the case where $\ell \leq k/2$. Quite recently, Glock et al. [7] showed that a δ-Dirac k-graph contains at least $(1 - o(1))^n n! c^n$ Hamilton ℓ-cycles, for all values ℓ and for some small constant $c > 0$. Our contribution is that (1.2) is the correct lower bound for all values of $0 \leq \ell < k - 1$ in any δ-Dirac k-graph.

Theorem 1.1 *Let $\ell, k \in \mathbb{N}$ be such that $0 \leq \ell < k - 1$, and let n be a sufficiently large integer which is divisible by $k - \ell$. Then the number of Hamilton ℓ-cycles in a δ-Dirac k-graph H on n vertices is at least*

$$(1 - o(1))^n \cdot \Psi_k(n, \ell) \cdot \delta^{\frac{n}{k-\ell}}.$$

2 Proof Outline

For $m \in \mathbb{N}$ and $X \subset V(H)$ we define an (ℓ, m, X)-*path-system* to be an ordered collection of m many vertex-disjoint ℓ-paths that cover X, and let $\mathcal{P}(\ell, m, X)$ be the collection of all (ℓ, m, X)-path-systems in H (see Definition 3.7). Our proof is largely based on the following three steps: (i) We remove a small subset $W \subset V$ with certain properties. This set will be used to tailor path-systems into Hamilton cycles. (ii) We show that, for an appropriate choice of m, $|\mathcal{P}(\ell, m, V \setminus W)|$ is at least as large as the number of Hamilton ℓ-cycles we eventually want. (iii) We show that each $P \in \mathcal{P}(\ell, m, V \setminus W)$ can be tailored into a Hamilton ℓ-cycle using the vertices in W in such a way that distinct path-systems correspond to distinct cycles. Clearly, it then follows that the number of Hamilton ℓ-cycles in H is at least the size of $\mathcal{P}(\ell, m, V \setminus W)$, as required.

Section 3.4 is dedicated to steps (i) and (iii), which mainly follow from other results (mostly stated in [7]). More specifically, in Lemma 3.13 we prove that our k-graph contains such a subset $W \subset V$, and in Lemma 3.16 we show how to tailor a path-system into a Hamilton ℓ-cycle using the set W.

Our main contribution is in step (ii), where the goal is to construct "many" path-systems, each of which covers all of the vertices in $V' := V \setminus W$. As mentioned above, we use W to tailor each path-system into exactly one Hamilton ℓ-cycle. Therefore, if we let $x := |W|$, then we clearly cannot have more than $(\Theta(n - x))^{n-x}$ many path-systems in V', as this is at most how many Hamilton cycles one can have on $n - x$ vertices. Since the desired lower bound on the number of Hamilton ℓ-cycles is of order $(\Theta(n))^n$, we require $(n-x)^{-x} = (1-o(1))^n$. Therefore, we must choose $x = o\left(\frac{n}{\log n}\right)$. Moreover, since we need to sew m paths together using vertices from W we also must have that $m = O(x)$.

For convenience we only count (and construct) path-systems in which all paths are of exactly the same length $s := \frac{n-x}{m}$ (so from the above discussion we must have $s = \omega(\log n)$). The process of constructing a path-system goes as follows. First, we choose an ordered equipartition $V' = V_1 \cup \ldots \cup V_s$. Second, we choose an "ordered" perfect matching $M_1 = (e_1, \ldots, e_m)$ in the k-partite k-graph induced by $V_1 \cup \ldots \cup V_k$, and for each $i \in [m]$ we let X_{e_i} be the vertices in e_i that are contained in the last ℓ parts $V_{k-\ell+1}, \ldots, V_k$. Next, we choose a perfect matching $M_2 = (f_1, \ldots, f_m)$ in the k-partite k-graph induced by $V_{k-\ell+1} \cup \ldots \cup V_{2k-\ell}$ in such a way that for each i we have $f_i \cap e_i = X_{e_i}$, and define X_{f_i}, analogously, to be the intersection of f_i with the last ℓ parts $V_{2(k-\ell)+1}, \ldots, V_{2k-\ell}$, for each $i \in [m]$. We repeat this, choosing a perfect matching $M_3 = (g_1, \ldots, g_m)$ in the k-partite k-graph induced by $V_{2(k-\ell)+1}, \ldots, V_{3k-2\ell}$ such that $g_i \cap f_i = X_{f_i}$ and define X_{g_i} analogously. We continue this way, considering the next k parts of the partition in steps of size $k-\ell$, until we cover V'. Clearly, the union of all the M_i's is an (ℓ, m, V')-path-system (the order of the paths is induced by the order on M_1).

Our goal is to show that this process yields many path-systems. Hence, the main building block in our counting argument will be finding many perfect matchings in k-partite k-graphs, where the intersection of each edge with the first ℓ parts is determined. Section 3.2 is dedicated to showing that this is possible when considering a δ-Dirac k-partite k-graph (where the Dirac property applies only for $(k-1)$-sets with vertices in distinct parts). We prove Lemma 3.4, and as a consequence we get Corollary 3.6, showing that many perfect matchings can indeed be found in each step of this process.

Having proved our building block in Corollary 3.6, we describe in detail the process of constructing many path-systems in Sect. 3.3. We first prove Lemma 3.8, showing that for an appropriate choice of the parameter m, most of the ordered equipartitions inherit the Dirac property of our k-graph (as s-partite induced k-graphs). We then prove Lemma 3.9, showing that for each such "good" equipartition we can construct many distinct path-systems by concatenating perfect matchings from each step in process described above. We conclude the section with Corollary 3.10, where we combine both lemmata to get the "correct" number of path-systems in our k-graph.

