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A DIOPHANTINE RAMSEY THEOREM

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Let $\mathsf{p}\in\mathbb{Z}[x]$ be any polynomial with $\mathsf{p}(0)=0$, $k\in\mathbb{N}$ and let $c_1,\dots,c_s\in\mathbb{Z}, s\geqslant k(k+1)$, be non-zero integers such that $\sum c_i=0$. We show that for a wide class of coefficients $c_1,\dots,c_s$ in every finite coloring $\mathbb{N}=A_1\cup\dots\cup A_r$ there is a monochromatic solution to the equation

$$c_1x_1^k+\dots+c_sx_s^k=\mathsf{p}(y).$$

1. Introduction

For a polynomial $P\in\mathbb{Z}[x_1,\dots,x_s]$ we call the equation $P(x_1,\dots,x_s)=0$ *regular* if in any finite partition $\mathbb{N}=A_1\cup\dots\cup A_r$ there is a non-trivial solution to this equation with $x_1,\dots,x_s\in A_i$ for some $1\leqslant i\leqslant r$. Throughout the course of the paper by a trivial solution we mean a solution with $x_1=\dots=x_s$. The study of regular equations was started by Schur [26], who showed that $x+y=z$ is regular. Later Rado [24] proved the following theorem that provides a complete characterization of regular linear equations.

Theorem 1 (Rado [24]). *Let $c_1,\dots,c_s\in\mathbb{Z}\setminus\{0\}$ and $s\geqslant 3$. Then the linear equation*

$$c_1x_1+\dots+c_sx_s=0$$

is regular if and only if there is a non-empty set $I\subseteq[s]$ such that $\sum_{i\in I}c_i=0$.

Recently, many various questions concerning regularity of equations were investigated [4,5,6,7,8,10,13,17,19,21,22]. Specifically, a number of authors attempted to find a Rado-type characterization for Diophantine equations [4,5,20]. The most general result was obtained by Chow, Lindqvist

Mathematics Subject Classification (2010): 11P99; 05D10 and Prendiville [5], who have proved a Rado criterion for k powers: if $s(k) \geq (1+o(1))k\log k$ then the equation

$$c_1x_1^k + \cdots + c_sx_s^k = 0 \tag{1}$$

is regular if and only if there is a non-empty set $I \subseteq [s]$ such that $\sum_{i\in I} c_i = 0$ (it is proven in [5] that one can even find such solution with distinct integers). The above result despite being very close to the best possible one it does not provide full characterization of regular equations for k powers. It is clear that we need a lower bound for the number of variables in terms of k, however it seems to be a very complicated matter related to Waring's problem, to find optimal value of $s(k)$. A result towards longstanding open problem concerning the regularity of the Pythagorean equation $x^2+y^2=z^2$ was proven in [5], specifically the equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2$$

is regular. Moreira [22] showed that the equation

$$c_1x_1^2 + \cdots + c_sx_s^2 = y$$

is regular provided that $\sum_i c_i = 0$ and $s \geqslant 2$. Another result was obtained by Bergelson [1] using the ergodic theory method, who showed regularity of the equations

$$x - y = \mathsf{p}(z), \tag{2}$$

where $\mathsf{p} \in \mathbb{Z}[x]$ is arbitrary polynomial with $\mathsf{p}(0) = 0$. In contrast, Green and Lindqvist [10] observe that the equation $x+y=z^2$ is not 3-regular (there is a 3-coloring of $\mathbb{N}$ without monochromatic solutions) and they used a very elaborate argument to prove that every 2-coloring of $\mathbb{N}$ has a monochromatic solution to this equation, see also [23]. It is also worth mentioning here that Khalfalah and Szemerédi [18] established regularity of the equation $x+y=n^2$, where $n \in \mathbb{N}$ can be arbitrary. Furthermore, many related Roth-type density results for higher powers were proven, see for example [3] and [16].

To formulate our main result we need a definition. We say that the equation

$$c_1x_1^k + \cdots + c_sx_s^k = \mathsf{p}(y)$$

contains two identical symmetric equations with $2h$ variables if $4h \leqslant s$ and (after possible permutation)

$$c_i = c_{h+i} = -c_{2h+i} = -c_{3h+i}$$

for every $1 \leqslant i \leqslant h$. Clearly, we can assume that $c_1, \ldots, c_h > 0$.

Theorem 2. *Let* $\mathsf{p} \in \mathbb{Z}[x]$ *be any polynomial with* $\mathsf{p}(0)=0$, $k \in \mathbb{N}$ *and let* $c_1,\ldots,c_s \in \mathbb{Z}\setminus\{0\}$. *Then the equation*

$$c_1x_1^k + \cdots + c_sx_s^k = \mathsf{p}(y) \tag{3}$$

is partition regular, provided that it contains two identical symmetric equations each with $2h \geqslant k(k+1)$.

Theorem 2 can be considered a partial synthesis of Chow, Lindqvist and Prendiville or Bergelson's results. It seems to be a first Ramsey type result for a Diophantine equation that may involve variables of different powers greater than 1. However, we have to impose some additional constraints on coefficients, which is a price we have to pay for inserting the polynomial on the right-hand side of (3). Let us also remark that a density version of the above result is not valid, as one can pick an appropriate arithmetic progression without any solution to (3).

Compared to the previous works our argument does not rely on the two deep techniques invented by Green in [12], namely the W-trick and the Fourier transference principle. Similarly as in [10] we will heavily use the arithmetic regularity lemma [11,14] and the restriction estimate for k powers. Our approach essentially adopts a classical Schur's scheme originally used for the equation $x+y=z$. Thus, assuming that an r-coloring of $\mathbb{N}$ has no solution to (3), we construct a sequence of additively rich sets B_i with the property that B_i omits at least i color classes. Clearly, such sequence can consists of at most r sets, which will lead to a contradiction.

Paper organization. In the next section we state the arithmetical regularity lemma and we make some necessary preparation for an application. The Section 3 is devoted to prove our main tool. Using the arithmetical regularity lemma and the restriction estimate we establish a version of Bogolyubov-Ruzsa lemma [25] for dense subsets of k powers. In Section 4, by applying Weyl's inequality and using a similar argument as in [10], we obtain a lower bound for the number of polynomial values in Bohr sets and we also prove some results for Bohr sets that will be used in the proof of the main result. The proof of Theorem 2 is concluded in Section 5.

Notation. For a set of integers A and $c_1,\ldots,c_s \in \mathbb{Z}$ we put

$$c_1A \dot{+} \ldots \dot{+} c_sA = \{c_1a_1 + \cdots + c_sa_s \colon a_1 \ldots, a_s \in A, a_i \neq a_j \text{ for all } i \neq j\},$$

though to avoid any confusion by $2T-2T$ we always mean $T+T-T-T$. We write $\mathbb{T}^d$ for $(\mathbb{R}/\mathbb{Z})^d$ and for $x \in \mathbb{T}^d$ put $\|x\|_{\mathbb{T}^d} = \max_{1\leqslant i\leqslant d} \min_{n\in\mathbb{Z}} |x_i - n|$. Put $e(x)=e^{2\pi i x}$ and for a function f (acting on $\mathbb{Z}$ or $\mathbb{T}^d$) we denote the Fourier transform of f by $\widehat{f}$.

