

## THE COLOURING NUMBER OF INFINITE GRAPHS

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We show that, given an infinite cardinal  $\mu$ , a graph has colouring number at most  $\mu$  if and only if it contains neither of two types of subgraph. We also show that every graph with infinite colouring number has a well-ordering of its vertices that simultaneously witnesses its colouring number and its cardinality.

## 1. Introduction

Our point of departure is a recent article of the third author [5] one of whose results addresses infinite graphs with infinite colouring number. Let us recall this notion introduced by Erdős and Hajnal in [2].

**Definition 1.1.** The *colouring number*  $\text{col}(G)$  of a graph  $G = (V, E)$  is the smallest cardinal  $\varkappa$  such that there exists a well-ordering  $<^*$  of  $V$  with

$$|N(v) \cap \{w : w <^* v\}| < \varkappa \quad \text{for all } v \in V,$$

where  $N(v)$  is the set of neighbours of  $v$ . We call such well-orderings *good*.

The result of [5] is that if the colouring number of a graph  $G$  is bigger than some infinite cardinal  $\mu$ , then  $G$  contains either a  $K_\mu$ , i.e.,  $\mu$  mutually adjacent vertices, or  $G$  contains for each positive integer  $k$  an induced copy of the complete bipartite graph  $K_{k,k}$ . This condition is not a characterisation:

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there are graphs, such as  $K_{\omega,\omega}$ , which have small colouring number but nevertheless include an induced  $K_{k,k}$  for each  $k$ .

Since having colouring number  $\leq \mu$  is closed not only under taking induced subgraphs but even under taking subgraphs, it seems easier to look first for a characterisation in terms of forbidden subgraphs. Our main result is that there is indeed a transparent characterisation of “having colouring number  $\leq \mu$ ” in terms of forbidden subgraphs. For some explicit graphs called  $\mu$ -obstructions, to be introduced in Definition 2.1 below, we shall prove the following.

**Theorem 1.2.** *Let  $G$  be a graph and let  $\mu$  denote some infinite cardinal. Then the statement  $\text{col}(G) > \mu$  is equivalent to  $G$  containing some  $\mu$ -obstruction as a subgraph.*

This result will also appear in the upcoming book [4] of the third author. The proof we describe has an interesting consequence.

**Theorem 1.3 (Erdős and Hajnal [2]).** *Every graph  $G$  whose colouring number is infinite possesses a good well-ordering of length  $|V(G)|$ .*

It is not hard to re-obtain the result mentioned above from our characterisation, Theorem 1.2, by inspecting whether the  $\mu$ -obstructions satisfy it. In fact, one can easily deduce the following strengthening.

**Theorem 1.4.** *If  $G$  is a graph with  $\text{col}(G) > \mu$ , where  $\mu$  denotes some infinite cardinal, then  $G$  contains either a  $K_\mu$  or, for each positive integer  $k$ , an induced  $K_{k,\omega}$ .*

We will also give an example in Section 5 demonstrating that the conclusion cannot be improved further to the presence of an induced  $K_{\omega,\omega}$ . Which complete bipartite graphs exactly one gets by this approach depends on which properties the relevant cardinals have in the partition calculus.

In our proof, we use the concept of ‘ $\mu$ -ladders’. They are somewhat equivalent to  $\mu$ -obstructions and we also prove the following variant of Theorem 1.2.

**Theorem 1.5.** *Let  $G$  be a graph and let  $\mu$  denote some infinite cardinal. Then the statement  $\text{col}(G) > \mu$  is equivalent to  $G$  containing some  $\mu$ -ladder as a subgraph.*

For standard set-theoretical background we refer to Kunen’s textbook [7].

## 2. Obstructions

Throughout this section, we fix an infinite cardinal  $\mu$ . There are two kinds of  $\mu$ -obstructions relevant for the condition  $\text{col}(G) > \mu$  in Theorem 1.2. They are introduced next.

**Definition 2.1.** (1) A  $\mu$ -obstruction of type I is a bipartite graph  $H$  with bipartition  $(A, B)$  such that for some cardinal  $\lambda \geq \mu$  we have

- $|A| = \lambda$ ,  $|B| = \lambda^+$ ,
- every vertex of  $B$  has at least  $\mu$  neighbours in  $A$ , and
- every vertex of  $A$  has  $\lambda^+$  neighbours in  $B$ .

