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A combinatorial proof of a sumset conjecture of Furstenberg

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Abstract

We give a new proof of a sumset conjecture of Furstenberg that was first proved by Hochman and Shmerkin in 2012: if $\log r / \log s$ is irrational and X and Y are $\times r$ - and $\times s$-invariant subsets of $[0, 1]$, respectively, then $\dim_H(X + Y) = \min(1, \dim_H X + \dim_H Y)$. Our main result yields information on the size of the sumset $\lambda X + \eta Y$ uniformly across a compact set of parameters at fixed scales. The proof is combinatorial and avoids the machinery of local entropy averages and CP-processes, relying instead on a quantitative, discrete Marstrand projection theorem and a subtree regularity theorem that may be of independent interest.

1 Introduction

Given $r \in \mathbb{N}$, a set $X \subseteq [0, 1]$ is $\times r$-*invariant* if it is closed and $T_r X \subseteq X$, where $T_r \colon [0, 1] \to [0, 1]$ is the map $x \mapsto rx \pmod 1$. In the late 1960's, Furstenberg conjectured[1] that if r and s are multiplicatively independent positive integers (that is,

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[1] To the authors' knowledge, the sumset conjecture (1.1) does not appear in print but was known to have originated with Furstenberg. The intersection conjecture (1.2), on the other hand, is one of several conjectures stated in [9].

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$\log r / \log s$ is irrational) and X and Y are $\times r$- and $\times s$-invariant, respectively, then

$$\dim_H (X + Y) = \min (1, \dim_H X + \dim_H Y), \text{ and} \tag{1.1}$$

$$\dim_H (X \cap Y) \leq \max (0, \dim_H X + \dim_H Y - 1). \tag{1.2}$$

The sumset conjecture (1.1) was resolved by Hochman and Shmerkin [12], who proved a more general result concerning the dimension of sums of invariant measures. It also follows by more recent results of Shmerkin [23] and Wu [24], who independently resolved a generalization of the intersection conjecture (1.2). We give a more detailed account of this recent history later in the introduction.

The purpose of this article is to give a new, combinatorial proof of Furstenberg's sumset conjecture (1.1). Denoting the unlimited γ-Hausdorff content by $\mathcal{H}^\gamma_{>0}$ (see Definition 2.2), our main theorem is as follows.

Theorem A *Let r and s be multiplicatively independent positive integers, and let $X, Y \subseteq [0, 1]$ be $\times r$- and $\times s$-invariant sets, respectively. Define $\overline{\gamma} = \min (\dim_H X + \dim_H Y, 1)$. For all compact $I \subseteq \mathbb{R} \setminus \{0\}$ and all $\gamma < \overline{\gamma}$,*

$$\inf_{\lambda, \eta \in I} \mathcal{H}^\gamma_{>0} (\lambda X + \eta Y) > 0. \tag{1.3}$$

Beyond implying (1.1), Theorem A gives finer quantitative information on the size of the sumset $\lambda X + \eta Y$ in terms of the unlimited γ-Hausdorff content uniformly over the parameters λ and η. The uniformity in the result, which does not appear to follow from [12], has found use in recent applications concerning digit problems; see, for example, [11] and [3]. See Remark 5.1 below for some further discussion on this uniformity.

Our proof of (1.1) differs from other proofs in the literature in that it completely avoids the machinery of CP-processes and local entropy averages. Instead, it features an elementary, combinatorial approach that builds on the work of Peres and Shmerkin in [22]. Important ingredients in the proof include a quantitative discrete Marstrand theorem (Theorem 3.2) and a subtree regularity theorem (Theorem 4.7), both of which may be of independent interest.

1.1 History and Context

In a highly influential work in geometric measure theory, Marstrand [18] related the Hausdorff dimension of a Borel set $E \subseteq \mathbb{R}^2$, $\dim_H E$, to the Hausdorff dimension of its images under orthogonal projections and its intersections with lines. More specifically, he showed that for almost every line $L \subseteq \mathbb{R}^2$, $\dim_H(\pi_L E) = \min (1, \dim_H E)$, where π_L is the orthogonal projection $\mathbb{R}^2 \to L$, and that for almost every line L intersecting E, $\dim_H(E \cap L) = \max (0, \dim_H E - 1)$.[2]

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[2] Marstrand considered sets of positive, finite Hausdorff measure. His results imply the ones stated here when combined with the fact that any Borel set of Hausdorff dimension $\gamma > 0$ can be approximated from above and below by sets with positive, finite $(\gamma \pm \epsilon)$-Hausdorff measure. For a more modern take on these theorems, see Corollary 9.4 and Theorem 10.10 in [19].

Images of a Cartesian product $X \times Y$ under orthogonal projections are, up to affine transformations which preserve dimension, sumsets of the form $\lambda X + \eta Y$, while intersections of $X \times Y$ with lines are affinely equivalent to sets of the form $\lambda X \cap (\eta Y + \sigma)$. Thus, Marstrand's theorems in the case $E = X \times Y$ imply the following.

Theorem 1.1 *([18, Theorems II and III]) Let X and Y be Borel subsets of* [0, 1]. *For Lebesgue-a.e.* $\lambda, \eta, \sigma \in \mathbb{R}$,

$$\dim_H \left(\lambda X + \eta Y \right) = \min \left(\dim_H (X \times Y), \ 1 \right), \ and \tag{1.4}$$

$$\dim_H \left(\lambda X \cap (\eta Y + \sigma) \right) \leq \max \left(0, \ \dim_H (X \times Y) - 1 \right). \tag{1.5}$$

Improving (1.4) and (1.5) by replacing the Lebesgue-typical projection or intersection of $X \times Y$ with a concrete projection or intersection is not possible in general [13] but can be done in special cases when the sets X and Y are structured. Furstenberg's conjectures (1.1) and (1.2) can be contextualized as such: when r and s are multiplicatively independent and $X \times Y$ is the product of a $\times r$- and a $\times s$-invariant set, results for the Lebesgue-typical projection and intersection should hold for the orthogonal projection to, and the intersection with, the line $x = y$. These conjectures join a host of results and conjectures by Furstenberg and others that aim to capture the independence between base-r and base-s structure when r and s are multiplicatively independent.

Conjectures (1.1) and (1.2) were recently resolved, both proven in more general forms. In the following theorem, we have combined special cases of the results by Hochman and Shmerkin [12], Shmerkin [23], and Wu [24] that are most relevant to this work. Note that $\overline{\dim}_M$ denotes the upper Minkowski dimension (see Definition 2.1).

Theorem 1.2 *([12] and [23, 24]) Let r and s be multiplicatively independent positive integers, and let* $X, Y \subseteq [0, 1]$ *be* $\times r$- *and* $\times s$-*invariant sets, respectively. For all* $\lambda, \eta \in \mathbb{R} \backslash \{0\}$ *and all* $\sigma \in \mathbb{R}$,

$$\dim_H \left(\lambda X + \eta Y \right) = \min \left(\dim_H X + \dim_H Y, \ 1 \right), \ and \tag{1.6}$$

$$\overline{\dim}_M \left(\lambda X \cap (\eta Y + \sigma) \right) \leq \max \left(0, \ \dim_H X + \dim_H Y - 1 \right). \tag{1.7}$$

A number of partial results preceded those in Theorem 1.2, both for multiplicatively invariant sets and for attractors of iterated function systems (IFSs). Carlos Moreira [20] considered sumsets of attractors of IFSs with certain irrationality and non-linearity conditions. Peres and Shmerkin [22] proved (1.6) for attractors of IFSs with rationally independent contraction ratios; this resolved (1.6) in the special case that X and Y are restricted digit Cantor sets with respect to multiplicatively independent bases. (This work of Peres and Shmerkin is particularly relevant to the arguments in this paper, as we explain in detail in Sect. 5.1.)

Hochman and Shmerkin [12] developed Furstenberg's CP processes [9] and introduced local entropy averages to prove (1.6) both for invariant sets and measures and for attractors of IFSs satisfying some general minimality conditions. Wu [24] combined the CP process machinery with Sinai's factor theorem from ergodic theory to

resolve (1.7) for invariant sets and attractors of regular, self-similar IFSs. Shmerkin [23] resolved (1.7) utilizing tools primarily from additive combinatorics, proving an inverse theorem for the decay of L^q norms of certain self-similar measures of dynamical origin. Yu [25] and Austin [1] gave dynamical proofs of (1.2), simplifying some aspects of earlier proofs.

The sumset and intersection theorems are closely related: fibers of orthogonal projections are precisely those lines with which intersections are considered. It is not surprising, then, that the intersection theorem can be used to deduce the sumset theorem. For example, if for arbitrary sets $X, Y \subseteq [0, 1]$ we know that for all $\gamma > \max\left(0, \dim_H X + \dim_H Y - 1\right)$, there exists $\delta_0 > 0$, for all $0 < \delta < \delta_0$, and for all balls B of diameter δ,

$$\mathcal{N}\left(X \cap (Y + B), \delta\right) \leq \delta^{-\gamma},$$

then we can deduce that $\dim_H(X+Y) = \min(1, \dim_H(X \times Y))$. This type of uniformity is made explicit in Shmerkin [23] and Yu [25] and may be implicit in the other proofs of the intersection conjecture. It is possible to deduce Theorem A from Shmerkin's main result in [23]; we explain the details in the course of another argument in [11]. Despite the fact that every proof of the intersection conjecture can be counted as a proof of the sumset conjecture, we believe our approach still has merit: it is the most elementary proof to date; it exposes uniformity important in certain number-theoretic applications; and it features tools which may be of independent interest.

Theorem A has a geometric formulation in terms of orthogonal projections; while we will not make particular use of the theorem in this form, it is worth formulating for its historical connection to the topic. Let $\pi_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal projection onto the line that contains the origin and forms an angle θ with the positive x-axis. The proof of the equivalence between Theorem A and Theorem B is standard and not needed in this work, so it is omitted.

Theorem B *Let r and s be multiplicatively independent positive integers, and let $X, Y \subseteq [0, 1]$ be $\times r$- and $\times s$-invariant sets, respectively. Define $\overline{\gamma} = \min\left(\dim_H X + \dim_H Y, 1\right)$. For all compact $I \subseteq (0, \pi)\backslash\{\pi/2\}$ and all $\gamma < \overline{\gamma}$, $\inf_{\theta \in I} \mathcal{H}_{>0}^\gamma\left(\pi_\theta(X \times Y)\right) > 0$.*

1.2 Overview of the Paper

The paper is organized as follows. In Sect. 2, we organize the terminology, notation, and basic facts we need from discrete and continuous fractal geometry, including some properties of $\times r$-invariant subsets of $[0, 1]$ and an equidistribution lemma. Section 3 contains a proof of Theorem 3.2, our discrete Marstrand projection theorem. Section 4 features notation and terminology for trees and the subtree regularity theorem, Theorem 4.7. Finally, we prove Theorem A in Sect. 5.

1.3 Acknowledgements

2 Preliminary Definitions and Results

The positive and non-negative integers are denoted by $\mathbb{N}$ and $\mathbb{N}_0$, respectively. For $x \in \mathbb{R}$, denote the fractional part by $\{x\}$ and the integer part (or floor) by $\lfloor x \rfloor$. The Lebesgue measure on the real line is denoted by Leb. Throughout the paper, $\mathbb{R}^d$ is equipped with the Euclidean norm which we denote by $|\cdot|$. Given two positive-valued functions f and g, we write $f \ll_{a_1,\ldots,a_k} g$ or $g \gg_{a_1,\ldots,a_k} f$ if there exists a constant $K > 0$, depending only on the quantities $a_1, \ldots, a_k$, for which $f(x) \leq Kg(x)$ for all x in the domain common to both f and g. We write $f \asymp_{a_1,\ldots,a_k} g$ if both $f \ll_{a_1,\ldots,a_k} g$ and $f \gg_{a_1,\ldots,a_k} g$.

2.1 Continuous and Discrete Fractal Geometry

In this section, we lay out the notation, tools, and results we need from continuous and discrete fractal geometry. A good general reference for the standard material in this section is [19, Ch. 4]. In the definitions that follow, $\rho, \gamma, c > 0$, $d \in \mathbb{N}$, and $X \subseteq \mathbb{R}^d$ is non-empty.

Definition 2.1 • The set X is *ρ-separated* if for all distinct $x_1, x_2 \in X$, $|x_1 - x_2| \geq \rho$.
• The *metric entropy of X at scale ρ* is

$$\mathcal{N}(X, \rho) = \sup \big\{ |X_0| \,\big|\, X_0 \subseteq X \text{ is } \rho\text{-separated} \big\}.$$

• The *lower Minkowski dimension* of X is

$$\underline{\dim}_{\mathrm{M}} X = \liminf_{\delta \to 0^+} \frac{\log \mathcal{N}(X, \delta)}{\log \delta^{-1}}. \tag{2.1}$$

The *upper Minkowski dimension*, $\overline{\dim}_{\mathrm{M}} X$, is defined analogously with a limit supremum in place of the limit infimum. If $\underline{\dim}_{\mathrm{M}} X = \overline{\dim}_{\mathrm{M}} X$, then this value is the *Minkowski dimension* of X, $\dim_{\mathrm{M}} X$.