Lastly, in Sect. 4 we tie everything together, showing that Lemma 3.13, Corollary 3.10, and Lemma 3.16 imply our result for an appropriate choice of parameters.

3 Auxiliary Results

3.1 Concentration Inequalities

We use two probabilistic tools. The first one is the known result by Chernoff, bounding the lower and the upper tails of the Binomial distribution (see [1, 10]).

Lemma 3.1 *(Chernoff bound) Let* $X \sim \text{Bin}(n, p)$ *and let* $\mathbb{E}[X] = \mu$. *Then*

- $\Pr[X < (1-\delta)\mu] < e^{-\delta^2\mu/2}$ *for every* $\delta > 0$;

- $\Pr[X > (1+\delta)\mu] < e^{-\delta^2\mu/3}$ *for every* $0 < \delta < 3/2$.

Remark 3.2 The above bounds also hold when X is a hypergeometric random variable.

Our second probabilistic tool is an application of a concentration inequality by McDiarmid [16], proved originally by Maurey [15] as one of the first uses of concentration inequalities outside of probability theory.

Theorem 3.3 *Let S_n be the group of permutations over a set of n elements, and let $h\colon S_n \to \mathbb{R}$. Assume that for some constant c we have that $|h(\pi) - h(\pi')| \le c$ for any $\pi, \pi' \in S_n$ which are obtained from one another by swapping two elements. Then for any $t \ge 0$ we have*

$$\Pr\left[h(\pi) \le \mathbb{E}(h(\pi)) - t\right] \le \exp\left(-\frac{t^2}{c^2 n}\right).$$

3.2 The Number of Perfect Matchings in *k*-Partite *k*-Graphs

In this section we show that in a δ-Dirac k-partite k-graph, one can find "many" perfect matchings, even when the intersection of each edge in each matching with the first ℓ parts is determined. This is our main building block for constructing many path-systems in the proof of the main result. We show this as a corollary of a more general statement, Lemma 3.4, which is a consequence of the concentration inequality given by Theorem 3.3. The focus of this section is proving Lemma 3.4, which allows us to reduce the problem to the bipartite case in graphs. This is a version of an idea from [4]. Then, by using a known result by Cuckler and Kahn [2], we deduce Corollary 3.6.

We start by introducing some further definitions and notation. Let H be a k-partite k-graph with parts $V_1, \dots, V_k$, all of size m. Let $\pi = (\pi_1, \dots, \pi_{k-1})$, where $\pi_i : [m] \to V_i$ is a permutation on the vertices in V_i for each $i \in [k-1]$. Let $\mathcal{M}_\pi$ be the collection of $(k-1)$-sets that intersect all $V_1, \dots, V_{k-1}$, induced by π. More precisely, let

$$\mathcal{M}_\pi := \{\{\pi_1(j), \dots, \pi_{k-1}(j)\} : \ j \in [m]\}.$$

Define the auxiliary graph $\mathcal{B}_\pi(H)$ to be the bipartite graph with parts $\mathcal{M}_\pi$ and V_k, and such that xv is an edge for $x \in \mathcal{M}_\pi$ and $v \in V_k$ if and only if $x \cup v \in E(H)$. If $\pi_k\colon [m] \to V_k$ is a permutation of the vertices in V_k, we say that the tuple of k permutations $(\pi_1, \dots, \pi_k)$ *induces* a perfect matching in H, if the set of edges

$$\{\{\pi_1(j), \dots, \pi_k(j)\} : \ j \in [m]\}$$

is a perfect matching in H.

When considering an s-partite k-graph H, for some $s \ge k$, it is simpler to use the following variant of the notion of minimum co-degree. Assume that $V_1, \dots, V_s$ are the parts of H. For $i \in [s]$ define

$$U_i := \bigcup V_{j_1} \times \cdots \times V_{j_{k-1}},$$

where the union goes over all $1 \leq j_1 < \ldots < j_{k-1} \leq s$ such that $j_1, \ldots, j_{k-1} \neq i$. Define further

$$\delta^*_{k-1}(H) := \min\{d(X, V_i) : i \in [s], \ X \in U_i\},$$

where $d(X, V_i)$ is the number of edges in H that contain X (viewed as a $(k-1)$-set) and intersect V_i non-trivially. That is, $\delta^*_{k-1}(H)$ is the minimum number of edges incident to a $(k-1)$-set that intersects exactly $k-1$ parts.

We can now state the main lemma of this section.

Lemma 3.4 *For every $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that the following holds for any integer $m \geq m_0$. Let H be a k-partite k-graph with parts $V_1 \cup \ldots \cup V_k$, all of size m. Suppose that $\delta^*_{k-1}(H) \geq \delta m$ for some $\varepsilon < \delta \leq 1$. For every $i \in [k-1]$ let $\pi_i : [m] \to V_i$ be a permutation on the vertices of V_i, such that $\pi_1, \ldots, \pi_{k-2}$ are fixed and π_{k-1} is chosen uniformly at random. Denote $\pi := (\pi_1, \ldots, \pi_{k-1})$. Then with high probability the bipartite graph $\mathcal{B}_\pi(H)$ has minimum degree at least $(\delta - \varepsilon)m$.*