(2) Let  $\varkappa > \mu$  be regular, and let  $G$  be a graph with  $V(G) = \varkappa$ . Define  $T_G$  to be the set of those  $\alpha \in \varkappa$  with the following properties:

- $\text{cf}(\alpha) = \text{cf}(\mu)$ .
- The order type of  $N(\alpha) \cap \alpha$  is  $\mu$ .
- The supremum of  $N(\alpha) \cap \alpha$  is  $\alpha$ .

If  $T_G$  is stationary in  $\varkappa$ , then  $G$  is a  $\mu$ -obstruction of type II. We also call graphs isomorphic to such graphs  $\mu$ -obstructions of type II.

Now we can directly proceed to the easier direction of Theorem 1.2. The case of type I obstructions was already mentioned in [9, Lemma 3.3].

**Proposition 2.2.** *If a graph  $G$  has a  $\mu$ -obstruction of either type as a subgraph, then  $\text{col}(G) > \mu$ .*

**Proof.** Suppose first that  $G$  contains a  $\mu$ -obstruction of type I, say with bipartition  $(A, B)$  as in Definition 2.1 above, and  $|A| = \lambda \geq \mu$ . Assume for a contradiction that there is a good well-ordering of  $G$  witnessing  $\text{col}(G) \leq \mu$ . Thus, every  $b \in B$  has a neighbour in  $A$  above it in that well-ordering. For  $a \in A$ , we denote by  $X_a$  the set of those neighbours of  $a$  that are below  $a$  in the well-ordering. Hence  $B = \bigcup_{a \in A} X_a$ . Since all the  $X_a$  have size less than  $\mu$ , we deduce that  $|B| \leq \lambda$ , which is the desired contradiction.

In the second case, we may without loss of generality assume that  $G$  itself is an obstruction of type II. Again we suppose for a contradiction that there is a good well-ordering  $<^*$  of  $V(G)$  witnessing  $\text{col}(G) \leq \mu$ . Notice that each  $\alpha \in T_G$  has a neighbour  $\beta < \alpha$  such that  $\alpha <^* \beta$ . Let  $f: T_G \rightarrow \varkappa$  be a function sending each  $\alpha$  to some such  $\beta$ . By Fodor's Lemma, there must be some  $\beta < \varkappa$  such that

$$T = \{\alpha \in T_G: f(\alpha) = \beta\}$$

is stationary. Now every element of  $T$  is a neighbour of  $\beta$ , and  $\beta$  comes after  $T$  in the ordering  $<^*$ , which in view of  $|T| = \varkappa > \mu$  contradicts our assumption that this ordering is good. ■

We say that a graph is  $\mu$ -unobstructed if it has no  $\mu$ -obstruction of either type as a subgraph. To complete the proof of Theorem 1.2 we still need to show that every  $\mu$ -unobstructed graph  $G$  satisfies  $\text{col}(G) \leq \mu$ . This will be the objective of Sections 3 and 4.

In the remainder of this section, we prove two results asserting that in order to find an obstruction in a given graph  $G$  it suffices to find something weaker.

**Definition 2.3.** A  $\mu$ -barricade is bipartite graph with bipartition  $(A, B)$  such that

- $|A| < |B|$ , and
- every vertex of  $B$  has at least  $\mu$  neighbours in  $A$ .

**Lemma 2.4.** *If  $G$  has a  $\mu$ -barricade as a subgraph, then it also has a  $\mu$ -obstruction of type I as a subgraph.*