It is a well-known fact which we will use without further mention that if $\rho < 1$, then $\underline{\dim}_{\mathrm{M}} X = \liminf_{N \to \infty} \log \mathcal{N}(X, \rho^{-N}) / \log \rho^N$ and $\overline{\dim}_{\mathrm{M}} X = \limsup_{N \to \infty} \log \mathcal{N}(X, \rho^{-N}) / \log \rho^N$.

Definition 2.2 • The *unlimited γ-Hausdorff content*[3] of X is

$$\mathcal{H}_{>0}^{\gamma}(X) = \inf\left\{\sum_{i\in I}\delta_i^{\gamma}\ \middle|\ X \subseteq \bigcup_{i\in I} B_i,\ B_i \text{ open ball of diameter } \delta_i\right\}.$$

Note that when X is compact, the index set I may be taken to be finite.
• The *Hausdorff dimension* of X is

$$\dim_{\mathrm{H}} X = \sup\{\gamma \in \mathbb{R} \mid \mathcal{H}_{>0}^{\gamma}(X) > 0\}$$
$$= \inf\{\gamma \in \mathbb{R} \mid \mathcal{H}_{>0}^{\gamma}(X) = 0\}.$$

In the following definition, we introduce two notions meant to capture the dimensionality of discrete sets.

Definition 2.3 • (cf. [15, Definition 1.2]) The set X is a $(\rho, \gamma)_c$-*set* if it is ρ-separated and for all $\delta \geq \rho$ and all open balls B of diameter δ,

$$|X \cap B| \leq c\left(\frac{\delta}{\rho}\right)^{\gamma}. \tag{2.2}$$

• The *discrete Hausdorff content of X at scale ρ and dimension γ* is

$$\mathcal{H}_{\geq\rho}^{\gamma}(X) = \inf\left\{\sum_{i\in I}\delta_i^{\gamma}\ \middle|\ X \subseteq \bigcup_{i\in I} B_i,\ B_i \text{ open ball of diameter } \delta_i \geq \rho\right\}.$$

Note that when X is compact, the index set I may be taken to be finite.

In the definition of a $(\rho, \gamma)_c$-set, we think of ρ as being positive and close to 0, $\gamma \in [0, d]$ as the "dimension" of the set, and $c > 0$ as an uninteresting parameter that exists only to make our arguments explicit. The inequality in (2.2) guarantees that the points of a $(\rho, \gamma)_c$-set cannot be too concentrated in any ball. It follows from that inequality that the maximum cardinality of a $(\rho, \gamma)_c$ set in $[0, 1]^d$ is on the order of $\rho^{-\gamma}$. A $(\rho, \gamma)_c$-set with cardinality $\gg \rho^{-\gamma}$ can be thought of as a discrete approximation to a set with Hausdorff dimension γ; this is made more precise in Remark 2.5 below and is realized in Lemma 2.13. In fact, if the discrete approximations of a set $X \subseteq \mathbb{R}^d$ at all scales $\rho > 0$ are $(\rho, \gamma)_c$-sets, then the *Assouad dimension* (cf. [7, Section 2.1]) of the set X is at most γ. More precisely, the Assouad dimension of X is the infimum of the set of γ's for which there exists $c > 0$ such that for all $\rho > 0$, the set X rounded to the lattice $\rho\mathbb{Z}^d$ is a $(\rho, \gamma)_c$-set.

The discrete Hausdorff content at scale ρ is a "ρ-resolution" analogue of the unlimited Hausdorff content. The discrete Hausdorff contents of two sets that look the same at scale ρ are approximately equal. The following lemma provides a connection between the discrete and the continuous regimes that will be useful in the proof of Theorem A.

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[3] The unlimited γ-Hausdorff content is sometimes denoted in the literature by $\mathcal{H}_{\infty}^{\gamma}$. We choose to use $\mathcal{H}_{>0}^{\gamma}$ because it is more consistent with the notation introduced in Definition 2.3.

Lemma 2.4 *Let $X \subseteq \mathbb{R}^d$ be compact. For all $\gamma \geq 0$,*

$$\lim_{\rho \to 0^+} \mathcal{H}^{\gamma}_{\geq \rho}(X) = \mathcal{H}^{\gamma}_{>0}(X). \tag{2.3}$$

Consequently, if $\lim_{\rho \to 0} \mathcal{H}^{\gamma}_{\geq \rho}(X) > 0$, then $\dim_H X \geq \gamma$.

Proof Let $\gamma \geq 0$. The limit in (2.3) exists because the function $\rho \mapsto \mathcal{H}^{\gamma}_{\geq \rho}(X)$ is non-increasing as ρ tends to 0^+ and is bounded from below by $\mathcal{H}^{\gamma}_{>0}(X)$. Equality in the limit follows from the fact that X is compact, allowing for the index set in the definition of $\mathcal{H}^{\gamma}_{>0}(X)$ to be taken to be finite. If $\lim_{\rho \to 0} \mathcal{H}^{\gamma}_{\geq \rho}(X) > 0$, then $\mathcal{H}^{\gamma}_{>0}(X) > 0$, and it follows from the definition of the Hausdorff dimension that $\dim_H X \geq \gamma$. □

Remark 2.5 It would be natural to define the *metric entropy at scale ρ and dimension γ* of the set X as

$$\mathcal{N}(X, (\rho, \gamma)_c) = \sup\left\{|X_0| \mid X_0 \subseteq X \text{ is a } (\rho, \gamma)_c\text{-set}\right\}.$$

Using a max flow, min cut argument similar to the one in [2, Ch. 3], it can be shown that for X compact,

$$\frac{\mathcal{N}\big(X, (\rho, \gamma)_c\big)}{\rho^{-\gamma}} \asymp_{c,d} \mathcal{H}^{\gamma}_{\geq \rho}(X). \tag{2.4}$$

Thus, $(\rho, \gamma)_c$-sets of cardinality $\gg \rho^{-\gamma}$ can be thought of as discrete fractal sets of dimension γ. We will not need (2.4); the interested reader can consult [6, Prop. A1] for some details.

The following is a discrete version of the well-known mass distribution principle, cf. [2, Lemma 1.2.8].

Lemma 2.6 *Let μ be a Borel probability measure on $\mathbb{R}^d$, and let $\rho, \kappa > 0$. If for all balls B of diameter $\delta \geq \rho$, $\mu(B) \leq \kappa \delta^{\gamma}$, then the support $\operatorname{supp} \mu$ of μ satisfies $\mathcal{H}^{\gamma}_{\geq \rho}(\operatorname{supp} \mu) \geq \kappa^{-1}$.*

Proof Let $\epsilon > 0$, and let $\{B_i\}_{i \in I}$ be a cover of $\operatorname{supp} \mu$ with ball B_i of diameter $\delta_i \geq \rho$ and with $\sum_{i \in I} \delta_i^{\gamma} \leq \mathcal{H}^{\gamma}_{\geq \rho}(\operatorname{supp} \mu) + \epsilon$. Then the conclusion follows because $\epsilon > 0$ was arbitrary. □

Denote by $[X]_{\delta}$ the closed δ-neighborhood of X:

$$[X]_{\delta} := \left\{z \in [0, 1] \mid \exists x \in X \text{ with } |z - x| \leq \epsilon\right\}.$$

Lemma 2.7 *Let $a \geq 1$ and $\rho > 0$. If $X, Y \subseteq \mathbb{R}$ are compact and $X \subseteq [Y]_{a\rho}$, then*

$$\mathcal{H}^{\gamma}_{\geq \rho}(X) \ll_a \mathcal{H}^{\gamma}_{\geq \rho}(Y).$$

Proof Let $\{B_i\}_{i \in I}$ be a collection of open balls covering Y and where B_i has diameter $r_i \geq \rho$ and $\sum_{i \in I} r_i^\gamma < 2\mathcal{H}_{\geq \rho}^\gamma(Y)$. Since $X \subseteq [Y]_{a\rho}$, it follows that $X \subseteq \bigcup_{i \in I} [B_i]_{a\rho}$ and $[B_i]_{a\rho}$ is a ball of diameter $r_i + 2a\rho \leq (2a+1)r_i$. Therefore $\mathcal{H}_{\geq \rho}^\gamma(X) \leq \sum_{i \in I} ((2a+1)r_i)^\gamma \leq 2(2a+1)\mathcal{H}_{\geq \rho}^\gamma(Y)$. □

2.2 Multiplicatively Invariant Subsets of the Reals and Their Finite Approximations

In this section, we record some basic facts about multiplicatively invariant subsets of [0, 1] and their discrete approximations.

Definition 2.8 Let $r \in \mathbb{N}$ and $X \subseteq [0, 1]$.

- The map $T_r : [0, 1] \to [0, 1]$ is defined by $T_r x = \{rx\}$, where $\{\cdot\}$ denotes the fractional part of a real number.
- The set X is $\times r$-*invariant* if it is closed and $T_r X \subseteq X$.

The Hausdorff and Minkowski dimensions of a multiplicatively invariant set coincide. As a consequence of this regularity, the Hausdorff dimension of products of such sets is also well-behaved. We record these facts here for later use.

Theorem 2.9 *([8, Proposition III.1]) If $X \subseteq [0, 1]$ is $\times r$-invariant, then $\dim_H X = \dim_M X$.*

Lemma 2.10 *If $X, Y \subseteq [0, 1]$ are $\times r$, $\times s$-invariant, respectively, then $\dim_H(X \times Y) = \dim_H X + \dim_H Y$.*

Proof This follows immediately from [19, Corollary 8.11] and the fact that $\dim_H X = \overline{\dim}_M X$. □

Since we will work almost exclusively with finite approximations to multiplicatively invariant sets, we establish some useful notation.

Definition 2.11 Let $X \subseteq [0, 1]$ be $\times r$-invariant. For $n \in \mathbb{N}_0$, the set X_n denotes the set X rounded down to the lattice $r^{-n}\mathbb{Z}$. That is, the point i/r^n is an element of X_n if and only if $X \cap [i/r^n, (i+1)/r^n)$ is non-empty.

The next results show that finite approximations to a multiplicatively invariant set are multiplicatively invariant and are discrete models of fractal sets as captured by Definition 2.3.

Lemma 2.12 *Let $X \subseteq [0, 1]$ be $\times r$-invariant. For all $n \in \mathbb{N}$, $T_r X_n \subseteq X_{n-1}$.*

Proof Let $n \in \mathbb{N}$, and let $i/r^n \in X_n$ with $i \in \{0, \ldots, r^n - 1\}$. Write $i = i_0 + d_{n-1} r^{n-1}$ with $i_0 \in \{0, \ldots, r^{n-1} - 1\}$ and $d_{n-1} \in \{0, \ldots, r - 1\}$. Note that $T_r(i/r^n) = i_0/r^{n-1}$ and $T_r((i+1)/r^n) = (i_0+1)/r^{n-1}$. We must show that $i_0/r^{n-1} \in X_{n-1}$.

Since $i/r^n \in X_n$, there exists $x \in X \cap [i/r^n, (i+1)/r^n)$. Since $T_r x \in X$, $T_r x \in X \cap [i_0/r^{n-1}, (i_0+1)/r^{n-1})$. It follows by the definition of X_{n-1} that $i_0/r^{n-1} \in X_{n-1}$, as was to be shown. □

Lemma 2.13 *Let $r \geq 2$, and let $X \subseteq [0, 1]$ be a $\times r$-invariant set. For all $\gamma > \dim_H X$, there exists $c > 0$ such that for all sufficiently large $N \in \mathbb{N}$, the set X_N is a $(r^{-N}, \gamma)_c$-set.*

Proof Let $\gamma > \dim_H X$. Because $\gamma > \overline{\dim}_M X$ (cf. Theorem 2.9), there exists $c_0 > 0$ such that for all $N \in \mathbb{N}$,

$$|X_N| \leq c_0 r^{N\gamma}. \tag{2.5}$$

using the fact that X is $\times r$-invariant, that T_r^n is injective on half-open intervals of length r^{-n}, Lemma 2.12, and the bound in (2.5), for all $0 \leq n \leq N$ and for all $i \in \{0, \dots, r^n - 1\}$,

$$\left| X_N \cap \left[\frac{i}{r^n}, \frac{i+1}{r^n} \right) \right| \leq |T_r^n X_N| \leq |X_{N-n}| \leq c_0 r^{(N-n)\gamma}. \tag{2.6}$$

Put $c = 2r^\gamma c_0$. To show that X_N is a $(r^{-N}, \gamma)_c$-set, let $B \subseteq \mathbb{R}$ be a ball of diameter $\delta \geq r^{-N}$. Put $n = \lfloor -\log_r \delta \rfloor$ so that $r^{-(n+1)} < \delta \leq r^{-n}$, and note that a union of two intervals of length r^n of the form above suffice to cover B. Therefore,

$$\left| X_N \cap B \right| \leq 2c_0 r^{(N-n)\gamma} \leq c \left(\frac{\delta}{r^{-N}} \right)^\gamma,$$

as was to be shown. □

Lemma 2.14 *Let $r \geq 2$, and let $X \subseteq [0, 1]$ be non-empty and $\times r$-invariant. For all $\gamma > \dim_H X$ and all sufficiently large $N \in \mathbb{N}$,*

$$r^{N \dim_H X} \leq |X_N| \leq r^{N\gamma}.$$

Proof Let $\gamma > \dim_H X$. Because $\gamma > \overline{\dim}_M X$ (cf. Theorem 2.9), we have that $|X_N| \leq r^{N\gamma}$ for all but finitely many $N \in \mathbb{N}$. It remains to show the lower bound.