2. The arithmetic regularity lemma

The arithmetic regularity lemma was invented by Green in [11], however the commonly applied version was proven in [14]. We recommend the readers who are not familiar with this technique to go through a brief self-contained exposition by Eberhard [9], where only the case of abelian groups and the U^2−norm are considered, however we will apply the arithmetic regularity lemma in that case.

To formulate the arithmetic regularity lemma we have to define an important notion of high irrationality. Let $N\in\mathbb{N}$ and $L>0$ be a real number. We say that $\theta\in\mathbb{T}^d$ is (L,N)-*irrational* if for every $\mathbf{r}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}$ with $\|\mathbf{r}\|_1\leqslant L$ we have $\|\mathbf{r}\cdot\theta\|\geqslant L/N$.

Lemma 3. *Let $\delta>0$ and let $\mathcal{F}\colon \mathbb{N}\to\mathbb{R}_+$ be an increasing function. Then there exists $M_{\max}\ll_{\delta,\mathcal{F}} 1$ such that for every function $f\colon [N]\to[0,1]$ there is an $M\leqslant M_{\max}$ and a decomposition $f=f_{\rm tor}+f_{\rm sml}+f_{\rm unf}$ into functions $f_{\rm tor}, f_{\rm sml}, f_{\rm unf}\colon [N]\to[-1,1]$ such that*

1. *$f_{\rm tor}(n)=F(n \pmod q), n/N, \theta n)$, where $F\colon \mathbb{Z}/q\mathbb{Z}\times[0,1]\times\mathbb{T}^d\to[0,1]$ for some $q,d\leqslant M$, F is M Lipschitz function and θ is $(\mathcal{F}(M),N)$-irrational,*
2. *$\|f_{\rm sml}\|_1\leqslant\delta$,*
3. *$\|\widehat{f_{\rm unf}}\|_\infty\leqslant N/\mathcal{F}(M)$.*

We also make here some standard preparations for application of Lemma 3. Let A be a subset of $[N]$ with $|A|=\gamma N$. We apply arithmetic regularity lemma with $f=1_A$, $\delta=c\gamma^3$ for some small constant $c>0$ and $\mathcal{F}$ to be specified later. We obtain positive integers $q,d\leqslant M$, $\theta\in\mathbb{T}^d$ and a function $F\colon \mathbb{Z}/q\mathbb{Z}\times[0,1]\times\mathbb{T}^d\to[0,1]$ satisfying conditions $1-3$ of Lemma 3 such that

$$1_A=f_{\rm tor}+f_{\rm sml}+f_{\rm unf}.$$

Denote by μ the product of the uniform probability on $\mathbb{Z}/q\mathbb{Z}$, Lebesgue measure on $\mathbb{R}$ and the normalized Lebesgue measure on $\mathbb{T}^d$. The next lemma is very similar to Lemma 4.3 in [10], so we omit its proof.

Lemma 4. *There is an absolute constant $C>0$ such that if $\mathcal{F}(M)\geqslant C\gamma^{-1}$ and $\delta\leqslant\gamma/4$ then*

$$\int F d\mu \ge \frac{1}{2}\gamma.$$

Thus, we have

$$\sum_{u=0}^{q-1} \frac{1}{q} \int_{[0,1]} \int_{\mathbb{T}^d} F(u,v,w)dvdw - \frac{\gamma}{4\delta} \sum_{u=0}^{q-1} \sum_{n \equiv u \pmod q} |f_{\text{sml}}(n)| \geqslant \gamma/4,$$

so there is $u \in \mathbb{Z}/q\mathbb{Z}$ such that

$$\int_{[0,1]} \int_{\mathbb{T}^d} F(u,v,w)dvdw \geq \gamma/4$$

and

$$\sum_{n \equiv u \pmod q} |f_{\text{sml}}(n)| \leqslant \frac{4\delta}{\gamma q} N.$$

For $v \in [0,1], \varepsilon, \varepsilon' > 0$ and $w \in \mathbb{T}^d$ define

$$B(v,w) = \left\{ n \in \mathbb{N} \colon n \equiv u \pmod q, |n/N - v|, \|\theta n - w\|_{\mathbb{T}^d} \leqslant \varepsilon \right\},$$
$$B'(v,w) = \left\{ n \in \mathbb{N} \colon n \equiv u \pmod q, |n/N - v|, \|\theta n - w\|_{\mathbb{T}^d} \leqslant \varepsilon' \right\},$$

and let

$$E(v,w) = \sum_{n \in [N] \cap B(v,w)} |f_{\text{sml}}(n)|, \qquad E'(v,w) = \sum_{n \in [N] \cap B'(v,w)} |f_{\text{sml}}(n)|.$$

Lemma 5. *There exists a choice of $v \in [0,1]$ and $w \in \mathbb{T}^d$ such that*
(4)
$$F(u,v,w) \geqslant \gamma/8, \ E(v,w) \leqslant (2\varepsilon)^{d+1} \frac{32\delta}{\gamma^2 q} N \text{ and } E'(v,w) \leqslant (2\varepsilon')^{d+1} \frac{32\delta}{\gamma^2 q} N.$$

Proof. We have

$$\int_0^1 \int_{\mathbb{T}^d} E(v,w)dvdw$$
$$= \sum_{n \equiv u \pmod q} |f_{\text{sml}}(n)| \int_0^1 1_{|n/N - x| \leqslant \varepsilon} dx \int_{\mathbb{T}^d} 1_{\|\theta n - y\|_{\mathbb{T}^d} \leqslant \varepsilon} dy$$
$$\leqslant (2\varepsilon)^{d+1} \frac{2\delta}{\gamma q} N,$$

and similarly

$$\int_0^1 \int_{\mathbb{T}^d} E'(v,w)dvdw \leqslant (2\varepsilon')^{d+1} \frac{2\delta}{\gamma q} N,$$

so

$$\int_0^1\int_{\mathbb{T}^d}\Big(F(u,v,w)-\frac{\gamma^2 q}{32(2\varepsilon)^{d+1}\delta N}E(v,w)-\frac{\gamma^2 q}{32(2\varepsilon')^{d+1}\delta N}E'(v,w)\Big)dvdw \geqslant \gamma/8.$$

Therefore, for certain v and w

$$F(u,v,w)\geqslant\gamma/8,\ E(v,w)\leqslant(2\varepsilon)^{d+1}\frac{32\delta}{\gamma^2 q}N \text{ and } E'(v,w)\leqslant(2\varepsilon')^{d+1}\frac{32\delta}{\gamma^2 q}N,$$

which concludes the proof. ∎

3. Bohr sets in sumsets of k powers

This section is devoted to establishing the main technical tool used in the proof of Theorem 2. Roughly speaking, we show that if s is sufficiently large in term of k then an appropriate s-fold sumset of a dense subset of k powers contains a Bohr set of the form

$$B=\{n\colon n\equiv u \ (\mathrm{mod}\ q), |n/N-v|\leqslant\varepsilon, \|\theta n-w\|_{\mathbb{T}^d}\leqslant\varepsilon\}.$$

Bohr sets have rich additive structure and with highly irrational θ they behave like convex bodies and therefore they are very useful.