**Proof.** Let  $H$  with bipartition  $(A, B)$  be a  $\mu$ -barricade which is a subgraph of  $G$ , chosen so that  $\lambda = |A|$  is minimal. By deleting some vertices of  $B$  if necessary, we may assume that  $B$  has cardinality  $\lambda^+$ . Let  $A'$  be the set of  $a \in A$  for which<sup>1</sup>  $N_B(a)$  is of size  $\lambda^+$ , and let  $B'$  be the set of elements of  $B$  with no neighbour in  $A \setminus A'$ . By  $|A| = \lambda$  and the definition of  $A'$ , there are at most  $\lambda$  edges  $ab$  with  $a \in A \setminus A'$  and  $b \in B$ . So  $B \setminus B'$  is of size at most  $\lambda$ . It follows that  $B'$  has cardinality  $\lambda^+$ . In particular, the subgraph  $H'$  of  $H$  on  $(A', B')$  is a  $\mu$ -barricade, so by minimality of  $|A|$  we have  $|A'| = \lambda$ . Since by construction every vertex of  $A'$  has  $\lambda^+$  neighbours in  $B$  and hence in  $B'$ , the subgraph  $H'$  is a  $\mu$ -obstruction of type I. ■

**Definition 2.5.** Let  $\varkappa > \mu$  be regular. A graph  $G$  with set of vertices  $\varkappa$  is said to be a  $\mu$ -ladder if there is a stationary set  $T$  such that each  $\alpha \in T$  has at least  $\mu$  neighbours in  $\alpha$ . Also, every graph isomorphic to such a graph is regarded as a  $\mu$ -ladder.

**Lemma 2.6.** *Every graph containing a  $\mu$ -ladder is  $\mu$ -obstructed.*

**Proof.** It suffices to prove that every  $\mu$ -ladder is  $\mu$ -obstructed. So let  $G$  with  $V(G) = \varkappa$  and the stationary set  $T$  be as described in the previous definition. For each  $\alpha \in T$  we let the sequence  $\langle \alpha_i \mid i < \mu \rangle$  enumerate the  $\mu$

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<sup>1</sup> Throughout we abbreviate  $N(a) \cap B$  simply by  $N_B(a)$ .

smallest neighbours of  $\alpha$  in increasing order and denote the limit point of this sequence by  $f(\alpha)$ . Clearly, we have  $f(\alpha) \leq \alpha$  and  $\text{cf}(f(\alpha)) = \text{cf}(\mu)$  for all  $\alpha \in T$ .

Let us first suppose that the set

$$T' = \{\alpha \in T : f(\alpha) < \alpha\}$$

is stationary in  $\varkappa$ . Then for some  $\gamma < \varkappa$  the set

$$B = \{\alpha \in T' : f(\alpha) = \gamma\}$$

is stationary and as  $|\gamma| < \varkappa = |B|$  the pair  $(\gamma, B)$  is a  $\mu$ -barricade in  $G$ . Due to Lemma 2.4 it follows that  $G$  contains a  $\mu$ -obstruction of type I.

So it remains to consider the case that

$$T'' = \{\alpha \in T : f(\alpha) = \alpha\}$$

is stationary in  $\varkappa$ . In that case we have  $N(\alpha) \cap \alpha = \{\alpha_i : i < \mu\}$  for all  $\alpha \in T''$ . So  $T_G$  is a superset of  $T''$  and thus stationary, meaning that  $G$  is a  $\mu$ -obstruction of type II. ■

Towards the converse implication of Lemma 2.6 we have the following.

**Lemma 2.7.** *Every  $\mu$ -obstruction is a  $\mu$ -ladder.*

**Proof.** Clearly, every  $\mu$ -obstruction of type II is a  $\mu$ -ladder. Let a  $\mu$ -obstruction of type I be given with bipartition  $(A, B)$ . Let  $\varkappa$  be a well-order of its vertex set such that  $A$  is an initial segment of that well-order. Then  $B$  is a club and hence stationary in that well-order. Thus this defines a  $\mu$ -ladder. ■

**Proof that Theorem 1.2 implies Theorem 1.5.** By Lemma 2.6 and Lemma 2.7 having a  $\mu$ -ladder is equivalent to having a  $\mu$ -obstruction. Hence Theorem 1.2 and Theorem 1.5 are equivalent. ■

### 3. Regular $\varkappa$

In this and the next section we shall prove the harder part of Theorem 1.2, in such a way that Theorem 1.3 is also immediate. To this end we shall show

**Theorem 3.1.** *Let  $G$  denote an infinite graph of order  $\varkappa$  and let  $\mu$  be an infinite cardinal. Then at least one of the following three cases occurs:*

- $G$  has a subgraph  $H$  with  $|V(H)| < |V(G)|$  and  $\text{col}(H) > \mu$ .
- $G$  is  $\mu$ -obstructed.