Let $M, N \in \mathbb{N}$. Since $\left[\frac{i}{r^N}, \frac{i+1}{r^N} \right), i = 0, 1, \dots, r^N - 1$, forms a partition of $[0, 1)$, we have

$$\left| X_{N+M} \right| = \sum_{i=0}^{r^N - 1} \left| X_{N+M} \cap \left[\frac{i}{r^N}, \frac{i+1}{r^N} \right) \right|.$$

Note that $X_{N+M} \cap \left[\frac{i}{r^N}, \frac{i+1}{r^N} \right)$ is non-empty if and only if $X \cap \left[\frac{i}{r^N}, \frac{i+1}{r^N} \right)$ is non-empty, which happens exactly when $i/r^N \in X_N$. Hence

$$\left| X_{N+M} \right| = \sum_{i/r^N \in X_N} \left| X_{N+M} \cap \left[\frac{i}{r^N}, \frac{i+1}{r^N} \right) \right|. \tag{2.7}$$

It follows from (2.6) that $\left| X_{N+M} \cap \left[\frac{i}{r^N}, \frac{i+1}{r^N} \right) \right| \leq |X_M|$, which combined with (2.7) shows that $|X_{N+M}| \leq |X_N||X_M|$. In view of this sub-additive property, it follows from Fekete's Lemma that the sequence $|X_N|^{1/N}$ converges to its infimum, i.e.,

$$\lim_{N \to \infty} |X_N|^{1/N} = \inf_{N \in \mathbb{N}} |X_N|^{1/N}.$$

It follows from $\dim_H X = \dim_M X$ that $r^{\dim_H X} = \lim_{N \to \infty} |X_N|^{1/N}$. Therefore, $r^{\dim_H X} = \inf_{N \in \mathbb{N}} |X_N|^{1/N}$, and hence $r^{N \dim_H X} \leq |X_N|$ for all $N \in \mathbb{N}$, as desired. $\square$

The following notation, borrowed from [22], allows us to easily compare powers of r and powers of s. This is useful when considering the finite approximations to the Cartesian product of a $\times r$- and a $\times s$-invariant set.

Definition 2.15 For $n \in \mathbb{N}_0$, we set $n' = \lfloor n \log r / \log s \rfloor$ to be the greatest integer so that $s^{n'} \leq r^n$. (The bases r and s do not appear in this notation but should always be clear from context.)

Recall from Definition 2.11 that X_N is the set X rounded to the lattice $r^{-N}\mathbb{Z}$. Extending this notation to Y, the set Y_N is the set Y rounded to the lattice $s^{-N}\mathbb{Z}$. Since r^{-N} is approximately equal to $s^{-N'}$ (where N' is as defined in Definition 2.15), the set $Y_{N'}$ is the discrete approximation to Y that is on a scale closest to the scale of X_N. Therefore, the sets X_N and $Y_{N'}$ will always be considered in the same context, as opposed to the sets X_N and Y_N.

Corollary 2.16 Let $2 \leq r < s$, let $X, Y \subseteq [0, 1]$ be non-empty $\times r$- and $\times s$-invariant sets. For all $\xi > \dim_H X + \dim_H Y$, there exist $c_1, c_2 > 0$ and $M_0 \in \mathbb{N}$ such that for all $N \geq M_0$, the sets $X_N \times Y_{N'}$ and $X_N \times Y_{N'+1}$ are $(c_1 r^{-N}, \xi)_{c_2}$-sets satisfying $r^{N(\dim_H X + \dim_H Y)} \leq |X_N \times Y_{N'}| \leq r^{N\xi}$ and $r^{N(\dim_H X + \dim_H Y)} \leq |X_N \times Y_{N'+1}| \leq r^{N\xi}$.

Proof Let $\xi > \dim_H X + \dim_H Y$. Let $g > \dim_H X$ and $h > \dim_H Y$ be such that

$$\dim_H(X \times Y) < g + h < \xi.$$

Applying Lemma 2.13 and Lemma 2.14, there exist $c, d > 0$ such that for sufficiently large $N \in \mathbb{N}$, the set X_N is a $(r^{-N}, g)_c$-set satisfying $r^{N \dim_H X} \leq |X_N| \leq r^{Ng}$ and $Y_{N'}$ is a $(s^{-N'}, h)_d$-set satisfying $s^{N' \dim_H Y} \leq |Y_{N'}| \leq s^{N'h}$. Since $r^{N(\dim_H X + \dim_H Y)} = r^{N \dim_H X} r^{N \dim_H Y} \geq r^{N \dim_H X} s^{N' \dim_H Y}$, $|X_N \times Y_N| \leq |X_N \times Y_{N+1}|$ and $g + h < \xi$, it follows that for sufficiently large $N \in \mathbb{N}$, one has $r^{N(\dim_H X + \dim_H Y)} \leq |X_N \times Y_{N'}| \leq r^{N\xi}$ and $r^{N(\dim_H X + \dim_H Y)} \leq |X_N \times Y_{N'+1}| \leq r^{N\xi}$.

Set $c_1 = s^{-1}$ and $c_2 = s^g cd$. Since $s^{N'} < r^N < s^{N'+1}$, the sets $X_N \times Y_{N'}$ and $X_N \times Y_{N'+1}$ are $c_1 r^{-N}$-separated. Since X_N is a $(r^{-N}, g)_c$-set, it is a $(c_1 r^{-N}, g)_{s^g c}$-set.[4] Let $B \subseteq \mathbb{R}^2$ be a ball of diameter $\delta \geq c_1 r^{-N}$. Note that

$$\left| (X_N \times Y_{N'}) \cap B \right| \leq s^g c \left(\frac{\delta}{c_1 r^{-N}} \right)^g d \left(\frac{\delta}{s^{-N'}} \right)^h$$

---

[4] More generally, if $0 < c_1 < 1$, then every $(\delta, \gamma)_c$-set is a $(c_1 \delta, \gamma)_{c_1^{-\gamma} c}$-set. This is a quick exercise left to the reader.

$$\leq s^g c \left(\frac{\delta}{c_1 r^{-N}}\right)^g dc_1^h \left(\frac{\delta}{c_1 r^{-N}}\right)^h \leq c_2 \left(\frac{\delta}{c_1 r^{-N}}\right)^\xi,$$

which shows that the set $X_N \times Y_{N'}$ is a $(c_1 r^{-N}, \xi)_{c_2}$-set. By a similar calculation,

$$\left|(X_N \times Y_{N'+1}) \cap B\right| \leq s^g c \left(\frac{\delta}{c_1 r^{-N}}\right)^g d \left(\frac{\delta}{s^{-(N'+1)}}\right)^h$$

$$\leq s^g c \left(\frac{\delta}{c_1 r^{-N}}\right)^g d \left(\frac{\delta}{c_1 r^{-N}}\right)^h \leq c_2 \left(\frac{\delta}{c_1 r^{-N}}\right)^\xi,$$

which shows that the set $X_N \times Y_{N'+1}$ is a $(c_1 r^{-N}, \xi)_{c_2}$-set. □

2.3 A Quantitative Equidistribution Lemma

The main result in this short section, Lemma 2.18, gives a lower bound on the number of visits of an equidistributed sequence to a set as a function only of the measure and topological complexity of the set's complement. This result is certainly not new; we state it explicitly here for convenience in a way that highlights the uniformity in the quantifiers.

For $U \in \mathbb{N}$, denote by $\mathcal{I}_U$ the collection of those subsets of $[0, 1)$ that are a union of no more than U disjoint intervals of the form $[a, b)$.

Lemma 2.17 *For any uniformly distributed sequence* $(x_n)_{n \in \mathbb{N}_0} \subseteq [0, 1)$, $U \in \mathbb{N}$, *and* $\epsilon > 0$, *there exists* $N_0 \in \mathbb{N}$ *such that for all* $N \geq N_0$ *and all* $B \in \mathcal{I}_U$,

$$\frac{1}{N}\left|\{0 \leq n \leq N - 1 \mid x_n \in B\}\right| \leq \mathrm{Leb}(B) + \epsilon.$$

Proof Let $(x_n)_{n \in \mathbb{N}_0} \subseteq [0, 1)$ be uniformly distributed, $U \in \mathbb{N}$, and $\epsilon > 0$. The discrepancy of $(x_n)_{n=0}^{N-1}$ (cf. [14, Ch. 2, Def. 1.1]) is

$$D_N = \sup_I \left|\frac{\{0 \leq n \leq N - 1 \mid x_n \in I\}}{N} - \mathrm{Leb}(I)\right|,$$

where the supremum is taken over all half-open intervals I in $[0, 1)$. Because $(x_n)_n$ is uniformly distributed, $D_N \to 0$ as $N \to \infty$ (cf. [14, Ch. 2, Thm. 1]). By the definition of discrepancy, for any half-open interval $I \subseteq [0, 1)$,

$$\frac{1}{N}\left|\{0 \leq n \leq N - 1 \mid x_n \in I\}\right| \leq \mathrm{Leb}(I) + D_N.$$

It follows that for every $B \in \mathcal{I}_U$,

$$\frac{1}{N}\left|\{0 \leq n \leq N - 1 \mid x_n \in B\}\right| \leq \mathrm{Leb}(B) + U D_N.$$

Let $N_0 \in \mathbb{N}$ be large enough so that for all $N \geq N_0$, $U D_N \leq \epsilon$. The conclusion follows. $\qquad\square$

Lemma 2.18 *Let $\beta > 0$. For any uniformly distributed sequence $(x_n)_{n\in\mathbb{N}_0} \subseteq [0, \beta)$ with respect to the Lebesgue measure, $U \in \mathbb{N}$, and $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ and all $J \subseteq [0, \beta)$ whose complement is covered by a union of no more than U many disjoint, half-open intervals of total Lebesgue measure less than $\epsilon\beta/2$,*

$$\frac{1}{N}\big|\{0 \leq n \leq N - 1 \mid x_n \in J\}\big| \geq 1 - \epsilon.$$

Proof Let $(x_n)_{n\in\mathbb{N}_0} \subseteq [0, \beta)$ be uniformly distributed, $U \in \mathbb{N}$, and $\epsilon > 0$. Let N_0 be from Lemma 2.17 with $(x_n/\beta)_{n\in\mathbb{N}_0}$, U, and $\epsilon/2$.

Let $N \geq N_0$ and $J \subseteq [0, \beta)$. Put $B = [0, \beta)\backslash J$, and note that by assumption, $B/\beta \in \mathcal{I}_U$ and $\mathrm{Leb}(B/\beta) < \epsilon/2$. It follows from Lemma 2.17 that

$$\frac{1}{N}\big|\{0 \leq n \leq N - 1 \mid x_n/\beta \in B/\beta\}\big| < \epsilon.$$

Therefore,

$$\frac{1}{N}\big|\{0 \leq n \leq N - 1 \mid x_n \in J\}\big| \geq 1 - \epsilon,$$

as was to be shown. $\qquad\square$

3 A Discrete Marstrand Projection Theorem

In this section, we prove a discrete analogue of Marstrand's projection theorem from geometric measure theory. The theorem – stated for sumsets in the introduction as Theorem 1.1 – says that for every Borel set $A \subseteq [0, 1]^2$, for Lebesgue-a.e. $\theta \in [0, \pi)$, $\dim_{\mathrm{H}} \pi_\theta A = \min(1, \dim_{\mathrm{H}} A)$, where $\pi_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is the orthogonal projection onto ℓ_θ, the line that contains the origin and forms an angle θ with the positive x-axis. Marstrand's theorem and its relatives have enjoyed much recent attention: we refer the interested reader to the survey [5] and to the end of this section where we put Theorem 3.2 into more context.

The key idea behind Marstrand's theorem is that of "geometric transversality" and is captured in the following lemma. The proof follows from a simple geometric argument and is left to the reader. An immediate consequence of the lemma is that there are not many projections which map two distant points close together.

