$(2K+1)$-CONNECTED TOURNAMENTS WITH LARGE MINIMUM OUT-DEGREE ARE K-LINKED

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Pokrovskiy conjectured that there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that any $2k$-strongly-connected tournament with minimum out and in-degree at least $f(k)$ is k-linked. In this paper, we show that any $(2k+1)$-strongly-connected tournament with minimum out-degree at least some polynomial in k is k-linked, thus resolving the conjecture up to the additive factor of 1 in the connectivity bound, but without the extra assumption that the minimum in-degree is large. Moreover, we show the condition on high minimum out-degree is necessary by constructing arbitrarily large tournaments that are $(2.5k-1)$-strongly-connected tournaments but are not k-linked.

1. Introduction

This paper is concerned with the relation between two central notions of connectivity in tournaments: strong-connectivity and linkedness. A directed graph is *strongly-connected* if for any pair of distinct vertices x and y there is a directed path from x to y, and is strongly k-connected if it has at least $k+1$ vertices and if it remains strongly-connected upon the removal of any set of at most $k-1$ vertices. We shall omit the use of the word 'strongly' with the understanding that we always mean strong connectivity. A directed graph G is *k-linked* if $|V(G)| \geq 2k$ and for any two disjoint sets of vertices $\{x_1,\ldots,x_k\}$ and $\{y_1,\ldots,y_k\}$ there are pairwise vertex disjoint directed paths $P_1,\ldots,P_k$ such that P_i has initial vertex x_i and terminal vertex y_i for every $i \in [k]$. Thus, G is 1-linked if and only if it is connected and nontrivial. Since

linkedness is a stronger notion than connectivity, it is natural to ask if a high enough connectivity is sufficient to guarantee linkedness. This is too much to hope for in general, as shown by Thomassen [14], who constructed digraphs of arbitrarily large connectivity, but which are not even 2-linked. This is in stark contrast to the situation for undirected graphs: Bollobás and Thomason [1] showed that any $22k$-connected graph is k-linked and using this result they confirmed a conjecture of Mader [9] and of Erdős and Hajnal [2] related to the smallest average degree that guarantees a subdivision of a clique on k vertices (this result was also proved independently by Komlós and Szemerédi [7]). The constant in the connectivity bound in Bollobás and Thomason's result has subsequently been improved by Thomas and Wollan [11]. They showed that $2k$-connectivity suffices for k-linkedness provided the graph has average degree at least $10k$. Thomassen [12] conjectured that any $(2k+2)$-connected graph is k-linked, though this is false if the graph does not have sufficiently many vertices (for example, K_{3k-1} minus a matching of size k is a counterexample. It is $(3k-3)$-connected, but not k-linked). If the amended conjecture is true, it would in fact be tight (see [3]).

The picture in the directed setting is more positive, however, if we restrict our attention to *tournaments*, those directed graphs obtained by orienting each edge of a complete graph in precisely one direction. Indeed, Thomassen [13] was the first to find a function $g(k)$ such that any $g(k)$-connected tournament is k-linked. The initial bounds on this function were poor: Thomassen proved the above result with $g(k) = 2^{Ck \log k}$, but there came a series of two major improvements. First, Kühn, Lapinskas, Osthus, and Patel [8] proved that it suffices to take $g(k) = 10^4 k \log k$ and conjectured that one could remove the logarithmic factor. Pokrovskiy [10], resolving this conjecture, showed that any $452k$-connected tournament is k-linked. Kang and Kim [6] proved an extension of this result, namely, that there exists an absolute constant C such that any Ck-connected tournament is k-linked, where the paths witnessing k-linkedness have prescribed lengths (provided the lengths are sufficiently large). Pokrovskiy went on to conjecture that one could push '452' down to '2' as long as the tournament has large minimum in/out-degree:

Conjecture 1.1 (Pokrovskiy [10]). *There is a function $f(k)$ such that any $2k$-connected tournament T with $\delta^0(T) \geq f(k)$ is k-linked, where $\delta^0(T) = \min\{\delta^+(T), \delta^-(T)\}$.*

This conjecture may be viewed as a directed analogue of several results in the undirected setting. In particular, as was mentioned earlier, $2k$-connectivity suffices provided one imposes some density condition on the

graph (like large average degree). Here, the natural 'density' condition for a tournament is large minimum in/out-degree.

In Section 3, we construct two families of tournaments that demonstrate that both conditions in Conjecture 1.1 are necessary: we show that for every $k \geq 2$ there exist infinitely many tournaments that are $(2k-1)$-connected with arbitrarily large minimum in/out-degrees, but which are not k-linked. Additionally, for every even $k \geq 6$ there are infinitely many tournaments that are $(2.5k-1)$-connected but are not k-linked.

The first and last authors [4] proved that the statement of Pokrovskiy's conjecture holds with '$2k$' replaced by '$4k$', without the assumption of large minimum in-degree.

Theorem 1.2 (Girão, Snyder [4]). *There is a function $f(k)$ such that any $4k$-connected tournament T with $\delta^+(T) \geq f(k)$ is k-linked.*

Our main result improves the above result in two ways, and nearly resolves Conjecture 1.1 in a stronger form. First, we are able to reduce the connectivity bound to $2k+1$. Second, we only impose a condition on the minimum out-degree, and the bound we obtain is significantly better than that obtained in Theorem 1.2. More precisely, while we proved Theorem 1.2 with $f(k)$ doubly-exponential, our main result shows that we may take $f(k)$ to be a polynomial.

Theorem 1.3. *There exists a polynomial f such that any $(2k+1)$-connected tournament T with $\delta^+(T) \geq f(k)$ is k-linked.*

An analysis of our proof shows that we may take $f(k) = Ck^{31}$ for some sufficiently large (but absolute) constant C. We have not made an attempt to optimize the power of k (see Section 4).

Our proof of Theorem 1.3 requires the notion of a subdivision. Recall that the complete digraph on k vertices, denoted by $\overrightarrow{K}_k$, is a directed graph in which every pair of vertices is connected by an edge in each direction. As usual, we say that a tournament T contains a subdivision of $\overrightarrow{K}_k$ if it contains a set B of k vertices and a collection of $2\binom{k}{2}$ pairwise internally vertex disjoint directed paths joining every ordered pair of vertices in B. We denote such a subdivision by $T\overrightarrow{K}_k$, and the vertices in B are called the *branch vertices* of the subdivision. Further, for a positive integer ℓ we denote by $T_\ell \overrightarrow{K}_k$ a subdivision of $\overrightarrow{K}_k$ where each edge is replaced by a directed path of length at most $\ell+1$ (i.e., each edge is subdivided at most ℓ times).

A central tool in the proof of Theorem 1.2 was to show that tournaments with sufficiently high minimum out-degree contain subdivisions of complete

digraphs. In particular, we showed that there is an absolute constant c such that any tournament T with $\delta^+(T) \geq 2^{2^{ck^2}}$ contains a $T\overrightarrow{K}_k$. Recently, the authors [5] improved considerably this result, reducing the bound on minimum out-degree to a quadratic.

1.1. Notation and organization

Our notation is standard. Thus, for a vertex v in a directed graph G, we let $N_G^+(v), N_G^-(v)$ denote the out-neighbourhood and in-neighbourhood of v, respectively. Moreover, we let $d_G^+(v) = |N_G^+(v)|$ denote the out-degree of v, and analogously $d_G^-(v)$ the in-degree of v. We often omit the subscript 'G' when the underlying digraph is clear. We denote by $\delta^+(G)$ the minimum out-degree of G; further, if $X \subset V(G)$, we write $\delta^+(X)$ to mean the minimum out-degree of $G[X]$. For a subset $X \subset V(G)$ we let $N^+(X)$ denote the set $\bigcup_{x \in X} N^+(x)$. If $X, Y \subset V(G)$, we write $X \to Y$ if every edge of G between X and Y is directed from X to Y. Whenever $X = \{x\}$ we simply write $x \to Y$ (and similarly if $Y = \{y\}$). If $P = x_1 \ldots x_\ell$ is a directed path, then we refer to x_1 as the *initial* vertex of P, and say that P *starts* at x_1. Similarly, we call x_ℓ the *terminal* vertex of P, and say that P *ends* at x_ℓ. We refer to both x_1 and x_ℓ as *endpoints* of P. The subpath of P excluding the initial and terminal vertices of P is called the *interior* of P, denoted by $\mathrm{int}(P)$.

