

CYCLE TRAVERSABILITY
FOR CLAW-FREE GRAPHS AND POLYHEDRAL MAPS

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Let G be a graph, and $v \in V(G)$ and $S \subseteq V(G) \setminus v$ of size at least k . An important result on graph connectivity due to Perfect states that, if v and S are k -linked, then a $(k-1)$ -link between a vertex v and S can be extended to a k -link between v and S such that the endvertices of the $(k-1)$ -link are also the endvertices of the k -link. We begin by proving a generalization of Perfect's result by showing that, if two disjoint sets S_1 and S_2 are k -linked, then a t -link ($t < k$) between two disjoint sets S_1 and S_2 can be extended to a k -link between S_1 and S_2 such that the endvertices of the t -link are preserved in the k -link.

Next, we are able to use these results to show that a 3-connected claw-free graph always has a cycle passing through any given five vertices, but avoiding any other one specified vertex. We also show that this result is sharp by exhibiting an infinite family of 3-connected claw-free graphs in which there is no cycle containing a certain set of six vertices but avoiding a seventh specified vertex. A direct corollary of our main result shows that a 3-connected claw-free graph has a topological wheel minor W_k with $k \leq 5$ if and only if it has a vertex of degree at least k .

Finally, we also show that a graph polyhedrally embedded in a surface always has a cycle passing through any given three vertices, but avoiding any other specified vertex. The result is best possible in the sense that the polyhedral embedding assumption is necessary, and there are infinitely many graphs polyhedrally embedded in surfaces having no cycle containing a certain set of four vertices but avoiding a fifth specified vertex.

1. Introduction

A graph G is Hamiltonian if G has a cycle containing all the vertices of G . Hamiltonicity of graphs is one of the major topics in graph theory. High

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connectivity does not guarantee the existence of a Hamilton cycle in a graph, but a highly connected graph does contain a long cycle. For example, given any k vertices of a k -connected graph G , there is a cycle containing all k of them (cf. [5]). Bondy and Lovász [1] proved an even stronger result which says that for any given vertex set S of size $k-1$ in a k -connected graph G , the cycles containing S generate the cycle space of G . Besides the Hamiltonicity problem, Chvátal [4] also considered the cyclability question for graphs, i.e., for a given set of vertices, does there exist a cycle through these vertices. Cyclability versus connectivity in graphs has been studied by a number of other authors (cf. [18,11,8]).

If one adds additional properties to the connectivity assumption, it is sometimes possible to guarantee higher cyclability. An example is the following nine-point theorem.

Theorem 1.1 (Holton, McKay, Plummer & Thomassen, [14]). *Let G be a 3-connected cubic graph. Then any set of nine vertices lies on a cycle.*

The cyclability problem has also been studied for 3-connected graphs in the presence of additional properties. Ellingham et al. [6] showed that a 3-connected cubic graph has a cycle which passes through any ten given vertices if and only if the graph is not contractible to the Petersen graph in such a way that the ten vertices each map to a distinct vertex of the Petersen graph.

A great deal of attention had been paid to cycles through specified edges as well. Lovász [20], and independently, Woodall [31] conjectured that every k -connected graph has a cycle through any k given independent edges unless these edges form an odd edge-cut. Häggkvist and Thomassen [12] proved a weak version of the Lovász–Woodall conjecture – that every k -connected graph has a cycle through any $k-1$ given independent edges, which was conjectured by Woodall [31]. A complete proof of the Lovász–Woodall conjecture was announced by Kawarabayashi [17], but a complete proof has yet to appear. Holton and Thomassen [16] conjectured that any cyclically $(k+1)$ -connected cubic graph has a cycle through any given k independent edges, and this still remains open.

The existence problem of a cycle through certain given edges in graphs is equivalent to the existence problem of a cycle through corresponding vertices in their line graphs with certain forbidden pairs of edges incident with these vertices. A graph is *claw-free* if it contains no induced subgraph isomorphic to $K_{1,3}$. Note that line graphs form a subfamily of claw-free graphs. Hence, it is interesting to study cyclability of claw-free graphs. An analogue of the above nine-point theorem has been obtained for claw-free graphs by the first two authors of this paper [10].

Theorem 1.2 ([10]). *Let G be a 3-connected claw-free graph and S be a set with $k \leq 9$ vertices. Then G has a cycle containing all the vertices of S .*

Recently, Chen [2] showed that a 3-connected claw-free graph G has a cycle through any given twelve vertices if the underlying graph of its closure (a line graph) cannot be contracted to the Petersen graph in certain ways. In [6], Ellingham et al. proved there is a cycle through any five vertices in a 3-connected cubic graph which avoids any specified edge.

In this paper, we consider the cyclability problem for graphs when certain sets of vertices are to be included and other sets are to be avoided. Let G be any connected graph containing a cycle and m and n , two non-negative integers with $1 \leq m + n \leq |V(G)|$. Then graph G is said to satisfy *property $C(m, n)$* (or simply, G is $C(m, n)$), if for any two disjoint sets S_1 and S_2 contained in $V(G)$ with $|S_1| = m$ and $|S_2| = n$, there is a cycle C in G such that $S_1 \subseteq V(C)$, but $S_2 \cap V(C) = \emptyset$. When $n = 0$, the maximum value of m such that there is a cycle through every set $S \subseteq V(G)$ with $|S| \leq m$ is known as the *cyclability* of G . Of course the case when $m = |V(G)|$ is just the well-studied Hamilton cycle problem. It has been shown in [13,15] that a graph G is k -connected if and only if G is $C(k-l, l)$ for all integer $0 \leq l \leq k-2$, and in [13,22] that a graph G is k -connected if and only if G is $C(2, k-2)$. So, in particular, a graph G is 3-connected if and only if G is $C(2, 1)$.

Cyclability problems and their near-relatives, have been widely studied for many graph classes. We make no attempt at a comprehensive listing of results in this area here, but instead refer the reader to the recent survey of Gould [9]. For the class of claw-free graphs there are many published results as well. We refer the interested reader to [7] which surveys results for claw-free graphs. Chudnovsky and Seymour recently published a deep analysis of claw-free graphs in a series of eight papers. (Cf. [3] for more information.)

Note that a k -connected graph may not have a cycle through any given k vertices which avoids a specified vertex. For example, the complete bipartite graph $K_{k,k}$ cannot have a cycle through all k vertices in one bipartite set which avoids a vertex from the other bipartite set. Hence, the graph $K_{3,3}$ shows that a 3-connected cubic graph may not be $C(3, 1)$. Similarly, the 3-cube (Q_3) demonstrates that a 3-connected plane graph may not be $C(4, 1)$. However, the connectivity of a graph does have a strong connection with the property $C(m, n)$.

Matthews and Sumner [21] conjectured that every 4-connected claw-free graph has cyclability $|V(G)|$, that is, every such graph has a Hamiltonian cycle. This conjecture still remains open. An immediate corollary of Theorem 1.2 guarantees that a 4-connected claw-free graph is $C(9, 1)$. But what about 3-connected claw-free graphs? Combining the results of Sections 3, 4

and 5, we prove the following theorem which is our main result on cyclability in 3-connected claw-free graphs.

Theorem 1.3. *Let G be a 3-connected claw-free graph and S be a set with $k \leq 6$ vertices. Then for every vertex of S , G has a cycle avoiding this vertex but containing all the remaining $k - 1$ vertices of S .*

The size of S in Theorem 1.3 is the best possible since there are infinitely many 3-connected claw-free graphs which has a vertex subset S of size 7 such that G does not have a cycle avoiding one vertex of S , but containing all the remaining $k - 1$ vertices of S . Two main tools that we use to prove Theorem 1.3 are Perfect's Theorem and a strengthened version of Perfect's Theorem which is proved in Section 2.

As a consequence of this result, we can easily guarantee small topological wheel minors W_k in 3-connected claw-free graphs as follows.