- $G$  has a good well-ordering of length  $\varkappa$  exemplifying  $\text{col}(G) \leq \mu$ .

Suppose for a moment that we know this. To deduce Theorem 1.2 we consider any graph with  $\text{col}(G) > \mu$ . Let  $G^*$  be subgraph of  $G$  with  $\text{col}(G^*) > \mu$  and subject to this with  $|V(G^*)|$  as small as possible. Then  $G^*$  is still infinite and when we apply Theorem 3.1 to  $G^*$  the first and third outcome are impossible, so the second one must occur. Thus  $G^*$  and hence  $G$  contains a  $\mu$ -obstruction, as desired. To obtain Theorem 1.3 we apply Theorem 3.1 to  $G$  with  $\mu = \text{col}(G)$ .

The proof of Theorem 3.1 itself is divided into two cases according to whether  $\varkappa$  is regular or singular. The former case will be treated immediately and the latter case is deferred to the next section. We would like to remark that the first case of the argument that follows is handled in the same way as Claim (\*) in [9, Theorem 2.4].

**Proof of Theorem 3.1 when  $\varkappa$  is regular.** Let  $V(G) = \varkappa$  and consider the set

$$T = \{\alpha < \varkappa : \text{some } \beta \geq \alpha \text{ has at least } \mu \text{ neighbours in } \alpha\}.$$

**First Case:**  $T$  is not stationary in  $\varkappa$ . We observe that  $0 \notin T$ . Let  $\langle \delta_i \mid i < \varkappa \rangle$  be a strictly increasing continuous sequence of ordinals with limit  $\varkappa$  starting with  $\delta_0 = 0$  and such that  $\delta_i \notin T$  holds for all  $i < \varkappa$ . Now if for some  $i < \varkappa$  the restriction  $G_i$  of  $G$  to the half-open interval  $[\delta_i, \delta_{i+1})$  has colouring number  $> \mu$ , then the first alternative (that is, the first bullet of Theorem 3.1) holds. Otherwise we may fix for each  $i < \varkappa$  a well-ordering  $<_i$  of  $V(G_i)$  that exemplifies  $\text{col}(G_i) \leq \mu$ . The concatenation  $<^*$  of all these well-orderings has length  $\varkappa$ , so it suffices to verify that it demonstrates  $\text{col}(G) \leq \mu$ .

To this end, we consider any vertex  $x$  of  $G$ . Let  $i < \varkappa$  be the ordinal with  $x \in G_i$ . The neighbours of  $x$  preceding it in the sense of  $<^*$  are either in  $\delta_i$  or they belong to  $G_i$  and precede  $x$  under  $<_i$ . Since  $x \geq \delta_i$  and  $\delta_i \notin T$ , there are less than  $\mu$  neighbours of  $x$  in  $\delta_i$ . Also, by our choice of  $<_i$ , there are less than  $\mu$  such neighbours in  $G_i$ .

**Second Case:**  $T$  is stationary in  $\varkappa$ .

Let us fix for each  $\alpha \in T$  an ordinal  $\beta_\alpha \geq \alpha$  with  $|N(\beta_\alpha) \cap \alpha| \geq \mu$ . A standard argument shows that the set

$$E = \{\delta < \varkappa : \text{if } \alpha \in T \cap \delta, \text{ then } \beta_\alpha < \delta\}$$

is club in  $\varkappa$ . Thus  $T \cap E$  is unbounded in  $\varkappa$ . Let the sequence  $\langle \eta_i \mid i < \varkappa \rangle$  enumerate the members of this set in increasing order. Then for each  $i < \varkappa$

the ordinal  $\xi_i = \beta_{\eta_i}$  is at least  $\eta_i$  and smaller than  $\eta_{i+1}$ , because the latter ordinal belongs to  $E$ . In particular, each of the equations  $\eta_i = \xi_j$  and  $\xi_i = \xi_j$  is possible only if  $i = j$ . Thus it makes sense to define

$$v_\alpha = \begin{cases} \alpha & \text{if } \alpha \neq \eta_i, \xi_i \text{ for all } i < \varkappa, \\ \xi_i & \text{if } \alpha = \eta_i \text{ for some } i < \varkappa, \\ \eta_i & \text{if } \alpha = \xi_i \text{ for some } i < \varkappa. \end{cases}$$

