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ORIGINAL PAPER

Polynomial Bounds for Chromatic Number. IV: A Near-polynomial Bound for Excluding the Five-vertex Path

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Abstract

A graph G is H-*free* if it has no induced subgraph isomorphic to H. We prove that a P_5-free graph with clique number $\omega \geq 3$ has chromatic number at most $\omega^{\log_2(\omega)}$. The best previous result was an exponential upper bound $(5/27)3^{\omega}$, due to Esperet, Lemoine, Maffray, and Morel. A polynomial bound would imply that the celebrated Erdős-Hajnal conjecture holds for P_5, which is the smallest open case. Thus, there is great interest in whether there is a polynomial bound for P_5-free graphs, and our result is an attempt to approach that.

Keywords Chromatic number · Induced subgraphs

1 Introduction

If G, H are graphs, we say G is H-*free* if no induced subgraph of G is isomorphic to H; and for a graph G, we denote the number of vertices, the chromatic number, the size of the largest clique, and the size of the largest stable set by $|G|$, $\chi(G)$, $\omega(G)$, $\alpha(G)$ respectively.

The k-vertex path is denoted by P_k, and P_4-free graphs are well-understood; every P_4-free graph G with more than one vertex is either disconnected or disconnected in the complement [24], which implies that $\chi(G) = \omega(G)$. Here we study how $\chi(G)$ depends on $\omega(G)$ for P_5-free graphs G.

The Gyárfás-Sumner conjecture [10, 25] says:

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1.1 Conjecture: *For every forest H there is a function f such that $\chi(G) \leq f(\omega(G))$ for every H-free graph G.*

This is open in general, but has been proved [10] when H is a path, and for several other simple types of tree ([3, 11–14, 17, 19]; see [18] for a survey). The result is also known if all induced subdivisions of a tree are excluded [17].

A class of graphs is *hereditary* if the class is closed under taking induced subgraphs and under isomorphism, and a hereditary class is said to be χ*-bounded* if there is a function f such that $\chi(G) \leq f(\omega(G))$ for every graph G in the class (thus, the Gyárfás-Sumner conjecture says that, for every forest H, the class of H-free graphs is χ-bounded). Louis Esperet [8] made the following conjecture:

1.2 (False) Conjecture: *Let $\mathcal{G}$ be a χ-bounded class. Then there is a polynomial function f such that $\chi(G) \leq f(\omega(G))$ for every $G \in \mathcal{G}$.*

Esperet's conjecture was recently shown to be false by Briański, Davies and Walczak [2]. However, this raises the further question: which χ-bounded classes are polynomially χ-bounded? In particular, the two conjectures 1.1 and 1.2 would together imply the following, which is still open:

1.3 Conjecture: *For every forest H, there exists $c > 0$ such that $\chi(G) \leq \omega(G)^c$ for every H-free graph G.*

This is a beautiful conjecture. In most cases where the Gyárfás-Sumner conjecture has been proved, the current bounds are very far from polynomial, and 1.3 has been only been proved for a much smaller collection of forests (see [5, 15, 16, 20–23]). In [22] we proved it for any P_5-free tree H, but it has not been settled for any tree H that contains P_5. In this paper we focus on the case $H = P_5$.

The best previously-known bound on the chromatic number of P_5-free graphs in terms of their clique number, due to Esperet, Lemoine, Maffray, and Morel [9], was exponential:

1.4 *If G is P_5-free and $\omega(G) \geq 3$ then $\chi(G) \leq (5/27)3^{\omega(G)}$.*

Here we make a significant improvement, showing a "near-polynomial" bound:

1.5 *If G is P_5-free and $\omega(G) \geq 3$ then $\chi(G) \leq \omega(G)^{\log_2(\omega(G))}$.*

(The cycle of length five shows that we need to assume $\omega(G) \geq 3$. Sumner [25] showed that $\chi(G) \leq 3$ when $\omega(G) = 2$.) Conjecture 1.3 when $H = P_5$ is of great interest, because of a famous conjecture due to Erdős and Hajnal [6, 7], that:

1.6 Conjecture: *For every graph H there exists $c > 0$ such that $\alpha(G)\omega(G) \geq |G|^c$ for every H-free graph G.*

This is open in general, despite a great deal of effort; and in view of [4], the smallest graph H for which 1.6 is undecided is the graph P_5. Every forest H satisfying 1.3 also satisfies the Erdős-Hajnal conjecture, and so showing that $H = P_5$ satisfies 1.3 would be a significant result. (See [1] for some other recent progress on this question.)

We use standard notation throughout. When $X \subseteq V(G)$, $G[X]$ denotes the subgraph induced on X. We write $\chi(X)$ for $\chi(G[X])$ when there is no ambiguity.

2 The Main Proof

We denote the set of nonnegative real numbers by $\mathbb{R}_+$, and the set of nonnegative integers by $\mathbb{Z}_+$. Let $f : \mathbb{Z}_+ \to \mathbb{R}_+$ be a function. We say

- f is *non-decreasing* if $f(y) \geq f(x)$ for all integers $x, y \geq 0$ with $y > x \geq 0$;
- f is a *binding* function for a graph G if it is non-decreasing and $\chi(H) \leq f(\omega(H))$ for every induced subgraph H of G; and
- f is a *near-binding* function for G if f is non-decreasing and $\chi(H) \leq f(\omega(H))$ for every induced subgraph H of G different from G.

In this section we show that if a function f satisfies a certain inequality, then it is a binding function for all P_5-free graphs. Then at the end we will give a function that satisfies the inequality, and deduce 1.5.

A *cutset* in a graph G is a set X such that $G \setminus X$ is disconnected. A vertex $v \in V(G)$ is *mixed* on a set $A \subseteq V(G)$ or a subgraph A of a graph G if v is not in A and has a neighbour and a non-neighbour in A. It is *complete* to A if it is adjacent to every vertex of A. We begin with the following:

2.1 *Let G be P_5-free, and let f be a near-binding function for G. Let G be connected, and let* X be a cutset of G. Then

$$\chi(G \setminus X) \leq f(\omega(G) - 1) + \omega(G) f(\lfloor \omega(G)/2 \rfloor).$$

Proof We may assume (by replacing X by a subset if necessary) that X is a minimal cutset of G; and so $G \setminus X$ has at least two components, and every vertex in X has a neighbour in $V(B)$, for every component B of $G \setminus X$. Let B be one such component; we will prove that $\chi(B) \leq f(\omega(G)-1)+\omega(G) f(\lfloor \omega(G)/2 \rfloor)$, from which the result follows.

Choose $v \in X$ (this is possible since G is connected), and let N be the set of vertices in B adjacent to v. Let the components of $B \setminus N$ be $R_1, \ldots, R_k, S_1, \ldots, S_\ell$, where $R_1, \ldots, R_k$ each have chromatic number more than $f(\lfloor \omega(G)/2 \rfloor)$, and $S_1, \ldots, S_\ell$ each have chromatic number at most $f(\lfloor \omega(G)/2 \rfloor)$. Let S be the union of the graphs $S_1, \ldots, S_\ell$; thus, $\chi(S) \leq f(\lfloor \omega(G)/2 \rfloor)$. For $1 \leq i \leq k$, let Y_i be the set of vertices in N with a neighbour in $V(R_i)$, and let $Y = Y_1 \cup \cdots \cup Y_k$.

(1) *For $1 \leq i \leq k$, every vertex in Y_i is complete to R_i.*
Let $y \in Y_i$. Thus, y has a neighbour in $V(R_i)$; suppose that y is mixed on R_i. Since R_i is connected, there is an edge ab of R_i such that y is adjacent to a and not to b. Now v has a neighbour in each component of $G \setminus X$, and since there are at least two such components, there is a vertex $u \in V(G) \setminus (X \cup V(B))$ adjacent to v. But then u-v-y-a-b is an induced copy of P_5, a contradiction. This proves (1).