Proof Recall that the graph $\mathcal{B}_\pi := \mathcal{B}_\pi(H)$ has parts $\mathcal{M}_\pi$ and V_k, and note that for every $x \in \mathcal{M}_\pi$ we have $d_{\mathcal{B}_\pi}(x) \geq \delta^*_{k-1}(H) \geq \delta m$. So it is left to show that the statement holds for vertices in V_k. Let $v \in V_k$ and consider $d_{\mathcal{B}_\pi}(v)$. Since π_{k-1} is chosen uniformly at random, for each $j \in [m]$ we have

$$\begin{aligned}
\mathbb{E}_{\pi_{k-1}}\left[\mathbb{1}_{\{\{\pi_1(j),\ldots,\pi_{k-1}(j),v\}\in E(H)\}}\right] &= \Pr\left[\{\pi_1(j), \ldots, \pi_{k-1}(j), v\} \in E(H)\right] \\
&= \frac{d_H\left(\{\pi_1(j), \ldots, \pi_{k-2}(j), v\}, V_{k-1}\right)}{m} \\
&\geq \delta.
\end{aligned}$$

Thus we have

$$\mathbb{E}_{\pi_{k-1}}\left[d_{\mathcal{B}_\pi}(v)\right] = \sum_{j\in[m]} \mathbb{E}_{\pi_{k-1}}\left[\mathbb{1}_{\{\{\pi_1(j),\ldots,\pi_{k-1}(j),v\}\in E(H)\}}\right] \geq \delta m.$$

Now note that swapping any two elements in π_{k-1} can change $d_{\mathcal{B}_\pi}(v)$ by at most 2. Thus, by Theorem 3.3 we get that

$$\Pr\left[d_{\mathcal{B}_\pi}(v) \leq (\delta - \varepsilon)m\right] \leq \Pr\left[d_{\mathcal{B}_\pi(v)} \leq \mathbb{E}\left[d_{\mathcal{B}_\pi}(v)\right] - \varepsilon m\right] \leq \exp\left(-\frac{\varepsilon^2 m}{4}\right).$$

Hence, after taking a union bound over all vertices in V_k, we see that the minimum degree in $\mathcal{B}_\pi(H)$ is at least $(\delta - \varepsilon)m$ with high probability. □

In [2] the authors provide a lower bound on the number of perfect matchings in Dirac graphs, given naturally by the lower bound on the number of Hamilton cycles in those graphs.

Theorem 3.5 (Theorems 1.5 and 3.1 in [2]) *Let G be a bipartite Dirac graph on parts of size n, and with minimum degree $d \geq n/2$. Then G contains at least*

$$(1-o(1))^n \cdot n! \cdot \left(\frac{d}{n}\right)^n$$

many perfect matchings.

We combine Lemma 3.4 and Theorem 3.5 to get the following corollary.

Corollary 3.6 *Let H be a k-partite k-graph with parts $V_1, \dots, V_k$, all of size m, and suppose that $\delta^*_{k-1}(H) \geq \delta m$, for some constant $1/2 < \delta \leq 1$. Let $0 \leq r \leq k-2$, and in case that $r \geq 1$ let further $\pi_1, \dots, \pi_r$ be such that $\pi_i : [m] \to V_i$ is a fixed permutation on the vertices in V_i, for each $i \in [r]$. Then there are at least*

$$(1-o(1))^m (m!)^{k-r} \delta^m$$

many tuples of permutations $(\pi_{r+1}, \dots, \pi_k)$ for which $\pi' = (\pi_1, \dots, \pi_k)$ induces a perfect matching in H.

Proof Let $0 < \varepsilon \leq \delta - 1/2$ and let m be sufficiently large. Fix a set of $k-2-r$ permutations, $\pi_{r+1}, \dots, \pi_{k-2}$ in the case $r \leq k-3$, and an empty set for $r = k-2$. There are $(m!)^{k-2-r}$ ways to choose this set of permutations. By Lemma 3.4 we know that there are at least $(1-o(1))^m m!$ permutations $\pi_{k-1} : [m] \to V_{k-1}$ for which, if $\pi = (\pi_1, \dots, \pi_{k-1})$, then the bipartite graph $\mathcal{B}_\pi(H)$ has minimum degree at least $(\delta - \varepsilon)m \geq m/2$. Consider one such π_{k-1}. By Theorem 3.5 we get that $\mathcal{B}_\pi(H)$ contains at least $(1-o(1))^m m! \delta^m$ perfect matchings, each of which can be encoded by a certain permutation $\pi_k : [m] \to V_k$. Moreover, each such perfect matching gives a perfect matching in H. In total we get that there are at least $(1-o(1))^m (m!)^{k-r} \delta^m$ many tuples $\pi' = (\pi_1, \dots, \pi_k)$ which induce a perfect matching in H (where one perfect matching may be induced by several tuples of permutations). □

3.3 Constructing Many (ℓ, m, V')-Path-Systems

This section is the heart of the argument. We show that we can cover a subset containing most of the vertices in H by the "correct" number of path-systems, that is, the number of Hamilton ℓ-cycles we aim to find in H. We start with the precise notion of a path-system. Similarly to an ℓ-cycle, an ℓ-path is a k-graph whose vertices may be ordered so that its edges are segments of length k and consecutive edges overlap in exactly ℓ vertices.

Definition 3.7 Let F be a k-graph and let $X \subset V(F)$ be a subset of vertices. We say that an ordered collection $\mathcal{P} = (P_1, \dots, P_m)$ is an (ℓ, m, X)-*path-system*, if

- P_i is an ℓ-path in F on $|X|/m$ vertices, for each $i \in [m]$,
- $\{P_i\}_{i \in [m]}$ are pairwise vertex-disjoint, and their union covers all of the vertices in X.

Note that two different (ℓ, m, X)-path-systems may consist of the same family of ℓ-paths but with different orderings. We distinguish between two such families, since they will eventually form two different Hamilton ℓ-cycles when we sew the paths to one another.

For a partition Π of the vertices of a k-graph H into s parts, we denote by $H[\Pi]$ the s-partite k-graph spanned by edges going between parts of Π. We use the following two lemmata to show that a subset containing most of the vertices in a δ-Dirac k-graph can be covered by many path-systems.