Lemma 3.1 *For all nonzero $x \in \mathbb{R}^2$ and all $\rho > 0$, the set of angles $\theta \in [0, \pi)$ for which $|\pi_\theta x| \leq \rho$ is contained in at most two balls of diameter $\ll \rho|x|^{-1}$.*

The results in this section add to a number of other discrete Marstrand-type theorems in the recent literature: [17, Lemma 5.2], [16, Prop. 3.2], [10, Lemma 3.8], [22,

Prop. 7], [21, Prop. 4.10] to name a few. Let us highlight some distinguishing features of LemmaA 3.1 and Theorem 3.2 that play an important role in this work. Analogues of Lemma 3.1 more commonly found in the literature, such as the one in [19, Lemma 3.11], bound the *measure* of the set of projections which map x close to 0. The result in Lemma 3.1 uses coverings to capture topological information on the set of projections. This information is carried into Theorem 3.2 and is important in the application to Theorem A. Another useful feature of Theorem 3.2 is the allowance of a subset A' in (3.1); this will allow us to treat sets in Theorem A that exhibit multiplicative invariance without necessarily being self-similar.

3.1 A Discrete Projection Theorem

Our discrete analogue of Marstrand's theorem, Theorem 3.2, reaches a conclusion similar to that of Marstrand's by quantifying the size of the set E of exceptional directions, those directions in which the image of the set A is small. On a first reading, it is safe to think of $\gamma < 1$, $n \approx \rho^{-\gamma}$, $\delta = 1$, and $m \approx \rho^{-(\gamma-\epsilon)}$. In this case, the set A is a discrete analogue of a set of Hausdorff dimension γ and the set E is the set of exceptional directions in which the set A loses at least a proportion ρ^ϵ of its points.

Theorem 3.2 *Let $\gamma, \rho, c > 0$. Put $\overline{\gamma} = \min(\gamma, 1)$. If $A \subseteq [0, 1]^2$ is a $(\rho, \gamma)_c$-set with $n := |A| > -\log c$, then for all $\delta > 0$ and all $0 \leq m \leq \delta^2 n / 4$, the set*

$$E = \left\{ \theta \in [0, \pi) \mid \exists A' \subseteq A, \ |A'| \geq \delta n, \ \mathcal{N}(\pi_\theta A', \rho) \leq m \right\} \tag{3.1}$$

satisfies

$$\mathcal{N}(E, \rho) \ll_{\gamma, c} \rho^{-1} \frac{m}{\delta^2 n} \begin{cases} n^{1 - \overline{\gamma}/\gamma} & \text{if } \gamma \neq 1 \\ \log n & \text{if } \gamma = 1 \end{cases}.$$

Proof Let $A \subseteq [0, 1]^2$ be a $(\rho, \gamma)_c$-set of cardinality $n > -\log c$. Let $\delta > 0$, and let $0 \leq m \leq \delta^2 n / 4$.

Define $S(\theta) = \left\{ (a_1, a_2) \in A^2 \mid |\pi_\theta(a_1 - a_2)| < \rho \right\}$. Let E' be a maximal ρ-separated subset of E; thus, $|E'| = \mathcal{N}(E, \rho)$. The goal is to bound $\sum_{\theta \in E'} |S(\theta)|$ from above and below to get the desired bound on $|E'|$.

Let $\theta \in E'$ and A' be the subset of A corresponding to θ. Since the set $\pi_\theta A'$ lies on a line and $\mathcal{N}(\pi_\theta A', \rho) \leq m$, there exists a collection $\{B\}_{B \in \mathcal{B}}$ of no more than $2m$ closed balls B of diameter ρ whose union covers $\pi_\theta A'$. By Cauchy-Schwarz,

$$(\delta n)^2 \leq |A'|^2 \leq \left(\sum_{B \in \mathcal{B}} \left| \{ a_0 \in A' \mid \pi_\theta a_0 \in B \} \right| \right)^2$$

$$\leq |\mathcal{B}| \sum_{B \in \mathcal{B}} \left| \{ a_0 \in A' \mid \pi_\theta a_0 \in B \} \right|^2$$

$$\leq 2m \sum_{B \in \mathcal{B}} \left| \{ a \in A \mid \pi_\theta a \in B \} \right|^2$$

$$= 2m \sum_{B \in \mathcal{B}} \left|\{(a_1, a_2) \in A^2 \mid \pi_\theta a_1, \pi_\theta a_2 \in B\}\right|$$
$$\leq 2m \left|S(\theta)\right|.$$

It follows that

$$\frac{\delta^2 n^2}{2m} |E'| \leq \sum_{\theta \in E'} \left|S(\theta)\right|. \tag{3.2}$$

Now we use Lemma 3.1 to bound the right hand side of (3.2) from above: for $a_1, a_2 \in [0, 1]^2$, the set

$$\Theta(a_1, a_2) = \left\{\theta \in [0, \pi) \mid \left|\pi_\theta(a_1 - a_2)\right| < \rho\right\}$$

is contained in at most two balls of diameter $\ll \rho/|a_1 - a_2|$. Therefore, $\mathcal{N}(\Theta(a_1, a_2), \rho) \ll 1/|a_1 - a_2|$, and using the fact that E' is ρ-separated, we see that

$$\sum_{\theta \in E'} 1_{S(\theta)}(a_1, a_2) = \sum_{\theta \in E'} 1_{\Theta(a_1,a_2)}(\theta) \leq K \frac{1}{|a_1 - a_2|}$$

for some constant K depending on the result in Lemma 3.1. It follows that

$$\sum_{\theta \in E'} \left|S(\theta)\right| = \sum_{\theta \in E'} \sum_{a_1, a_2 \in A} 1_{S(\theta)}(a_1, a_2)$$
$$= n|E'| + \sum_{\substack{a_1, a_2 \in A \\ a_1 \neq a_2}} \sum_{\theta \in E'} 1_{\Theta(a_1,a_2)}(\theta)$$
$$\leq n|E'| + K \sum_{\substack{a_1, a_2 \in A \\ a_1 \neq a_2}} |a_1 - a_2|^{-1},$$

and so we are left to bound the second term from above.

For $\ell \in \mathbb{N}_0$, let $H_\ell = \{x \in \mathbb{R}^2 \mid |x| \in [\rho e^\ell, \rho e^{\ell+1})\}$. Breaking up the sum $\sum |a_1 - a_2|^{-1}$ by fixing a_1 and partitioning the a_2's by shells, and using the fact that A is ρ-separated, we see

$$\sum_{\substack{a_1, a_2 \in A \\ a_1 \neq a_2}} |a_1 - a_2|^{-1} = \sum_{a_1 \in A} \sum_{\ell=0}^{\infty} \sum_{a_2 \in A \cap (a_1 + H_\ell)} |a_1 - a_2|^{-1}$$
$$\leq \rho^{-1} \sum_{a_1 \in A} \sum_{\ell=0}^{\infty} e^{-\ell} \left|A \cap (a_1 + H_\ell)\right|.$$

Since A is a $(\rho, \gamma)_c$-set, for all $\ell \geq 0$, $\left|A \cap (a_1 + H_\ell)\right| \leq c \left(2\rho e^{\ell+1}/\rho\right)^\gamma$. On the other hand, $\sum_{\ell=0}^\infty \left|A \cap (a_1 + H_\ell)\right| = |A| - 1$. It follows then from the fact that $\ell \mapsto e^{-\ell}$ is decreasing that $\sum_{\ell=0}^\infty e^{-\ell}\left|A \cap (a_1 + H_\ell)\right| \leq \sum_{\ell=0}^{\ell_0} 2^\gamma c e^{\ell(\gamma-1)+\gamma}$, where $\ell_0 = \lceil \log((n/c)^{1/\gamma}) \rceil$ is the smallest value such that the set A could be contained in a ball of diameter ρe^{ℓ_0} about a_1. Therefore,

$$\rho^{-1} \sum_{a_1 \in A} \sum_{\ell=0}^\infty e^{-\ell}\left|A \cap (a_1 + H_\ell)\right| \ll_{\gamma,c} \rho^{-1} \sum_{a_1 \in A} \sum_{\ell=0}^{\ell_0} \left(e^{\gamma-1}\right)^\ell$$

$$\ll_{\gamma,c} \rho^{-1} n \begin{cases} n^{1-\overline{\gamma}/\gamma} & \text{if } \gamma \neq 1 \\ \log n & \text{if } \gamma = 1 \end{cases}.$$

Combining the upper and lower bounds on $\sum_{\theta \in E'} \left|S(\theta)\right|$, we see that there exists a constant K depending on the result in Lemma 3.1, γ, and c such that

$$\frac{\delta^2 n^2}{2m}|E'| \leq n|E'| + K\rho^{-1} n \begin{cases} n^{1-\overline{\gamma}/\gamma} & \text{if } \gamma \neq 1 \\ \log n & \text{if } \gamma = 1 \end{cases}.$$

Dividing both sides by n and using the fact that $m \leq \delta^2 n/4$, we see that

$$\frac{\delta^2 n}{4m}|E'| \leq \left(\frac{\delta^2 n}{2m} - 1\right)|E'| \leq K\rho^{-1} \begin{cases} n^{1-\overline{\gamma}/\gamma} & \text{if } \gamma \neq 1 \\ \log n & \text{if } \gamma = 1 \end{cases},$$

which rearranges to the desired conclusion. $\qquad\square$

3.2 A Corollary for Oblique Projections

The proof of Theorem A will feature oblique projections instead of orthogonal ones. The following corollary concerns oblique projections and is stated in a way that will make it immediately applicable in the proof of Theorem A.

Denote by $\Pi_t : \mathbb{R}^2 \to \mathbb{R}$ the oblique projection $\Pi_t(x, y) = x + ty$. Let $\varphi : (0, \pi/2) \to \mathbb{R}$ be the diffeomorphism $\varphi(\theta) = \log \tan \theta$. Note that $\Pi_{e^{\varphi(\theta)}}$ is the oblique projection that is the "continuation" of the orthogonal projection π_θ, meaning that the points (x, y), $(\Pi_{e^{\varphi(\theta)}}(x, y), 0)$, and $\pi_\theta(x, y)$ are collinear.

Corollary 3.3 *Let $0 < \gamma_1 < \gamma_2 < \gamma_3 < \gamma_4$ be such that $\gamma_1 < 1$ and*

$$2(\gamma_4 - \gamma_2) < \gamma_3 - \gamma_1. \tag{3.3}$$

For all compact $I \subseteq \mathbb{R}$, all $\epsilon, c_1, c_2, c_3 > 0$, all sufficiently small $\rho > 0$ (depending on all previous quantities), and all $(c_1\rho, \gamma_4)_{c_2}$-sets $A \subseteq [0, 1]^2$ with $|A| \geq \rho^{-\gamma_3}$, there exists $T \subseteq I$ with the following properties:

(I) *the set $I \backslash T$ can be covered by a disjoint union of not more than $\epsilon \rho^{-1}/2$-many half-open intervals of length ρ, a cover of total Lebesgue measure less than ϵ.*

(II) *for all $t \in T$ and all $A' \subseteq A$ with $|A'| \geq \rho^{-\gamma_2}$, there exists a subset $A'_t \subseteq A'$ with $|A'_t| \geq \rho^{-\gamma_1}$ such that the points of $\Pi_{e^t} A'_t$ are distinct and $c_3 \rho$-separated.*

Proof Let $I \subseteq \mathbb{R}$ be compact and $\epsilon, c_1, c_2, c_3 > 0$. Let $\sigma \in \left(\gamma_4 - \gamma_2, (\gamma_3 - \gamma_1)/2 \right)$. Let $\rho > 0$ be sufficiently small (to be specified later, but depending only on the quantities introduced thus far). Let $A \subseteq [0,1]^2$ be a $(c_1 \rho, \gamma_4)_{c_2}$-set with $|A| \geq \rho^{-\gamma_3}$. Put $\overline{\gamma_4} = \min(1, \gamma_4)$, $n = |A|$, $\delta = \rho^\sigma$, and $m = 2c_3 \rho^{-\gamma_1}$. Note that since A is a $(c_1 \rho, \gamma_4)_{c_2}$-set contained in a ball of diameter $\sqrt{2}$, $n \leq 2c_2(c_1 \rho)^{-\gamma_4}$.