(2) $\chi(Y) \leq (\omega(G) - 1) f(\lfloor \omega(G)/2 \rfloor)$.
Let $1 \leq i \leq k$. Since $f(\lfloor \omega(G)/2 \rfloor) < \chi(R_i) \leq f(\omega(R_i))$, and f is non-decreasing, it follows that $\omega(R_i) > \omega(G)/2$. By (1), $\omega(G[Y_i])+\omega(R_i) \leq \omega(G)$, and so $\omega(G[Y_i]) < \omega(G)/2$. Consequently $\chi(Y_i) \leq f(\lfloor \omega(G)/2 \rfloor)$, for $1 \leq i \leq k$. Choose $I \subseteq \{1, \ldots, k\}$ minimal such that $\bigcup_{i \in I} Y_i = Y$. From the minimality of I, for each $i \in I$ there exists $y_i \in Y_i$ such that for each $j \in I \setminus \{i\}$ we have that $y_i \notin Y_j$; and so the vertices y_i $(i \in I)$

are all distinct. For each $i \in I$ choose $r_i \in V(R_i)$. For all distinct $i, j \in I$, if y_i, y_j are nonadjacent, then r_i-y_i-v-y_j-r_j is isomorphic to P_5, a contradiction. Hence the vertices y_i $(i \in I)$ are all pairwise adjacent, and adjacent to v; and so $|I| \le \omega(G) - 1$. Thus, $\chi(Y) = \chi(\bigcup_{i \in I} Y_i) \le (\omega(G) - 1) f(\lfloor \omega(G)/2 \rfloor)$. This proves (2).

All the vertices in $N \setminus Y$ are adjacent to v, and so $\omega(G[N \setminus Y]) \le \omega(G) - 1$. Moreover, for $1 \le i \le k$, each vertex of R_i is adjacent to each vertex in Y_i, and $Y_i \ne \emptyset$ since B is connected, and so $\omega(R_i) \le \omega(G) - 1$. Since there are no edges between any two of the graphs $G[N \setminus Y], R_1, \dots, R_k$, their union ($Z$ say) has clique number at most $\omega(G) - 1$ and so has chromatic number at most $f(\omega(G) - 1)$. But $V(B)$ is the union of Y, $V(S)$ and $V(Z)$; and so

$$\chi(B) \le f(\omega(G) - 1) + (\omega(G) - 1) f(\lfloor \omega(G)/2 \rfloor) + f(\lfloor \omega(G)/2 \rfloor).$$

This proves 2.1. □

2.2 *Let $\Omega \ge 1$, and let $f : \mathbb{Z}_+ \to \mathbb{R}_+$ be non-decreasing, satisfying the following:*

- f is a binding function for every P_5-free graph H with $\omega(H) \le \Omega$; and
- $f(w - 1) + (w + 2) f(\lfloor w/2 \rfloor) \le f(w)$ for each integer $w > \Omega$.

Then f is a binding function for every P_5-free graph G.

Proof We prove by induction on $|G|$ that if G is P_5-free then f is a binding function for G. Thus, we may assume that G is P_5-free and f is near-binding for G. If G is not connected, or $\omega(G) \le \Omega$, it follows that f is binding for G, so we assume that G is connected and $\omega(G) > \Omega$. Let us write $w = \omega(G)$ and $m = \lfloor w/2 \rfloor$. If $\chi(G) \le f(w)$ then f is a binding function for G, so we assume, for a contradiction, that:

(1) $\chi(G) > f(w - 1) + (w + 2) f(m)$.

We deduce that:

(2) *Every cutset X of G satisfies $\chi(X) > 2f(m)$.*
If some cutset X satisfies $\chi(X) \le 2f(m)$, then since $\chi(G \setminus X) \le f(w - 1) + wf(m)$ by 2.1, it follows that $\chi(G) \le f(w - 1) + (w + 2) f(m)$, contrary to (1). This proves (2).