Lemma 3.8 *Let $H = (V, E)$ be a δ-Dirac k-graph on n vertices, and let $V' \subset V$ be a fixed subset of vertices of size $n' = n - o(n)$. Let $\Pi = (V_1, \ldots, V_{n'/m})$ be an equipartition of V' chosen uniformly at random, into parts of size $m := m(n) = \omega(\log n)$. Then with high probability $H[\Pi]$ satisfies*

$$\delta^*_{k-1}(H[\Pi]) \geq (\delta - o(1))m. \tag{3.1}$$

Proof Let $0 < \varepsilon < \delta/2$ and let n be sufficiently large. Since V' contains all but $o(n)$ many vertices in H, and since $\delta_{k-1}(H) \geq \delta n$, we get

$$\delta_{k-1}\left(H[V']\right) \geq (\delta - \varepsilon)\, n'.$$

Let $i \in [n'/m]$ and let $X \in V_{j_1} \times \cdots \times V_{j_{k-1}}$ for some $1 \leq j_1 < \ldots < j_{k-1} \leq n'/m$ with $j_1, \ldots, j_{k-1} \neq i$. We have

$$\mu := \mathbb{E}\left[d(X, V_i)\right] = |V_i| \frac{d_{H[V']}(X)}{n'} \geq (\delta - \varepsilon)m.$$

Since $d(X, V_i)$ has hypergeometric distribution for each i, by Lemma 3.1 we get that

$$\Pr\left[d(X, V_i) < (\delta - 2\varepsilon)m\right] \leq \Pr\left[d(X, V_i) < (1 - \varepsilon)\mu\right] \leq \exp\left(-\frac{1}{2}\varepsilon^2(\delta - \varepsilon)\omega(\log n)\right).$$

Taking a union bound over all $i \in [n'/m]$ and $X \in \binom{V'}{k-1}$ we get

$$\Pr\left[\exists X, \text{ and } i : d(X, V_i) < (\delta - 2\varepsilon)m\right] \leq n^{k-\omega(1)} = o(1),$$

and the statement follows. □

In other words, Lemma 3.8 shows that almost all partitions of $H[V']$ into n'/m parts inherit the relative minimum co-degree from H as induced (n'/m)-partite k-graphs. In particular, if H is δ-Dirac, then most of these partitions inherit this property (perhaps with a slightly smaller value of δ). Now we show that we may cover those induced k-graphs given by "good" partitions with many path-systems.

Lemma 3.9 *Let $H = (V, E)$ be a δ-Dirac k-graph on n vertices, and let $n' = n - o(n)$ and $m := m(n) = \omega(\log n)$ be integers such that $m | n'$ and $\frac{n'}{m} \equiv k (\bmod k - \ell)$. Let $V' \subset V$ be a fixed subset of vertices of size $n' = n - o(n)$, and $\Pi = (V_1, \ldots, V_{n'/m})$*

be an equipartition of the vertices of V' into parts of size $m := m(n) = \omega(\log n)$, satisfying (3.1). Then the $\frac{n'}{m}$-partite k-graph $H[\Pi]$ can be covered by at least

$$(1-o(1))^{n'} \cdot (m!)^{\frac{n'}{m}} \cdot \delta^{\frac{n'}{k-\ell}}$$

many distinct (ℓ, m, V')-path-systems.

Proof For the first step we consider the k-partite subhypergraph of H induced by the first k parts of Π, that is $H_1 := H[V_1, \dots, V_k]$. By Corollary 3.6 with $r = 0$ we get that there are at least

$$(1-o(1))^m \, (m!)^k \, \delta^m \tag{3.2}$$

many $\pi^1 = (\pi_1^1, \dots, \pi_k^1)$ which give a perfect matching in H_1. Fix one such $\pi^1 = (\pi_1^1, \dots, \pi_k^1)$, and let M_1 be the ordered perfect matching induced by π^1, ordered according to π_1^1. That is, $M_1 = (e_1^1, \dots, e_m^1)$, where $e_j^1 = \{\pi_1^1(j), \dots, \pi_k^1(j)\}$. Moreover, for each $j \in [m]$, let X_j^1 be the intersection of e_j^1 with parts $V_{k-\ell+1}, \dots, V_k$, that is,

$$X_j^1 := e_j^1 \cap (V_{k-\ell+1}, \dots, V_k) = \left(\pi_{k-\ell+1}^1(j), \dots, \pi_k^1(j)\right).$$

For the second step, consider the k consecutive parts in Π beginning with the last ℓ parts of H_1. More precisely, we look at the k-partite induced k-graph $H_2 := H[V_{k-\ell+1}, \dots, V_{2k-\ell}]$, and we find there a perfect matching $M_2 = (e_1^2, \dots, e_m^2)$ (ordered according to $\pi_{k-\ell+1}^1$) that extends M_1 in the sense that $e_j^2 \cap e_j^1 = X_j^1$ for every $j \in [m]$. We do this as follows. We let $(\pi_1^2, \dots, \pi_\ell^2) = (\pi_{k-\ell+1}^1, \dots, \pi_k^1)$ be our fixed ℓ permutations on $V_{k-\ell+1}, \dots, V_k$, respectively. Then, by Corollary 3.6 with $r = \ell$, there are at least

$$(1-o(1))^m (m!)^{k-\ell} \delta^m \tag{3.3}$$

many tuples $(\pi_{\ell+1}^2, \dots, \pi_k^2)$ for which $\pi^2 = (\pi_1^2, \dots, \pi_k^2)$ induces a perfect matching in H_2. Note further that two distinct such tuples induce two distinct perfect matchings in H_2, since π_1^2 is fixed. Hence, we get that the number of perfect matchings in H_2 which agree with M_1 on the first ℓ parts is at least what is given in (3.3). Let $M_2 = (e_1^2, \dots, e_m^2)$ be one such perfect matching, and for every $j \in [m]$, let X_j^2 to be the intersection of e_j^2 with the last ℓ parts of H_2, that is

$$X_j^2 := e_j^2 \cap (V_{2(k-\ell)+1}, \dots, V_{2k-\ell}) = \left(\pi_{k-\ell+1}^2(j), \dots, \pi_k^2(j)\right).$$

Note that indeed we have $e_j^2 \cap e_j^1 = X_j^1$ for every $j \in [m]$.