The remainder of this paper is organized as follows. In Section 2, we give the proof of our main theorem. In Section 3, we present two families of constructions showing the necessity of both conditions in Theorem 1.3. Finally, we close the paper with some open problems in Section 4.

2. Proof of the main result

2.1. Preliminaries

We need the following simple lemma from [5]. To state it, we say that a subset $B \subset V(T)$ is *C-nearly-regular* if either $d^-(v) \leq d^+(v) \leq Cd^-(v)$ for every $v \in B$, or $d^+(v) \leq d^-(v) \leq Cd^+(v)$ for every $v \in B$. Further, B is *(C, m, t)-nearly-regular* if it is C-nearly-regular and additionally $d^-(v) \in [m - 10t, m + 10t]$ for every $v \in B$. The following lemma allows us to find $(4, m, t)$-nearly-regular t-element subsets in tournaments. We include the short proof for the reader's convenience.

Lemma 2.1. *Any tournament T contains a 4-nearly-regular subset of size $|T|/10$, and a $(4, m, t)$-nearly-regular subset of size t provided $|T| \geq t$, for some m.*

Proof. We first claim that T contains a 4-nearly-regular subset of size at least $|T|/10$. Indeed, let $|T|=n$ and let $R \subset V(T)$ be the vertices for which either the ratio between the out-neighbourhood and in-neighbourhood (or vice-versa) is between 1 and 4. If $|R| \geq n/5$, then we are done, as we may pass to a subset $A \subset R$ of at least half the size for which the property is satisfied for one or the other. If not, then let $T'=T \setminus R$, so that $|T'| \geq 4n/5$. Let T_1' be the set of vertices $v \in V(T')$ for which $d_T^+(v) > 4d_T^-(v)$ and T_2' be those vertices $v \in V(T')$ for which $d_T^-(v) > 4d_T^+(v)$. Suppose without loss of generality that $|T_1'| \geq |T_2'|$, so that $|T_1'| \geq 2n/5$. This implies that there is a vertex u in T_1' which has in-degree inside T_1' at least $n/5$. But then

$$n/5 \leq d_T^-(u) < \frac{1}{4}d_T^+(u) \leq n/5,$$

a contradiction.

Thus, we can always find a 4-nearly-regular subset A of size at least $|T|/10$. Partition the interval $[1,\ldots,|T|]$ into consecutive intervals of size $10t$, and distribute the vertices of A according to their in-degrees in T. By the pigeonhole principle, there must exist at least

$$10t \cdot \frac{|A|}{|T|} \geq 10t \cdot \frac{1}{10} = t$$

vertices in the same interval. These t vertices form a $(4,m,t)$-nearly-regular subset for some m. ∎

We also need the following well-known result by Erdős and Szekeres.

Proposition 2.2. *Let S be a finite set of order n and suppose there exist ℓ total orderings $<_1,\ldots,<_\ell$ on S. Then there exists a subset $S' \subset S$ of size at least $t=n^{1/2^{\ell-1}}$ and an ordering of $S' = \{s_1,s_2,\ldots,s_t\}$ such that, in this ordering, S' forms an increasing chain in $<_1$ and an increasing chain or decreasing chain in $<_i$ for every $i \in \{2,\ldots,\ell\}$.*

2.2. Finding a (k,ℓ)-good family

We need some terminology to state our main lemma precisely. Let T be a tournament and let $X \subset V(T)$. We say that a subdivision contained in T *sits on* X if its branch vertex set is some subset of X.

Definition 2.3. Let T be a tournament and suppose k,ℓ are positive integers. A family $\mathcal{F}$ of pairwise disjoint subsets of $V(T)$ is (k,ℓ)-*good* in T if it satisfies the following properties:

- For each $A \in \mathcal{F}$, if there is no $T_2 \overrightarrow{K}_\ell$ sitting on A, then $|A| = 12k^2$.
- For each distinct $A, B \in \mathcal{F}$, if there is no $T_2 \overrightarrow{K}_\ell$ sitting on A and no $T_2 \overrightarrow{K}_\ell$ sitting on B, then either

$$A \to B \quad \text{or} \quad B \to A \quad \text{in} \quad T.$$

Moreover, we denote by $S(\mathcal{F})$ the *subdivision sets* in $\mathcal{F}$: those $A \in \mathcal{F}$ such that there exists a $T_2 \overrightarrow{K}_\ell$ sitting on A.

The first part of the definition is there for technical reasons later in the proof. The important point is that the size of A is quadratic in k.

Here is our main lemma.

Lemma 2.4. *Let $k \leq \ell$ be positive integers and suppose T is a tournament such that $V(T) = \bigcup_{i \in [k]} W_i$ for pairwise disjoint subsets $W_1, W_2, \ldots, W_k$ with $|W_i| \geq 12k^{22}\ell^2$ for each i. Then there exists a family of sets $\mathcal{F} = \{S_1, \ldots, S_k\}$ that is (k, ℓ)-good in T with $S_i \subset W_i$ for $i = 1, \ldots, k$. Furthermore, there is a family $\mathcal{T}$ of pairwise vertex disjoint copies of $T_2 \overrightarrow{K}_\ell$ such that for each $S_i \in S(\mathcal{F})$ there is some subdivision $\mathcal{S} \in \mathcal{T}$ that sits on S_i.*

Proof. We can assume that each W_i has size precisely $12k^{22}\ell^2$ by, when necessary, passing to smaller subsets of exactly that size. We proceed by induction on k; for $k = 1$ there is nothing to show, so assume $k \geq 2$ and the result holds for smaller values. Applying Lemma 2.1 with $t = k\ell$ to T, we find a $(4, m, k\ell)$-nearly-regular subset $A \subset \bigcup_{i \in [k]} W_i$ of size $k\ell$. Without loss of generality, we may assume that $d^-(v) \leq d^+(v) \leq 4d^-(v)$ for every $v \in A$. Now, there is a subset B of A size at least $|A|/k = \ell$ contained in some W_i, say W_1, without loss of generality. We now break the proof up into two cases, depending on whether or not there is a $T_2 \overrightarrow{K}_\ell$ sitting on B. Suppose there exists such a subdivision sitting on B, say $\mathcal{S}$. Then remove the non-branch vertices of $\mathcal{S}$ from T to form the subtournament $T' = \bigcup_{i=2}^{k} W_i'$. We have removed at most $2\binom{\ell}{2} < \ell^2$ vertices from T, so $|W_i'| > 12k^{22}\ell^2 - \ell^2 \geq 12(k-1)^{22}\ell^2$, and so we are done by induction applied to T' .

Therefore, assume that no subdivision sits on B. In order to apply induction, we require the following claim, which allows us to partition the W_i's in a particularly nice way.

Claim 1. *Let $W_1, \ldots, W_k$ and B be as above with no $T_2 \overrightarrow{K}_\ell$ sitting on B. Then there is a partition $I \cup J = [k]$ and families $\mathcal{F}_1 = \{W_i : i \in I\}$, $\mathcal{F}_2 = \{W_j : j \in J\}$ satisfying:*

(1) $|\mathcal{F}_i| \geq k/10$ for $i = 1, 2$.

(2) *There exists $W_i' \subset W_i$ with $|W_i'| \ge |W_i|/10$ for each $i \in [k]$ such that*

$$\bigcup_{i \in I} W_i' \to \bigcup_{j \in J} W_j'.$$

Proof. Suppose we try to embed greedily a $T_2 \overrightarrow{K}_\ell$ with branch vertex set B. Since by assumption we cannot succeed, there exists a partial subdivision $\mathcal{S}$ and two distinct vertices $x, y \in B$ such that

$$N^+(x) \cap N^-(y) \subset \mathcal{S} \quad \text{and} \quad N^-(y) \setminus \mathcal{S} \to N^+(x) \setminus \mathcal{S}.$$

As B is $(4, m, k\ell)$-nearly-regular for some m we have that $d^-(x), d^-(y) \in [m - 10k\ell, m + 10k\ell]$. Further, $|N^-(y) \setminus N^-(x)| = |N^+(x) \cap N^-(y)| \le |\mathcal{S}| \le 2\binom{\ell}{2} = \ell^2$, so we obtain

$$|N^-(x) \setminus N^-(y)| \le |N^-(y) \setminus N^-(x)| + 10k\ell \le \ell^2 + 10k\ell \le 11\ell^2,$$

where the last inequality follows since $k \le \ell$, by assumption. Recall that B is 4-nearly-regular, and as such, $d^-(v) \le d^+(v) \le 4d^-(v)$ for every $v \in B$. This implies, in particular, that $d^-(v), d^+(v) \ge |T|/5$ for each v. Letting $X = N^+(x) \setminus \mathcal{S}$ and $Y = N^-(y) \setminus \mathcal{S}$, we obtain that $|X|$ and $|Y|$ are both at least

$$\ge |T|/5 - |\mathcal{S}| - |N^+(x) \cap N^-(y)| - |N^-(x) \cap N^+(y)|$$
$$\ge |T|/5 - \ell^2 - \ell^2 - 11\ell^2$$
$$= |T|/5 - 13\ell^2.$$