(3) *If P, Q are cliques of G, both of cardinality at least $w/2$, then $G[P \cup Q]$ is connected.*
Suppose not; then there is a minimal subset $X \subseteq V(G) \setminus (P \cup Q)$ such that P, Q are subsets of different components (A, B say) of $G \setminus X$. From the minimality of X, every vertex $x \in X$ has a neighbour in $V(A)$ and a neighbour in $V(B)$. If x is mixed on A and mixed on B, then since A is connected, there is an edge $a_1 a_2$ of A such that x is adjacent to a_1 and not to a_2; and similarly there is an edge $b_1 b_2$ of B with x adjacent to b_1 and not to b_2. But then a_2-a_1-x-b_1-b_2 is an induced copy of P_5, a contradiction; so every $x \in X$ is complete to at least one of A, B. The set of vertices in X complete to A is also complete to P, and hence has clique number at most m, and hence has chromatic number at most $f(m)$; and the same for B. Thus, $\chi(X) \le 2f(m)$, contrary to (2). This proves (3).

If $v \in V(G)$, we denote its set of neighbours by $N(v)$, or $N_G(v)$. Let $a \in V(G)$, and let B be a component of $G\backslash(N(a)\cup\{a\})$; we will show that $\chi(B) \leq (w-m+2)f(m)$.

A subset Y of $V(B)$ is a *joint* of B if there is a component C of $B\backslash Y$ such that $\chi(C) > f(m)$ and Y is complete to C. If $\emptyset$ is not a joint of B then $\chi(B) < f(m)$ and the claim holds, so we may assume that $\emptyset$ is a joint of B; let Y be a joint of B chosen with Y maximal, and let C be a component of $B \setminus Y$ such that $\chi(C) > f(m)$ and Y is complete to C.

(4) *If $v \in N(a)$ has a neighbour in $V(C)$, then $\chi(V(C) \setminus N(v)) \leq f(m)$.*
Let $N_C(v)$ be the set of neighbours of v in $V(C)$, and $M = V(C)\setminus N_C(v)$; and suppose that $\chi(M) > f(m)$. Let C' be a component of $G[M]$ with $\chi(C') > f(m)$, and let Z be the set of vertices in $N_C(v)$ that have a neighbour in $V(C')$. Thus, $Z \neq \emptyset$, since $N_C(v), V(C') \neq \emptyset$ and C is connected. If some $z \in Z$ is mixed on C', let p_1p_2 be an edge of C' such that z is adjacent to p_1 and not to p_2; then a-v-z-p_1-p_2 is an induced copy of P_5, a contradiction. So every vertex in Z is complete to $V(C')$; but also every vertex in Y is complete to $V(C)$ and hence to $V(C')$, and so $Y \cup Z$ is a joint of B, contrary to the maximality of Y. This proves (4).

(5) $\chi(Y) \leq f(m)$ and $\chi(C) \leq (w-m+1)f(m)$.
Let X be the set of vertices in $N(a)$ that have a neighbour in $V(C)$. Since C is a component of $B \setminus Y$ and hence a component of $G\backslash(X \cup Y)$, and a belongs to a different component of $G \setminus (X \cup Y)$, it follows that $X \cup Y$ is a cutset of G. By (2), $\chi(X \cup Y) > 2f(m)$. Since $\omega(C) \geq m+1$ (because $\chi(C) > f(m)$, and f is near-binding for G) and every vertex in Y is complete to $V(C)$, it follows that $\omega(G[Y]) \leq w-m-1 \leq m$, and so has chromatic number at most $f(m)$ as claimed; and so $\chi(X) > f(m)$. Consequently there is a clique $P \subseteq X$ with cardinality $w-m$. The subgraph induced on the set of vertices of C complete to P has clique number at most m, and so has chromatic number at most $f(m)$; and for each $v \in P$, the set of vertices of C nonadjacent to v has chromatic number at most $f(m)$ by (4). Thus, $\chi(C) \leq (|P|+1)f(m) = (w-m+1)f(m)$. This proves (5).