We then repeat the above procedure, where in each step we consider the next k parts, overlapping with the last ℓ parts from the preceding step. We do this $\frac{n'/m-k}{k-\ell}$ times, until we cover all parts in Π.

More formally, for $2 \leq s \leq \frac{n'/m-k}{k-\ell}$ we do the following in the s-th step. Consider the k-partite induced k-graph $H_s := H[V_{(s-1)(k-\ell)+1}, \ldots, V_{sk-(s-1)\ell}]$, and ℓ fixed permutations $(\pi_1^s, \ldots, \pi_\ell^s) = (\pi_{k-\ell}^{s-1}, \ldots, \pi_k^{s-1})$ on $V_{(s-1)(k-\ell)+1}, \ldots, V_{(s-1)k-s\ell}$ given by step $s-1$, respectively. By Corollary 3.6 with $r = \ell$, the number of tuples $(\pi_{\ell+1}^s, \ldots, \pi_k^s)$ for which $\pi^s = (\pi_1^s, \ldots, \pi_k^s)$ induces a perfect matching in H_s is at least what is given in (3.3). Again, any two distinct such tuples induce two distinct perfect matchings in H_s, as we have fixed π_1^s. We get that the number of perfect matchings M_s in H_s which agree with M_{s-1} on the first ℓ parts is as at least what is given in (3.3). Let $M_s = (e_1^s, \ldots, e_m^s)$ be one such perfect matching (ordered according to π_1^s). For every $j \in [m]$ let

$$X_j^s := e_j^s \cap (V_{s(k-\ell)+1}, \ldots, V_{sk-(s-1)\ell}) = \left(\pi_{k-\ell+1}^s, \ldots, \pi_k^s\right),$$

and note that we have $e_j^s \cap e_j^{s-1} = X_j^{s-1}$.

The statement then follows by multiplying by the number of options to extend the paths in each step. That is, multiplying (3.2) by (3.3) raised to the power of $\frac{n'/m-k}{k-\ell}$, we get that there are at least

$$(1-o(1))^{n'} \cdot (m!)^{\frac{n'}{m}} \cdot \delta^{\frac{n'}{k-\ell}}$$

many (ℓ, m, V')-path-systems that cover $H[\Pi]$. □

Combining the counting in Lemmas 3.8 and in 3.9, we get the following corollary, which gives us the required number of (ℓ, m, V')-path-systems covering $H[V']$.

Corollary 3.10 *Let $H = (V, E)$ be a δ-Dirac k-graph on n vertices. Let $V' \subset V$ be a subset of size $n' = n - o\left(\frac{n}{\log n}\right)$, and let $m = \omega(\log n)$ be an integer such that $m | n'$ and $\frac{n'}{m} \equiv k (mod k - \ell)$. Then $H[V']$ can be covered by at least*

$$(1-o(1))^n \cdot \Psi_k(n, \ell) \cdot \delta^{\frac{n}{k-\ell}} \tag{3.4}$$

many (ℓ, m, V')-path-systems.

Proof By Lemma 3.8 there are at least

$$(1-o(1)) \cdot \frac{n'!}{(m!)^{n'/m}}$$

many equipartitions Π which satisfy (3.1). By Lemma 3.9, each such partition can be covered by at least $(m!)^{\frac{n'}{m}} \delta^{\frac{n'}{k-\ell}} (1-o(1))^{n'}$ many distinct (ℓ, m, V')-path-systems. However, for some values of ℓ, the same path-system can be obtained from many different orderings of certain vertices within the paths. We denote by $c_k(\ell)$ the number of ways to reorder the first $k-\ell$ vertices in any edge such that the path-system is not changed, so we get $(c_k(\ell))^{\frac{n'}{k-\ell}}$ many such reorderings. Considering this double counting, and recalling that $n' = n - o\left(\frac{n}{\log n}\right)$, we get that in total $H[V']$ can be covered by at least

$$(1-o(1))^{n'} \cdot n'! \cdot \left(\frac{\delta}{c_k(\ell)}\right)^{\frac{n'}{k-\ell}} = (1-o(1))^{n} \cdot n! \cdot \left(\frac{\delta}{c_k(\ell)}\right)^{\frac{n}{k-\ell}}$$

many (ℓ, m, V')-path-systems. In order to show that this is precisely the expression in (3.4), it is sufficient to prove the following claim.

Claim 3.11 $\Psi_k(n,\ell) = (1-o(1))^n \cdot n! \cdot c_k(\ell)^{-\frac{n}{k-\ell}}$, given that $(k-\ell)|n$.