The map  $\pi$  sending each  $\alpha < \varkappa$  to  $v_\alpha$  is a permutation of  $\varkappa$ . If  $\alpha$  belongs to the stationary set  $T \cap E$ , then  $v_\alpha = \xi_i$  for some  $i < \varkappa$  and therefore  $v_\alpha$  has at least  $\mu$  neighbours in  $\eta_i$  and all of these are of the form  $v_\beta$  with  $\beta < \alpha$ . So  $\pi$  gives an isomorphism between  $G$  and a  $\mu$ -ladder, and in the light of Lemma 2.6 we are done. ■

### 4. Singular $\varkappa$

Next we consider the case that  $\varkappa$  is a singular cardinal. Except for the fact that we aim for a *good* well-ordering in the third bullet of Theorem 3.1, the required result was obtained by Shelah in [8, Conclusion 2.3]. It turns out that Shelah’s proof actually yields such a good well-ordering. But, as the considerably greater generality of [8] adds an extra burden of technical detail for the reader, we provide a self-contained verification of this fact here.

Throughout this section, sets of size at least  $\mu$  will be referred to as *big* and sets of size less than  $\mu$  will be said to be *small*. We will often consider  $\subseteq$ -increasing sequences  $\langle X_i \mid i < \gamma \rangle$  of vertex-sets for which each  $N_{X_i}(v)$  is small. In such cases we would like to conclude that also  $N_{\bigcup_{i < \gamma} X_i}(v)$  is small. We can do this as long as  $\gamma$  and  $\mu$  have different cofinalities. So we fix the notation  $\varpi$  for the rest of the argument to mean the least infinite cardinal whose cofinality is not equal to  $\text{cf}(\mu)$ . Thus  $\varpi$  is either  $\omega$  or  $\omega_1$ .

**Definition 4.1.** A set  $X$  of vertices of a graph  $G$  is *robust* if for any  $v \in V(G) \setminus X$  the neighbourhood  $N_X(v)$  is small.

**Remark 4.2.** Let  $\langle X_i \mid i < \varpi \rangle$  be a  $\subseteq$ -increasing sequence of robust sets. Then  $\bigcup_{i < \varpi} X_i$  is also robust.

**Lemma 4.3.** *Let  $G$  be a  $\mu$ -unobstructed graph and let  $X$  be an uncountable set of vertices of  $G$ . Then there is a robust set  $Y$  of vertices of  $G$  which includes  $X$  and is of the same cardinality.*

**Proof.** Let  $\lambda$  be the cardinality of  $X$ . We build a  $\subseteq$ -increasing sequence  $\langle X_i \mid i < \varpi \rangle$  of sets recursively by letting  $X_0 = X$ , taking  $X_{i+1} = X_i \cup \{v \in V(G) : N_{X_i}(v) \text{ is big}\}$  in the successor step and  $X_\ell = \bigcup_{i < \ell} X_i$  for  $\ell$  a limit ordinal. Finally, we set  $Y = \bigcup_{i < \varpi} X_i$ . Since by construction  $Y$  is robust and includes  $X$ , it remains to prove that  $|Y| = \lambda$ .

To do this, we prove by induction on  $i$  that each  $X_i$  is of size  $\lambda$ . The cases where  $i$  is 0 or a limit are clear, so suppose  $i = j + 1$ . By the induction hypothesis,  $|X_j| = \lambda$ . If  $|X_{j+1}|$  were greater than  $\lambda$ , then the induced bipartite subgraph on  $(X_j, X_{j+1} \setminus X_j)$  would be a  $\mu$ -barricade, which is impossible by Lemma 2.4. Thus  $|X_{j+1}| = \lambda$ , as required. ■

**Remark 4.4.** Lemma 4.3 also holds when  $X$  is countably infinite, but the proof is more involved and so we have omitted it (unlike in the above proof, we need that there are no type II obstructions).