Over the course of the proof of Proposition 8 we will use some smooth approximants. The next two lemmas were established in [10], however we have to slightly adapt some constants appearing in those lemmas. Condition 3 in Lemma 6 was originally stated in [10], however with a different constant (1/2 instead of 0.99). It follows easily from the proof of Lemma A.3 in [10] that we can increase the constant at the expense of decreasing ε' and increasing L. It is not formulated explicitly in [10], but one can prove Lemma 7 using analogous majorant of the ε-ball in $\mathbb{T}^d$ (see Lemma 12) and essentially the same argument to obtain an upper bound for the size of Bohr set provided that θ is sufficiently irrational. For $\varepsilon>\varepsilon'>0$ small compared to ε we put

$$B_-=\{n\colon n\equiv u \ (\mathrm{mod}\ q), |n/N-v|, \|\theta n-w\|_{\mathbb{T}^d}\leqslant\varepsilon-\varepsilon'\}.$$

Lemma 6. *Let $d,q\in\mathbb{N}$, $0<100\varepsilon' d<\varepsilon<1$, and $\theta\in\mathbb{T}^d$. Then there is an $L=L(\varepsilon,\varepsilon',d,q)$ with the following property. Suppose that $\theta\in\mathbb{T}^d$ is (L,N)-irrational and $N\geqslant N(\varepsilon,\varepsilon',d,q,L)$. Then there exists a function $\beta\colon\mathbb{Z}\to[0,1]$ satisfying*

1. $1_{B_-}(n)\leqslant\beta(n)\leq 1_B(n)$ *for all n,*

2. $\|\widehat{\beta}\|_1 = O_{\varepsilon,\varepsilon',d,q}(1)$,
3. $\sum_n \beta(n) \geqslant 0.99(2\varepsilon)^{d+1}q^{-1}N$.

Let us mention that high irrationality of θ allows to asymptotically find the size of the correspondent Bohr set, not just to bound it, however it does not bring any improvement to the main result.

Lemma 7. *Let $d,q\in\mathbb{N}$, $0<\varepsilon<1$, and $\theta\in\mathbb{T}^d$. Then there is an $L=L(\varepsilon,d,q)$ such that if $\theta\in\mathbb{T}^d$ is (L,N)-irrational and $N\geqslant N(\varepsilon,d,q,L)$, then*

$$0.99(2\varepsilon)^{d+1}q^{-1}N \leqslant |B| \leqslant 1.01(2\varepsilon)^{d+1}q^{-1}N.$$

We state now the main result of this section.

Proposition 8. *Let $\gamma > 0$ and let $\mathcal{L}\colon \mathbb{N}^2 \times \mathbb{R}_{>0} \to \mathbb{N}$ be a nondecreasing function in each variable, which may depend on γ. Suppose that $T \subseteq [N]$, $|T| = \gamma N$ and $N > N(\mathcal{L})$. Then there are $q, d \ll_{\mathcal{L}} 1$, $\varepsilon \gg_{\mathcal{L}} 1$ and $(\mathcal{L}(q,d,1/\varepsilon),N)$-irrational $\theta\in\mathbb{T}^d$ such that the Bohr set*

$$B = \big\{n \in \mathbb{N}\colon n \equiv 0 \pmod q, |n/N|, \|\theta n\|_{\mathbb{T}^d} \leqslant \varepsilon\big\}$$

is contained in $2T-2T$.

Proof. We apply Lemma 3 with $f=1_T$,

$$\delta = c\gamma^3 \tag{5}$$

for a small absolute constant $c>0$ and the function $\mathcal{F}$ depending on $\mathcal{L}$ which will be specified later. This gives integers $q,d \leqslant M \leqslant M_{\max}(\delta,\mathcal{F})$, $\theta\in\mathbb{T}^d$, a function $F\colon \mathbb{Z}/q\mathbb{Z}\times[0,1]\times\mathbb{T}^d\to[0,1]$ and a decomposition

$$1_T = f_{\rm tor} + f_{\rm sml} + f_{\rm unf}$$

satisfying conditions of Lemma 3. Let $u\in\mathbb{Z}/q\mathbb{Z}$, $v\in\mathbb{R}$ and $w\in\mathbb{T}^d$ be given by Lemma 5. Put

$$\varepsilon = \frac{\gamma}{8M} \tag{6}$$

and

$$B_0 = \big\{n\colon n \equiv u \pmod q, |n/N - v|, \|\theta n - w\|_{\mathbb{T}^d} \leqslant \varepsilon\big\}.$$

Then using the fact that F is M-Lipschitz function we deduce that for every integer $n\in B$ we have

$$F(u,v,w) + \varepsilon M \geqslant f_{\rm tor}(n) \geqslant F(u,v,w) - \varepsilon M \geqslant \gamma/8. \tag{7}$$

We prove that large portion of the Bohr set

$$B_1 := \{n\colon n \equiv 2u \pmod q), |n/N - 2v|, \|\theta n - 2w\|_{\mathbb{T}^d} \leqslant \varepsilon\}$$

belongs to $T+T$. To show this we will use further Bohr sets

$$B_- = \{n\colon n \equiv u \pmod q), |n/N - v|, \|\theta n - w\|_{\mathbb{T}^d} \leqslant \varepsilon - \varepsilon'\},$$
$$B' = \{n\colon n \equiv u \pmod q), |n/N - v|, \|\theta n - w\|_{\mathbb{T}^d} \leqslant \varepsilon'\}$$

and

$$B'_- = \{n\colon n \equiv u \pmod q), |n/N - v|, \|\theta n - w\|_{\mathbb{T}^d} \leqslant \varepsilon' - \varepsilon''\},$$

where $\varepsilon' = \varepsilon/100d$ and $\varepsilon'' = \varepsilon'/100d$. Let β be a smooth minorant for 1_B given by Lemma 6 such that $\beta(n) = 1$ for every $n \in B_-$ and let β' be a minorant for $1_{B'}$, given by the same lemma applied with ε' and ε'', such that $\beta'(n) = 1$ for every $n \in B'_-$. We can apply Lemma 6 provided that $\mathcal{L}(q,d,1/\varepsilon) \geqslant \max(L(q,d,\varepsilon,\varepsilon'), L(q,d,\varepsilon',\varepsilon''))$. Let $m \in B_1$ then

$$\begin{aligned} 1_T * 1_T(m) &\geqslant (1_T 1_B) * (1_T 1_{B'})(m) \\ &\geqslant (1_T \beta) * (1_T \beta')(m), \end{aligned}$$

thus, to estimate $1_T * 1_T(m)$ it is sufficient to bound $f_1\beta * f_2\beta'(m)$ for every possible choice of $f_i \in \{f_{\rm tor}, f_{\rm sml}, f_{\rm unf}\}$. We will start with $f_1 = f_2 = f_{\rm tor}$. Note that we have

$$B_- + B' \subseteq B_1$$

and since $|B_-| \geqslant 0.9|B_1|$ by Lemma 7 it follows that at least 80% of elements of B_1 have at least $0.12|B'|$ representations in the form $m = m_1 + m_2$ with $m_1 \in B_-$ and $m_2 \in B'$. Denote by $X \subseteq B$ the set of all m with

$$1_{B_-} * 1_{B'}(m) \geqslant 0.12|B'| \geqslant 0.1(2\varepsilon')^{d+1} q^{-1} N.$$