Corollary 1.4. *Let G be a 3-connected claw-free graph. Then the following hold:*

- (1) *G has a topological wheel minor W_k with $k \leq 5$ if and only if G has a vertex of degree at least k ;*
- (2) *For any vertex z of degree $k \leq 5$, G has a topological wheel minor W_k with z as its hub;*
- (3) *For any given six vertices, G has a subdivision of W_3 (or K_4) containing any five of the six vertices on its rim and the remaining vertex as its hub.*

In Section 6, we consider graphs embedded in closed surfaces. An embedding of a graph in a surface is *polyhedral* if every face is bounded by a cycle and the boundaries of every two faces meet properly, i.e., their intersection is either empty, or a single vertex or an edge. An immediate consequence of Theorem 6.1 is the following result for graphs polyhedrally embedded in surfaces.

Theorem 1.5. *Let G be a graph polyhedrally embedded in a closed surface. Then G has a cycle through any given three vertices which avoids any other specified vertex.*

The above result is the best possible in the sense that there are infinitely many graphs polyhedrally embedded in a closed surface which have no cycle passing through a certain set of four vertices but avoiding a fifth specified vertex. We describe such an infinite class in Section 6. Since $K_{3,3}$ has a closed 2-cell embedding in the projective plane, the polyhedral embedding assumption in Theorem 1.5 is necessary.

2. Perfect's theorem and its generalizations

We now introduce a theorem on disjoint paths in graphs, due to Perfect [25], which deserves to be more widely known in graph theory. Let G be a graph and S be a subset of $V(G)$. For clarity, use “ \setminus ” to denote the deletion operation for sets, but use “ $-$ ” to denote deletion operation for graphs. In other words, $V(G) \setminus S$ is the vertex subset of $V(G)$ with no vertices in S , but $G - S$ is the subgraph of G obtained by deleting all vertices in S from G together with all edges incident to the vertices of S .

Let G be a graph and S be a non-empty subset of $V(G)$. Suppose $v \in V(G) \setminus S$. We say that v and S are k -linked in G if there exist k internally disjoint paths joining v and k distinct vertices of S such that each of the k paths meets S in exactly one vertex.

Theorem 2.1 (Perfect's Theorem [25]). *Let G be a graph, and let $x \in V(G)$ and $S \subseteq V(G) \setminus \{x\}$ such that x and S are k -linked in G . If S has a subset T of size $k-1$ such that x and T are $(k-1)$ -linked, then there exists a vertex $s \in S \setminus T$ such that x and $T \cup \{s\}$ are k -linked.*

The following result strengthens Perfect's Theorem, and will be particularly useful in our work in this paper. Two disjoint subsets S_1 and S_2 are k -linked if there exist k vertex disjoint paths joining k distinct vertices of S_1 to k distinct vertices of S_2 such that each of the k paths meets S_i in exactly one vertex for $i \in \{1, 2\}$.

Theorem 2.2. *Let G be a graph and let S_1 and S_2 be two disjoint subsets of $V(G)$ such that S_1 and S_2 are k -linked. If each S_i has a subset T_i of size $k-1$ for $i \in \{1, 2\}$ such that T_1 and T_2 are $(k-1)$ -linked, then there is a vertex $s_i \in S_i \setminus T_i$ for $i \in \{1, 2\}$ such that $T_1 \cup \{s_1\}$ and $T_2 \cup \{s_2\}$ are k -linked.*

Proof. Add two new vertices x_1 and x_2 to the graph G , and join x_i to all vertices of S_i for each $i \in \{1, 2\}$. Then identify all vertices in $S_i \setminus T_i$ and denote the resulting vertex by y_i . Let G' be the resulting graph. Let U be a minimum vertex-cut of G' which separates x_1 and x_2 . We claim that:

Claim. $|U| \geq k$.

Proof of Claim. Suppose to the contrary that $|U| < k$. If $y_i \in U$ for some $i \in \{1, 2\}$, then $U \setminus \{y_i\}$ is a cut of size at most $k-2$ which separates x_1 and x_2 in $G' - \{y_1, y_2\}$. So there are at most $k-2$ internally disjoint paths from x_1 to x_2 in $G' - \{y_1, y_2\}$. Hence, there are at most $k-2$ disjoint paths joining vertices of T_1 and vertices of T_2 , contradicting the assumption that

T_1 and T_2 are $(k-1)$ -linked. Hence $U \cap \{y_1, y_2\} = \emptyset$. It follows that U is a cut separating the subgraphs x_1y_1 and x_2y_2 .

So G' has at most $k-1$ disjoint paths from $T_1 \cup \{y_1\}$ to $T_2 \cup \{y_2\}$. On the other hand, since S_1 and S_2 are k -linked, there are k disjoint paths between S_1 and S_2 . So there are k internally disjoint paths between $T_1 \cup \{y_1\}$ and $T_2 \cup \{y_2\}$, a contradiction. This contradiction implies that $|U| \geq k$ and the proof of the Claim is complete. ■

By the above Claim and Menger's Theorem, the graph G' has k internally disjoint paths joining x_1 and x_2 . Among these k internally disjoint paths from x_1 to x_2 , one passes through y_1 and one passes through y_2 . Deleting x_1 and x_2 from these k internally disjoint paths generates k disjoint paths between $T_1 \cup \{y_1\}$ and $T_2 \cup \{y_2\}$ in G' , and hence k disjoint paths between S_1 and S_2 in G . For $i \in \{1, 2\}$, let $s_i \in S_i$ be the endvertex of the path of G' corresponding to the path of G' passing through y_i . Note that $|T_1| = |T_2| = k-1$. So each of these k disjoint paths of G between S_1 and S_2 meets S_i exactly one vertex. Hence $T_1 \cup \{s_1\}$ and $T_2 \cup \{s_2\}$ are k -linked. This completes the proof. ■

The above result cannot be further strengthened so as to guarantee that the extended k -link contains a $(k-1)$ -link between T_1 and T_2 . For example, the graph in Figure 1 does not have three disjoint paths between S_1 and S_2 which contain two disjoint paths between T_1 and T_2 . But T_1 and T_2 are 2-linked, and S_1 and S_2 are 3-linked.

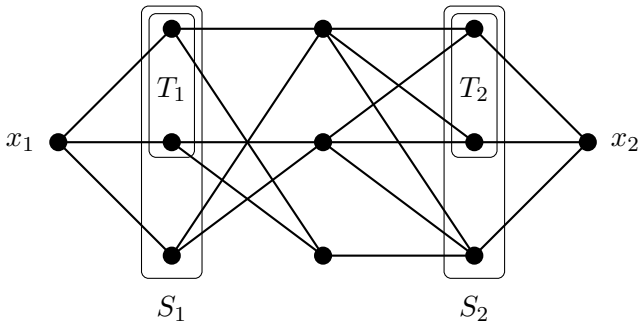


Figure 1. A 2-link cannot be extended to a 3-link which contains a 2-link maintaining the initial and terminal vertex sets of the original 2-link

It turns out that Theorem 2.2 admits the following ‘self-refining’ version. Although we will not use it, we include it as it seems to be of some independent interest.

Theorem 2.3. *Let G be a graph. Assume S_1 and S_2 are two disjoint subsets of $V(G)$ such that S_1 and S_2 are k -linked. If each S_i has a subset T_i of size $k-t$ for $i \in \{1, 2\}$ such that T_1 and T_2 are $(k-t)$ -linked, then there is a subset $T'_i \subseteq S_i \setminus T_i$ of size t for $i \in \{1, 2\}$ such that $T_1 \cup T'_1$ and $T_2 \cup T'_2$ are k -linked.*

Proof. The proof is by induction on t . If $t = 1$, the result follows directly from Theorem 2.2. So suppose that $t \geq 2$ and that the result holds for all $t' < t$. Let T_i be a $(k-t)$ -subset of S_i such that T_1 and T_2 are $(k-t)$ -linked. We need to show that, for $i = 1$ and $i = 2$, there exists a subset $T'_i \subseteq S_i \setminus T_i$ of size t such that $T_1 \cup T'_1$ and $T_2 \cup T'_2$ are k -linked.