**Proof of Theorem 3.1 when  $\varkappa$  is singular.** If  $G$  is  $\mu$ -obstructed, then we are done, so we suppose that it is not. Let us fix any bijective enumeration  $\langle v_i \mid i < \varkappa \rangle$  of the set of vertices and a continuous increasing sequence  $\langle \varkappa_i \mid i < \text{cf}(\varkappa) \rangle$  of cardinals with limit  $\varkappa$ , where  $\varkappa_0 > \text{cf}(\varkappa)$  is uncountable.

We begin by building a family  $\langle X_{i,j} \mid i < \text{cf}(\varkappa), j < \varpi \rangle$  of robust sets of vertices of  $G$ , with  $X_{i,j}$  of size  $\varkappa_i$ . This will be done by nested recursion on  $i$  and  $j$ . When we come to choose  $X_{i,j}$ , we will already have chosen all  $X_{i',j'}$  with  $j' < j$  or with both  $j' = j$  and  $i' < i$ . Whenever we have just selected such a set  $X_{i,j}$ , we fix immediately an arbitrary enumeration  $\langle x_{i,j}^k \mid k < \varkappa_i \rangle$  of this set. We impose the following conditions on this construction:

- (1)  $\{v_k : k < \varkappa_i\} \subseteq X_{i,0}$  for all  $i < \text{cf}(\varkappa)$ .
- (2)  $\bigcup_{i' < i, j' \leq j} X_{i',j'} \subseteq X_{i,j}$  for all  $i < \text{cf}(\varkappa)$  and  $j < \varpi$ .
- (3)  $\{x_{i',j}^k : k < \varkappa_{i'}\} \subseteq X_{i,j+1}$  for all  $i < i' < \text{cf}(\varkappa)$  and  $j < \varpi$ .

These three conditions specify some collection of  $\varkappa_i$ -many vertices which must appear in  $X_{i,j}$ . By Lemma 4.3 we can extend this collection to a robust set of the same size and we take such a set as  $X_{i,j}$ . This completes the description of our recursive construction.

The purpose of condition (3) is to ensure that we have

- (4)  $X_{\ell,j} \subseteq \bigcup_{i < \ell} X_{i,j+1}$  whenever  $\ell < \text{cf}(\varkappa)$  is a limit non-zero ordinal and  $j < \varpi$ .

Indeed, for any  $x \in X_{\ell,j}$  there is some index  $k < \varkappa_\ell$  with  $x = x_{\ell,j}^k$ , owing to the continuity of the  $\varkappa_i$  there is some ordinal  $i < \ell$  with  $k < \varkappa_i$ , and condition (3) yields  $x \in X_{i,j+1}$  for any such  $i$ .

Now for  $i < \text{cf}(\varkappa)$  the set  $X_i = \bigcup_{j < \varpi} X_{i,j}$  is robust by Remark 4.2. We claim that for any limit ordinal  $\ell < \text{cf}(\varkappa)$  we have  $X_\ell = \bigcup_{i < \ell} X_i$ . That each



$X_i$  with  $i < \ell$  is a subset of  $X_\ell$  is clear by condition (2) above. The other inclusion follows by taking the union over all  $j < \varpi$  in (4).

Each vertex must lie in some set  $X_i$  by condition (1) above, and it follows from what we have just shown that the least such  $i$  can never be a limit. That is,  $X_0$  together with all the sets  $X_{i+1} \setminus X_i$  gives a partition of the vertex set. If the induced subgraph of  $G$  on any of these sets has colouring number  $> \mu$ , then the first alternative of Theorem 3.1 holds. Otherwise all of these induced subgraphs have good well-orderings. Since each  $X_i$  is robust, the well-ordering obtained by concatenating all of these well-orderings is also good, so that the third alternative of Theorem 3.1 holds. ■

### 5. A necessary condition

In this section we show that we can now easily deduce Theorem 1.4 from Theorem 1.2. We shall rely on the following result of Dushnik, Erdős, and Miller from [1].

**Theorem 5.1.** *For each infinite cardinal  $\lambda$  we have  $\lambda \longrightarrow (\lambda, \omega)$ . This means that if the edges of a complete graph on  $\lambda$  vertices are coloured red and green, then there is either a red clique of order  $\lambda$ , or a green clique of order  $\omega$ .*

By restricting the attention to the red graph, one realises that this means that every infinite graph  $G$  either contains a clique of order  $|V(G)|$  or an infinite independent set. When used in this formulation, we refer to the above theorem as DEM.