Let $m \in X$ then by (7)

$$\begin{aligned} (f_{\rm tor}\beta) * (f_{\rm tor}\beta')(m) &\geqslant (\gamma/8)^2 \beta * \beta'(m) \\ &\geqslant 0.1(\gamma/8)^2 (2\varepsilon')^{d+1} q^{-1} N. \end{aligned} \tag{8}$$

Next, we will bound from above the convolutions $f_1\beta * f_2\beta'(m)$, when $f_1 = f_{\rm sml}$ and $f_2 \in \{f_{\rm tor}, f_{\rm sml}\}$. Then by Lemma 5, Lemma 7, (5) and (7) we have

$$\begin{aligned}\sum_{m\in X} f_{\mathrm{sml}}\beta * f_2\beta'(m) &= \sum_{m\in X}\sum_{m_1} f_{\mathrm{sml}}\beta(m_1) f_2\beta'(m-m_1)\\ &= \sum_{m_1\in B} f_{\mathrm{sml}}\beta(m_1)\sum_{m} f_2\beta'(m-m_1)\\ &\leqslant 20\delta\gamma^{-1}(2\varepsilon)^{d+1}(2\varepsilon')^{d+1}(N/q)^2\\ &\leqslant 20c\gamma^2(2\varepsilon')^{d+1}q^{-1}N|B_1|,\end{aligned}\tag{9}$$

and similarly if $f_1\in\{f_{\mathrm{tor}}, f_{\mathrm{sml}}\}$ and $f_2 = f_{\mathrm{sml}}$ then

$$\sum_{m\in X} 1_T\beta f_{\mathrm{sml}}\beta'(m) \leqslant 20c\gamma^2(2\varepsilon')^{d+1}q^{-1}N|B_1|.\tag{10}$$

Now assume that $f_1 = f_{\mathrm{unf}}$ and $f_2\in\{f_{\mathrm{tor}}, f_{\mathrm{sml}}, f_{\mathrm{unf}}\}$. Observe that

$$\widehat{f_{\mathrm{unf}}\beta}(t) = \int_0^1 \widehat{f_{\mathrm{unf}}}(t')\widehat{\beta}(t-t')dt'.$$

hence by Lemma 6 for every $t\in[0,1]$ we have

$$|\widehat{f_{\mathrm{unf}}\beta}(t)| \le \|\widehat{f_{\mathrm{unf}}}\|_\infty\|\widehat{\beta}\|_1 \ll_M N/\mathcal{F}(M),$$

and therefore by the Cauchy–Schwarz inequality, Parseval's formula and (4) one has

$$\begin{aligned}\sum_{m\in X} f_{\mathrm{unf}}\beta * f_2\beta'(m) &= \int_0^1 \widehat{f_{\mathrm{unf}}\beta}(t)\widehat{f_2\beta'}(t)\widehat{1_X}(-t)dt\\ &\ll_M \frac{N}{\mathcal{F}}\left(\int_0^1 |\widehat{f_2\beta'}(t)|^2dt\right)^{1/2}\left(\int_0^1 |\widehat{1_X}(-t)|^2dt\right)^{1/2}\\ &\leqslant c\gamma^2(2\varepsilon')^{d+1}q^{-1}N|B_1|,\end{aligned}\tag{11}$$

and again for $f_1\in\{f_{\mathrm{tor}}, f_{\mathrm{sml}}, f_{\mathrm{unf}}\}$ and $f_2 = f_{\mathrm{unf}}$ similarly we have

$$\sum_{m\in X} f_1 * f_{\mathrm{unf}}\beta'(m) \leqslant c\gamma^2(2\varepsilon')^{d+1}q^{-1}N|B_1|.\tag{12}$$

Thus, by (9),(10), (11), (12) and assuming that c is sufficiently small, there are at most $0.1|B_1|$ elements of B_1 that violate at least one of the inequalities

$$f_1\beta * f_2\beta'(m) \leqslant 0.01(\gamma/8)^2(2\varepsilon')^{d+1}q^{-1}N,$$

for some $(f_1,f_2)\neq(f_{\rm tor},f_{\rm tor})$. Hence by (8), there is a set $Y\subseteq B_1$ of size at least $0.7|B_1|$ such that all $m\in Y$ satisfy

$$1_T*1_T(m)\geqslant 0.1(2\varepsilon')^{d+1}q^{-1}N.$$

Now we are able to define a Bohr set contained in $2T-2T$, let

$$B=\{n\colon n\equiv 0 \pmod q), |n/N|, \|\theta n\|_{\mathbb{T}^d}\leqslant \varepsilon'\}.$$

Put

$$B_1^+:=\{n\colon n\equiv 0 \pmod q), |n/N|, \|\theta n\|_{\mathbb{T}^d}\leqslant \varepsilon+\varepsilon'\},$$

then by Lemma 7, $|B_1^+|\leqslant 1.1|B_1|$. Observe that for every $m\in B$ we have $Y+m\subseteq B_1^+$, so

$$|(Y+m)\cap Y|\geqslant 2|Y|-|B_1^+|\geqslant 0.3|B_1|,$$

hence $m\in Y-Y$ and furthermore, m has at least $0.3|B_1|$ representations as $m=y-y'$ for $y,y'\in Y$. Thus, for each $m\in B$

$$1_T*1_T*1_{-T}*1_{-T}(m)\gg(2\varepsilon)^{d+1}(2\varepsilon')^{3d+3}(N/q)^3,$$

which concludes the proof.

It is only left to choose an appropriate function $\mathcal{F}$. Since $\varepsilon,\varepsilon',\varepsilon''\gg_M 1$ and $q,d\leqslant M$, then clearly we have

$$\|\widehat{\beta}\|_1,\|\widehat{\beta'}\|_1, L(q,d,\varepsilon,\varepsilon'), L(q,d,\varepsilon',\varepsilon'')\ll_M 1.$$

Furthermore to satisfy the inequality (11) we need

$$\mathcal{F}(M)\gg\gamma^{-2}(200\gamma^{-1}M^2)^{-d-1}M.$$

Finally, we may choose $\mathcal{F}(M)$ to be the maximum of the above functions and $\mathcal{L}(M,M,8\gamma^{-1}M)$. ∎

To prove the main result of this section we will need a mean value theorem for k powers. The next lemma follows from the resolution of the main conjecture of Vinogradov's mean value theorem by Bourgain, Demeter and Guth [2] and Theorem 4.1 in [27] (for $k\geq 3$, the case $k=2$ was known before).