Moreover, $|X \cup Y| \ge |T| - 12\ell^2$; without loss of generality, assume $|Y| \ge |X|$. Then $|Y| \ge |T|/2 - 6\ell^2$, and $|X| \ge |T|/5 - 13\ell^2$. In summary, we have obtained large sets X and Y with $Y \to X$, and such that $X \cup Y$ covers most of T. In particular, as $V(T) = \bigcup_{i \in [k]} W_i$, for each $i \in [k]$ we have that either $|W_i \cap Y| \ge |W_i|/2 - 6\ell^2 \ge |W_i|/10$ or $|W_i \cap X| \ge |W_i|/2 - 6\ell^2 \ge |W_i|/10$. Now, partition $[k]$ into I and J such that for every $i \in I$ we have $|W_i \cap X| \ge |W_i|/10$, for every $j \in J$ we have $|W_j \cap Y| \ge |W_j|/10$, and $|I|, |J|$ are as equal as possible. Finally, set $\mathcal{F}_1 = \{W_i : i \in I\}$ and $\mathcal{F}_2 = \{W_j : j \in J\}$ and let $W_i' = W_i \cap Y$ if $i \in I$ and $W_i' = W_i \cap X$ if $i \in J$. Then property (2) of the claim certainly holds by definition of the sets W_i' and the fact that $Y \to X$, so we just need to check that $|I|$ and $|J|$ are large according to (1). Suppose for contradiction that this is not the case and, say, $|I| < k/10$. This means that $|J| > \frac{9}{10}k$ and for every $j \in J$ we have $|X \cap W_j| < |W_j|/10$, as otherwise we could move j from J to I, decreasing the distance between $|I|$ and $|J|$ and hence contradicting the assertion that $|I|, |J|$ are as equal as possible.

Therefore, $|W_j \cap Y| \geq |W_j| - |W_j \cap X| - 12\ell^2 > \frac{9|W_j|}{10} - 12\ell^2$ for every $j \in J$. It follows that $|Y| > \frac{9}{10}k \cdot \left(\frac{9|T|}{10k} - 12\ell^2\right) \geq \frac{81}{100}|T| - 12k\ell^2$, which implies that $|X| < \frac{19}{100}|T| + 12k\ell^2$. Together with the fact that $|X| \geq |T|/5 - 13\ell^2$, we have

$$|T| < 100(12k\ell^2 + 13\ell^2) \leq 2500k\ell^2 < 12 \cdot 2^{21}k\ell^2 \leq 12k^{22}\ell^2,$$

where we used the assumption that $k \geq 2$. This contradicts the fact that $|T| \geq |W_1| \geq 12k^{22}\ell^2$. ∎

Let $W_1', \ldots, W_k'$, and $\mathcal{F}_1, \mathcal{F}_2$ be given as in the conclusion of Claim 1. As $\min\{|\mathcal{F}_1|, |\mathcal{F}_2|\} \geq k/10$, and since for each i we have

$$|W_i'| \geq \frac{12k^{22}}{10}\ell^2 \geq 12(9k/10)^{22}\ell^2,$$

apply induction to $T_1 = T[\bigcup_{i \in I} W_i']$ and to $T_2 = T[\bigcup_{j \in J} W_j']$. This yields a (k, ℓ)-good family $\mathcal{F} = \{S_1, \ldots, S_k\}$ in T, and a family $\mathcal{T}$ of copies of $T_2 \overrightarrow{K}_\ell$, as required. Indeed, if there is no subdivision sitting on S_i nor S_j, then if $i, j \in I$ (or $i, j \in J$), the required property is satisfied by induction, and if $i \in I$, $j \in J$, then $S_i \to S_j$ by construction of the families $\mathcal{F}_1$ and $\mathcal{F}_2$. Moreover, as T_1 and T_2 are disjoint, all subdivisions in $\mathcal{T}$ are pairwise vertex disjoint. ∎

2.3. Utilizing a (k, ℓ)-good family

Let T be a $(2k+1)$-connected tournament with minimum out-degree at least $k \cdot 12k^{22}\ell^2 + 2k$, where $\ell \geq 3k + 10^4 k^3 + 2 \cdot 10^{13} k^4$. We may assume that $k \geq 2$, since the result is immediate for $k = 1$. Suppose $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_k\}$ are vertex disjoint k-sets of vertices and that we wish to link x_i to y_i in T for each $i \in [k]$. First, because of the large minimum out-degree we can find pairwise vertex disjoint sets $W_i \subset (N^+(x_i) \setminus (X \cup Y))$ such that $|W_i| = 12k^{22}\ell^2$ for each $i = 1, \ldots, k$. Consider now the subtournament

$$T_0 = T[\bigcup_{i \in [k]} W_i].$$

By Lemma 2.4 applied to T_0, find a (k, ℓ)-good family $\mathcal{F} = \{S_1, \ldots, S_k\}$ with $S_i \subset W_i$ for each i, and a family $\mathcal{T}$ of pairwise disjoint copies of $T_2 \overrightarrow{K}_\ell$ according to the lemma. Recall that $S(\mathcal{F})$ denotes the subdivision sets in $\mathcal{F}$: those $S_i \in \mathcal{F}$ such that there exists a subdivision from $\mathcal{T}$ sitting on S_i. We shall assume that each such S_i consists of precisely those branch vertices in the corresponding subdivision that sits on it (by possibly removing some

vertices from S_i). Further, any set in $\mathcal{F}$ that is not a subdivision set we shall call a *non-subdivision* set.

Define an auxiliary digraph H with vertex set $[k]$ in the following way. For a pair $i, j \in [k]$ with $i \neq j$:

- If S_i and S_j are non-subdivision sets, then orient i to j if $S_i \rightarrow S_j$ in T, and vice-versa if $S_j \rightarrow S_i$ (here we are using that $\mathcal{F}$ is a (k, ℓ)-good family).
- If S_i is a non-subdivision set but S_j is a subdivision set, then orient i to j if (a) at least $|S_i|/2$ vertices in S_i have at least $|S_j|/2$ out-neighbours in S_j, and orient j to i if (b) at least $|S_i|/2$ vertices in S_i have at least $|S_j|/2$ in-neighbours in S_j.
- If both S_i and S_j are subdivision sets, orient i to j if at least $|S_i|/4$ vertices in S_i have at least $|S_j|/4$ out-neighbours in S_j, and orient j to i if at least $|S_j|/4$ vertices in S_j have at least $|S_i|/4$ out-neighbours in S_i.

Note that H is a semicomplete digraph (i.e., it is a tournament with some potential double edges created between subdivision sets in the third case above). Let H^s denote the subdigraph of H induced on $\{i \in [k]: S_i$ is a subdivision set$\}$, and let H^{ns} denote the subdigraph induced on the remaining vertices of H. Observe that it is possible that one of H^s, H^{ns} is empty. Since H^s is a tournament up to some possible double edges, it contains a Hamiltonian path P^s. Similarly, H^{ns} contains a Hamiltonian path P^{ns}. With some abuse of notation we write

$$P^s = S_{i_1} \ldots S_{i_r} \text{ and } P^{ns} = S_{j_1} \ldots S_{j_t},$$

where $\{i_1, \ldots, i_r\} \cup \{j_1, \ldots, j_t\} = [k]$, to emphasize that these paths in H^s and H^{ns} correspond to sequences of the sets S_i. Further, we shall at times write $\text{init}(P^s)$ and $\text{ter}(P^s)$ for S_{i_1} and S_{i_r}, respectively (and similarly for P^{ns}).

For technical reasons we discard the vertices in S_{j_t} which have fewer than $|S_{i_r}|/2$ out-neighbours in S_{i_r} in case (a), or fewer than $|S_{i_r}|/2$ in-neighbours in S_{i_r} in case (b).