We want to apply Theorem 3.2 with γ_4 as γ, $c_1 \rho$ as ρ, c_2 as c, and with A, n, δ, and m as they are. We see that the inequality $n > -\log c_2$ holds for ρ sufficiently small, as does $m \leq \delta^2 n/4$ since $\sigma < (\gamma_3 - \gamma_1)/2$. Since the conditions of Theorem 3.2 hold, the set $E \subseteq [0, \pi)$ defined in (3.1) satisfies

$$
\begin{aligned}
\mathcal{N}(E, \rho) &\ll_{\gamma_4, c_2} \rho^{-1} \frac{m}{\delta^2 n} n^{1 - \overline{\gamma_4}/\gamma_4} \log n \\
&\ll_{\gamma_4, c_1, c_2, c_3} \rho^{-1} \frac{\rho^{-\gamma_1}}{\rho^{2\sigma} \rho^{-\gamma_3 \overline{\gamma_4}/\gamma_4}} \log \left(\rho^{-\gamma_4} \right).
\end{aligned}
\tag{3.4}
$$

Let $J = \varphi^{-1}(I)$, and put $T = I \backslash \varphi|_J(E)$. Since the map $\varphi|_J$ is bi-Lipschitz,

$$
\mathcal{N}(\varphi|_J(E), \rho) \asymp_I \mathcal{N}(E, \rho).
$$

Combining this with (3.4) and the fact that $\sigma < (\gamma_3 - \gamma_1)/2$, we have that for sufficiently small ρ, $\mathcal{N}(I \backslash T, \rho) \leq \epsilon \rho^{-1}/6$. It follows that the set $I \backslash T$ can be covered by a disjoint union of not more than $\epsilon \rho^{-1}/2$-many half-open intervals of length ρ, a cover of total measure less than ϵ. This establishes (I).

To prove (II), let $t \in T$, and let $A' \subseteq A$ with $|A'| \geq \rho^{-\gamma_2}$. Since $n \leq 2c_2(c_1 \rho)^{-\gamma_4}$ and $\sigma > \gamma_4 - \gamma_2$, for sufficiently small ρ, $\rho^{-\gamma_2} \geq \delta n$. It follows that $|A'| \geq \delta n$. Because $\theta := \varphi^{-1}(t) \notin E$, $\mathcal{N}(\pi_\theta A', \rho) \geq m$. It follows that $\mathcal{N}(\pi_\theta A', c_3 \rho) \geq \rho^{-\gamma_1}$. By choosing points in A' in each fiber of a maximally ρ-separated set of the projection, we see that there exists a subset $A'_t \subseteq A'$ of cardinality at least $\rho^{-\gamma_1}$ such that the orthogonal projection of the points in A'_θ onto ℓ_θ are disjoint and $c_3 \rho$-separated. Since the oblique projection Π_{e^t} increases distances between points that lie on ℓ_θ, the images of points of A'_t under Π_{e^t} are $c_3 \rho$-separated. $\qquad \square$

4 Trees and a Subtree Regularity Theorem

Trees are combinatorial objects that are convenient for describing fractal sets. We will be concerned solely with finite trees throughout this work. After giving the main definitions, we motivate their importance by explaining how they will be used in the proof of Theorem A. We move then to prove the main result in this section.

4.1 Preliminary Definitions

The following definitions describe the familiar notion of a rooted tree, a graph with no cycles whose vertices can be arranged on levels and whose edges only connect vertices on adjacent levels.

Definition 4.1

- A *tree of height* $N \in \mathbb{N}_0$ is a finite set of *nodes* Γ together with a partition $\Gamma = \Gamma_0 \cup \cdots \cup \Gamma_N$ with $|\Gamma_0| = 1$ and a *parent* function $P : \Gamma \backslash \Gamma_0 \to \Gamma \backslash \Gamma_N$ such that for every $n \in \{1, \ldots, N\}$, $P(\Gamma_n) = \Gamma_{n-1}$.
- The nodes in Γ_n have *height* n. The single node with height 0 is the *root* and the nodes with height N are called *leaves*.
- The node Q is the *parent* of each of its *children*, nodes in the set $C_\Gamma(Q) := P^{-1}(Q)$.
- If Q is a node of height n, the *induced tree based at* Q is the tree $\Gamma_Q := \cup_{i=0}^{N-n} C_\Gamma^i(Q)$ of height $N - n$ with root Q and the same parent function as Γ, restricted to the set Γ_Q.
- A *subtree* of Γ is a tree $\Gamma' \subseteq \Gamma$ of the same height as Γ with parent function $P|_{\Gamma' \backslash \Gamma'_0}$. (A subtree is uniquely determined by its non-empty set of leaves $\Gamma'_N \subseteq \Gamma_N$.)

Continuing with terminology inspired by genealogy trees, the ancestors of a node Q are those nodes that lie between Q and the root. For the reasons described below in Remark 4.4, it will be important to count the number of ancestors of Q that have many children. To this end, we introduce the following terminology and notation.

Definition 4.2 Let Γ be a tree, $c > 0$, and $\omega \in [0, 1]$.

- The *ancestry* of $Q \in \Gamma_n$ is the set

$$\mathcal{A}_\Gamma(Q) := \{P^k(Q) \mid 1 \le k \le n\}.$$

  Note that $|\mathcal{A}_\Gamma(Q)|$ is equal to the height of Q.
- The node Q is *c-fertile* if $|C_\Gamma(Q)| \ge c$. The set of *c*-fertile ancestors of Q is denoted

$$\mathcal{F}_{\Gamma,c}(Q) := \{A \in \mathcal{A}_\Gamma(Q) \mid A \text{ is } c - \text{fertile}\}.$$

  A node Q has *(c, ω)-fertile ancestry* if $|\mathcal{F}_{\Gamma,c}(Q)| \ge \omega |\mathcal{A}_\Gamma(Q)|$.

The following definitions allow us to capture the dimension of a finite tree by giving costs to the nodes and measuring the cost of the least expensive cut.

Definition 4.3 Let Γ be a tree, $r \in \mathbb{N}$, $r \ge 2$, and $\gamma > 0$.

- A *cut* of Γ is a subset $\mathcal{C} \subseteq \Gamma$ such that for every leaf L of Γ, $(\{L\} \cup \mathcal{A}_\Gamma(L)) \cap \mathcal{C} \ne \emptyset$.
- The *γ-Hausdorff content of* Γ *with base* r is

$$\mathcal{H}_r^\gamma(\Gamma) := \min \left\{ \sum_{Q \in \mathcal{C}} r^{-\text{height}(Q)\gamma} \;\middle|\; \mathcal{C} \text{ is a cut of } \Gamma \right\}.$$

The main result in this section, Theorem 4.7, says, roughly speaking, that any tall enough tree with Hausdorff content bounded from below and with a uniform upper bound on the number of children of any node has a subtree in which most nodes have fertile ancestry. Before making this statement precise and beginning with the details of the proof, let us make two observations about the concept of fertile ancestry that will help explain why it will be useful later on in the proof of Theorem A.

Remark 4.4

(I) The property of having fertile ancestry is preserved under a type of tree thinning process that we will employ in the proof of Theorem A. More specifically, suppose that Γ is a tree in which every node has either one child or at least c many children and in which every node has (c, ω)-fertile ancestry. Suppose further that for every node Q, there exists a subset $\tilde{C}(Q) \subseteq C_\Gamma(Q)$ of the children of Q with $|\tilde{C}(Q)| \geq \min\left(\tilde{c}, |C_\Gamma(Q)|\right)$. These subsets naturally give rise to a subtree $\tilde{\Gamma}$ obtained by thinning the tree Γ: the subtree $\tilde{\Gamma}$ is uniquely defined by the property that if Q is a node of $\tilde{\Gamma}$, then $C_{\tilde{\Gamma}}(Q) = \tilde{C}(Q)$. It is not hard to see that every node in $\tilde{\Gamma}$ has $(\tilde{c}, \omega)$-fertile ancestry, regardless of how the subsets of children $\tilde{C}(Q)$ were chosen.

(II) A tree in which every node has fertile ancestry necessarily has large Hausdorff content. This is a simple consequence of the mass distribution principle (or the max flow-min cut theorem) for trees, the real analogue of which is stated in Lemma 2.6. More specifically, let Γ be a tree, and consider a "flow" through Γ of magnitude 1 starting at the root that splits equally amongst children. The value of the flow at any node Q with fertile ancestry can be bounded from above using the fact that many times, much of the flow is split amongst a large set of children before reaching Q. If all nodes of Γ have fertile ancestry, then the flow is not concentrated too highly at any node. According to the mass distribution principle, the Hausdorff content of a tree that supports such a flow is high.

4.2 A Subtree Regularity Theorem

We now proceed with the main results in this subsection. In the next two results, fix $r \geq 2$ and $0 < \gamma_2 < \gamma_3 < \gamma_4$ such that setting

$$A := \gamma_4 - \gamma_3 + \log_r 2,$$
$$B := \gamma_3 - \gamma_2 - \log_r 2,$$

ensures the quantity B is positive. The following lemma describes the fundamental dichotomy behind Theorem 4.7.

Lemma 4.5 *If Γ is a tree with the property that*

$$\text{every node in the tree has at most } r^{\gamma_4} \text{ many children,} \tag{4.1}$$

then at least one of the following holds:

(I) *there are at least r^{γ_2} many children Q of the root, each of which satisfies*

$$\mathcal{H}_r^{\gamma_3}(\Gamma_Q) \geq \mathcal{H}_r^{\gamma_3}(\Gamma)r^{-A};$$

(II) *there is at least one child Q of the root satisfying*

$$\mathcal{H}_r^{\gamma_3}(\Gamma_Q) \geq \mathcal{H}_r^{\gamma_3}(\Gamma)r^{B}.$$

Proof Let Γ be a tree satisfying (4.1). Let Q_1, Q_2, ..., Q_I be the children of the root of Γ, ordered so that $\mathcal{H}_r^{\gamma_3}(\Gamma_{Q_i}) \geq \mathcal{H}_r^{\gamma_3}(\Gamma_{Q_{i+1}})$. If neither (I) nor (II) holds, then $\mathcal{H}_r^{\gamma_3}(\Gamma_{Q_1}) < \mathcal{H}_r^{\gamma_3}(\Gamma)r^B$ and $\mathcal{H}_r^{\gamma_3}(\Gamma_{Q_{\lceil r^{\gamma_2}\rceil}}) < \mathcal{H}_r^{\gamma_3}(\Gamma)r^{-A}$. It follows by the ordering of the Q_i's and the definition of the Hausdorff content and induced trees that

$$\mathcal{H}_r^{\gamma_3}(\Gamma) \leq r^{-\gamma_3} \sum_{i=1}^{I} \mathcal{H}_r^{\gamma_3}(\Gamma_{Q_i})$$

$$= \sum_{i=1}^{\lfloor r^{\gamma_2}\rfloor} r^{-\gamma_3}\mathcal{H}_r^{\gamma_3}(\Gamma_{Q_i}) + \sum_{i=\lceil r^{\gamma_2}\rceil}^{I} r^{-\gamma_3}\mathcal{H}_r^{\gamma_3}(\Gamma_{Q_i})$$

$$< r^{\gamma_2}r^{-\gamma_3}\mathcal{H}_r^{\gamma_3}(\Gamma)r^B + r^{\gamma_4}r^{-\gamma_3}\mathcal{H}_r^{\gamma_3}(\Gamma)r^{-A} = \mathcal{H}_r^{\gamma_3}(\Gamma),$$

a contradiction. $\qquad\square$

Lemma 4.6 *Every finite tree Γ that satisfies (4.1) has a subtree Γ' with the property that for all nodes Q in Γ',*

$$\left|\mathcal{F}_{\Gamma',r^{\gamma_2}}(Q)\right| \geq \frac{|\mathcal{A}_{\Gamma'}(Q)|B + \log_r \mathcal{H}_r^{\gamma_3}(\Gamma)}{A + B}. \tag{4.2}$$

Proof We will prove the lemma by induction on the height N of the tree Γ. To verify the base case, let Γ be the tree of height $N = 0$: a single node with no children. Taking $\Gamma' = \Gamma$, the inequality (4.2) for this single node follows from the fact that $\log_r \mathcal{H}_r^{\gamma_3}(\Gamma) = 0$.

Suppose that $N \in \mathbb{N}$ is such that the theorem holds for all trees of height $N-1$. Let Γ be a tree of height N that satisfies (4.1). By Lemma 4.5, at least one of Case (I) or Case (II) holds.

Suppose Case (I) of Lemma 4.5 holds. Let Q be any one of the r^{γ_2}-many children guaranteed by Case (I). By the induction hypothesis, there exists a subtree Γ'_Q of Γ_Q in which every node satisfies (4.2) with Γ_Q in place of Γ and Γ'_Q in place of Γ'. Define the subtree Γ' of Γ to be the root node of Γ with the collection of at least r^{γ_2} many children Q, each of those children followed by its subtree Γ'_Q.