Note that S_1 and S_2 are k -linked and hence also $(k - (t - 1))$ -linked. By Theorem 2.2, there exist vertices $s_i \in S_i \setminus T_i$ for $i = 1$ and 2 such that $T_1 \cup \{s_1\}$ and $T_2 \cup \{s_2\}$ are $(k - (t - 1))$ -linked. By the induction hypothesis, there are subsets $T''_i \subseteq S_i \setminus (T_i \cup \{s_i\})$ for $i = 1$ and 2 of size $t - 1$ such that $(T_1 \cup \{s_1\}) \cup T''_1$ and $(T_2 \cup \{s_2\}) \cup T''_2$ are k -linked.

Let $T'_i = T''_i \cup \{s_i\}$. Then $|T'_i| = |T''_i| + 1 = t$ because $s_i \notin T''_i$. Hence T'_1 and T'_2 are the desired subsets of $S_1 \setminus T_1$ and $S_2 \setminus T_2$, respectively. This completes the proof. ▀

3. Technical lemmas and property $C(3, 1)$ for claw-free graphs

Let G be a graph and let C be a cycle of G where we arbitrarily adopt one direction for the traversal of C and call it “clockwise” and call the opposite direction “counterclockwise”. Suppose x and y are two vertices of G . Use $C[x, y]$ to denote the segment of C from x to y in the clockwise direction, and $C^{-1}[x, y]$ to denote the segment of C from x to y in counterclockwise direction. Furthermore, let $C(x, y)$ denote $C[x, y] - x$ and $C(x, y)$ denote $C[x, y] - \{x, y\}$. For a connected subgraph Q of G , the new graph obtained from G by contracting all edges in Q is denoted by G/Q .

The strategy for proving our Theorem 1.3 is: first assume that G has a cycle C which contains $k - 1$ vertices from a k -vertex set S , but avoids a k^{th} vertex in $V(G) \setminus S$, and then apply Perfect’s Theorem to the cycle C and the vertex in S which is not in C . Sometimes, we may use Perfect’s Theorem more than once in order to find enough paths to help form a suitable cycle containing all k vertices of S , but avoiding the given vertex.

We often encounter the following situation: C is a cycle, and P_1 and P_2 are two internally disjoint paths from x and y which end at the same vertex u on C . Then u has degree at least 4, and has two neighbors u_1 and u_2 on C and a neighbor u_3 on P_1 and u_4 on P_2 . Since G is claw-free, either $u_1u_2 \in E(G)$ or one of u_1u_4 and u_2u_4 is an edge of $E(G)$ (see Figure 2). Then we use the following “Jumper Operations”:

(J1) If $u_3u_4 \in E(G)$ (dashed edge in Figure 2), then $(P_1 \cup P_2 - \{u\}) \cup \{u_3u_4\}$ is a path joining x and y .

(J2) If $u_3u_4 \notin E(G)$, then G must contain an edge joining a vertex $u_i \in \{u_1, u_2\}$ and a vertex $u_j \in \{u_3, u_4\}$ (dotted edges in Figure 2). Otherwise, G has a claw. If $j = 3$, then replace P_1 by $P'_1 = (P_1 - \{u\}) \cup \{u_3u_i\}$ which is a path from x to C ending at $u_i \neq u$. If $j = 4$, then replace P_2 by $P'_2 = (P_2 - \{u\}) \cup \{u_4u_i\}$ which is a path from y to C ending at $u_i \neq u$. Then for both $j=3$ and 4 , the graph G has disjoint paths from x and y to C ending at different vertices.

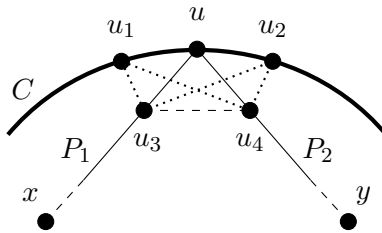


Figure 2. Jumpers at the vertex u

By (J1) and (J2), the above circumstance can always be converted to one of the following cases:

(C1) a cycle C and a path joining x and y ; or

(C2) a cycle C and two disjoint paths from x and y to C ending at different vertices.

So, for the proofs contained in this paper, we simply say “by Jumper Operations, we assume that P_2 does not end at the endpoint of P_1 on C ”.

We now proceed to prove our main theorem by supposing that G is a minimum counterexample with respect to the number of vertices. That is, let G be a 3-connected claw-free graph containing a set $S \subseteq V(G)$ consisting of vertices x_1, x_2, \dots, x_k , where $k \leq 5$, and let z be a vertex not in S such that G has no cycle which contains S , but misses z . Moreover, let G be a such graph with the smallest number of vertices.

Lemma 3.1. *Let v be a vertex of a minimum counterexample G such that $v \notin S$. Then v is contained in a 3-cut of G .*

Proof. Since G is claw-free, the subgraph $G - v$ is also claw-free. If v is not contained in a 3-cut, then $G - v$ is 3-connected and claw-free. Since G is a minimum counterexample, $G - v$ has a cycle C passing through all vertices

of S , but avoiding z . But then, the cycle C is a cycle in G of the type we desire, thus contradicting the assumption that G is a counterexample. ■

Lemma 3.2. *Let G be a 3-connected claw-free graph, and T be a 3-cut of G . Then $G - T$ has exactly two components Q_1 and Q_2 such that each Q_i does not contain a cutvertex.*

Proof. First, we prove that $G - T$ has precisely two components. Assume to the contrary that $G - T$ has at least three components, say Q_1, Q_2 and Q_3 . Since G is 3-connected, every vertex of T has a neighbor in each of these components. So this vertex of T , together with these neighbors, induce a claw, contradicting the assumption that G is claw-free. So $G - T$ has exactly two connected components Q_1 and Q_2 as claimed.

In the following, we show that each Q_i does not contain a cutvertex. Assume, to the contrary, that v is a cutvertex of Q_i . Then Q_i has exactly two blocks B_1 and B_2 (i.e., maximal 2-connected subgraphs) separated by v since G is claw-free. Note that, for any vertex $v_j \in T$, the vertex v_j has neighbors in both Q_1 and Q_2 since G is 3-connected. It follows that the neighbors of v_j in Q_i induce a clique because G is claw-free. So all neighbors of v_j in Q_i belong to the same block of Q_i . Let

$$U_t = \{v_j | v_j \in T \text{ and } (V(B_t) \cap (N(v_j)) \setminus v) \neq \emptyset\} \text{ for } t \in \{1, 2\},$$

where $N(v_j)$ is the set of all neighbors of v_j in G . Then $U_1 \cap U_2 = \emptyset$ and $|U_1| + |U_2| = |T| = 3$. Without loss of generality, assume $|U_1| \geq |U_2|$. Then $U_2 \cup \{v\}$ is a vertex cut of G separating B_2 from the remaining subgraph of G and has size at most two, which contradicts the assumption that G is 3-connected. This completes the proof. ■

Lemma 3.3. *Let $T = \{z, w_1, w_2\}$ be a 3-cut of a minimum counterexample G . Then $G - T$ has a component which is a singleton vertex from S .*

Proof. By Lemma 3.2, $G - T$ has two components Q_1 and Q_2 . If $Q_i \cap S = \emptyset$ for some $i \in \{1, 2\}$, let $G' = (G - V(Q_i)) \cup \{zw_1w_2z\}$ where zw_1w_2z is a 3-cycle. Then G' is claw-free. By Lemma 3.2, G' is 3-connected. So G' has a cycle C passing through all the vertices in S , but avoiding z . If C contains w_1w_2 and w_1w_2 is not an edge of G , use a path P of $G - V(Q_j)$ ($j \neq i$) joining w_1 and w_2 to replace the edge w_1w_2 in C to get a cycle C' in G of the type desired. Therefore, both $Q_1 \cap S \neq \emptyset$ and $Q_2 \cap S \neq \emptyset$ hold.