Our aim is to apply Menger's theorem to find $k+1$ vertex disjoint paths from either $\text{ter}(P^s)$ or $\text{ter}(P^{ns})$ to $Y \cup \{v\}$, where v is some vertex in $\cup_{q=1}^t S_{j_q}$ (initially, we choose $v \in \text{init}(P^{ns})$ to have high out-degree in $\text{init}(P^{ns})$). We call the initial set of $k+1$ vertices of these paths the *origin*, and denote it by O, and we call the vertex v the *special vertex*. The choice of O depends on the following circumstances:

(1) If $P^s = \emptyset$, choose $O \subset \text{ter}(P^{ns})$; similarly, if $P^{ns} = \emptyset$, choose $O \subset \text{ter}(P^s)$.
(2) If $i_r j_t \in E(H)$, choose $O \subset S_{j_t}$.
(3) If $j_t i_r \in E(H)$, choose $O \subset S_{i_r}$.

In each case, we initially let the special vertex v be an element of $\text{init}(P^{ns})$ with the largest out-degree in $\text{init}(P^{ns})$, except of course when $P^{ns} = \emptyset$: in that case we choose no special vertex and we let $O \subset \text{ter}(P^s)$ be a set of size k. In fact, we shall assume that (1) does not occur since the proof in this case follows from the arguments for cases (2) and (3) (and is simpler).

So choose O in accordance with (2) or (3) and let v be the special vertex in $\text{init}(P^{ns})$. Since T is $(2k+1)$-connected, Menger's theorem implies that, upon the removal of X, there exists a family $\mathcal{Q}$ of $k+1$ pairwise vertex disjoint directed paths from O to $Y \cup \{v\}$. Let $\mathcal{Q}$ be chosen to minimize $|\bigcup \mathcal{Q}|$, the total number of vertices used in the paths. We refer to the path in $\mathcal{Q}$ ending at v as the *special path* in $\mathcal{Q}$. In general, during the course of the proof we shall make modifications to the family $\mathcal{Q}$. If $\mathcal{Q}'$ denotes another collection of paths from O to $Y \cup \{v'\}$ for some $v' \notin Y$, then we refer to the path ending at v' as the *special path* of $\mathcal{Q}'$.

We would like to do the following for each $i = 1, \ldots, k$: starting with x_i, form a directed path to y_i by first choosing an out-neighbour of x_i in S_i, then travelling along one of the paths P^s or P^{ns} (depending on whether S_i happens to be a subdivision or non-subdivision set) to the corresponding vertex in O. Finally, we use the paths from $\mathcal{Q}$ to reach y_i. We need to ensure these paths are chosen disjointly, but more importantly, we need to ensure that the initial paths in $\mathcal{Q}$ do not obstruct our goal. In other words, we need to make sure that the paths in $\mathcal{Q}$ do not intersect the S_i's in too many places.

Our first lemma in this regard asserts that if S_i is a subdivision set, then the paths from $\mathcal{Q}$ do not intersect S_i in many places.

Lemma 2.5. *Suppose that $S = S_i$ is a subdivision set. Then at most $10^4 k^3$ vertices of S belong to $\bigcup \mathcal{Q}$.*

Proof. As S is a subdivision set, there is a copy $\mathcal{S}$ of $T_2 \overrightarrow{K}_\ell$ in T_0 with branch vertex set S. For each ordered pair of vertices $a, b \in S$ write P_{ab} for the path in $\mathcal{S}$ from a to b. Suppose the lemma is false, so that there is some path $Q := Q_j \in \mathcal{Q}$ which intersects S in $m = 10^4 k^3/(k+1) \geq 4 \cdot 10^3 k^2$ vertices. Denote these vertices by $u_1, \ldots, u_m$ in the ordering they appear along the path Q. A subdivision path P_{ab} is *free* if no path of $\mathcal{Q}$ intersects the interior $\text{int}(P_{ab})$. In the remainder of this proof, we write $P_{i,j}$ for $P_{u_i u_j}$. If any of the paths in $\mathcal{P} := \{P_{i,m-i+1} : i = 1, \ldots, 10^3 k^2\}$ are free, then we reach a contradiction with the minimality of $\mathcal{Q}$: each path $P_{i,j}$ has length at most 3, so we can replace Q with a shorter path by simply taking a shortcut through one of the free paths. It follows that each of these $10^3 k^2$ paths contains at least one vertex from $\bigcup \mathcal{Q}$.

Given $M, N \in (\mathcal{Q} \cup \{\emptyset\})$ we call (M, N) the *intersection pattern* of $P \in \mathcal{P}$ if M either intersects P in the second vertex or is empty and there is no path in $\mathcal{Q}$ which intersects P in the second vertex; and similarly N either intersects P in the third vertex or is empty and there is no path in $\mathcal{Q}$ which intersects P in the third vertex. It is easy to see that there are at most $(k+1)^2$ intersection patterns. Indeed, if $P \in \mathcal{P}$ is given, then there are $k+1$ choices for each internal vertex: one of the k paths in $\mathcal{Q}$, or none. By the pigeonhole principle, there is a collection of paths $\mathcal{P}' \subset \mathcal{P}$ of size at least $|\mathcal{P}|/(k+1)^2 = 10^3 k^2/(k+1)^2 \geq 100$ such that all have the same intersection pattern. Suppose the intersection pattern is given by $M, N \in \mathcal{Q}$, not both empty. In other words, M and N intersect the second and third vertex, respectively, of each path in $\mathcal{P}'$. Our aim now is to create a new family of paths from O to $Y \cup \{v\}$ which uses fewer vertices, contradicting the minimality of $\mathcal{Q}$.

Suppose first that $M = N$. If for any $P_{i,j} \in \mathcal{P}'$ we have that the third vertex of P comes before the second vertex of $P_{i,j}$ along the path M, then we can perform the following rerouting of Q and M. Let x and y be the second vertex and the third vertex of $P_{i,j}$, respectively. Form the path Q' by following Q to u_i, go to x, and continue via M to the terminal vertex of M. Then, form the path M' by following M to y, go to u_j, and continue via Q to the terminal vertex of Q. Observe that replacing Q and M with Q' and M' results in path system which has fewer vertices than $\mathcal{Q}$ (as we are not using the vertices of M which came after y but before x), which contradicts the minimality of $\mathcal{Q}$. We may then assume for all $P_{i,j}$ the third vertex comes after than the second vertex along M. In that case, we may also assume that these vertices appear consecutively along M by minimality.

Otherwise we can find three paths $P_1, P_2, P_3 \in \mathcal{P}'$ such that, according to the ordering given by M, the internal vertices of P_1 come before the internal vertices of P_2 and the internal vertices of P_2 come before the internal vertices of P_3. Form the path Q' by following Q to the first vertex of P_3, go to the second vertex of P_3, and continue via M to the terminal vertex of M. Then, form the path M' by following M to the third vertex of P_1, go to the fourth vertex of P_1, and continue via Q to the terminal vertex of Q. Observe that replacing Q and M with Q' and M' results in path system which has fewer vertices than $\mathcal{Q}$ (as we are not using the vertices of P_2 anymore), which contradicts the minimality of $\mathcal{Q}$.

From now on, we shall assume that M and N are distinct and nonempty paths, as the case when one of them is empty follows a similar (and simpler) analysis. We define two total orderings $<_1, <_2$ on the paths in $\mathcal{P}'$. Indeed, we say $P_{i,j} <_1 P_{i',j'}$ if the second vertex of $P_{i,j}$ comes before the second vertex

of $P_{i',j'}$ in the path M, and similarly $P_{i,j} <_2 P_{i',j'}$ if the third vertex of $P_{i,j}$ comes before the third vertex of $P_{i',j'}$ in the path N.

Applying Proposition 2.2, we may pass to a subcollection $\mathcal{P}'' \subset \mathcal{P}'$, where $|\mathcal{P}''| \geq |\mathcal{P}'|^{1/4} \geq (100)^{1/4} \geq 3$. Let $\mathcal{P}'' = \{P_{i_1,m-i_1+1}, P_{i_2,m-i_2+1}, P_{i_3,m-i_3+1}\}$ (where $i_1 < i_2 < i_3$) such that it forms an increasing or decreasing chain in each of the total orderings $<_1$ and $<_2$.

There are four cases to consider:

(1) $\mathcal{P}''$ forms an increasing chain in both $<_1$ and $<_2$;
(2) $\mathcal{P}''$ forms an increasing chain in $<_1$ and decreasing chain in $<_2$;
(3) $\mathcal{P}''$ forms a decreasing chain in $<_1$ and an increasing chain in $<_2$;
(4) $\mathcal{P}''$ forms a decreasing chain in both $<_1$ and $<_2$.