We will now verify that (4.2) holds for all nodes of Γ'. Let Q be any node of Γ'. If Q is the root node of Γ', then (4.2) holds because $\log_r \mathcal{H}_r^{\gamma_3}(\Gamma) \leq 0$. (Indeed, that $\mathcal{H}_r^{\gamma_3}(\Gamma) \leq 1$ follows by considering the cut $\mathcal{C}:=\{Q\}$ of Γ.) If Q is a non-root node of

Γ', then it belongs to one of the subtrees Γ'_S for some child S of the root of Γ'. By property (4.2) for the subsubtree Γ'_S, we see

$$
\begin{aligned}
|\mathcal{F}_{\Gamma',r^{\gamma_2}}(Q)| - 1 &= |\mathcal{F}_{\Gamma'_S,r^{\gamma_2}}(Q)| \\
&\geq \frac{|\mathcal{A}_{\Gamma'_S}(Q)|B + \log_r \mathcal{H}_r^{\gamma_3}(\Gamma_S)}{A + B} \\
&\geq \frac{(|\mathcal{A}_{\Gamma'}(Q)| - 1)B + \log_r \mathcal{H}_r^{\gamma_3}(\Gamma) - A}{A + B}.
\end{aligned}
$$

This simplifies to the inequality in (4.2), verifying the inductive step if Case (I) of Lemma 4.5 holds.

Suppose Case (II) of Lemma 4.5 holds. Let Q be the child guaranteed by Case (II). By the induction hypothesis, there exists a subtree Γ'_Q of Γ_Q in which every node satisfies (4.2) with Γ_Q in place of Γ and Γ'_Q in place of Γ'. Define the subtree Γ' of Γ to be the root of Γ with only the child Q followed by its subtree Γ'_Q.

We will now verify that (4.2) holds for all nodes of Γ'. Let Q be any node of Γ'. If Q is the root node of Γ', then (4.2) holds because $\log_r \mathcal{H}_r^{\gamma_3}(\Gamma) \leq 0$. If Q is a non-root node of Γ', then by property (4.2) for the subtree containing Q, we see

$$
|\mathcal{F}_{\Gamma',r^{\gamma_2}}(Q)| \geq \frac{(|\mathcal{A}_{\Gamma'}(Q)| - 1)B + \mathcal{H}_r^{\gamma_3}(\Gamma) + B}{A + B}.
$$

This simplifies to the inequality in (4.2), verifying the inductive step if Case (II) of Lemma 4.5 holds. The proof of the inductive step is complete, and the lemma follows. $\qquad\square$

Theorem 4.7 *For all $0 < \epsilon < 1$, for all $0 < \gamma_2 < \gamma_3 < \gamma_4 < \gamma_3 + \epsilon(\gamma_3 - \gamma_2)$, for all sufficiently large $r \in \mathbb{N}$, and for all $V > 0$, there exists $N_0 \in \mathbb{N}$ for which the following holds. For all $N \geq N_0$ and for all trees Γ of height N with $\mathcal{H}_r^{\gamma_3}(\Gamma) \geq V$ that satisfy (4.1), there exists a subtree Γ' of Γ such that all nodes $Q \in \Gamma'$ with height at least N_0 have $(r^{\gamma_2}, 1 - \epsilon)$-fertile ancestry in Γ'.*

Proof Let $0 < \epsilon < 1$ and $0 < \gamma_2 < \gamma_3 < \gamma_4 < \gamma_3 + \epsilon(\gamma_3 - \gamma_2)$. Let $r \in \mathbb{N}$ be sufficiently large so that $\gamma_3 - \gamma_2 - \log_r 2 > (1-\epsilon)(\gamma_4 - \gamma_2)$. Define $A = \gamma_4 - \gamma_3 + \log_r 2$ and $B = \gamma_3 - \gamma_2 - \log_r 2$, and note by the inequality in the previous sentence, $B/(A + B) > (1 - \epsilon)$. Let $V > 0$. Choose $N_0 \in \mathbb{N}$ such that

$$
\frac{N_0 B + \log_r V}{N_0(A + B)} > 1 - \epsilon, \tag{4.3}
$$

and note that for all $N \geq N_0$, the inequality in (4.3) holds with N_0 replaced by N.

Let $N \geq N_0$, and let Γ be a tree of height N with $\mathcal{H}_r^{\gamma_3}(\Gamma) \geq V$ that satisfies (4.1). By Lemma 4.6, there exists a subtree Γ' of Γ such that for all nodes Q of Γ', the inequality in (4.2) holds.

Let Q be a node of Γ' with height at least N_0. By (4.2) and (4.3), we see that

$$\frac{|\mathcal{F}_{\Gamma', r^{\gamma_2}}(Q)|}{|\mathcal{A}_{\Gamma'}(Q)|} \geq \frac{|\mathcal{A}_{\Gamma'}(Q)|B + \log_r V}{|\mathcal{A}_{\Gamma'}(Q)|(A + B)} > 1 - \epsilon.$$

It follows that Q has $(r^{\gamma_2}, 1 - \epsilon)$-fertile ancestry in Γ', as was to be shown. $\qquad\square$

5 Proof of the Sumsets Theorem

In this section, we prove Theorem A, the main theorem in this work. We restate it here for the reader's convenience.

Theorem A Let r and s be multiplicatively independent positive integers, and let $X, Y \subseteq [0, 1]$ be $\times r$- and $\times s$-invariant sets, respectively. Define $\overline{\gamma} = \min\big(\dim_{\mathrm{H}} X + \dim_{\mathrm{H}} Y, 1\big)$. For all compact $I \subseteq \mathbb{R}\setminus\{0\}$ and all $\gamma < \overline{\gamma}$,

$$\inf_{\lambda, \eta \in I} \mathcal{H}^{\gamma}_{>0}\big(\lambda X + \eta Y\big) > 0. \tag{5.1}$$

Several auxiliary results go into the proof: the discrete version of Marstrand's projection theorem in Sect. 3, the subtree regularity theorem for finite trees in Sect. 4, and the quantitative equidistribution result in Sect. 2.3. We outline the proof of Theorem A in Sect. 5.1 before presenting the full details in Sects. 5.2 and 5.3.

Remark 5.1 It is natural to ask about the value of the infimum $C := \inf_{\lambda, \eta \in I} \mathcal{H}^{\gamma}_{>0}\big(\lambda X + \eta Y\big)$ that appears in (5.1), or, more precisely, how it depends on X and Y. The value of C must depend on r, s, γ, $\overline{\gamma}$, and I, but also on X and Y, at least to the extent that it accounts for the Hausdorff content of $X \times Y$. It follows from the proof of Theorem A below that this is essentially the only sense in which C depends on X and Y.

More precisely, there exist $\gamma_3, \gamma_4 > 0$ (depending only on γ and $\dim_{\mathrm{H}}(X \times Y)$) with $\gamma_3 < \dim_{\mathrm{H}}(X \times Y) < \gamma_4$ such that taking $M_0 \in \mathbb{N}$ and $c_1, c_2 > 0$ as given by Corollary 2.16 when applied with γ_4 as ξ, the quantity C depends only on M_0, c_1, c_2, $r, s, I, \gamma, \dim_{\mathrm{H}}(X \times Y)$, and $\mathcal{H}^{\gamma_3}_{>0}(X \times Y)$, but otherwise not on X and Y.[5]

5.1 Outline of the Proof of Theorem A

Before beginning with the details of the proof of Theorem A, we explain the main ideas behind it. To understand the argument, it helps to begin by assuming that the set $X \times Y$ is self-similar in the sense that for every $n \in \mathbb{N}_0$, it is a union of approximately $r^{n(\dim_{\mathrm{H}} X + \dim_{\mathrm{H}} Y)}$ many translates of the set $r^{-n}X \times s^{-n'}Y$. (Recall that $n' = \lfloor n \log r / \log s \rfloor$ so that $s^{-n'} \approx r^{-n}$.) This is the case, for example, if X and Y are both restricted digit Cantor sets. In this case, Peres and Shmerkin [22] proved

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[5] As will be evident from the proof, our argument requires from γ_3 only that $\mathcal{H}^{\gamma_3}_{>0}(X \times Y) > 0$. Appealing to a result such as [4, Theorem 3.1] it is possible to take $\gamma_3 = \dim_{\mathrm{H}}(X \times Y)$. This improvement would eliminate a parameter but ultimately would not simplify the argument.

that for all $\lambda, \eta \in \mathbb{R}\backslash\{0\}$, $\dim_H(\lambda X + \mu Y) = \overline{\gamma}$. Our argument follows along the same lines as theirs.

Recall that $\Pi_t : \mathbb{R}^2 \to \mathbb{R}$ is the oblique projection $\Pi_t(x, y) = x + ty$. A quick calculation shows that

$$\Pi_{e^t}(r^{-n}X \times s^{-n'}Y) = r^{-n}\Pi_{e^t r^n/s^{n'}}(X \times Y),$$

which implies that the images of the translates of $r^{-n}X \times s^{-n'}Y$ under the map Π_{e^t} are affinely equivalent to the image of the full set $X \times Y$ under the map $\Pi_{e^t r^n/s^{n'}}$. It follows that the set $\Pi_{e^t}(X \times Y)$ contains affine images of the sets $\Pi_{e^t r^n/s^{n'}}(X \times Y)$ and hence that

$$\dim_H \Pi_{e^t}(X \times Y) \geq \sup_{n \in \mathbb{N}_0} \dim_H \Pi_{e^t r^n/s^{n'}}(X \times Y).$$

Thus, to bound $\dim_H \Pi_{e^t}(X \times Y)$ from below, it suffices to show that there is some $n \in \mathbb{N}_0$ for which $e^t r^n/s^{n'}$ is a "good angle" for $X \times Y$, in the sense that $\dim_H \Pi_{e^t r^n/s^{n'}}(X \times Y) > \overline{\gamma} - \epsilon$. It follows from Marstrand's theorem that the set of such "good angles" for $X \times Y$ (indeed, for any set) has full measure in $\mathbb{R}$, and it will be shown that the sequence $n \mapsto \log(e^t r^n/s^{n'})$ has image in $[t, t + \log s)$ and is the orbit of t under the irrational $x \mapsto x + \log r \pmod{\log s}$ translated by t. When combined, these facts fall just short of allowing us to conclude the existence of $n \in \mathbb{N}_0$ for which $e^t r^n/s^{n'}$ is a good angle: it is possible that the image of an equidistributed sequence misses a set of full measure.

To make use of the above outline, one needs to gain some topological information on the set of good angles from Marstrand's theorem. This can be accomplished by moving the argument to a discrete setting. Discretizing introduces a number of technical nuisances, but the core of the argument remains the same. Recall that X_n and $Y_{n'}$ are the sets X and Y rounded to the lattices $r^{-n}\mathbb{Z}$ and $s^{-n'}\mathbb{Z}$, respectively. The discrete analogue of Marstrand's theorem in Theorem 3.2 tells us that the complement of the set of "good angles" for a finite set such as $X_n \times Y_{n'}$ can be covered by a disjoint union of few half-open intervals. This topological information combines with the equidistribution of the irrational rotation described above to allow us to find many $n \in \mathbb{N}_0$ for which $e^t r^n/s^{n'}$ is a good angle for $X_n \times Y_{n'}$.

The argument described thus far is essentially due to Peres and Shmerkin in [22] and allows them to conclude that for all $t \in \mathbb{R}\backslash\{0\}$, $\dim_H \Pi_{e^t}(X \times Y) = \overline{\gamma}$. We will now describe the two primary modifications we make to this argument in the course of the proof of Theorem A.

The first modification allows us to show that the discrete Hausdorff content of $\Pi_{e^t}(X \times Y)$ at all small scales is uniform in t. Ultimately, this uniformity stems from the fact that the irrational rotation described above is uniquely ergodic: changing t in the argument above changes only the point whose orbit we consider. Exposing the uniformity in the argument after this is then mainly a matter of taking care with the quantifiers in the auxiliary results.

The second modification allows us to handle sets X and Y which are only assumed to be $\times r$- and $\times s$-invariant. Such sets need not be self-similar, but they do exhibit

some "near self similarity" in the following sense. Consider the discrete set X_m for some large $m \in \mathbb{N}$. Because X is $\times r$-invariant, the set $X_{(n+1)m} \cap \left[i/r^{nm}, (i+1)/r^{nm}\right)$, when dilated by r^{mn} and considered modulo 1, is a subset of X_m. While this set is generally not equal to X_m, it is, by an averaging argument, very often of cardinality greater than $r^{-\epsilon}|X_m|$. This is profitably re-interpreted in the language of trees: in the tree with levels $X_{nm} \times Y_{(nm)'}$, $n \in \mathbb{N}_0$, many nodes have nearly the maximum allowed number of children. The tree thinning result in Theorem 4.7 exploits this abundance by finding a sufficiently "regular" subtree on which we focus our attention. Then, we invoke our discrete analogue of Marstrand's theorem – which provides information on the set of angles that are good not only for the original set $X_m \times Y_{m'}$, but also for large subsets of it – to further thin the subtree. Following the reasoning given in Remark 4.4, the resulting subtree has fertile ancestry and hence has large Hausdorff content. By the construction of the subtree, its image under Π_{e^t} is large, and this yields the lower bound on the Hausdorff dimension in the conclusion of the theorem.