Proof In the complete k-graph on n vertices, any cyclical ordering of the vertices yields a Hamilton ℓ-cycle (where n, k, and ℓ satisfy the appropriate divisibility conditions). However, several different vertex-orderings may give rise to the same cycle. For any fixed ordering, we may of course cyclically permute the edges in $\frac{n}{k-\ell}$ ways or reverse their ordering and leave the cycle unchanged. Additionally, by the definition of $c_k(\ell)$, we may reorder the first $k-\ell$ vertices within each edge in $c_k(\ell)$ ways per edge and obtain the same cycle. So we have

$$\Psi_k(n,\ell) = n! \cdot \frac{k-\ell}{n} \cdot \frac{1}{2} \cdot \left(\frac{1}{c_k(\ell)}\right)^{\frac{n}{k-\ell}}.$$

To complete the proof, we also describe the quantity $c_k(\ell)$ explicitly. Let $0 \le r < k-\ell$ be the remainder when dividing k by $k-\ell$. That is, r satisfies $k = (k-\ell)\left\lfloor \frac{k}{k-\ell} \right\rfloor + r$. Then we have

$$c_k(\ell) = r!(k-\ell-r)!. \tag{3.5}$$

For example, for $\ell < k/2$ we have $r = \ell$, which gives $c_k(\ell) = \ell!(k-2\ell)!$.

Note that, in particular, we have $\Psi_k(n,\ell) = n^{\Theta(n)}$. □

All in all, we get that there are at least

$$(1-o(1))^n \cdot \Psi_k(n,\ell) \cdot \delta^{\frac{n}{k-\ell}}$$

many (ℓ, m, V')-path-systems covering $H[\Pi]$, as required. □

3.4 Turning (ℓ, m, V')-Path-Systems into Hamilton ℓ-Cycles

In this section we show how to form a Hamilton ℓ-cycle from an (ℓ, m, V')-path-system. We do this in three steps. First, we put aside a special subset of vertices. Second, we cover the remaining vertices with a path-system. Finally, we use our special subset to tailor together the paths in the path-system so that they form a Hamilton ℓ-cycle.

3.4.1 Finding a $(\delta - o(1), \ell, m, t)$-Connecting-Set in a Dirac Hypergraph

Let us first make explicit the properties of the special set to put aside. We will actually put aside a family of subsets—one subset for each path in the path-system.

Definition 3.12 Let F be a k-graph, $\eta \in (0, 1)$, and ℓ, m, t be integers such that $1 \leq \ell \leq k-1$ and $1 \leq m \leq t$. We say that an ordered collection $\mathcal{W} = (W_1, \dots, W_m)$ is an (η, ℓ, m, t)-*connecting-system* in F, if

(i) $\{W_i\}_{i\in[m]}$ are pairwise disjoint subsets of vertices in F,
(ii) $\sum_{i\in[m]} |W_i| = t$,
(iii) $|W_i| \equiv -k(mod k - \ell)$ for all $i \in [m]$,
(iv) $\big||W_i| - |W_j|\big| \leq k - \ell$ for all $i, j \in [m]$, and
(v) for all $i \in [m]$ and for all $X \in \binom{V(F)}{k-1}$ we have $d_F(X, W_i) \geq \eta|W_i|$.

We say that a subset $W \subset V(F)$ is an (η, ℓ, m, t)-*connecting-set* if it admits a partition $W = W_1 \cup \dots \cup W_m$ such that $(W_1, \dots, W_m)$ is an (η, ℓ, m, t)-connecting-system.

In other words, an (η, ℓ, m, t)-connecting-system consists of t vertices (property (ii)), broken into m parts (property (i)) of roughly equal size (property (iv)), where each $(k-1)$-set in F completes to an edge with an η-fraction of each part (property (v)). We will see later that property (iii) ensures that some divisibility conditions are met when we create our ℓ-cycles.

We now show that we can find such a set W in H with $\eta \approx \delta$.

Lemma 3.13 *Let $H = (V, E)$ be a k-graph on n vertices with minimum co-degree δn for some constant $0 < \delta \leq 1$, and let $1 \leq \ell \leq k-1$ be an integer. Then there exists a $(\delta - o(1), \ell, m, t)$-connecting-system $\mathcal{W} = (W_1, \dots, W_m)$ in H, for any integers $1 \leq m \leq t \leq n$ satisfying $\frac{t}{m} = \omega(\log n)$ and $t \equiv -mk(mod k - \ell)$.*

Proof Let $1 \leq \ell \leq k$ and $1 \leq m \leq t \leq n$ be integers satisfying the conditions of the statement, and write $t = T(k-\ell) - mk$ for some integer T. Let $\mathcal{W} := (W_1, \dots, W_m)$ be an ordered collection of pairwise-disjoint subsets of V such that $|W_i| \in \left\{\left\lfloor \frac{T}{m} \right\rfloor (k-\ell) - k, \left\lceil \frac{T}{m} \right\rceil (k-\ell) - k\right\}$ for every $i \in [m]$ and $\sum_{i\in[m]} |W_i| = t$, chosen uniformly at random. We show that with high probability $\mathcal{W}$ is a $(\delta - o(1), \ell, m, t)$-connecting-system in H.

The proof is similar to that of Lemma 3.8. For every $X \in \binom{V}{k-1}$ and for every $i \in [m]$, the co-degree $d(X, W_i)$ is a hypergeometrically distributed random variable with

$$\mu := \mathbb{E}\left[d(X, W_i)\right] = \frac{d_H(X)}{n}|W_i| \geq \delta|W_i| = \Theta\left(\frac{t}{m}\right) = \omega(\log n).$$

Fix $X \in \binom{V}{k-1}$, and $i \in [m]$. By Lemma 3.1 with $0 < \varepsilon < \delta$ and by the above, we have

$$\begin{aligned}\Pr\left[d(X, W_i) < (\delta - \varepsilon)|W_i|\right] &\leq \Pr\left[d(X, W_i) < (1-\varepsilon)\mu\right] \\ &\leq \exp\left(-\frac{\varepsilon^2\delta}{2} \cdot \omega(\log n)\right).\end{aligned}$$