**Proof of Theorem 1.4.** By Theorem 1.2 it remains to show that every graph with an obstruction of type I or II has a  $K_\mu$  subgraph or an induced  $K_{k,\omega}$ .

First we check this for obstructions of type I. Let  $(A, B)$  be the bipartition of that obstruction. By DEM, we may assume that the neighbourhood  $N(b)$  of every  $b \in B$  contains an independent set  $Y_b$  of size  $k$ . Let  $f$  be the function mapping  $b$  to  $Y_b$ . There must be a  $k$ -element subset  $Y$  of  $A$  such that  $|f^{-1}[Y]| = |B|$ . By DEM again, we may assume that  $f^{-1}[Y]$  contains an infinite independent set  $B'$ . Then  $G[B' \cup Y]$  is isomorphic to  $K_{k,\omega}$ .

Hence, it remains to show that every obstruction  $G$  of type II has a  $K_\mu$  subgraph or an induced  $K_{k,\omega}$ . For every  $\alpha \in T_G$ , we may assume by DEM that  $N(\alpha) \cap \alpha$  contains an independent set  $Y_\alpha$  of size  $k$ . For each  $i$  with  $1 \leq i \leq k$ , let  $f_i: T \rightarrow \aleph$  be the function mapping  $\alpha$  to the  $i$ -th smallest element of  $Y_\alpha$ . By Fodor's Lemma, there is some stationary  $T' \subseteq T_G$  at which  $f_1$  is constant, and some stationary  $T'' \subseteq T'$  at which  $f_2$  is constant. Proceeding like this,

we find some stationary  $S \subseteq T_G$  at which all the  $f_i$  are constant. Let  $X$  be the set of these  $k$  constants. By DEM, we may assume that  $S$  contains a countably infinite independent set  $I$ . Then  $G[X \cup I]$  is isomorphic to  $K_{k,\omega}$ . ■

In the following example, we show that if we replace ‘ $K_{k,\omega}$ ’ by ‘ $K_{\omega,\omega}$ ’ in Theorem 1.4, then it becomes false.

**Example 5.2.** Let  $A$  be the set of finite 0-1-sequences, and let  $B$  be the set of 0-1-sequences with length  $\omega$ . We define a bipartite graph  $G$  with vertex set  $A \cup B$  by adding for each  $a \in A$  and  $b \in B$  the edge  $ab$  if  $a$  is an initial segment of  $b$ . Since  $G$  is bipartite, it cannot contain a  $K_\omega$ . It cannot contain a  $K_{\omega,\omega}$  either, since any two vertices in  $B$  have only finitely many neighbours in common. On the other hand,  $\text{col}(G) > \aleph_0$ , since  $G$  is an  $\aleph_0$ -barricade.

**Remark 5.3.** The proof of Theorem 1.4 actually shows something slightly stronger: in order to have  $\text{col}(G) \leq \mu$  it is enough to have no  $K_\mu$ -subgraph and for some natural number  $k$  no independent set of size  $k$  such that these  $k$  vertices are the left bipartition set of a  $K_{k,\mu^+}$ -subgraph. If  $\mu = \omega$ , then DEM implies that for  $\text{col}(G) \leq \mu$  it is enough to forbid  $K_\mu$  and an induced  $K_{k,\mu^+}$ . On the other hand if  $\varkappa = 2^\omega$  and  $\mu = \omega_1$ , it may happen that the bipartite graph (of the proof of Theorem 1.4) contains neither a  $K_\mu$  nor an induced  $K_{k,\omega_1}$  by Sierpiński’s theorem from [10], which says that

$$2^\omega \not\rightarrow (\omega_1)_2^2.$$

Our characterisation simplifies the study of many questions about colouring numbers, since they can often be reduced to questions about the properties of our obstructions.

For instance, Halin showed in [3] that if  $\lambda$  is infinite and a graph  $G$  has colouring number greater than  $\lambda$ , then  $G$  includes a subdivision of  $K_\lambda$ . A quick proof of this result based on Theorem 1.2 can be found in [6].

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