5.2 Proof of Theorem A

In this section and the next, let r, s, X, Y, and $\overline{\gamma}$ be given as in the statement of Theorem A. The proof of Theorem A begins with a number of reductions, the last of which in Claim 5.2 is a statement about the existence of measures on the images of the discrete product sets under oblique projections. We prove Claim 5.2 in the next subsection.

By Lemma 2.10, $\dim_H(X \times Y) = \dim_H X + \dim_H Y$. Note that if $\dim_H X = 0$, then the conclusion is clear by considering, for any $x \in X$, images of the set $\{x\} \times Y$. The same is true if $\dim_H Y = 0$. Thus, we will proceed under the assumption that $\dim_H X, \dim_H Y > 0$. Note that the set $1 - X$ is $\times r$-invariant and that $-\lambda X + \eta Y$ is a translate of the set $\lambda(1 - X) + \eta Y$. The analogous statement holds for Y. Combining these facts, it is easy to see that it suffices to prove Theorem A in the case that $I \subseteq (0, \infty)$.

The next step is to formulate a statement sufficient to prove Theorem A in terms of oblique projections of discrete sets. Recall that $n' = \lfloor n \log r / \log s \rfloor$ and that X_n, $Y_{n'}$ are the sets X and Y rounded to the lattices $r^{-n}\mathbb{Z}$ and $s^{-n'}\mathbb{Z}$, respectively. For $n \in \mathbb{N}_0$, define

$$\mathcal{Q}_n = X_n \times Y_{n'} \qquad \text{and} \qquad \widetilde{\mathcal{Q}}_n = X_n \times Y_{n'+1}. \tag{5.2}$$

Claim 5.2 For all compact $I \subseteq \mathbb{R}$ and all $0 < \gamma < \overline{\gamma}$, there exists m, $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ and all $t \in I$, there exists a probability measure μ supported on the finite set $\Pi_{e^t} \mathcal{Q}_{Nm}$ with the property that for all balls $B \subseteq \mathbb{R}$ of diameter $\delta \geq r^{-Nm}$, $\mu(B) \leq r^{N_0 m} \delta^{\gamma}$.

To deduce Theorem A from Claim 5.2, let $I \subseteq (0, \infty)$ be compact and $0 < \gamma < \overline{\gamma}$. Apply Claim 5.2 with $\tilde{I} := \left\{ \log(\eta/\lambda) \mid \eta, \lambda \in I \right\}$ as I and γ as it is. Let m, $N_0 \in \mathbb{N}$ be as guaranteed by Claim 5.2.

Note that by Lemma 2.4 and the fact that the function $\rho \mapsto \mathcal{H}_{\geq\rho}^{\gamma}(\lambda X + \eta Y)$ is non-increasing (as ρ decreases),

$$\inf_{\lambda,\eta\in I} \mathcal{H}_{>0}^{\gamma}(\lambda X + \eta Y) = \lim_{\rho\to 0}\inf_{\lambda,\eta\in I} \mathcal{H}_{\geq\rho}^{\gamma}(\lambda X + \eta Y).$$

The limit in the final expression exists because $\inf_{\lambda,\eta\in I} \mathcal{H}_{\geq\rho}^{\gamma}(\lambda X + \eta Y)$ is non-increasing and is bounded from below by zero.

Therefore, to show that (5.1) holds, it suffices to prove that

$$\lim_{N\to\infty}\inf_{\lambda,\eta\in I} \mathcal{H}_{\geq r^{-Nm}}^{\gamma}(\lambda X + \eta Y) > 0. \tag{5.3}$$

It follows from the fact that

$$d_H(\lambda X_{Nm} + \eta Y_{(Nm)'}, \lambda X + \eta Y) \ll_{I,r,s} r^{-Nm},$$

where d_H is the Hausdorff metric, and Lemma 2.7 that for all $\lambda, \eta \in I$,

$$\begin{aligned}
\mathcal{H}_{\geq r^{-Nm}}^{\gamma}(\lambda X + \eta Y) &\asymp_{I,r,s} \mathcal{H}_{\geq r^{-Nm}}^{\gamma}(\lambda X_{Nm} + \eta Y_{(Nm)'}) \\
&\asymp_{I,r,s} \mathcal{H}_{\geq r^{-Nm}}^{\gamma}(X_{Nm} + e^{\log(\eta/\lambda)} Y_{(Nm)'}).
\end{aligned} \tag{5.4}$$

Therefore, to show (5.3), it suffices to prove that

$$\lim_{N\to\infty}\inf_{t\in\tilde{I}} \mathcal{H}_{\geq r^{-Nm}}^{\gamma}(\Pi_{e^t} \mathcal{Q}_{Nm}) > 0. \tag{5.5}$$

Combining the conclusion of Claim 5.2 with Lemma 2.6, we see that for all $N \geq N_0$ and $t \in \tilde{I}$, $\mathcal{H}_{\geq r^{-Nm}}^{\gamma}(\Pi_{e^t} \mathcal{Q}_{Nm}) \geq r^{-N_0 m}$. This shows that the limit in (5.5) is positive and completes the deduction of Theorem A from Claim 5.2.

5.3 Proof of Claim 5.2

Choosing the parameter m and scale ρ. Recall that r, s, X, Y, and $\overline{\gamma}$ are as given in the statement of Theorem A. Without loss of generality, we can assume that $r < s$. Put $\beta = \log s$, let $0 < \gamma < \overline{\gamma}$, and define $\epsilon := \overline{\gamma} - \gamma$ and $\gamma_0 := \gamma$.

We claim that there exist γ_1, γ_2, γ_3, and γ_4 such that

  (I) $0 < \gamma_0/(1 - \epsilon/2) < \gamma_1 < \gamma_2 < \gamma_3 < \dim_H X + \dim_H Y < \gamma_4$;
 (II) $\gamma_4 < \gamma_3 + \epsilon(\gamma_3 - \gamma_2)/6$;
(III) $\gamma_1 < 1$;
(IV) $2(\gamma_4 - \gamma_2) < \gamma_3 - \gamma_1$ (this is the inequality in (3.3)).

To see why, note that if we put $\gamma_1 = \gamma_0/(1 - \epsilon/2)$, $\gamma_4 = \gamma_3 = \dim_H X + \dim_H Y$, and $\gamma_2 = \gamma_1/3 + 2\gamma_3/3$, then the inequalities in (I) holds with "$<$" replaced by "$\leq$", while the inequalities in (II), (III), and (IV) hold as written. It follows that γ_1 and γ_4 can be increased and γ_3 can be decreased (with the corresponding change to $\gamma_2 = \gamma_1/3 + 2\gamma_3/3$) so that all of the strict inequalities hold.

Let c_1, c_2, and M_0 be the constants guaranteed by Corollary 2.16, when applied with γ_4 as ξ. Let $I \subseteq (0, \infty)$ be compact, and define $I_\beta = I + [0, \beta]$. Let $P > 0$ be a Lipschitz constant for all of the maps Π_{e^t}, $t \in I_\beta$, and let $c_3 = 4Ps^{-1} + 1$. Choose $m \in \mathbb{N}$ large enough so that we can apply

- Theorem 4.7 with $\epsilon/6$ as ϵ and r^m as r;
- Corollary 3.3 with I_β as I, $\epsilon\beta/12$ as ϵ and r^{-m} as ρ;
- Corollary 2.16 with m as N (i.e., $m \geq M_0$).

Put $\rho = r^{-m}$.

A uniformly distributed sequence. Let $\alpha = \log\left(r^m/s^{m'}\right)$ and let $R : [0, \beta) \to [0, \beta)$ be the transformation $R : x \mapsto x + \alpha \pmod{\beta}$. As $\beta = \log s$ and $m' = \lfloor m \log r / \log s \rfloor$, we have

$$\alpha/\beta = m \log r / \log s - m' = \left\{m \log r / \log s\right\}. \tag{5.6}$$

Since $\log r / \log s$ is irrational, we conclude that α/β is irrational, whereby the sequence $(R^n(0))_{n \in \mathbb{N}_0}$ is uniformly distributed on $[0, \beta)$.

Claim 5.3 For all $n \in \mathbb{N}_0$,

(V) $R^n(0) + (nm)' \log s = nm \log r$;
(VI)

$$((n+1)m)' = \begin{cases} (nm)' + m' & \text{if } R^n(0) + \alpha < \beta \\ (nm)' + m' + 1 & \text{if } R^n(0) + \alpha > \beta \end{cases}.$$

Proof Since for all $n \in \mathbb{N}$, $R^n(0) = n\alpha \pmod{\beta}$, using (5.6), we can write $R^n(0)/\beta = \left\{n\alpha/\beta\right\} = \left\{n\{m \log r/\beta\}\right\} = \left\{nm \log r/\beta\right\}$. Recalling that $(nm)' = \lfloor nm \log r/\beta \rfloor$, this establishes (V).

Next, note that for any real numbers x, y,

$$\lfloor x + y \rfloor = \begin{cases} \lfloor x \rfloor + \lfloor y \rfloor & \text{if } \{x\} + \{y\} < 1 \\ \lfloor x \rfloor + \lfloor y \rfloor + 1 & \text{if } \{x\} + \{y\} \geq 1 \end{cases}.$$

The equality in (VI) follows from this by substituting $x = nm \log r/\beta$ and $y = m \log r/\beta$ and using $R^n(0)/\beta = \left\{nm \log r/\beta\right\}$ and (5.6). □

Choosing the parameter N_0. From Corollary 2.16, the sets $\mathcal{Q}_m$ and $\widetilde{\mathcal{Q}}_m$ (defined in (5.2)) are $(c_1\rho, \gamma_4)_{c_2}$-sets and satisfy

$$\rho^{-\gamma_3} \leq |\mathcal{Q}_m|, |\widetilde{\mathcal{Q}}_m| \leq \rho^{-\gamma_4}. \tag{5.7}$$

Let T_1 (resp. T_2) be the subset of I_β obtained from applying Corollary 3.3 with I_β as I, $\epsilon\beta/12$ as ϵ and $\mathcal{Q}_m$ (resp. $\widetilde{\mathcal{Q}}_m$) as A. Put $T = T_1 \cap T_2$. It follows from Corollary 3.3 that $I_\beta \setminus T$ is covered by a disjoint union of not more than $U := \lceil 2\epsilon\beta\rho^{-1}/24 \rceil$ many half-open intervals of Lebesgue measure less than $\epsilon\beta/6$.

Let $N_0 \in \mathbb{N}$ be the larger of

- the N_0 from Theorem 4.7 with $\epsilon/6$ as ϵ, r^m as r, and $2^{-\gamma_3}\mathcal{H}_{>0}^{\gamma_3}(X \times Y)$ as V;
- the N_0 from Lemma 2.18 with $(R^n(0))_{n \in \mathbb{N}_0}$ as $(x_n)_{n \in \mathbb{N}_0}$ and $\epsilon/3$ as ϵ.

Fixing the parameters N and t. To prove Claim 5.2, we will show that for all $N \geq N_0$ and all $t \in I$ there exists a probability measure μ supported on the set $\Pi_{e^t} \mathcal{Q}_{Nm}$ with the property that for all balls $B \subseteq \mathbb{R}$ of diameter $\delta \geq \rho^N$, $\mu(B) \leq \rho^{-N_0}\delta^\gamma$. Let $N \geq N_0$ and $t \in I$. From this point on, all new quantities and objects can depend on N and t.

Constructing the tree Γ. Let Γ be the tree (see Definition 4.1) of height N with node set at height $n \in \{0, 1, \ldots, N\}$ equal to $\mathcal{Q}_{nm}$. Associating the point $(i/r^{mn}, j/s^{(mn)'}) \in \mathcal{Q}_{nm}$ with the rectangle

$$\left[\frac{i}{r^{mn}}, \frac{i+1}{r^{mn}}\right) \times \left[\frac{j}{s^{(mn)'}}, \frac{j+1}{s^{(mn)'}}\right),$$

parentage in the tree Γ is determined by containment amongst associated rectangles. Denote by $C_\Gamma(Q)$ the children of the node Q in Γ. Denote by $\odot : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ the binary operation of pointwise multiplication.

Claim 5.4 Let $n < N$ and $Q \in \mathcal{Q}_{nm}$.

(VII) If $R^n(0) + \alpha < \beta$, then $C_\Gamma(Q) \subseteq Q + (r^{-nm}, s^{-(nm)'}) \odot \mathcal{Q}_m$.
(VIII) If $R^n(0) + \alpha > \beta$, then $C_\Gamma(Q) \subseteq Q + (r^{-nm}, s^{-(nm)'}) \odot \widetilde{\mathcal{Q}}_m$.
(IX) $\mathcal{H}_{r^m}^{\gamma_3}(\Gamma) \geq 2^{-\gamma_3}\mathcal{H}_{>0}^{\gamma_3}(X \times Y)$.