We shall see how to proceed in Cases (1) and (2). The other cases follow by a symmetric argument.

So first assume that $\mathcal{P}''$ forms an increasing chain in both $<_1$ and $<_2$. We are going to perform the following rerouting of the paths Q, M, N using the paths in $\mathcal{P}''$. For brevity, write P_{i_j} for $P_{i_j,m-i_j+1}$ for $i=1,2,3$. Form the path M' by following Q to u_{i_3}, go to the second vertex of P_{i_3} (which belongs to M), and continue via M to the terminal vertex of M. Form the path N' by following M from the initial vertex of M to the second vertex of P_{i_2} (which we can do, since this vertex appears in M before the second vertex of P_{i_3}), following P_{i_2} to the third vertex of P_{i_2} (which belongs to N), and continue via N to the terminal vertex of N. Finally, form Q' by following N from its initial vertex to the third vertex of P_{i_1}, continue to the last vertex of P_{i_1}, and then continue via Q to the terminal vertex of Q. It is not hard to check that in this process we gain 3 new directed edges, but loose at least 5. Thus, letting $\mathcal{Q}' = (\mathcal{Q} \setminus \{Q,M,N\}) \cup \{Q',M',N'\}$, we see that $|\bigcup \mathcal{Q}'| < |\bigcup \mathcal{Q}|$, contradicting the minimality of $\mathcal{Q}$.

Let us consider Case (2), that is, $\mathcal{P}''$ is an increasing chain in $<_1$ and a decreasing chain in $<_2$. Now we may form Q' by following N to the third vertex of P_{i_3}, following P_{i_3} to the last vertex of P_{i_3} (which is $u_{m-i_3+1} \in Q$), and then continuing along Q to the terminal vertex of Q. Form M' by following Q from its initial vertex to u_{i_2}, then following P_{i_2} to its second vertex (which is in M), and then continuing along M to the terminal vertex of M. Lastly, form N' following M from its initial vertex to the second vertex of P_{i_1}, following P_{i_1} to its third vertex (which is in N), and then continuing via N to the terminal vertex of N. As before, we gain 3 new edges but lose at least 5, so the path system obtained by replacing Q, M, N with Q', M', N', respectively, has a fewer total number of vertices, contradicting minimality. ∎

We also need to show that for each subdivision set, 'many' of the subdivision paths connecting branch vertices do not intersect the path system $\mathcal{Q}$. This is the content of the following lemma.

Lemma 2.6. *Let $S = S_i$ be a subdivision set and let $\mathcal{S}$ denote the subdivision sitting on S. Let $x \in S$ with $x \notin \bigcup \mathcal{Q}$. Then there are at most $10^{13}k^4$ paths in $\mathcal{S}$ with one endpoint x and another endpoint in $S \backslash \bigcup \mathcal{Q}$ that intersect $\bigcup \mathcal{Q}$.*

Proof. This proof follows a very similar argument as in the proof of Lemma 2.5. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_{10^{13}k^4}\}$ be a collection of paths from the subdivision $\mathcal{S}$ each of which has non empty intersection with $\bigcup \mathcal{Q}$. Moreover, suppose each path of $\mathcal{P}$ has $x \in S \backslash \bigcup \mathcal{Q}$ as an initial or terminal vertex.

We may assume at least half of these paths, say $\mathcal{P}' := \{P_1, \ldots, P_{10^{12}k^4}\}$, start at x (the other case is symmetric). Note that each path in $\mathcal{P}'$ has length at most 3, which implies that each such path can have at most 2 vertices that belong to $\bigcup \mathcal{Q}$. As argued before, there are at most $2(k+1)^2$ possible patterns regarding intersections with $\bigcup \mathcal{Q}$. We therefore may pass to a sub-collection of $\mathcal{P}'$ of size at least $|\mathcal{P}'|/2(k+1)^2 \geq 10^{12}k^4/8k^2 > 10^{11}k^2$ where all paths have the same intersection pattern. With a slight abuse of notation we shall still denote this collection of paths by $\mathcal{P}'$. Suppose the intersection pattern is given by $M, N \in \mathcal{Q}$, not both empty. In other words, M, N intersect the second and third vertex, respectively, of each path in $\mathcal{P}'$. In the following we identify a path in $\mathcal{P}'$ by its second and third vertices. Thus if $P \in \mathcal{P}'$ has second vertex a and third vertex b, then we shall write P_{ab} for P. In this case, we have $a \in M$ and $b \in N$ (one of M, N could be empty). Now, we apply the same procedure to the paths from $\mathcal{S}$ that start at the terminal vertices of the paths in $\mathcal{P}'$ and which end at x. Indeed, let this collection of paths be denoted by $\mathcal{R}'$. As before, since there are at most $2(k+1)^2$ possible intersection patterns between a path in $\mathcal{R}'$ and the collection $\bigcup \mathcal{Q}$, we may pass to a sub-collection of $\mathcal{R}'$, say $\mathcal{R}''$ all of whose paths have intersection pattern $M', N' \in \mathcal{Q}$, where now both M' and N' could be empty. Clearly $|\mathcal{R}''| \geq |\mathcal{P}'|/2(k+1)^2 \geq 10^{10}$. Let $\mathcal{P}''$ be the set of paths in $\mathcal{P}$ which end at an initial vertex of some path in $\mathcal{R}''$.

We define two total orderings $<_1, <_2$ on the paths of $\mathcal{P}''$: $P_{ab} <_1 P_{cd}$ if a comes before c in the path M, and $P_{ab} <_2 P_{cd}$ if b comes before d in the path N. By Proposition 2.2 applied to the two orderings $<_1$, $<_2$, we may pass to a subset of $\mathcal{P}''$ of size $(10^{10})^{1/2} \geq 10^5$ with an ordering of the paths such that they form an increasing chain in $<_1$ and an increasing or decreasing chain in $<_2$. For simplicity, denote this collection again by $\mathcal{P}''$.

Likewise, we may define two total orderings on the paths of $\mathcal{R}''$ (restricted to those paths with the same endpoints as paths in $\mathcal{P}''$ and with the induced

ordering given by $<_1$): define $P_{ab} <_3 P_{cd}$ if a comes before c in the path M', and $P_{ab} <_4 P_{cd}$ if b comes before d in the path N'. Applying Proposition 2.2 to $\mathcal{R}''$, with the orderings $<_1, <_3, <_4$, we obtain a collection of paths of size at least $(10^5)^{1/4} > 12$ which forms an increasing chain in $<_1$ and an increasing or decreasing chain in $<_3$ and $<_4$. With a slight abuse of notation we shall still denote this sub-collection by $\mathcal{R}''$. Let $\mathcal{R}'' = \{R_1, R_2, \ldots, R_{12}\}$ and $\mathcal{P}'' = \{P_1, P_2, \ldots, P_{12}\}$ be the corresponding paths in $\mathcal{P}'$ where the endpoint of P_i is the initial vertex of R_i, for every $i \in [12]$. We shall assume that each of the paths M, N, M', N' are nonempty (otherwise, the following rerouting argument only becomes simpler).

There are eight cases to consider, depending on whether or not $\mathcal{P}''$ is an increasing or decreasing chain in $<_2$, and whether or not $\mathcal{R}''$ is an increasing or decreasing chain in $<_3$ and $<_4$. We consider two of these cases (the others follow by a similar arguments):

(1) $\mathcal{P}''$ forms an increasing chain in both $<_1$ and $<_2$; $\mathcal{R}''$ forms increasing chain in both $<_3$ and $<_4$;

(2) $\mathcal{P}''$ is increasing in $<_1$ and decreasing in $<_2$; $\mathcal{R}''$ is increasing in $<_3$ and increasing in $<_4$.

Let us consider now Case (1). We are going to make the following rerouting of the paths M, N, M' and N'. More precisely, whenever the path M hits P_4, then it goes to the third vertex of P_4 and continues via the sub-path of N which starts at the third vertex of P_4. Whenever the path N hits P_3 (which is before hitting P_4, by assumption), then it goes to the second vertex of R_3 through the terminal vertex of P_3 and continues via M'. Similarly, the path M' is altered in the following way: whenever it hits R_2 (which is before hitting R_3), then it goes to the third vertex of R_2 and continues via the path N'. Finally, whenever N' hits R_1, then it goes to x and then to the second vertex of P_{12} which belongs to M and continues via M. Call this new collection of paths $\mathcal{Q}'$. Note that we have added at most 6 more edges in total by using the paths in $\mathcal{P}''$ and $\mathcal{R}''$, but we now miss all second vertices of the paths $P_5, \ldots, P_{11}$. Therefore we decreased $|\bigcup \mathcal{Q}'|$, which is a contradiction.