If $|V(Q_i)| \geq 2$ for both $i = 1$ and 2 , consider $G_i = (G/Q_i) \cup \{zw_2, zw_1\}$. Note that G_i is 3-connected and claw-free. Since G is a minimum counterexample, both G_1 and G_2 are not counterexamples. Let q_i be the vertex of G_i obtained by contracting Q_i . Let $S_i = (S \setminus V(Q_i)) \cup \{q_i\}$. Then $|S_i| \leq |S|$, and

hence G_i has a cycle C_i containing all vertices of S_i , but avoiding z . Then $(C_1 - q_1) \cup (C_2 - q_2)$ is a cycle of G containing all vertices of S , but avoiding z , and this contradicts the assumption that G is a counterexample. Therefore, one of the two components is a singleton vertex from S . ■

Theorem 3.4. *Let G be a 3-connected claw-free graph. Then G has the property $C(3,1)$.*

Proof. The proof is by contradiction. So suppose G is a minimum counterexample; i.e., suppose there exist three vertices x_1, x_2, x_3 and a fourth vertex z in G such that there is no cycle in G containing $S = \{x_1, x_2, x_3\}$ which avoids z .

By Lemmas 3.1 and 3.3, z belongs to a 3-cut $T = \{z, v_1, v_2\}$ such that $G - T$ consists of precisely two components, one of which is a singleton from S , say x_1 . Since $G - z$ is 2-connected, x_1 is contained in a cycle which must contain v_1 and v_2 . If $\{v_1, v_2\} = \{x_2, x_3\}$, then any cycle of $G - z$ containing x_1 is a cycle of the type sought, contradicting the assumption that G is a counterexample. So assume that $x_2 \notin \{v_1, v_2\}$.

By Menger’s Theorem, there are three internally disjoint paths from x_2 to three distinct vertices of T , two of which end at v_1 and v_2 , respectively. These two paths, together with $v_2x_1v_1$, form a cycle C containing both x_1 and x_2 , but not z . Denote the third path from x_2 to z by P' . If $x_3 \in V(C)$, then C is a cycle of the type desired, a contradiction again. So in the following, assume that $x_3 \notin V(C)$. Then by Menger’s Theorem, there are three internally disjoint paths P_1, P_2 and P_3 from x_3 to three distinct vertices of $V(C)$.

If none of the paths P_1, P_2 and P_3 contains z , then all three end in the segment $C[v_1, v_2]$. Then one of the segments $C[v_1, x_2]$ and $C[x_2, v_2]$ contains two of the endvertices of P_1, P_2 and P_3 . Without loss of generality, assume $C[v_1, x_2]$ contains the endvertices of P_1 and P_2 , denoted by u_1 and u_2 , which appear in clockwise order on C , respectively. Then $C[u_2, u_1]$ contains both x_1 and x_2 . Hence $C' = P_1 \cup P_2 \cup C[u_2, u_1]$ is a cycle containing all vertices of S , but not z , again a contradiction. So, in the following, suppose that

- (*) among any three internally disjoint paths P_1, P_2 and P_3 from x_3 to three distinct vertices of C , there is always one containing z .

Without loss of generality, assume that P_3 contains z . Let P'' be the subpath of P_3 joining x_3 and z . Let w be the first vertex in $P'' \cap P'$ encountered when traversing P'' from x_3 to z .

If $w \neq z$, let P'_3 be the path obtained by traversing P'' from x_3 to w and then traversing P' from w to x_2 . If $w = z$, then there is an edge e joining the

vertex of P'' in $N(z) \setminus (T \cup x_1)$ and the vertex of P' in $N(z) \setminus (T \cup x_1)$ since G is claw-free and hence the neighbors of z induce a clique in $G - (T \cup \{x_1\})$. Then let $P'_3 = ((P'' \cup P') - z) \cup e$. So, no matter whether $w = z$ or $w \neq z$, P'_3 is a path from x_3 to x_2 which avoids z . By Jumper Operations, we may assume that P_1, P_2 and P'_3 end at different vertices of C .

By (*), it follows that $(P_1 \cup P_2) \cap P'_3 \neq \emptyset$. Since P_1 and P_2 are internally disjoint from P_3 and P_3 contains P'' as a subpath, it follows that $(P_1 \cup P_2) \cap P' \neq \emptyset$. Let w'' be the first vertex in $(P_1 \cup P_2) \cap P'$ encountered when traversing P' from x_2 to z . Without loss of generality, assume $w'' \in P_1$. Let P'_1 be the path from x_3 to C obtained by traversing P_1 from x_3 to w'' and then along P' to x_2 . Because of the choice of w'' , P'_1 and P_2 are two internally disjoint paths from x_3 to C ending at two distinct vertices in the same closed segment of C determined by x_1 and x_2 . Then x_3 can be inserted into C by using P'_1 and P_2 to generate a new cycle C' containing all the vertices of S , but not z , where $C' = C[x_1, x_2] \cup P'_1 \cup P_2 \cup C[u_2, x_1]$, and this again contradicts G being a counterexample. This completes the proof. ■

In the rest of this section, we derive two more properties of 3-cuts of a minimum counterexamples to $C(k, 1)$ with $k \leq 5$.

Lemma 3.5. *Let G be a minimum counterexample to $C(k, 1)$ with $k \leq 5$ and let T and T' be two distinct 3-cuts of G such that $|T \cap T'| = 2$. Then $G - (T \cup T')$ has two components. Furthermore, if $z \in T \cap T'$, then $G - z$ has a Hamiltonian cycle.*

Proof. Assume that $T \cap T' = \{v_1, v_2\}$, $T \setminus (T \cap T') = \{u\}$ and $T' \setminus (T \cap T') = \{u'\}$.

Since T is a 3-cut, one of the components of $G - T$ does not contain u' which we will denote by Q_1 . Note that Q_1 is also a component of $G - (T \cup T')$. Similarly, the component Q_2 of $G - T'$ not containing u is also a component of $G - (T \cup T')$. Since G is 3-connected, both v_1 and v_2 have neighbors in Q_1 and Q_2 . It follows, since G is claw-free, that both v_1 and v_2 have no neighbors in components of $G - (T \cup T')$ other than Q_1 and Q_2 . If $G - (T \cup T')$ has a component different from Q_1 and Q_2 , then the component is separated by $\{u, u'\}$ from the remaining subgraph, which contradicts the assumption that G is 3-connected. This contradiction implies that $G - (T \cup T')$ has exactly two components Q_1 and Q_2 .

Now, assume that $z \in T \cap T'$ and, without loss of generality, assume $z = v_1$. By Lemma 3.3, both Q_1 and Q_2 are single vertices from S , denoted by x and y . Since G is 3-connected, there is an edge joining u and u' . But then $xv_2yu'ux$ is a Hamiltonian cycle of $G - z$. ■

Lemma 3.6. *Let G be a minimum counterexample such that G has exactly one 3-cut T containing z , and T separates a single vertex $x_1 \in S$ from $Q = G - (T \cup \{x_1\})$. Then for some vertex $x \in V(Q)$ and a connected subgraph H of $G - z$ with at least three vertices in $G - \{z, x_1\}$, the vertex x and H are 3-linked in $G - z$.*

Proof. Let H be a connected subgraph of $G - z$ containing at least three vertices of $G - \{z, x_1\}$. Suppose to the contrary that $G - z$ does not contain three internally disjoint paths joining x to three distinct vertices of H . In other words, $G - z$ has a 2-cut separating x and H by Menger’s Theorem. This 2-cut, together with z , forms a 3-cut of G which we shall denote by T' . By Lemma 3.3, T' separates a single vertex of S from the remaining subgraph of G . This single vertex must be x since H has more than one vertex. Note that $x \in V(Q) = V(G) \setminus (T \cup \{x_1\})$. So $T \neq T'$, a contradiction to the assumption that G has exactly one 3-cut T containing z . ■

4. Property $C(4,1)$ for claw-free graphs

In this section, we will show that every 3-connected claw-free graph is $C(4,1)$.

Lemma 4.1. *Let G be a minimum counterexample to the $C(4,1)$ property. That is, suppose G is a smallest 3-connected claw-free graph containing a vertex z and four additional vertices x_1, x_2, x_3 and x_4 such that G has no cycle containing all x_i vertices, but avoiding z . Then z is contained in exactly one 3-cut in G .*

Proof. Let G be as stated above and let $S = \{x_1, x_2, x_3, x_4\}$. Suppose to the contrary that z is contained in at least two 3-cuts. By Lemma 3.3 and the fact that G is claw-free, z is not contained in three different 3-cuts. So assume that z is contained in exactly two 3-cuts T and T' . By Lemma 3.5, we have $T \cap T' = \{z\}$.