(6) $\chi(B) \leq (w-m+2)f(m)$.
By (3), every clique contained in $V(B)\backslash(V(C) \cup Y)$ has cardinality less than $w/2$ (because it is anticomplete to the largest clique of C) and so

$$\chi(B \setminus (V(C) \cup Y)) \leq f(m);$$

and hence $\chi(B \setminus Y) \leq (w-m+1)f(m)$ by (5), since there are no edges between C and $V(B)\setminus(V(C)\cup Y)$. But $\chi(Y) \leq f(m)$ by (5), and so $\chi(B) \leq (w-m+2)f(m)$. This proves (6).

By (6), $G \setminus N(a)$ has chromatic number at most $(w-m+2)f(m)$. But $G[N(a)]$ has clique number at most $w-1$ and so chromatic number at most $f(w-1)$; and so $\chi(G) \leq f(w-1)+(w-m+2)f(m)$, contrary to (1). This proves 2.2. □

Now we deduce 1.5, which we restate:

2.3 *If G is P_5-free and $\omega(G) \geq 3$ then $\chi(G) \leq \omega(G)^{\log_2(\omega(G))}$.*

Proof Define $f(0) = 0$, $f(1) = 1$, $f(2) = 3$, and $f(x) = x^{\log_2(x)}$ for every real number $x \geq 3$. Let G be P_5-free. If $\omega(G) \leq 2$ then $\chi(G) \leq 3 = f(2)$, by a result of Sumner [25]; if $\omega(G) = 3$ then $\chi(G) \leq 5 \leq f(3)$, by an application of the result 1.4 of Esperet, Lemoine, Maffray, and Morel [9]; and if $\omega(G) = 4$ then $\chi(G) \leq 15 \leq f(4)$, by another application of 1.4. Consequently every P_5-free graph G with clique number at most four has chromatic number at most $f(\omega(G))$.

We claim that

$$f(x-1) + (x+2)f(\lfloor x/2 \rfloor) \leq f(x)$$

for each integer $x > 4$. If that is true, then by 2.2 with $\Omega = 4$, we deduce that $\chi(G) \leq f(\omega(G))$ for every P_5-free graph G, and so 1.5 holds. Thus, it remains to show that

$$f(x-1) + (x+2)f(\lfloor x/2 \rfloor) \leq f(x)$$

for each integer $x > 4$. This can be verified by direct calculation when $x = 5$, so we may assume that $x \geq 6$.

The derivative of $f(x)/x^4$ is

$$(2\log_2(x) - 4)x^{\log_2(x)-5},$$

and so is nonnegative for $x \geq 4$. Consequently

$$\frac{f(x-1)}{(x-1)^4} \leq \frac{f(x)}{x^4}$$

for $x \geq 5$. Since $x^2(x^2 - 2x - 4) \geq (x-1)^4$ when $x \geq 5$, it follows that

$$\frac{f(x-1)}{x^2 - 2x - 4} \leq \frac{f(x)}{x^2},$$

that is,

$$f(x-1) + \frac{2x+4}{x^2} f(x) \leq f(x),$$

when $x \geq 5$. But when $x \geq 6$ (so that $f(x/2)$ is defined and the first equality below holds), we have

$$f(\lfloor x/2 \rfloor) \leq f(x/2) = (x/2)^{\log_2(x/2)} = (x/2)^{\log_2(x)-1} = (2/x)(x/2)^{\log_2(x)} = (2/x^2)f(x),$$

and so

$$f(x-1) + (x+2)f(\lfloor x/2 \rfloor) \leq f(x)$$

when $x \geq 6$. This proves 2.3. □

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