Finally, consider Case (2). We perform the following rerouting. Whenever N hits the third vertex of P_3, follow P_3 to its terminal vertex, then to the second vertex of R_3, and continues via M'. Now, starting from M', follow M' to the second vertex of R_2, follow R_2 to its third vertex, then continue via N'. Starting from N', follow N' to the third vertex of R_1, continue along R_1 to x, follow P_{12} to its second vertex, and then continue via M. Finally, starting from M, follow M to the second vertex of P_1, follow P_1 to its third vertex, and then continue via N. Denoting this new collection by $\mathcal{Q}'$, we note

that $\mathcal{Q}'$ uses 6 new edges from the paths in $\mathcal{P}''$, $\mathcal{R}''$, but avoids the second vertices of $P_2, \ldots, P_{11}$. It follows that $|\bigcup \mathcal{Q}'| < |\bigcup \mathcal{Q}|$, a contradiction. ∎

Our next goal is to show that our path system $\mathcal{Q}$ can be modified such that it does not intersect non-subdivision sets in too many places. Roughly speaking, we will show that, if some path in $\mathcal{Q}$ intersects some non-subdivision set S_i in many places, then we can transform $\mathcal{Q}$ into another collection of paths that (1) are vertex disjoint and go from O to $Y \cup \{v^*\}$, where v^* is some vertex in $\bigcup_{q=1}^{t} S_{j_q}$, (2) does not intersect subdivision sets in more places than $\mathcal{Q}$ did, and (3) intersects non-subdivision sets in 'few' vertices. We formalize this in the following lemma. To state it precisely, we make the following definitions. Let $\mathcal{P}$ be some path system constructed in the above process from O to $Y \cup \{z\}$ with special vertex $z \in \bigcup_{q=1}^{t} S_{j_q}$. A vertex $s \in S_{j_q}$ is $\mathcal{P}$-*free* if no path in $\mathcal{P}$ intersects s. Furthermore, we say that S_{j_q} is $(\mathcal{P}, l)$-*free* if it contains at least l free vertices.

Lemma 2.7. *There exists a family $\mathcal{Q}^*$ of vertex disjoint directed paths from O to $Y \cup \{v^*\}$, where $v^* \in \bigcup_{q=1}^{t} S_{j_q}$ satisfying the following properties:*

(1) *S_{j_q} is $(\mathcal{Q}^*, q)$-free for $1 \leq q \leq t$.*
(2) *The paths in $\mathcal{Q}^*$ do not intersect subdivision sets in more vertices than paths in $\mathcal{Q}$ do.*

Proof. We consider the sets S_{j_q} in order from $q = 1$ to t, and show that we can incrementally free vertices in each set along the way. The process terminates with the desired path system $\mathcal{Q}^*$. To simplify notation during the course of this proof, we write S_q for S_{j_q} for $q = 1, \ldots, t$. To begin, consider S_1 together with the original path system $\mathcal{Q}$ from O to $Y \cup \{v\}$ with $v \in S_1$. We may assume that v was chosen to be a vertex in S_1 with out-degree at least $(|S_1| - 1)/2$ in S_1. Since the S_q's are a part of a (k, ℓ)-good family, we have $|S_q| = 12k^2$ (see Definition 2.3). Moreover, we discarded at most half of the vertices of the last non-subdivision set S_t, so $|S_t| \geq 12k^2/2 = 6k^2$. Now, if $\bigcup \mathcal{Q}$ intersects S_1 in at most $|S_1| - 2$ vertices, then S_1 is $(\mathcal{Q}, 2)$-free. Otherwise, $\bigcup \mathcal{Q}$ intersects S_1 in at least $|S_1| - 1$ vertices, and hence intersects $N := N_{S_1}^+(v)$ in at least $|N| - 1 \geq |S_1|/2 - 1 \geq 3k^2 - 1$ vertices. Thus, some path $P \in \mathcal{Q}$ intersects N in at least $(3k^2 - 1)/(k+1) > 2k - 1$ vertices. Let $u_1, \ldots, u_l$ be the vertices in the intersection in their order along P with $l \geq 2k$. If P is the special path in $\mathcal{Q}$, then replace P with $P' = Pu_1$, so that u_1 is the new special vertex. Then the vertices $u_2, \ldots, u_l$ are free. Otherwise, P is not a special path. Let $Q \neq P$ denote the special path in $\mathcal{Q}$ with terminal vertex $v \in S_1$. Replace Q with Pu_1 and let u_1 be the new special vertex. Replace P with the path P' defined by following Q to v, going along the edge vu_l, then

following P to its endpoint in Y. We have thus freed vertices $u_2, \ldots, u_{l-1}$, and since $l \geq 2k \geq 4$, we have freed at least 2 vertices.

In any case, we denote the resulting collection of paths and special vertex by $\mathcal{Q}^1$ and $v^1 \in S_1$, respectively. Observe that S_1 is now $(\mathcal{Q}^1, 2)$-free.

Now, suppose $2 \leq p < t$, and that we have already constructed a family of paths $\mathcal{Q}^{p-1}$ with special vertex $v^{p-1} \in S_z$, where $1 \leq z \leq p-1$ satisfying the following properties:

- S_q is $(\mathcal{Q}^{p-1}, q)$-free for all $2 \leq q \leq z-1$,
- S_q is $(\mathcal{Q}^{p-1}, q+1)$-free for all $z \leq q \leq p-1$.

We show how to construct $\mathcal{Q}^p$. If the $k+1$ paths in $\mathcal{Q}^{p-1}$ intersect S_p in less than $5k^2$ vertices, then, recalling that each non-subdivision set has size at least $6k^2$, there are at least $6k^2 - 5k^2 \geq k+1 \geq p+1$ free vertices in S_p. Therefore, we may set $\mathcal{Q}^p = \mathcal{Q}^{p-1}$ and set $v^p = v^{p-1}$, and note that S_p is $(\mathcal{Q}^p, p+1)$-free.

Otherwise, there is some path P which intersects S_p in at least $5k^2/(k+1) \geq 2k+1$ vertices. Write these vertices in the intersection as $u_1, \ldots, u_l$ in the order they appear in P with $l \geq 2k+1$. If P is the special path of $\mathcal{Q}^{p-1}$, then $\mathcal{Q}^p$ is simply formed by setting $v^p = u_1$ (the first intersection with S_p) and following P to v^p. Thus, we may assume that P is not the special path of $\mathcal{Q}^{p-1}$. Now construct the following new paths. Follow the special path in $\mathcal{Q}^{p-1}$ to $v^{p-1} \in S_z$. As each S_q is a non-subdivision set, and these sets are part of a (k, ℓ)-good family, by definition we have $S_z \to \ldots \to S_p$. Accordingly, we may go from v^{p-1} to u_l using free vertices, and then follow P to its endpoint in Y. Call this path P'. Our new special path is formed by following P to u_1 and setting $v^p = u_1$. Let $\mathcal{Q}^p$ be the resulting family of paths. We have thus freed vertices $u_2, \ldots, u_{l-1}$ for a total of $l-2 \geq 2k-1 \geq k+1 \geq p+1$ vertices in S_p. Thus $\mathcal{Q}^p$ satisfies the following properties:

- S_q is $(\mathcal{Q}^p, q)$-free for all $1 \leq q \leq p-1$,
- $v^p \in S_p$ and S_p is $(\mathcal{Q}^p, p+1)$-free.

Indeed, the second item above is clear by construction, and the first item holds because S_q is $(\mathcal{Q}^{p-1}, q+1)$-free for $z \leq q \leq p-1$, and the new path P' uses precisely one free vertex from each of these sets. Thus $\mathcal{Q}^p$ satisfies the desired properties.

Finally, if the origin set O was chosen as a subset of S_t, then note that O remains invariant in this process and therefore is maintained as a free set of $k+1$ vertices in S_t. Therefore, we terminate this process with $v^* = v^{t-1}$ and $\mathcal{Q}^* = \mathcal{Q}^{t-1}$. On the other hand, if $O \subset \text{ter}(P^s)$, then we repeat the above procedure to the set S_t yielding $v^* = v^t$ and $\mathcal{Q}^* = \mathcal{Q}^t$. By the same argument

we can guarantee at least $k+1 \geq t+1$ free vertices in S_t, completing the proof of the lemma. ∎

2.4. Finishing the proof

In the previous subsection we showed that there is a system $\mathcal{Q}$ of pairwise vertex disjoint paths from O to $Y \cup \{v\}$, for some $v \in \bigcup_{q=1}^{t} S_{j_q}$, that do not intersect any of the S_i's in many vertices. We shall use these *free* vertices to extend the paths in $\mathcal{Q}$ and obtain the desired pairwise vertex disjoint paths from x_i to y_i, for $i \in [k]$, thus finishing the proof of Theorem 1.3.