Lemma 9. *Let $k\geqslant 2$ be an integer. Then for every $h\geqslant k(k+1)/2$ we have*

$$\int_{\mathbb{T}}\left|\sum_{n=1}^{N}e(n^kt)\right|^{2h}dt\ll_{h,k}N^{2h-k}.$$

For a finite set of integers A let us denote by $D(A,\mathbf{c})=2T-2T$, where $T:=c_1A^k\dot{+}\ldots\dot{+}c_hA^k$. Let us recall that we have assumed that $c_1,\ldots,c_h>0$, so $T\subseteq\left[(\sum c_i)\,N^k\right]$.

Corollary 10. *Let $\gamma>0$, $h,k\in\mathbb{N}$, $h\geqslant k(k+1)/2$ and let $\mathcal{L}\colon \mathbb{N}^2\times\mathbb{R}_{>0}\to\mathbb{N}$ be a nondecreasing function in each variable, which may depend on γ,k,h and $\mathbf{c}$. Suppose that $A\subseteq[N]$, $|A|=\gamma N$ and that $N>N(\mathcal{L})$. Then there are $q,d\ll_{\mathcal{L}}1$, $\varepsilon\gg_{\mathcal{L}}1$ and $(\mathcal{L}(q,d,1/\varepsilon),N^k)$-irrational $\theta\in\mathbb{T}^d$ such that the Bohr set*

$$B=\left\{n\in\mathbb{N}\colon n\equiv 0 \pmod q, |n/N^k|, \|\theta n\|_{\mathbb{T}^d}\leqslant\varepsilon\right\}$$

is contained in $D(A,\mathbf{c})$.

Proof. For $k=1$ the result follows immediately from Proposition 8. Suppose next that $k\geqslant 2$. By the Hölder inequality and Lemma 9 we have

$$\int_{\mathbb{T}}\prod_{i=1}^{s}\left|\sum_{n\in A}e(c_in^kt)\right|^2dt\leq\int_{\mathbb{T}}\left|\sum_{n\in A}e(n^kt)\right|^{2h}dt$$
$$\leq\int_{\mathbb{T}}\left|\sum_{n=1}^{N}e(n^kt)\right|^{2h}dt\ll_{h,k}N^{2h-k}.$$

Let $\rho(n)$ be the number of representations of n in the form $c_1a_1^k+\cdots+c_ha_h^k$: $a_1\ldots,a_h\in A$, $a_i\neq a_j$ for all $i\neq j$. Then

$$\sum_{n\in T}\rho(n)=\binom{|A|}{h}$$

and

$$\sum_{n\in T}\rho(n)^2\leq\int_{\mathbb{T}}\prod_{i=1}^{s}\left|\sum_{n\in A}e(c_in^kt)\right|^2dt.$$

Thus, by the Cauchy–Schwarz inequality

$$|T|\gg_{h,k}|A|^{2h}N^{-2h+k}=:\gamma'\left(\sum c_i\right)N^k.$$

Now the assertion follows by applying Proposition 8 with

$$T=c_1A^k\dot{+}\ldots\dot{+}c_hA^k\subseteq\left[\left(\sum c_i\right)N^k\right]$$

and the function $\mathcal{L}$. ∎

4. Polynomial values in Bohr sets

The next important step of our approach is to show that every Bohr set contains many polynomial values $\mathsf{p}(n)$, $n \in \mathbb{N}$ provided that $\mathsf{p}(0) = 0$. An upper bound for this quantity was proven in [10]. However, it turns out that essentially the same argument provides also a lower bound. Thus, the proof of Lemma 13 is similar to the proof of Lemma 4.5 in [10]. Here we also use an ε-free version of Weyl's inequality established in [15] (Corollary 3.3in [10]) and a smooth minorant for a ball in $\mathbb{T}^d$ (Lemma A.2 in [10]).

Lemma 11. *For every integer $k \geqslant 2$ there exists a positive constant c_k such that the following holds. Let $\mathsf{p} \in \mathbb{Z}[x]$ be a polynomial of degree k with leading coefficient α. Suppose that $\theta \in \mathbb{T}^d$ is (L, N) irrational and that $\mathbf{r} \in \mathbb{Z}^d \setminus \{0\}$. Then*

$$\left| \sum_{n=1}^{N^{1/k}} e(\mathbf{r} \cdot \theta \mathsf{p}(n)) \right| \leqslant N^{1/k} |\alpha| \|\mathbf{r}\|_1 L^{-c_k}.$$

We denote by $\mathcal{B}_\varepsilon$ the ball with radius ε centered at 0 in $\mathbb{T}^d$.

Lemma 12. *There is a minorant ψ_ε^- for $\mathcal{B}_\varepsilon$ in $\mathbb{T}^d$ satisfying*

1. $\frac{1}{2}(2\varepsilon)^d \leqslant \int_{\mathbb{T}^d} \psi_\varepsilon^-(t) dt \leqslant 2(2\varepsilon)^d$,
2. $\sum_{\mathbf{r} \in \mathbb{Z}^d \setminus \{0\}} \left| \widehat{\psi_\varepsilon^-}(\mathbf{r}) \right| \|\mathbf{r}\|_1 \ll_{d,\varepsilon} 1$.

Let us remark that $\sum_{\mathbf{r} \in \mathbb{Z}^d \setminus \{0\}} \left| \widehat{\psi_\varepsilon^-}(\mathbf{r}) \right| \|\mathbf{r}\|_1$ can be bounded from above by a function $K(1/\varepsilon, d)$ that is nondecreasing in each variable.

Lemma 13. *Let $B = \{n \colon n \equiv 0 \pmod q), |n/N|, \|\theta n\| \leqslant \varepsilon\}$, where $\theta \in \mathbb{T}^d$ is (L, N)-irrational. Suppose that $\mathsf{p} \in \mathbb{Z}[x]$ is a polynomial of degree $l \geqslant 1$ and the leading coefficient $\alpha \in \mathbb{Z} \setminus \{0\}$ such that $\mathsf{p}(0) = 0$. Then*

$$\left| \left\{ n \in [N^{1/l}] \colon \mathsf{p}(n) \in B \right\} \right| \geqslant \frac{1}{4} (2\varepsilon/|\alpha|)^d q^{-1} (\varepsilon N / 2|\alpha|)^{1/l}$$

provided that $L \geqslant \left(4q^l |\alpha| (2\varepsilon)^{-d} K(\varepsilon, d) \right)^{1/c_l}$.