Let $x_1 \in S$ be separated by T and $x_2 \in S$ be separated by T' . Denote $G - (T \cup T' \cup \{x_1, x_2\})$ by Q . If Q has at least two components Q_1 and Q_2 , then at least one vertex in $(T \cup T') \setminus \{z\}$ has neighbors in both Q_1 and Q_2 since G is 3-connected. This vertex together with its neighbors form a claw, a contradiction. Hence Q is connected.

Since G is claw-free, z has no neighbor in Q . Suppose Q contains at most one vertex x_3 from S . Then x_4 belongs to either T or T' . Since G has property $C(3,1)$, G has a cycle C containing x_1, x_2 and x_3 , but avoiding z . Note that the cycle C must contain all the vertices of $(T \cup T') \setminus \{z\}$. Then C contains all the vertices of S , a contradiction to the assumption that G is a counterexample.

So in the following, assume that Q contains both x_3 and x_4 . Since G has the $C(3,1)$ -property, G has a cycle C containing x_1, x_2, x_3 , but avoiding z . Without loss of generality, assume that $x_3 \in C(x_2, x_1)$. By Menger's Theorem, G has three internally disjoint paths P_1, P_2 and P_3 joining x_4 and three distinct vertices of C which do not contain z . If two of these three paths end on the same closed segment of C determined by x_1, x_2 and x_3 , then we can insert x_4 into the cycle C to obtain a cycle containing all the vertices of S , but not containing z , a contradiction to the assumption that G is a counterexample. Hence, we may assume the three paths P_1, P_2 and P_3 end at u_1, u_2 and u_3 , respectively, such that $u_i \in C(x_i, x_{i+1})$ for $i = 1, 2$, and $u_3 \in C(x_3, x_1)$.

Let $H = C[u_3, u_2] \cup P_2 \cup P_3 \cup P_1$. Since G is 3-connected, there are three internally disjoint paths from x_3 ending at three different vertices of H . Hence x_3 and H are 3-linked. By applying Perfect's Theorem to x_3 and the subgraph H , there are three internally disjoint paths P'_1, P'_2 and P'_3 joining x_3 and three distinct vertices of H which do not contain z such that P'_2 and P'_3 end at u_2 and u_3 , respectively. Let w be the vertex at which P'_1 ends. Note that $w \notin P_2 \cup P_3 \cup C[x_2, u_2] \cup C[u_3, x_1]$. For otherwise, one could insert x_3 into the cycle $C[u_3, u_2] \cup P_2 \cup P_3$ using the path P'_1 and one of P'_2 and P'_3 to form a cycle containing all the vertices from S , but avoiding z . Therefore, we may assume that either $w \in P_1$ or $w \in C(x_1, x_2)$.

Note that, if $w = u_1$, we could use Jumper Operations to modify the path P_1 or the cycle C at u_1 to reduce the case $w = u_1$ to $w \neq u_1$.

If $w \in P_1 - u_1$, then there are two disjoint paths from x_4 to the segment of $C[x_2, x_3]$ (one is P_2 and the other is from x_4 to w along P_1 then to x_3 along P'_1), which could be used to insert x_4 into the cycle C to get a cycle through all vertices in S , but avoiding z , a contradiction.

So, in the following, assume that $w \in C(x_1, u_1)$ or $w \in C(u_1, x_2)$. By symmetry, it suffices to consider $w \in C(x_1, u_1)$. Then $C' = C[x_1, w] \cup P'_1 \cup P'_2 \cup C^{-1}[u_2, u_1] \cup P_1 \cup P_3 \cup C[u_3, x_1]$ is a cycle of the type we seek, a contradiction to the assumption that G is a counterexample. Hence z is contained in exactly one 3-cut and the proof is complete. ■

Theorem 4.2. *Let G be a 3-connected claw-free graph. Then G satisfies property $C(4,1)$.*

Proof. Let G be a minimum counterexample. Then G has four vertices x_1, x_2, \dots, x_4 and an additional vertex z such that $G - z$ has no cycle through x_1, x_2, x_3 and x_4 . By Lemma 4.1, z is contained in exactly one 3-cut T . By Lemma 3.3, T separates a single vertex from S , say x_1 . Let Q be a component of $G - T$ not containing x_1 .

Claim. $T \cap S = \emptyset$.

Proof of the Claim. If not, assume that $|T \cap S| \geq 1$. Then $Q \cap S$ consists of at most two vertices, say x_2 and x_3 . So $x_4 \in T$. Since G satisfies property $C(3, 1)$, G has a cycle containing x_1, x_2 and x_3 , but avoiding z . Since T is a 3-cut separating x_1 , it follows that both vertices of $T \setminus \{z\}$ are contained in this cycle, which implies that the cycle contains all the vertices of S , a contradiction of the assumption that G is a counterexample. Hence $T \cap S = \emptyset$ as claimed. ■

By the claim, $S \setminus \{x_1\} \subseteq V(Q)$. By the property $C(3, 1)$, let C be a cycle of G containing x_1, x_2 and x_3 , but avoiding z . Since G is a counterexample, x_4 is not on C . Since z is contained in exactly one 3-cut, by Lemma 3.6, x_4 and C are 3-linked in $G - z$. Therefore, there are three internally disjoint paths P_1, P_2 and P_3 joining x_4 and three distinct vertices of C each of which avoids z .

Assume that P_1, P_2 and P_3 end on C at u_1, u_2 and u_3 , respectively. Note that no two of these paths end in the same closed segment of C determined by x_1, x_2 and x_3 , or otherwise we could insert x_4 into C using the two paths to get a cycle of the type desired, a contradiction to the assumption that G is a counterexample. Without loss of generality, assume that $u_i \in C(x_i, x_{i+1})$ for each $i \in \{1, 2\}$ and $u_3 \in C(x_3, x_1)$. Let $H = C[x_1, u_1] \cup P_1 \cup P_2 \cup C[u_2, x_1] \cup P_3$. By Lemma 3.6, x_2 and H are 3-linked in $G - z$. Applying Perfect's Theorem to x_2 and H in $G - z$, there are three internally disjoint paths P'_1, P'_2 and P'_3 joining x_2 and three distinct vertices of H such that P'_1 and P'_2 end at u_1 and u_2 , respectively. Assume that P'_3 ends at w . Then w does not belong to $C[x_1, u_1] \cup P_1 \cup P_2 \cup C[u_2, x_3]$. Otherwise, we could insert x_2 into the cycle $C[x_1, u_1] \cup P_1 \cup P_2 \cup C[u_2, x_1]$ which already contains x_1, x_4 and x_3 to generate a cycle of the type desired, a contradiction to the assumption that G is a counterexample.

Note that the case $w = u_3$ can be converted to the case $w \neq u_3$ by using Jumper Operations at the vertex $w = u_3$. So it suffices to consider the cases when $w \in P_3 - \{x_4, u_3\}$, when $w \in C(u_3, x_1)$ or when $w \in C(x_3, u_3)$.

If $w \in P_3 - \{x_4, u_3\}$, then x_4 can be inserted into the cycle $C[u_2, u_1] \cup P'_1 \cup P'_2$ by replacing P'_1 with the path from u_1 to x_4 along P_1 and to w along P_3 and then to x_2 along P'_3 . Therefore, G has a cycle containing all the vertices of S , but avoiding z , again a contradiction.

If $w \in C(u_3, x_1)$, then $C' = C[x_1, u_1] \cup P_1 \cup P_3 \cup C^{-1}[u_3, u_2] \cup P'_2 \cup P'_3 \cup C[w, x_1]$ is a cycle containing all the vertices of S , but avoiding z , a contradiction. If $w \in C(x_3, u_3)$, then $C' = C[x_1, x_2] \cup P'_3 \cup C^{-1}[w, u_2] \cup P_2 \cup P_3 \cup C[u_3, x_1]$ is

a cycle containing all the vertices of S , but avoiding z , a contradiction yet again. This completes the proof of the theorem. ■

5. Property $C(5, 1)$ for claw-free graphs

We are now prepared to prove the last case $C(5, 1)$ of our main result on claw-free graphs. As we shall see, Theorem 2.2 will play an important role in our proof.