A union bound over all possible $X \in \binom{V}{k-1}$ and $i \in [m]$ gives

$$\Pr\left[\exists X, \text{ and } i : d(X, W_i) < (\delta - \varepsilon)|W_i|\right] \leq n^{k-\omega(1)} = o(1).$$

Hence, property (v) is satisfied, so with high probability $\mathcal{W}$ is a $(\delta - \varepsilon, \ell, m, t)$-connecting-system in H. □

3.4.2 Forming a Hamilton Cycle Using a $(\delta - o(1), \ell, m, t)$-Connecting-System

It is left to show how to use a $(\delta - o(1), \ell, m, t)$-connecting-system in H, for appropriate parameters m, t, to form a Hamilton ℓ-cycle from an (ℓ, m, V')-path-system that covers the rest of the vertices in H. For this we need Lemma 3.15, which follows from Lemma 3.6 in [7], and the notion of ℓ-Hamiltonian connectedness. Given a Hamilton ℓ-path $P = (x_1, \ldots, x_n)$, we say that $(x_1, \ldots, x_\ell)$ and $(x_{n-\ell+1}, \ldots, x_n)$ are the *ℓ-end-tuples* of P.

Definition 3.14 A k-graph F is *ℓ-Hamiltonian connected* if, for every two disjoint ordered subsets of ℓ vertices $\overrightarrow{X}, \overrightarrow{Y} \in V(F)^\ell$ there exists a Hamilton ℓ-path in F with $\overrightarrow{X}$ and $\overrightarrow{Y}$ as ℓ-end-tuples.

The authors of [7] show that all sufficiently dense (in the sense of co-degree) k-graphs on sufficiently many vertices are $(k-1)$-Hamiltonian connected (provided that divisibility conditions are met). Their proof implies the next lemma which is slightly more general.

Lemma 3.15 *For every $\varepsilon > 0$ there exists n_0 such that every k-graph F on $n \geq n_0$ vertices, where $n \equiv k (mod k - \ell)$, and with minimum co-degree at least $(1/2 + \varepsilon)n$ is ℓ-Hamiltonian connected.*

Now we show that when we put aside a connecting-system and cover the remaining vertices with a path-system, we may connect the paths into Hamilton ℓ-cycles in such a way that distinct path-systems yield distinct cycles.

Lemma 3.16 *Let $H = (V, E)$ be a δ-Dirac k-graph on n vertices. Let $W \subset V$ be a $(\delta - o(1), \ell, m, t)$-connecting-set in H, for some integers ℓ, m, t satisfying $1 \leq \ell \leq k - 1$ and $\frac{t}{m} = \omega(1)$. Denote $V' := V \setminus W$, and let $\mathcal{P} = (P_1, \ldots, P_m)$ be an (ℓ, m, V')-path-system covering $H[V']$. Then there exists a Hamilton ℓ-cycle C in H, containing $\{P_i\}_{i\in[m]}$ as segments, according to their ordering in $\mathcal{P}$. Moreover, if $\mathcal{P}_1, \mathcal{P}_2$ are two distinct (ℓ, m, V')-path-systems, then the Hamilton ℓ-cycles C_1, C_2 obtained from them are distinct as well.*

Proof Let $W = W_1 \cup \cdots \cup W_m$ be a partition of W such that $\mathcal{W} = (W_1, \ldots, W_m)$ is a $(\delta - \varepsilon, \ell, m, t)$-connecting-system in H, for some $0 < \varepsilon < \delta/2 - 1/4$. We use these parts to tailor the ℓ-paths in $\mathcal{P}$ to one another. For every $i \in [m]$ let $\overrightarrow{X_i}$ and $\overrightarrow{Y_i}$ be the ℓ-end-tuples of the path P_i, and let X_i, Y_i be their unordered sets of vertices, respectively. By Definition 3.12 (v), for each $i \in [m]$, the induced subgraph $H_i := H[W_i \cup Y_i \cup X_{i+1}]$ (where we replace $i+1$ by 1 when $i = m$) has minimum co-degree at least $(\delta-2\varepsilon)|H_i| > \frac{1}{2}|H_i|$. By items (ii) and (iv) of Definition 3.12 we get that $|H_i| = \Theta\left(\frac{t}{m}\right) = \omega(1)$ for each $i \in [m]$. Moreover, since $|W_i| \equiv -k(mod k-\ell)$ and $|X_{i+1}| = |Y_i| = \ell$, we get that $|H_i| \equiv k(mod k-\ell)$ for every $i \in [m]$. Hence, by Lemma 3.15 it is ℓ-Hamiltonian connected. Let Q_i be a Hamilton ℓ-path in H_i with $\overrightarrow{Y_i}$ and $\overrightarrow{X}_{i+1}$ as ℓ-end-tuples. Then we get that $C := (P_1, Q_1, \dots, P_m, Q_m)$ is a Hamilton ℓ-cycle in H.

Note that in C, any two paths P_i, P_{i+1} are separated by the vertices of W_i, where $\mathcal{W}$ is fixed in the process. Hence, given $\mathcal{W}$, any two distinct (ℓ, m, V')-path-systems form two distinct Hamilton ℓ-cycles. □

4 Proof of Theorem 1.1

We can now put all of the ingredients together and quickly derive the proof of our main theorem.