Proof We first prove parts (VII) and (VIII). By Lemma 2.12, $rX_n \subseteq X_{n-1} \pmod 1$ and $sY_{n'} \subseteq Y_{n'-1} \pmod 1$. By (VI), if $R^n(0) + \alpha < \beta$, then $\big((n+1)m\big)' = (nm)' + m'$, and hence $(r^{nm}, s^{(nm)'}) \odot \mathcal{Q}_{(n+1)m} \subseteq \mathcal{Q}_m \pmod 1$, and in particular $(r^{nm}, s^{(nm)'}) \odot C_\Gamma(Q) \subseteq \mathcal{Q}_m \pmod 1$. If $R^n(0) + \alpha > \beta$, then $\big((n+1)m\big)' = (nm)' + m' + 1$, and hence $(r^{nm}, s^{(nm)'}) \odot \mathcal{Q}_{(n+1)m} \subseteq \widetilde{\mathcal{Q}}_m \pmod 1$, and in particular $(r^{nm}, s^{(nm)'}) \odot C_\Gamma(Q) \subseteq \widetilde{\mathcal{Q}}_m \pmod 1$.

Write $Q = (i/r^{nm}, j/s^{(nm)'})$ and let $Q' \in C_\Gamma(Q)$. Because Q' is a child of Q, we can write $Q' = Q + (i_0/r^{(n+1)m}, j_0/s^{((n+1)m)'})$ where $0 \leq i_0 < r^m$ and $0 \leq j_0 < s^{m'}$. It follows that $(r^{nm}, s^{(nm)'}) \odot (C_\Gamma(Q) - Q) \subseteq \mathcal{Q}_m$ (in the first case $R^n(0) + \alpha < \beta$) or $(r^{nm}, s^{(nm)'}) \odot (C_\Gamma(Q) - Q) \subseteq \widetilde{\mathcal{Q}}_m$ (in the second case $R^n(0) + \alpha > \beta$), where the containment now is understood without reducing modulo 1.

To prove (IX), take a cut $\{Q_1, \ldots, Q_\ell\} \subseteq \Gamma$ of Γ with node Q_i at height n_i. Then, by construction of Γ, there exists a cover $X \times Y \subseteq \cup_{i=1}^\ell B_i$ where ball B_i has diameter at most $2\rho^{n_i}$. Since the cut was arbitrary, it follows that $\mathcal{H}_{r^m}^{\gamma_3}(\Gamma) \geq 2^{-\gamma_3}\mathcal{H}_{>0}^{\gamma_3}(X \times Y)$. $\square$

Constructing the tree Γ'. Combining (5.7) with (VII) and (VIII), it follows that $|C_\Gamma(Q)| \leq r^{m\gamma_4}$ for every non-leaf node Q of Γ. The tree Γ has now been shown to satisfy all the hypothesis of Theorem 4.7 (with $\epsilon/6$ as ϵ, r^m as r, and $2^{-\gamma_3}\mathcal{H}_{>0}^{\gamma_3}(X \times Y)$ as V), thus there exists a subtree Γ' of Γ with the property that every node with height at least N_0 has $(r^{m\gamma_2}, 1 - \epsilon/6)$-fertile ancestry in Γ'.

Constructing the tree Γ''. Now we will use Corollary 3.3, the corollary to the discrete version of Marstrand's theorem, to further thin out the tree Γ'; an outline for this step was described in Remark 4.4 (I). For each non-leaf node $Q \in \Gamma'$, we will define a subset $C^m_{\Gamma'}(Q)$ of $C_{\Gamma'}(Q)$. Define $J = (T - t) \cap [0, \beta)$. Since $I_\beta \setminus T$ is covered by at most U many half-open intervals of measure less than $\epsilon\beta/6$, the same is true for the set $[0, \beta) \setminus J$. Define $\mathcal{J} = \{0 \leq n \leq N - 1 \mid R^n(0) \in J\}$. Note that for all $n \geq N_0$, by Lemma 2.18, $|\mathcal{J} \cap \{0, \ldots, n-1\}| \geq (1 - \epsilon/3)n$.

Let Q be a non-leaf node of Γ', and let $n \in \{0, \ldots, N-1\}$ be the height of Q. Consider the following cases:

(X) $n \notin \mathcal{J}$ or $|C_{\Gamma'}(Q)| < \rho^{-\gamma_2}$. Select a single child Q' of Q and put $C^m_{\Gamma'}(Q) = \{Q'\}$.

(XI) $n \in \mathcal{J}$, $|C_{\Gamma'}(Q)| \geq \rho^{-\gamma_2}$, and $R^n(0) + \alpha < \beta$. By Theorem 4.7 and (VII), the set $A' := (r^{nm}, s^{(nm)'}) \odot (C_{\Gamma'}(Q) - Q)$ is a subset of $\mathcal{Q}_m$ of cardinality at least $\rho^{-\gamma_2}$. Since $n \in \mathcal{J}$, we have that $t + R^n(0) \in T$. Applying Corollary 3.3 (II) with $t + R^n(0)$ in the role of t, there exists a subset $A'_t \subseteq A'$ with $|A'_t| \geq \rho^{-\gamma_1}$ and such that the points of $\Pi_{e^t + R^n(0)} A'_t$ are distinct and $c_3\rho$-separated. Define $C^m_{\Gamma'}(Q) = Q + (r^{-nm}, s^{-(nm)'}) \odot A'_t$ so that $(r^{nm}, s^{(nm)'}) \odot (C^m_{\Gamma'}(Q) - Q) = A'_t$.

(XII) $n \in \mathcal{J}$, $|C_{\Gamma'}(Q)| \geq \rho^{-\gamma_2}$, and $R^n(0) + \alpha > \beta$. We do exactly as in (XI) with $\mathcal{Q}_m$ replaced by $\widetilde{\mathcal{Q}}_m$ and using (VIII) to get the set $C^m_{\Gamma'}(Q)$.

Let Γ'' be the subtree of Γ' with the property that if Q is a non-leaf node of Γ'', then $C_{\Gamma''}(Q) = C^m_{\Gamma'}(Q)$. We claim that

$$\text{every node of } \Gamma'' \text{ with height at least } N_0 \text{ has} (r^{m\gamma_1}, 1 - \epsilon/2) - \text{fertile ancestry.}$$
$$(5.8)$$

Indeed, let Q be a node of Γ'' with height $n \geq N_0$. The ancestry of Q in Γ' is $(r^{m\gamma_2}, 1 - \epsilon/6)$-fertile. Each $r^{m\gamma_2}$-fertile ancestor of Q in Γ' with height in the set $\mathcal{J}$ is an $r^{m\gamma_1}$-fertile ancestor of Q in Γ''. Since $\left|\mathcal{J} \cap \{0, \ldots, n-1\}\right| \geq (1 - \epsilon/3)n$, there are at least $(1 - \epsilon/2)n$ many $r^{m\gamma_2}$-fertile ancestor of Q in Γ' with height in the set $\mathcal{J}$. It follows that Q has $(r^{m\gamma_1}, 1 - \epsilon/2)$-fertile ancestry in Γ''.

Claim 5.5 If L_1 and L_2 are two distinct leaves of Γ'' and n is maximal such that L_1 and L_2 have a common ancestor at height n, then $|\Pi_{e^t} L_1 - \Pi_{e^t} L_2| \geq \rho^{n+1}$.

Proof Let Q be the common ancestor of L_1 and L_2 in Γ'' of height n. Note that by the definition of Γ'' and maximality of n, it must be that Q has more than one child and hence that $n \in \mathcal{J}$. Let Q_1 and Q_2 be the children of Q in Γ'' that are ancestors of L_1 and L_2, respectively. Note that $Q_1 \neq Q_2$ but that Q_i may be equal to L_i.

We will show first that $\Pi_{e^t} Q_1$ and $\Pi_{e^t} Q_2$ are $c_3\rho^{n+1}$-separated. Write $Q = (p, q)$ and $Q_i = (p_i, q_i)$. Suppose that $R^n(0) + \alpha < \beta$. It follows from (V) that

$$\begin{aligned}
\Pi_{e^t} Q_i &= r^{-nm} \left(r^{nm} \Pi_{e^t}(Q_i - Q)\right) + \Pi_{e^t} Q \\
&= \rho^n \left(r^{nm}(p_i - p) + e^{t + R^n(0)} s^{(nm)'}(q_i - q)\right) + \Pi_{e^t} Q \qquad (5.9) \\
&= \rho^n \left(\Pi_{e^{t + R^n(0)}}\left((r^{nm}, s^{(nm)'}) \odot (Q_i - Q)\right)\right) + \Pi_{e^t} Q.
\end{aligned}$$

By (XI), the points of $\Pi_{e^t + R^n(0)}\big((r^{nm}, s^{(nm)'}) \odot (Q_i - Q)\big), i = 1, 2$, are $c_3 \rho$-separated. It follows then from (5.9) that the points of $\Pi_{e^t} Q_i, i = 1, 2$, are $c_3 \rho^{n+1}$-separated. A similar argument works to reach the same conclusion if $R^n(0) + \alpha > \beta$ using (XII).

By the definition of the $\mathcal{Q}_{nm}$ sets, $|Q_i - L_i| \leq 2\, s^{-1} \rho^{n+1}$. By the triangle inequality and the fact that $c_3 = 4Ps^{-1} + 1$,

$$
\begin{aligned}
\big|\Pi_{e^t} L_1 - \Pi_{e^t} L_2\big| &\geq \big|\Pi_{e^t} Q_1 - \Pi_{e^t} Q_2\big| - \big|\Pi_{e^t}(Q_1 - L_1)\big| - \big|\Pi_{e^t}(Q_1 - L_1)\big| \\
&\geq (4Ps^{-1} + 1)\rho^{n+1} - 4Ps^{-1}\rho^{n+1} \geq \rho^{n+1}.
\end{aligned}
$$

It follows that $\big|\Pi_{e^t} L_1 - \Pi_{e^t} L_2\big| \geq \rho^{n+1}$, as was to be shown. □

Constructing the measure μ. The proof of Claim 5.2 will be concluded by demonstrating that 1) the fertile ancestry property of Γ'' in (5.8) guarantees that Γ'' supports a "measure" which is not too concentrated on any node (an outline for this step was described in Remark 4.4 (II)); and 2) by Claim 5.5, the projection of this measure is not too concentrated on any ball.

Let $\nu : \Gamma'' \to [0, 1]$ be the unique function that takes 1 on the root of Γ'' and has the properties that for all non-leaf nodes Q of Γ'', ν is constant on $\mathcal{C}_{\Gamma''}(Q)$ and $\nu(Q) = \sum_{C \in \mathcal{C}_{\Gamma''}(Q)} \nu(C)$. (Colloquially, a mass of 1 begins at the root of Γ'' and spreads down the tree by splitting equally amongst the children of each node.) Let ν_N be the function ν restricted to Γ''_N, the set of leaves of Γ''. By the defining properties of ν, the function ν_N is a probability measure on Γ''_N.

Since $\Gamma''_N \subseteq \mathcal{Q}_{Nm}$, the measure $\mu = \Pi_{e^t} \nu_N$, the push-forward of ν_N through the map Π_{e^t}, is a probability measure supported on the set $\Pi_{e^t} \mathcal{Q}_{Nm}$. We will conclude the proof of Claim 5.2 by verifying that for all balls $B \subseteq \mathbb{R}$ of diameter $\delta \geq \rho^N$, $\mu(B) \leq \rho^{-N_0} \delta^{\gamma_0}$. (Recall that $\gamma_0 = \gamma$.)

Let $B \subseteq \mathbb{R}$ be an interval of length $\delta \geq \rho^N$. Put $n = \lfloor \log_\rho \delta \rfloor + 1$ and note that $\rho^n < \delta \leq \rho^{n-1}$. It follows from Claim 5.5 that there exists a node Q of Γ'' with height at least n with the property that if L is a leaf of Γ'' with $\Pi_{e^t} L \in B$, then Q is an ancestor of L. This implies that $\mu(B) \leq \nu(Q)$, and so it suffices to show that

$$
\nu(Q) \leq \rho^{-N_0} \delta^{\gamma_0}. \tag{5.10}
$$

If $n \leq N_0$, then $\rho^{-N_0} \delta^{\gamma_0} > 1$ and (5.10) holds trivially. If $n > N_0$, then by the definition of ν and the fact that Q has $(r^{m\gamma_1}, 1 - \epsilon/2)$-fertile ancestry (cf. (5.8)),

$$
\nu(Q) \leq \frac{1}{r^{m\gamma_1(1-\epsilon/2)n}} = \rho^{\gamma_1(1-\epsilon/2)n} \leq \rho^{-N_0} \delta^{\gamma_0},
$$

since $(1 - \epsilon/2)\gamma_1 > \gamma_0$. This verifies (5.10), completing the proof of Claim 5.2 and hence of Theorem A.

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