Proof. We start with the case $l>1$. Let $I=[1,\lfloor q^{-1}(\varepsilon N/2|\alpha|)^{1/l}\rfloor]$ be a discrete interval and put $J=\{qi\colon i\in I\}$. Since $\mathsf{p}(0)=0$ it follows that

$$\left|\left\{n\in[N^{1/l}]\colon \mathsf{p}(n)\in B\right\}\right|\geqslant\sum_{n\in J}\psi_\varepsilon^-(\theta\mathsf{p}(n)). \tag{13}$$

By Fourier expansion it may be written as

$$\sum_{\mathbf{r}\in\mathbb{Z}^d}\widehat{\psi_\varepsilon^-}(\mathbf{r})\sum_{n\in J}e(\mathbf{r}\cdot\theta\mathsf{p}(n))=\sum_{\mathbf{r}\in\mathbb{Z}^d}\widehat{\psi_\varepsilon^-}(\mathbf{r})\sum_{n\in I}e(\mathbf{r}\cdot\theta\mathsf{p}(qn)).$$

The contribution from $\mathbf{r}=\mathbf{0}$ is

$$|I|\int_{\mathbb{T}^d}\psi_\varepsilon^-(t)dt\geqslant\frac{1}{2}(2\varepsilon)^d q^{-1}(\varepsilon N/2|\alpha|)^{1/l}. \tag{14}$$

The contribution from $\mathbf{r}\neq\mathbf{0}$ can be bounded from above by

$$\sum_{\mathbf{r}\in\mathbb{Z}^d\setminus\{0\}}\left|\widehat{\psi_\varepsilon^-}(\mathbf{r})\right|\left|\sum_{n\in I}e(\mathbf{r}\cdot\theta\mathsf{p}(qn))\right|.$$

By Lemma 11 we have

$$\left|\sum_{n\in I}e(\mathbf{r}\cdot\theta\mathsf{p}(qn))\right|\leqslant|I||\alpha|q^l\|\mathbf{r}\|_1L^{-c_l}$$

and therefore

$$\begin{aligned}\sum_{\mathbf{r}\in\mathbb{Z}^d\setminus\{0\}}\left|\widehat{\psi_\varepsilon^-}(\mathbf{r})\right|\left|\sum_{n\in I}e(\mathbf{r}\cdot\theta\mathsf{p}(qn))\right|&\leqslant|I||\alpha|q^lL^{-c_l}\sum_{\mathbf{r}\in\mathbb{Z}^d\setminus\{0\}}|\widehat{\psi_\varepsilon^-}(\mathbf{r})|\|\mathbf{r}\|_1\\&\leqslant\frac{1}{4}(2\varepsilon)^dq^{-1}(\varepsilon N/2|\alpha|)^{1/l},\end{aligned} \tag{15}$$

provided that

$$L\geqslant\Big(4q^l|\alpha|(2\varepsilon)^{-d}K(\varepsilon,d)\Big)^{1/c_l}. \tag{16}$$

Now the required estimate follows by (14) and (15).

Next, assume that $l=1$. Then, clearly

$$\big\{n\colon n\equiv 0\ (\mathrm{mod}\ q),|n/N|,\|\theta n\|\leqslant\varepsilon/|\alpha|\big\}\subseteq\big\{n\in[N]\colon\mathsf{p}(n)\in B\big\},$$

hence by Lemma 6

$$|\{n\in[N]\colon\mathsf{p}(n)\in B\}|\geqslant\frac{1}{4}(2\varepsilon/|\alpha|)^{d+1}q^{-1}N,$$

which completes the proof. ∎

In the proof of Theorem 2 we will deal with possibly two different powers of variables that appear in (3). This will result in very different sizes of obtained Bohr sets and if we restrict a Bohr set to a shorter interval we may lose high irrationality of θ with respect to this interval. First we will need a simple general lower estimate for the size of a Bohr set, as we can not apply Lemma 6 and Lemma 7 due to the lack of high irrationality of θ. Then in Lemma 15 we show that every Bohr set contains another Bohr set with highly irrational θ. Our argument is rather not efficient, but it is simple and sufficient for its purpose.

Lemma 14. *Let* $B=\big\{n\colon n\equiv 0 \pmod q), |n/N|, \|\theta n\|\leqslant\varepsilon\big\}$ *be a Bohr set, where* $\theta\in\mathbb{T}^d$. *Then*

$$|B|\geqslant 2\varepsilon^d\lfloor\varepsilon N/2q\rfloor.$$

Proof. Put $I=[-\lfloor\varepsilon N/2q\rfloor,\lfloor\varepsilon N/2q\rfloor]$ and observe that

$$\begin{aligned}|B| &\geqslant \varepsilon^{-d}|I|^{-1}\sum_{|n|\leqslant\varepsilon N/2q} 1_{\mathcal{B}_{\varepsilon/2}}*1_{\mathcal{B}_{\varepsilon/2}}(\theta qn)1_I*1_I(n)\\ &=\varepsilon^{-d}|I|^{-1}\sum_{\mathbf{r}\in\mathbb{Z}^d}|\widehat{1_{\mathcal{B}_{\varepsilon/2}}}(\mathbf{r})|^2\sum_{|n|\leqslant\varepsilon N/2q}1_I*1_I(n)e(\mathbf{r}\cdot\theta qn)\\ &=\varepsilon^{-d}\sum_{\mathbf{r}\in\mathbb{Z}^d}|\widehat{1_{\mathcal{B}_{\varepsilon/2}}}(\mathbf{r})|^2F_{\lfloor\varepsilon N/2q\rfloor}(\mathbf{r}\cdot\theta q)\\ &\geqslant\varepsilon^{-d}|\widehat{1_{\mathcal{B}_{\varepsilon/2}}}(\mathbf{0})|^2F_{\lfloor\varepsilon N/2q\rfloor}(0)=2\varepsilon^d\lfloor\varepsilon N/2q\rfloor,\end{aligned}$$

where the last inequality follows from positivity of the Fejér kernel. ∎

Lemma 15. *Let* $B'=\{n\colon n\equiv 0 \pmod{q'}), |n/N'|, \|\theta' n\|\leqslant\varepsilon'\}$ *be a Bohr set, where* $0<\varepsilon'\leqslant 1/2$, q',d',N' *are integers and* $\theta'\in\mathbb{T}^{d'}$. *Suppose that* $\mathcal{L}\colon\mathbb{N}^2\times\mathbb{R}_{>0}\to\mathbb{N}$ *is a nondecreasing function in each variable, which may depend on* q',d',ε', *and suppose that* $N'>N>N(\mathcal{L})$. *Then there are* $q,d\ll_{\mathcal{L}}1$, $\varepsilon\gg_{\mathcal{L}}1$ *and a* $(\mathcal{L}(q,d,1/\varepsilon),N)$*-irrational* $\theta\in\mathbb{T}^d$ *such that*

$$\begin{aligned}&\big\{n\colon n\equiv 0 \pmod q), |n/N|, \|\theta n\|\leqslant\varepsilon\big\}\\ &\qquad\subseteq\big\{n\colon n\equiv 0 \pmod{q'}), |n/N'|, \|\theta' n\|\leqslant\varepsilon'\big\}.\end{aligned}$$

Proof. We put

$$B=\big\{n\colon n\equiv 0 \pmod{q'}), |n/N|, \|\theta' n\|\leqslant\frac{1}{4}\varepsilon'\big\}\subseteq B'$$

and note that by the previous lemma

$$|B|:=\gamma N\geqslant\frac{1}{2}(\varepsilon'/4)^{d'}N/q'$$

provided that N is large enough. Since $N>N(\mathcal{L})$ we may apply Proposition 8 with $T=B$ and $\mathcal{L}$. Thus, there are $q,d\ll_{\mathcal{L}}1$, $\varepsilon\gg_{\mathcal{L}}1$ and a

$(\mathcal{L}(q,d,1/\varepsilon),N)$-irrational $\theta\in\mathbb{T}^d$ such that

$$\big\{n\colon n\equiv 0 \pmod{q}, |n/N|, \|\theta n\| \leqslant \varepsilon\big\} \subseteq 2B-2B \subseteq B',$$

and this concludes the proof. ∎

5. Proof of Theorem 2

Before we begin the proof let us remark that it is enough to find a nontrivial solution to (3) such that

$$x_{4h+1}=\cdots=x_s=y.$$

Thus, we replace the polynomial on the right-hand side of (3) by $\mathsf{p}(y)-\big(\sum_{i=4h+1}^{s} c_i\big)\,y^k$ and call it again p. Hence we may assume $s=4h$ and the left-hand side of (3) consists of two identical symmetric equations. Non-triviality of obtained solution will follow from the definition of $D(A,\mathbf{c})$.