Recall that we have assumed that $\ell \geq 3k + 10^4 k^3 + 2 \cdot 10^{13} k^4$. For each $i \in [k]$, let z_i be the vertex in O such that there is a path in $\mathcal{Q}$ starting at z_i and ending at y_i, and let $S_i' = S_i \setminus \bigcup \mathcal{Q}$. First, we need the following lemma, which says that the set S_{i_r}' is, in a certain way, highly linked.

Lemma 2.8. *For any two disjoint sets of vertices $\{u_1, \ldots, u_k\}$ and $\{v_1, \ldots, v_k\}$ in S_{i_p}', we can find pairwise vertex disjoint paths joining each u_i to v_i using only vertices of the subdivision sitting on S_{i_p}' and avoiding the vertices of $\mathcal{Q}$.*

Proof. It follows from Lemmas 2.5 and 2.6 that for any two vertices u and v in S_{i_p}' there are at least $\ell - 2k - 10^4 k^3 - 2 \cdot 10^{13} k^4 \geq k$ vertices w in $S_{i_p}' \setminus \{u_1, \ldots, u_k, v_1, \ldots, v_k\}$, such that we can go from u to w and from w to v via two paths of the subdivision and avoiding the vertices of $\mathcal{Q}$. Therefore, we can find a system of pairwise vertex disjoint paths by greedily choosing paths with the desired properties. ∎

Our goal now is to find a system of pairwise vertex disjoint paths $\mathcal{Q}'$ joining x_i to z_i, for each $i \in [k]$, that do not use any vertices of $\mathcal{Q}$ or Y. It is easy to see that if we find such a system, then we are done: for each i we can simply go from x_i to z_i via a path of $\mathcal{Q}'$ and then from z_i to y_i using a path from $\mathcal{Q}$, thus obtaining a path starting at x_i and ending at y_i. To achieve this goal, we have to consider two cases depending on whether O is in $\mathrm{ter}(P^{ns})$ or in $\mathrm{ter}(P^s)$.

Case 1. $O \subset \mathrm{ter}(P^{ns})$ Recall that for every $i \in [k]$ we have $S_i \subset W_i \subset (N^+(x_i) \setminus (X \cup Y))$. Hence, using the fact that, for each $q \in [t]$, $|S_{j_q}'| \geq q$ (by Lemma 2.7), and the property that $S_{j_q}' \to S_{j_{q+1}}'$ for each $q < t$, we can greedily find pairwise vertex disjoint paths from x_{j_q} to z_{j_q} using only vertices in $\bigcup_{q \in [t]} S_{j_q}'$.

On the other hand, to find a path from x_{i_p} to z_{i_p} where $p \in [r]$, we shall use the property that z_{i_p} has many in-neighbours in S'_{i_r}. Indeed, as $O \subset \mathrm{ter}(P^{ns})$ we must have $i_r j_t \in E(H)$ in the auxiliary digraph H, by construction. Recall that we have discarded those vertices in S_{j_t} which have few in-neighbours in S_{i_r}, which means that each of the vertices $z_{i_1}, \ldots, z_{i_r} \in O$ have at least

$$\ell/2 - 10^4 k^3 \geq k$$

in-neighbours in S'_{i_r} (where we have applied Lemma 2.5). Thus, we may find r distinct vertices $z'_{i_1}, \ldots, z'_{i_r} \in S'_{i_r}$ such that $z'_{i_1} \to z_{i_1}, \ldots, z'_{i_r} \to z_{i_r}$.

Now, it follows from Lemma 2.5 and the definition of the auxiliary digraph H that at least $\ell/4 - 10^4 k^3 \geq 2k$ vertices in S'_{i_p} have at least $\ell/4 - 10^4 k^3 \geq 2k$ out-neighbours in $S'_{i_{p+1}}$, and hence, with a help of Lemma 2.8 which we use to arrive to vertices in S'_{i_p} with high out-degrees in $S'_{i_{p+1}}$, we can greedily find pairwise vertex disjoint paths from $\{x_{i_1}, \ldots, x_{i_r}\}$ to Z, for some $Z \subseteq S'_{i_r} \setminus \{z'_1, \ldots, z'_r\}$, again using only vertices in $\bigcup_{p \in [r]} S'_{i_p}$. Finally, using Lemma 2.8 we can appropriately link Z to $z'_{i_1}, \ldots, z'_{i_r}$ to obtain a system of pairwise vertex disjoint paths from x_{i_p} to z'_{i_p} for every $p \in [r]$. Using the fact that $z'_{i_p} \to z_{i_p}$, we obtain the desired paths from x_{i_p} to z_{i_p} for each $p \in [r]$.

Case 2. $O \subset \mathrm{ter}(P^s)$ In this case, we must have $j_t i_r \in E(H)$, so each vertex in S'_{j_t} has at least $\ell/2 - 10^4 k^3 \geq 2k$ out-neighbours in S'_{i_r}. As before, we can greedily find pairwise vertex disjoint paths from $x_1, \ldots, x_k$ to Z, for some $Z \subseteq S'_{i_r} \setminus \{z_1, \ldots, z_k\}$, using only vertices in $\bigcup_{i \in [k]} S'_i$. Again, using Lemma 2.8 we can appropriately link Z with $\{z_1, \ldots, z_k\}$ and obtain a system of pairwise vertex disjoint paths from x_i to z_i for every $i \in [k]$.

In each case, we have found the required collection of vertex disjoint directed paths linking x_i to y_i for each $i \in [k]$. This completes the proof of Theorem 1.3. ∎

3. Constructions

3.1. There exist $(2k-1)$-connected tournaments with large minimum out-degree which are not k-linked

For all integers $k \geq 2$ and $m \geq 2k$, we construct a tournament T on n vertices ($n \geq 100m$) which is $(2k-1)$-connected and whose minimum out-degree and in-degree is at least m, but which is not k-linked.

Indeed, let T be a tournament on vertex set $V = A \cup B \cup X \cup Y \cup C$, where $X = \{x_1, \ldots, x_k\}$, $Y = \{y_1, \ldots, y_k\}$, $|C| = k - 1$ and $|X| = |Y| = (n - 3k + 1)/2$ and whose edges are oriented in the following way.

(1) The edges within A, B, C are oriented arbitrarily.
(2) The edges within X and Y are oriented so that $T[X]$ and $T[Y]$ form a $2m$-connected tournament.
(3) All edges are oriented from Y to X, from A to C, from C to B, from X to C, from C to Y, from Y to A and from B to X.
(4) All edges are oriented from A to B except for edges between x_i and y_i, for each $i \in [k]$.
(5) The edges between A and X are oriented in such a way that every vertex in A sends at least m out-edges to X and m in-edges to X. Similarly, the edges between B and Y are oriented in such a way that every vertex in A sends at least m out-edges to Y and m in-edges to Y.
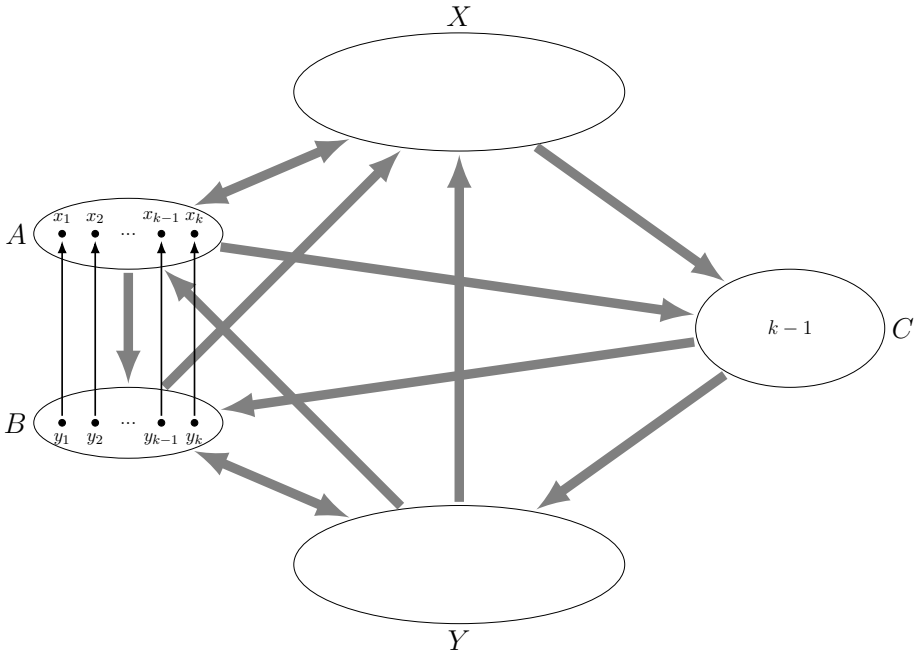

Figure 1. A $(2k-1)$-connected tournament with large minimum in/out-degree that is not k-linked

We need to prove the following three properties of T.