Lemma 5.1. *Let G be a minimum counterexample to $C(5, 1)$. That is, suppose G is a smallest 3-connected claw-free graph containing a vertex z and five additional vertices x_1, x_2, x_3, x_4 and x_5 such that G has no cycle containing all the x_i 's, but avoiding z . Then z belongs to exactly one 3-cut.*

Proof. Let G be as stated above and let $S = \{x_1, x_2, x_3, x_4, x_5\}$. Suppose to the contrary that z belongs to at least two different 3-cuts, say T and T' . By Lemma 3.3 and the fact that G is claw-free, T and T' are the only two 3-cuts of G containing z . Without loss of generality, assume that T separates the single vertex $x_1 \in S$ and T' separates the single vertex $x_2 \in S$ by Lemma 3.3. Let $Q = G - (T \cup T' \cup \{x_1, x_2\})$. An argument similar to that used in the proof of Lemma 4.1 shows that Q is connected and z has no neighbors in Q .

By Theorem 4.2, G has a cycle C containing x_1, x_2 and two other vertices from S , say x_3 and x_4 , but avoiding z .

Claim 1. *Either $C(x_1, x_2) \cap S = \emptyset$ or $C(x_2, x_1) \cap S = \emptyset$.*

Proof of Claim 1. Suppose the Claim is false and assume that $x_3 \in C(x_1, x_2) \cap S$ and $x_4 \in C(x_2, x_1) \cap S$. By Menger's Theorem, there are three internally disjoint paths P_1, P_2 and P_3 joining x_5 and three distinct vertices of C , say u_1, u_2 and u_3 , respectively. Since $x_5 \in Q$, but x_1 and x_2 are separated from Q by the two 3-cuts T and T' , it follows that $u_i \notin \{x_1, x_2\}$ for $i \in \{1, 2, 3\}$. Note that P_1, P_2 and P_3 do not contain z because $x_4 \in Q$ and z has no neighbors in Q . Note that x_1, x_2, x_3 and x_4 separate C into four segments, none of which contains two vertices of $\{u_1, u_2, u_3\}$. For otherwise, one could use the two paths with endvertices in the same segment to insert x_5 into C to generate a cycle of the type we seek. It follows that:

(**) *any three internally disjoint paths from x_5 to C end in three different segments of C determined by x_1, \dots, x_4 .*

By symmetry, we may assume that $u_1 \in C(x_1, x_3]$, $u_2 \in C(x_2, x_4)$ and $u_3 \in C(x_4, x_1)$. Now, applying Perfect's Theorem to x_4 and $H = C[u_3, u_2] \cup (\bigcup P_i)$,

we obtain three internally disjoint paths joining x_4 and three distinct vertices of H , and two of them, say P'_1 and P'_2 , end at u_2 and u_3 . Assume the third path P'_3 from x_4 to H ends at $w \notin \{x_1, x_2\}$. By $(**)$ and Jumper Operations, we may assume that $w \notin P_1$. The vertex w does not belong to $P_2 \cup P_3 \cup C[u_3, x_1] \cup C(x_2, u_2]$, for otherwise x_4 could be inserted into the cycle $C[u_3, u_2] \cup P_2 \cup P_3$ by two of the three paths from P'_1, P'_2 and P'_3 to generate the cycle sought and again we would have a contradiction. Hence, $w \in C(x_1, u_1)$ or $C(u_1, x_3]$ or $C(x_3, x_2)$.

If $w \in C(x_1, u_1)$, then

$$C' = C[x_1, w] \cup P'_3 \cup P'_1 \cup C^{-1}[u_2, u_1] \cup P_1 \cup P_3 \cup C[u_3, x_1]$$

is a cycle of the type desired and again we have a contradiction. If $w \in C(u_1, x_3]$, then cycle $C' = C[x_1, u_1] \cup P_1 \cup P_2 \cup C^{-1}[u_2, w] \cup P'_3 \cup P'_2 \cup C[u_3, x_1]$ again yields a contradiction. So suppose that $w \in C(x_3, x_2)$. Then applying Perfect's Theorem to x_3 and $H = C[u_3, u_1] \cup C[w, u_2] \cup (\bigcup P_i) \cup (\bigcup P'_i)$, we obtain three internally disjoint paths P''_1, P''_2 and P''_3 from x_3 to H ending at u_1, w and a third vertex $w' \notin \{x_1, x_2\}$, respectively. By Jumper Operations, $w' \notin \{u_2, u_3, x_4, x_5\}$. Again, none of these three paths contains z because $x_3 \in Q$ and z has no neighbors in Q . Note that $w' \notin C(x_1, u_1) \cup C(w, x_2) \cup P_1 \cup P'_3$. Otherwise, x_3 can be inserted into the cycle $C[u_3, u_1] \cup P_1 \cup P_2 \cup C^{-1}[u_2, w] \cup P'_3 \cup P'_2$ by using two paths from among the P'_i 's to generate a cycle of the type desired, a contradiction. Similarly, $w' \notin P_2 \cup P_3$ (otherwise, x_5 could be inserted to the cycle $C[u_3, u_1] \cup P''_1 \cup P''_2 \cup C[w, u_2] \cup P'_1 \cup P'_2$ to yield a contradiction) and $w' \notin P'_1 \cup P'_2$ (otherwise, x_4 could be inserted to the cycle $C[u_3, u_1] \cup P''_1 \cup P''_2 \cup C[w, u_2] \cup P_2 \cup P_3$ to yield a contradiction). Therefore, $w' \in C(x_2, u_2)$ or $C(u_3, x_1)$. If $w' \in C(x_2, u_2)$, then $C' = C[x_1, u_1] \cup P''_1 \cup P''_3 \cup C^{-1}[w', w] \cup P'_3 \cup P'_1 \cup P_2 \cup P_3 \cup C[u_3, x_1]$ is a cycle of the type desired, a contradiction. If $w' \in C(u_3, x_1)$, then $C' = C[x_1, u_1] \cup P_1 \cup P_3 \cup P'_2 \cup P'_1 \cup C^{-1}[u_2, w] \cup P''_2 \cup P''_3 \cup C[w', x_1]$ is a cycle of the type desired, a contradiction again. This completes the proof of Claim 1. ■

By Claim 1, every cycle C containing x_1, x_2 and two other vertices from S satisfies the property that one of $C(x_1, x_2)$ and $C(x_2, x_1)$ does not contain vertices from S . Without loss of generality, assume that $C(x_1, x_2)$ fails to contain vertices from S . So then $C(x_2, x_1)$ must contain two vertices from S , say x_3 and x_4 . Assume that x_1, x_2, x_3 and x_4 appear clockwise around the cycle C . By Menger's Theorem, there are three disjoint paths P_1, P_2 and P_3 from x_5 to C ending at three distinct vertices u_1, u_2 and u_3 , respectively. Since all three paths belong to Q , they do not contain the vertex z . Moreover,

each of the segments $C[x_1, x_2]$, $C[x_2, x_3]$, $C[x_3, x_4]$ and $C[x_4, x_1]$ contains at most one vertex from $\{u_1, u_2, u_3\}$.

Claim 2. *The segment $C[x_3, x_4]$ does not contain any vertex from $\{u_1, u_2, u_3\}$.*

Proof of Claim 2. Assume, to the contrary, that $C[x_3, x_4]$ does contain a vertex from $\{u_1, u_2, u_3\}$.

First, assume that $C[x_1, x_2]$ does not contain a vertex of $\{u_1, u_2, u_3\}$. Without loss of generality, we further assume that $u_1 \in C(x_2, x_3)$, $u_2 \in C(x_3, x_4)$ and $u_3 \in C(x_4, x_1)$, respectively. Now apply Perfect's Theorem to x_3 and $H = C[x_1, u_1] \cup P_1 \cup P_2 \cup C[u_2, x_1] \cup P_3$. There are three internally disjoint paths joining x_3 and three distinct vertices of H such that two of them, say P'_1 and P'_2 , end at u_1 and u_2 , and the third path P'_3 ends at some vertex w . Using Jumper Operations, we may assume that $w \notin \{u_3, x_5\}$. A routine check shows that w must belong to $C(x_1, x_2)$, for otherwise, there exists a cycle through x_1, \dots, x_5 which avoids z , yielding a contradiction.