Let N be a large positive integer, let $[N^k]=A_1\cup\cdots\cup A_r$ be any partition and assume that there are no monochromatic solutions to (3). Recall that

$$D(A,\mathbf{c}) := 2T-2T,$$

where $T:=c_1A^k\dot{+}\ldots\dot{+}c_hA^k$.

We apply repetitively Corollary 10 and Lemma 13 to obtain a sequence of Bohr sets

$$B_i=B(q_i,N_i,\theta_i,\varepsilon_i) := \big\{n\colon n\equiv 0 \pmod{q_i}, |n/N_i|, \|\theta_i n\| \leqslant \varepsilon_i\big\},$$

satisfying the following properties:

- $B_i\subseteq D(A_i,\mathbf{c})$, for all $1\leqslant i\leqslant r$,
- $B_{i+1}\subseteq B_i$, for all $1\leqslant i\leqslant r-1$,
- $q_i,d_i\ll_{\mathcal{L}_i}1$, $\varepsilon_i\gg_{\mathcal{L}_i}1$ and $\theta_i\in\mathbb{T}^{d_i}$ is $(\mathcal{L}_i(q_i,d_i,1/\varepsilon_i),N_i)$-irrational, where $\mathcal{L}_i$ is a nondecreasing function in each variable which may depend on $q_{i-1},d_{i-1},\varepsilon_{i-1}$ for $i\geqslant 2$, satisfying

$$\mathcal{L}_i(q_i,d_i,1/\varepsilon_i) \geqslant \Big(4q_i^l|\alpha|(2\varepsilon_i)^{-d_i}K(1/\varepsilon_i,d_i)\Big)^{1/c_l}$$

  and $N_i>N(\mathcal{L}_i)$. Furthermore, $N_1\geq\cdots\geq N_r$.

Recall that $K(1/\varepsilon,d)$ was defined after Lemma 12 and that K is non-decreasing in each variable. To prove this claim we proceed with induction on i. Suppose that $|A_1\cap[N]|=\gamma_1 N\geqslant N/r$ and apply Corollary 10 with the set $A_1\cap[N]$ and a function $\mathcal{L}_1(q,d,1/\varepsilon)\geqslant\Big(4q^l|\alpha|(2\varepsilon)^{-d}K(1/\varepsilon,d)\Big)^{1/c_l}$ that may depend on γ_1. Hence, there are integers $q_1,d_1\ll_{\mathcal{L}_1}1$, $\varepsilon_1\gg_{\mathcal{L}_1}1$ and a $(\mathcal{L}_1(q_1,d_1,1/\varepsilon_1),N^k)$-irrational $\theta_1\in\mathbb{T}^{d_1}$ such that for $N_1=N^k$ we have

$$B_1=B(q_1,N_1,\theta_1,\varepsilon_1)\subseteq D(A_1,\mathbf{c}).$$

Next, assume that we have already defined Bohr sets $B_1,\dots,B_i$ with the required properties for some $1\leqslant i\leqslant r-1$. Since θ_i is (L_i,N_i)-irrational for some $L_i\geqslant\Big(4q_i^l|\alpha|(2\varepsilon_i)^{-d_i}K(1/\varepsilon_i,d_i)\Big)^{1/c_l}$ it follows by Lemma 13 that

$$\Big|\Big\{n\in[N_i^{1/l}]\colon \mathsf{p}(n)\in B_i\Big\}\Big|\geqslant\frac{1}{4}(2\varepsilon_i/|\alpha|)^{d_i}q_i^{-1}(\varepsilon_iN_i/2|\alpha|)^{1/l}.$$

Let $A:=\{n\in[N_i^{1/l}]\colon \mathsf{p}(n)\in B_i\}$ and note that by the second property for every $n\in A$ we have

$$\mathsf{p}(n)\in B_i\subseteq\cdots\subseteq B_1,$$

which in view of the first property implies that

$$A\subseteq A_{i+1}\cup\cdots\cup A_r.$$

Therefore for some $j>i$ we have $|A\cap A_j|\geqslant|A|/r$. We may assume that $j=i+1$ and put $X=A\cap A_{i+1}$. Averaging over $\vartheta\in\mathbb{T}^{d_i}$, there is a specific choice of ϑ and a set $Y\subseteq X\subseteq A_{i+1}$ such that

$$|Y|\geqslant(2\varepsilon_i/\sigma)^{d_i}|X|,$$

and

$$\|\theta_in^k-\vartheta\|_{\mathbb{T}^{d_i}}\leq\varepsilon_i/\sigma \text{ for all } n\in Y, \tag{17}$$

where $\sigma=\sum|c_i|$. Thus, $Y\subseteq[N_i^{1/l}]$ and

$$|Y|=:\gamma_{i+1}N_i^{1/l}\geqslant\frac{1}{4r}(2\varepsilon_i^2/|\alpha|\sigma)^{d_i}q_i^{-1}(\varepsilon_iN_i/2|\alpha|)^{1/l}.$$

Next, we apply Corollary 10 to the set Y and a function $\mathcal{L}_{i+1}(q,d,1/\varepsilon)\geqslant\Big(4q^l|\alpha|(2\varepsilon)^{-d}K(1/\varepsilon,d)\Big)^{1/c_l}$ that may depend on γ_{i+1}. There exist q',

$d' \ll_{\mathcal{L}_{i+1}} 1$, $\varepsilon' \gg_{\mathcal{L}_{i+1}} 1$ and $(L', N_i^{k/l})$-irrational $\theta' \in \mathbb{T}^{d'}$ for some $L' \geqslant \Big(4(q')^l|\alpha|(2\varepsilon')^{-d}K(1/\varepsilon', d')\Big)^{1/c_l}$ such that

$$B' = B(q', N_i^{k/l}, \theta', \varepsilon') \subseteq D(Y, \mathbf{c}) \subseteq D(A_{i+1}, \mathbf{c}).$$