(1) T is $(2k-1)$-connected.

(2) T has minimum out- and in-degree at least $m/2$.

(3) There do not exist k vertex-disjoint paths joining x_i to y_i, for each $i \in [k]$.

Proof. Suppose T is not $(2k-1)$-connected. Then there exists a subset $W \subset T$ of size at most $2k-2$ such that $T \setminus W$ is not connected. First, we show that $C \subseteq W$. If not, there must exist $z \in C \setminus W$ and it is not hard to see that every vertex can reach z within $T \setminus W$ and every vertex can be reached from z, which is a contradiction. Hence, we may assume $C \subset W$. Note then that $|(A \cup B) \cap W| \leq k-1$, which, in particular, implies that neither A nor B can be fully contained within W. Let $x_i \in A \setminus W$ and $y_{j_1}, y_{j_2} \in B \setminus W$. It is easy to see that x_i can reach every vertex in $T \setminus W$, since it can certainly reach $X \setminus W, C \setminus W$. Moreover, it can reach either y_{j_1} or y_{j_2}, and then via one of these vertices, it can reach $Y \setminus W$. A similar argument shows that any vertex can reach x_i, which is a contradiction.

It is easy to see that every vertex x has $d^+(x) \geq m/2$. Finally, we need to show there do not exist k vertex disjoint paths joining x_i to y_i. Observe that any path from x_i to y_i can not use any vertex of $(B \setminus y_i) \cup (A \setminus x_i)$ and therefore, it must use a vertex of C. But since $|C| < k$ this is not possible. ∎

3.2. There exist $(2.5k-1)$-connected tournaments which fail to be k-linked

We shall now show that for each $k \geq 3$ and any sufficiently large n there exist $(5k-1)$-connected tournaments on n vertices that are not $2k$-linked, which shows that the minimum out-degree condition in our theorem is necessary.

Let T be a tournament on vertex set $V = X \cup Y \cup S \cup W$, where $X = \{x_1, \ldots, x_k\}$, $Y = \{y_1, \ldots, y_k\}$, $|S| = 4k-1$ and $|W| = n-6k+1$, and whose edges are oriented in the following way.

1. The edges inside each of X, Y, S, W, and between S and W are oriented in such a way that $T[S \cup W]$ is $(5k-1)$-connected (for large enough n, a random configuration of edges in $S \cup W$ will have this property), and both $T[X]$ and $T[Y]$ are strongly connected.
2. All edges are oriented from X to Y, from Y to W, and from W to X.
3. For every i, all the edges are oriented from x_i to S except for the edge between x_i and y_i' for some unique vertex y_i'.
4. For every i, all the edges are oriented from S to y_i except for the edge between x_i' and y_i for some unique vertex $x_i' \notin \{y_1', \ldots, y_k'\}$.
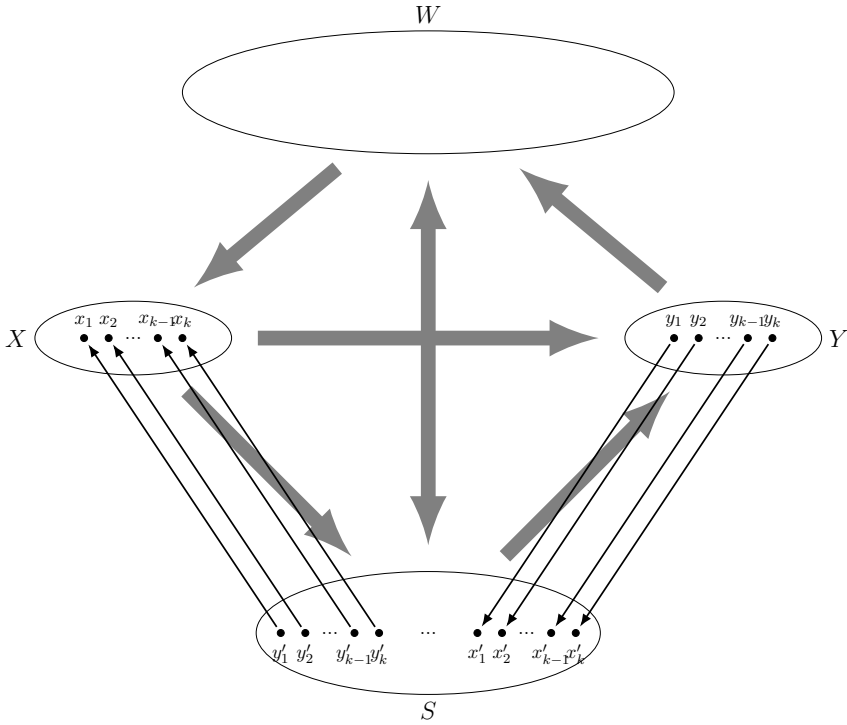
Claim 2. T is not $2k$-linked.

Figure 2. Example of a $(5k-1)$-connected tournament that is not $2k$-linked

Proof. Observe that for each i any path joining x_i to y'_i must use an extra vertex from $X \cup S \cup Y$. The same holds from paths going from x'_i to y_i. Hence, any system of disjoint paths joining x_i to y'_i and x'_i to y_i, for every i, uses at least $6k$ vertices in $X \cup Y \cup S$. But this cannot happen, as by construction $|X \cup Y \cup S| = 6k - 1$. ∎

Claim 3. T is $(5k-1)$-connected.

Proof. Let T' be a tournament obtained by removing any $5k-2$ vertices from T. We shall show that T' is still connected. Let us write X', Y', W', S' for $X \cap T'$, $Y \cap T'$, $W \cap T'$, $S \cap T'$, respectively. By construction $W \cup S$ is $(5k-1)$-connected, therefore $W' \cup S'$ is still connected and hence every vertex in T' can be reach via a directed path from $W' \cup S'$. Therefore, it remains to show that (1) there is a path between any vertex in X and $W' \cup S'$, and (2) a path between $W' \cup S'$ and any vertex in Y. We will only prove (1) as the proof of (2) is symmetrical.

Take any $x_i \in X'$. Observe that if $Y' \neq \emptyset$ or $S' \neq \{y'_i\}$, then we can easily find a path from x_i to $W' \cup S'$. We can therefore assume that $Y' = \emptyset$ and

$S' = \{y_i\}$, and therefore $X' = X$. By construction x_i is the only vertex in $X' = X$ which does not send an out-edge to y_i', hence x_i can reach y_i' using any other vertex in $X' = X$. ∎

4. Final remarks

An analysis of our methods shows that there is an absolute constant $C > 0$ such that any $(2k + 1)$-connected tournament with minimum out-degree at least Ck^{31} is k-linked. We remark that we did not make a strong effort to optimize the power of k in the minimum out-degree condition. While we believe that we could bring its value down, we were unable to obtain a linear bound, which we conjecture is the truth.

Conjecture 4.1. *There exists a constant $C > 0$ such that every $(2k + 1)$-connected tournament with minimum out-degree at least Ck is k-linked.*

In Subsection 3.1, we showed that one cannot replace $2k + 1$ by $2k - 1$ in Theorem 1.3, as there exist arbitrarily large tournaments which are $(2k - 1)$-connected with large minimum out and in-degree, but fail to be $2k$-linked. We have not ruled out the possibility that the connectivity condition can be relaxed to $2k$, however. It is therefore natural to ask the following.

Question 4.2. Does Theorem 1.3 still hold if we replace $2k + 1$ by $2k$?

Note that an affirmative answer to this question would completely resolve Conjecture 1.1 in a stronger form, in the sense of not additionally requiring large minimum in-degree.

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