Now apply Perfect's Theorem to x_4 and $H' = C[u_3, u_1] \cup (\bigcup P_i) \cup (\bigcup P'_i)$. There are three internally disjoint paths joining x_4 and three distinct vertices of H' such that two of them, say P''_1 and P''_2 , end at u_2 and u_3 and the third path P''_3 ends at some vertex w' . By Jumper Operations, $w' \notin \{x_3, x_5, u_1, w\}$. By symmetry of x_3 and x_4 , the vertex w' must belong to $C(x_1, x_2)$ or P'_3 . If $w' \in P'_3$, then $C' = C[x_1, u_1] \cup P'_1 \cup P'_3[x_3, w'] \cup P''_3 \cup P''_1 \cup P_2 \cup P_3 \cup C[u_3, x_1]$ is a cycle of the type we need, a contradiction. Therefore, $w' \in C(x_1, x_2)$. Now let $C' = C[x_1, w] \cup P'_3 \cup P'_2 \cup P''_1 \cup P''_3 \cup C[w', u_1] \cup P_1 \cup P_3 \cup C[u_3, x_1]$ if $w' \in C(w, x_2)$, or let $C' = C[x_1, w'] \cup P''_3 \cup P''_1 \cup P'_2 \cup P'_3 \cup C[w, u_1] \cup P_1 \cup P_3 \cup C[u_3, x_1]$ if $w' \in C(x_1, w)$. Then C' is a cycle containing all five vertices from S , but avoiding z , a contradiction. This contradiction implies that $C[x_1, x_2]$ does contain a vertex of $\{u_1, u_2, u_3\}$.

So in the following, assume that $u_1 \in C(x_1, x_2)$ and $u_2 \in C[x_3, x_4]$. Then u_3 belongs to either $C(x_2, x_3)$ or $C(x_4, x_1)$. By symmetry, without loss of generality, let us assume that $u_3 \in (x_4, x_1)$. Now, apply Perfect's Theorem to x_4 and $H = C[u_3, u_2] \cup (\bigcup P_i)$. There are three internally disjoint paths all avoiding z joining x_4 and three distinct vertices of H such that two of them, say P'_1 and P'_2 , end at u_2 and u_3 , and the third path P'_3 ends at some vertex w . By Jumper Operations, $w \notin \{x_5, u_1\}$. Note that w does not belong to $P_2, P_3, C(x_3, u_2)$ or $C(u_3, x_1)$ because, otherwise, x_4 could be inserted into the cycle $C[u_3, u_2] \cup P_2 \cup P_3$ by using two of the three paths P'_1, P'_2 and P'_3 to generate a cycle of the type desired, a contradiction. If $w \in P_1$, then x_5 can be inserted into the cycle $C[u_3, u_2] \cup P'_1 \cup P'_2$ by the paths P_2, P_1 and P'_3 to generate a cycle of the type desired, a contradiction again.

If $w \in C(x_1, u_1)$, then $C' = C[x_1, w] \cup P'_3 \cup P'_1 \cup C^{-1}[u_2, u_1] \cup P_1 \cup P_3 \cup C[u_3, x_1]$ yields a cycle of the type sought, a contradiction. If $w \in C(u_1, x_2)$, then $C' = C[x_1, u_1] \cup P_1 \cup P_2 \cup C^{-1}[u_2, w] \cup P'_3 \cup P'_2 \cup C[u_3, x_1]$ yields a cycle of the type we want, a contradiction again. So $w \in C(x_2, x_3)$.

Now apply Perfect's Theorem to x_3 and $H'' = C[u_3, w] \cup (\cup P_i) \cup (\cup P'_i)$ to obtain three internally disjoint paths avoiding z which join x_3 and three distinct vertices of H'' such that two of them, say P''_1 and P''_2 , end at w and u_2 , respectively, and the third path P''_3 ends at some vertex $w' \notin \{u_1, u_3, x_4, x_5\}$ by Jumper Operations. Note that $w' \notin C[x_2, w] \cup P'_3 \cup P_2 \cup P'_1$ since, otherwise, x_3 could be inserted into $C[u_3, w] \cup P'_3 \cup P'_1 \cup P_2 \cup P_3$ to generate a cycle of the type desired, a contradiction. Similarly, $w' \notin P_1 \cup P_3$ (otherwise, x_5 could be inserted into $C[u_3, w] \cup P''_1 \cup P''_2 \cup P'_1 \cup P'_2$) and $w' \notin P'_2$ (otherwise, x_4 could be inserted into $C[u_3, w] \cup P''_1 \cup P''_2 \cup P_2 \cup P_3$ by P'_3, P''_3 and a subpath of P'_2). If $w' \in C(x_1, x_2)$, then replace the subpath of $C(x_1, x_2)$ joining w' and u_1 by $P_1 \cup P_2 \cup P''_2 \cup P''_3$ (which contains both x_3 and x_5) in the cycle $C[u_3, w] \cup P'_3 \cup P'_2$ to generate a cycle that we need, a contradiction. So $w' \in C(u_3, x_1)$. However, the cycle $C' = C[x_1, w] \cup P'_3 \cup P'_2 \cup P_3 \cup P_2 \cup P''_2 \cup P''_3 \cup C[w', x_1]$ is then of the type we desired, a contradiction again. This completes the proof of Claim 2. ■

By Claim 2, we may assume that $u_1 \in C(x_1, x_2)$, $u_2 \in C(x_2, x_3)$ and $u_3 \in C(x_4, x_1)$. Now, we need Theorem 2.2 which is a stronger version of Perfect's Theorem. The two disjoint paths $C[x_4, u_3]$ and $C[u_2, x_3]$ is a 2-link between $C[x_3, x_4]$ and $H_1 = C[u_3, u_2] \cup (\cup P_i)$. If $C[x_3, x_4]$ contains at least three vertices, Menger's Theorem implies that there are three disjoint paths between $C[x_3, x_4]$ and H_1 . Hence, $C[x_3, x_4]$ and H_1 are 3-linked, and then apply Theorem 2.2 to $C[x_3, x_4]$ and H_1 . There are three internally disjoint paths P'_1, P'_2 and P'_3 joining $\{x_3, x_4, w\} \subseteq V(C[x_3, x_4])$ and three distinct vertices $\{u_2, u_3, w'\} \subseteq V(H_1)$. If $C[x_3, x_4]$ is an edge x_3x_4 , replace x_3 by a clique K with $d_G(x_3) \geq 3$ vertices such that the edge of G incident with x_3 is incident with exactly one vertex of K . Then the resulting graph G' is 3-connected. Now let $C'[x_3, x_4]$ denote a path containing x_4 and all vertices of the clique K such that x_4 is an endvertex. Then $C'[x_3, x_4]$ has at least four vertices, and $C'[x_3, x_4]$ and H_1 are 3-linked in G' . Apply Theorem 2.2 to $C'[x_3, x_4]$ and H_1 . There are three internally disjoint paths joining x_4 and two vertices of K to three distinct vertices $\{u_2, u_3, w'\} \subset V(H_1)$. The two disjoint paths joining two vertices of K and two vertices from $\{u_2, u_3, w'\}$ in G' correspond to two internally disjoint paths of G joining x_3 and two distinct vertices from $\{u_2, u_3, w'\}$. Therefore, without loss of generality, assume that w, x_3 and x_4 are endvertices of P'_1, P'_2 and P'_3 (note that w may be equal to x_3), respectively.

By symmetry of the two vertices x_3 and x_4 , and the symmetry of H_1 , the only two possibilities for the other endvertices of P'_1, P'_2 and P'_3 are:

- (1) $w' \in P'_1, u_2 \in P'_2$ and $u_3 \in P'_3$; or
- (2) $u_2 \in P'_1, w' \in P'_2$ and $u_3 \in P'_3$.