Proof of Theorem 1.1 Let N be the largest integer satisfying $N \equiv k(mod k-\ell)$ such that $N\log^2 n \le n - \log^4 n$. Let $m := \left\lfloor \frac{n-\log^4 n}{N} \right\rfloor$ and let $n' := mN$ and $t := n - n'$. Then we have $m = \Theta\left(\log^2 n\right)$, $t = \Theta\left(\log^4 n\right)$, and moreover $t \equiv -mk(mod k-\ell)$ and $\frac{n'}{m} \equiv k(mod k-\ell)$. Hence, by Lemma 3.13 there exists a subset $W \subset V$ which is a $(\delta - o(1), \ell, m, t)$-connecting-set in H. Consider the remaining set of vertices $V' := V \setminus W$. Since $|W| = t$ we get that $|V'| = n'$, so by Corollary 3.10, there are at least

$$(1-o(1))^n \cdot \Psi_k(n, \ell) \cdot \delta^{\frac{n}{k-\ell}}$$

many (ℓ, m, V')-path-systems covering $H[V']$. By Lemma 3.16, each such (ℓ, m, V')-path-system can be completed to a Hamilton ℓ-cycle in H using the vertices in W, such that no two distinct (ℓ, m, V')-path-system form the same cycle. Hence the statement is proved.

5 Concluding Remarks and Open Problems

We highlight the natural barrier in our approach for extending our main result to tight Hamilton cycles. The main obstacle is extending Corollary 3.6, a critical component of our proof, to the case where $r = k-1$. More precisely, in the setting of Corollary 3.6, we cannot use Theorem 3.3 to find many perfect matchings in $\mathcal{B}_\pi(H)$ for fixed $\pi = (\pi_1, \dots, \pi_{k-1})$ because we no longer have a "free" part where we can permute vertices at random. In order to resolve the tight case, we believe that a different approach is necessary. However, it seems reasonable to believe that the number of tight Hamilton cycles one can find in a δ-Dirac k-graph is consistent with our result for all other values of ℓ. Hence we state here the following conjecture, which is a slight generalization of Conjecture 7.1 in [7].

Conjecture 5.1 *A δ-Dirac k-graph H on n vertices contains at least $(1-o(1))^n \cdot n! \cdot \delta^n$ many tight Hamilton cycles.*

Recall that for those values of ℓ for which $(k-\ell) \nmid k$ the necessary minimum co-degree in a k-graph on n vertices to guarantee ℓ-Hamiltonicity is $\mu^*_{k-1}(\ell, n) := \frac{n}{\lceil k/(k-\ell) \rceil (k-\ell)} < \frac{n}{2}$. It makes sense then to ask for the number of Hamilton ℓ-cycles in k-graphs with minimum co-degree larger than this threshold.

Question 5.2 Let ℓ, k, n be integers satisfying $(k-\ell) \nmid k$. Let H be a k-graph on n vertices with $\delta_{k-1} \geq \delta n$ for some $\delta > \mu^*_{k-1}(\ell, n)/n$. What is the number of Hamilton ℓ-cycles in H?

One can also study Dirac-type problems in hypergraphs with respect to other notions of degrees. For a k-graph $H = (V, E)$ and a subset of d vertices, $X \in \binom{V}{d}$, for some $1 \leq d \leq k-1$, we define the *d-degree* of X to be the number of edges in E containing X. For example, one can ask what is the minimum co-degree condition which enforces a perfect matching in k-graph. For integers n, k, d satisfying $1 \leq d \leq k-1$ we let $m_d(k, n)$ be the smallest integer m such that any k-graph H on n vertices with $\delta_d(H) \geq m$ contains a perfect matching. $m_d(n, k)$ is unknown for most values of of d. Define further

$$\mu_d(k) := \lim_{n\to\infty} m_d(k, n) / \binom{n-d}{k-d}$$

to be the parameter that encodes the asymptotic behaviour of $m_d(k, n)$. Although true, it is not obvious that the limit exists, as was proved by Ferber and Kwan [5]. However, $\mu_d(k)$ is unknown for most values of d, k (for example, even the case $d = 1$ and $k = 6$ is open).

Given a k-graph H on n vertices satisfying the minimum d-degree condition for perfect matchings, one can ask how many of them can be found in H. For $d = k-1$ the answer is the same as the expected number of perfect matchings in a random graph with the same edge-density, but it is not clear if the same phenomenon also occurs in cases where $d < k-1$. Indeed, as was pointed out to the first author by Lisa Sauermann, this is not the case already for minimum 1-degree in 3-graphs, as can be shown in the following construction. Consider the bipartite 3-graph H on parts $X \cup Y$, where $|X| = n/3 - 1$ and $|Y| = 2n/3 + 1$, and all possible edges which intersect X in at least one vertex. We note that $\delta_1(H) < \frac{5}{9}\binom{n}{2}$ which was proved in [8] to be the correct threshold for minimum 1-degree in 3-graphs. Clearly H does not contain a perfect matching, since every set of disjoint edges has size at most $n/3 - 1$. Now, for $\varepsilon > 0$ let H_ε be the bipartite 3-graph defined similarly to H, but with parts of size $|X| = (1/3+\varepsilon)n$ and $|Y| = (2/3-\varepsilon)n$. H_ε satisfies the minimum 1-degree condition and hence contains a perfect matching. A calculation shows that for sufficiently small $\varepsilon > 0$ H_ε cannot contain more than $(1+o_\varepsilon(1))^n \frac{n!}{(n/3)!(3!)^{n/3}} \cdot \left(\frac{12}{27}\right)^{n/3}$ perfect matchings. Note that H_ε has edge-density at least $5/9$, so the above number is smaller than the expected number of perfect matchings in a random 3-graph with the same edge-density.

For $1 \leq d < k$, define the function $f(d, k, n)$ to be the number of perfect matchings in a k-graph H on n vertices with $\delta_d(H) \geq \delta\binom{n-d}{k-d}$ where $\delta > \mu_d(k)$. It would be interesting to understand the behavior of this function for all values $0 < d < k$.

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