Since each $b \in B'$ can be written as $c_1 n_1^k + \cdots + c_s n_s^k$ for some $n_1, \ldots, n_s \in Y$, and $\sum c_j = 0$, it follows by (17) that

$$\begin{aligned} \|\theta_i b\|_{\mathbb{T}^{d_i}} &= \|\theta_i(c_1 n_1^k + \cdots + c_s n_s^k)\|_{\mathbb{T}^{d_i}} \\ &\leqslant \sum_{j=1}^{s} \|\theta_i c_j (n_j^k - \vartheta)\|_{\mathbb{T}^{d_i}} \leqslant \varepsilon_i. \end{aligned} \tag{18}$$

If $k < l$ then we put $q_{i+1} = q' q_i, d_{i+1} = d', \theta_{i+1} = \theta'$ and $N_{i+1} = N_i^{k/l}$, so

$$B_{i+1} := B(q_{i+1}, N_{i+1}, \theta_{i+1}, \varepsilon_{i+1}) \subseteq B' \subseteq D(A_{i+1}, \mathbf{c}).$$

To keep high irrationality of θ_{i+1} for $k > l$, we apply Lemma 15 with $N' = N_i$, $N = N^{k/l}$, B' and a function $\mathcal{L}'_{i+1}(q, d, 1/\varepsilon) \geqslant \Big(4q^l|\alpha|(2\varepsilon)^{-d}K(1/\varepsilon, d)\Big)^{1/c_l}$ that may depend on q', d' and ε'. Thus, there are q_{i+1}, $d_{i+1} \ll_{\mathcal{L}'_{i+1}} 1$, $\varepsilon_{i+1} \gg_{\mathcal{L}_{i+1}} 1$ and (L_{i+1}, N_{i+1})-irrational $\theta_{i+1} \in T^{d_{i+1}}$ such that

$$B(q_{i+1}, N_{i+1}, \theta_{i+1}, \varepsilon_{i+1}) \subseteq B(q, N_{i+1}, \theta', \varepsilon') \subseteq B' \subseteq D(A_{i+1}, \mathbf{c}),$$

which completes the proof of this inductive step.

To finish the proof of Theorem 2 it is sufficient to observe that by Lemma 13 and Lemma 6 we have

$$\Big|\Big\{n \in [N_r^{1/l}] \colon \mathsf{p}(n) \in B_r\Big\}\Big| \geqslant \frac{1}{4}(2\varepsilon_r/|\alpha|)^{d_r} q_r^{-1} (\varepsilon_r N_r / 2|\alpha|)^{1/l} \gg_{r,\alpha,\sigma} N^{1/l^r}.$$

Thus, if N is sufficiently large then the set $\big\{n \in [N_r^{1/l}] \colon \mathsf{p}(n) \in B_r\big\}$ contains a positive integer n such that

$$\mathsf{p}(n) \in B_r \subseteq \cdots \subseteq B_1.$$

This gives a non-trivial monochromatic solution to (3), which is a contradiction. ∎

References

[1] V. Bergelson: Ergodic Ramsey theory – an update, in: *Ergodic theory of* $\mathbb{Z}^d$ *actions (Warwick, 1993–1994),* 1–61, London Math. Soc. Lecture Note Ser., 228, Cambridge Univ. Press, Cambridge, 1996.

[2] J. Bourgain, C. Demeter and L. Guth: Proof of the main conjecture in Vinogradov's mean value theorem for degrees higher than three, *Ann. of Math.* **184** (2016), 633–682.

[3] T. D. Browning and S. Prendiville: A transference approach to a Roth-type theorem in the squares, *Int. Math. Res. Not. IMRN* **7** (2017), 2219–2248.

[4] S. Chow: Roth–Waring–Goldbach, *Int. Math. Res. Not. IMRN* **8** (2018), 2341–2374.

[5] S. Chow, S. Lindqvist and S. Prendiville: Rado's criterion over squares and higher powers, *Journal of the European Mathematical Society*, to appear.

[6] P. Csikvári, K. Gyarmati and A. Sárközy: Density and ramsey type results on algebraic equations with restricted solution sets, *Combinatorica* **32** (2012), 425–449.

[7] K. Cwalina and T. Schoen: Tight bounds on additive Ramsey-type numbers, *J. London Math. Soc.* **96** (2017) 601–620.

[8] M. Di Nasso and L. Luperi Baglini: Ramsey properties of nonlinear Diophantine equations, *Adv. Math.* **324** (2018), 84–117.

[9] S. Eberhard: *The abelian arithemtic regularity lemma,* arXiv:1606.09303.

[10] B. Green and S. Lindqvist: Monochromatic solutions to $x + y = z^2$, *Canad. J. Math.*, **71** (2019), 579–605.

[11] B. Green: A Szemerédi-type regularity lemma in abelian groups, with applications, *Geom. Funct. Anal.* **15** (2005), 340–376.

[12] B. Green: Roth's theorem in the primes, *Ann. of Math.* **161** (2005), 1609–1636.

[13] B. Green and T. Sanders: Monochromatic sums and products, *Discrete Anal.* (2016), Paper No. 5.

[14] B. Green and T. Tao: An arithmetic regularity lemma, an associated counting lemma, and applications, in: *An irregular mind*, 261–334, Bolyai Soc. Math. Stud., 21, János Bolyai Math. Soc., Budapest, 2010.

[15] B. Green and T. Tao: The quantitative behaviour of polynomial orbits on nilmanifolds, *Ann. of Math.* **175** (2012), 465–540.

[16] K. Henriot: Logarithmic bounds for translation-invariant equations in squares, *Int. Math. Res. Not. IMRN* **23** (2015), 12540–12562.

[17] N. Hindman: Partitions and sums and products of integers, *Trans. Amer. Math. Soc.* **247** (1979), 227–245.

[18] A. Khalfalah and E. Szemerédi: On the number of monochromatic solutions of $x+y=z^2$, *Combin. Probab. Comput.* **15** (2006), 213–227.

[19] T. H. Lê: Partition regularity and the primes, *C. R. Math. Acad. Sci. Paris* **350** (2012), 439–441.

[20] H. Lefmann: On partition regular systems of equations, *J. Combin. Theory Ser. A* **58** (1991), 35–53.

[21] S. Lindqvist: Partition regularity of generalised Fermat equations. *Combinatorica* **38** (2018), 1457–1483.

[22] J. Moreira: Monochromatic sums and products in $\mathbb{N}$, *Ann. of Math.* **185** (2017), 1069–1090.

[23] P. Pach: Monochromatic solutions to $x+y=z^2$ in the interval $[N, cN^4]$, *Bull. Lond. Math. Soc.* **50** (2018), 1113–1116.

[24] R. Rado: Studien zur Kombinatorik, *Math. Z.* **36** (1933), 424–470.
[25] I. Z. Ruzsa: Generalized arithmetical progressions and sumsets, *Acta Math. Hungar.* **65** (1994), 379–388.
[26] I. Schur: Über die Kongruenz $x^m + y^m \equiv z^m \pmod p$, *Jahresber. Dtsch. Math.- Ver.* **25** (1916), 114–117.
[27] T. Wooley: The asymptotic formula in Waring's problem, *Int. Math. Res. Not.* **7** (2012), 1485–1504.

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