First, assume (1) holds, that $w' \in P'_1, u_2 \in P'_2$ and $u_3 \in P'_3$. By Jumper Operations, $w' \notin \{u_1, x_5\}$. If $w' \in C(x_1, x_2)$, then G has a cycle

$$C[x_1, w'] \cup P'_1 \cup C^{-1}[w, x_3] \cup P'_2 \cup C^{-1}[u_2, u_1] \cup P_1 \cup P_3 \cup C[u_3, x_1]$$

(if $w' \in C(x_1, u_1)$) or

$$C[x_1, u_1] \cup P_1 \cup P_2 \cup C^{-1}[u_2, w'] \cup P'_1 \cup C[w, x_4] \cup P'_3 \cup C[u_3, x_1]$$

(if $w' \in C(u_1, x_2)$), either of which violates Claim 1. If $w' \in \cup_{i=1}^3 P_i$, then x_5 and the cycle $C[u_3, u_2] \cup P'_2 \cup C[x_3, x_4] \cup P'_3$ violates Claim 2 because G has three disjoint paths joining x_5 and the cycle such that one of the path ends at $w \in C[x_3, x_4]$.

So $w' \in C(x_2, u_2)$ or $w' \in C(u_3, x_1)$. By symmetry, we may assume that $w' \in C(x_2, u_2)$. Then applying Perfect's Theorem to x_4 and $H_2 = H_1 \cup C[x_3, w] \cup P'_2 \cup P'_1$, there are three internally disjoint paths joining x_4 and three distinct vertices of H_2 such that two of the paths, say P''_1 and P''_2 , end at w and u_3 , and the third path P''_3 ends at some vertex w'' of H_2 . By Jumper Operations, $w'' \notin \{w', u_1, u_2, x_5\}$. An argument similar to that used in the proof of $w' \notin C(x_1, x_2)$ shows that $w'' \notin C(x_1, x_2)$. By Claim 2, $w'' \notin \cup_{i=1}^3 P_i$ and $w'' \notin P'_2$ (where we consider x_5 and the cycle $C[x_1, w'] \cup P'_1 \cup C^{-1}[w, x_3] \cup P'_2[x_3, w''] \cup P''_3 \cup P''_2 \cup C[u_3, x_1]$ for the last case). Clearly, $w'' \notin C(u_3, x_1)$ or $C[x_3, w]$ or P'_1 . Otherwise, x_4 could be inserted into the cycle $C[u_3, w'] \cup P'_1 \cup C^{-1}[w, x_3] \cup P'_2 \cup P_2 \cup P_3$, which yields a contradiction. So $w'' \in C(x_2, w')$ or $w'' \in C(w', u_2)$. If $w'' \in C(x_2, w')$, then $C' = C[x_1, w''] \cup P''_3 \cup P''_1 \cup C^{-1}[w, x_3] \cup P'_2 \cup P_2 \cup P_3 \cup C[u_3, x_1]$ is a cycle of the type desired, yielding a contradiction. If $w'' \in C(w', u_2)$, then $C' = C[x_1, w''] \cup P''_3 \cup P''_1 \cup C^{-1}[w, x_3] \cup P'_2 \cup P_2 \cup P_3 \cup C[u_3, x_1]$ also gives a contradiction. This contradiction implies that (1) does not happen.

So, in the following, assume that $u_2 \in P'_1, w' \in P'_2$ and $u_3 \in P'_3$. By Jumper Operations, $w' \notin \{u_1, x_5\}$. If $w' \in C(x_1, u_1)$, then G contains the cycle $C[u_3, w'] \cup P'_2 \cup C[x_3, w] \cup P'_1 \cup C^{-1}[u_2, u_1] \cup P_1 \cup P_3$, violating Claim 1. If $w' \in C(u_1, x_2)$, then G contains the cycle $C[x_1, u_1] \cup P_1 \cup P_2 \cup C^{-1}[u_2, w'] \cup P'_2 \cup C[x_3, x_4] \cup P'_3 \cup C[u_3, x_1]$ of the type desired, yielding a contradiction. By Claim 2, $w' \notin \cup_{i=1}^3 P_i$. If $w' \in C(u_3, x_1)$, then $C' = C[x_1, u_2] \cup P_2 \cup P_3 \cup P'_3 \cup C^{-1}[x_4, x_3] \cup P'_2 \cup C[w', x_1]$ is a cycle of the type desired, a contradiction. So $w' \in C(x_2, u_2)$. Then consider the vertex

x_4 and the cycle $C'' = C[u_3, w'] \cup P'_2 \cup C[x_3, w] \cup P'_1 \cup P_2 \cup P_3$. By Perfect's Theorem, there are three internally disjoint paths avoiding z which join x_4 and three distinct vertices of the cycle C'' such that one of the paths ends at $w \in C''[x_3, x_5]$, but this contradicts Claim 2 (by interchanging the labels of x_4 and x_5). This completes the proof of Lemma 5.1. ■

We are now equipped to complete the proof of our main result on cyclability in 3-connected claw-free graphs – Theorem 1.3.

Proof of Theorem 1.3. Let G be a minimum counterexample. Hence, G has a vertex z and $S = \{x_1, x_2, \dots, x_5\}$ such that G has no cycle containing all vertices of S , but avoiding z . By Lemma 5.1, the vertex z belongs to exactly one 3-cut T of G . By Lemma 3.3, T separates a single vertex from S , say x_1 . By Theorem 4.2, G has a cycle C containing x_1 and three other vertices from S , but avoiding z . Without loss of generality, assume that C contains x_1, x_2, x_3 and x_4 in clockwise order.

By Lemma 3.6, x_5 and C are 3-linked in $G - z$ since z is contained in exactly one 3-cut T . Hence, there are three internally disjoint paths P_1, P_2 and P_3 joining x_5 and three distinct vertices of C . Let u_i be the endvertex of P_i on C for $i \in \{1, 2, 3\}$. Note that none of the segments of C determined by x_1, x_2, x_3 and x_4 contains two endvertices of the three paths from x_5 to C . Otherwise, x_5 can be inserted into C to give a cycle of the type we seek, contradicting the assumption that G is a counterexample.

Claim. *The segment $C(x_1, x_2)$ contains exactly one vertex of $\{u_1, u_2, u_3\}$.*

Proof of the Claim. Suppose not. Without loss of generality, assume that $u_1 \in C[x_2, x_3], u_2 \in C[x_3, x_4]$ and $u_3 \in C[x_4, x_1]$. Let $H = C[u_2, u_1] \cup (\bigcup P_i)$. By Lemma 3.6, x_3 and H are 3-linked in $G - z$. Now apply Perfect's Theorem to x_3 and H to obtain three internally disjoint paths joining x_3 and three distinct vertices of H , two of which, say P'_1 and P'_2 , end at u_1 and u_2 , respectively. Assume that the third path P'_3 from x_3 to H ends at w . By Jumper Operations, we assume that $w \notin \{u_1, u_2, u_3\}$. If w belongs to any of the $P_i - u_i$'s, then there are two internally disjoint paths joining x_5 and either $\{u_1, x_3\}$ or $\{u_2, x_3\}$ which can be used to insert x_5 into the cycle $C[u_2, u_1] \cup P'_1 \cup P'_2$ to yield a cycle of the type desired, a contradiction. Similarly, $w \notin C[x_2, u_1] \cup C[u_2, x_4]$, for otherwise x_3 can be inserted into the cycle $C[u_2, u_1] \cup P_1 \cup P_2$ by using two paths from among the three P'_i 's to yield a cycle of the type desired, a contradiction. If $w \in C(u_3, x_1)$, the cycle $C' = C[x_1, u_1] \cup P_1 \cup P_3 \cup C^{-1}[u_3, u_2] \cup P'_2 \cup P'_3 \cup C[w, x_1]$ is of the type desired, a contradiction. If $w \in C(x_4, u_3)$, the cycle

$C' = C[x_1, u_1] \cup P'_1 \cup P'_3 \cup C^{-1}[w, u_2] \cup P_2 \cup P_3 \cup C[u_3, x_1]$ is of the type desired and yet again we have a contradiction. Therefore, $w \in C(x_1, x_2)$.

Let $H' = C[u_2, w] \cup (\bigcup P_i) \cup (\bigcup P'_i)$. By Lemma 3.6, x_2 and H' are 3-linked in $G - z$. Then apply Perfect's Theorem to obtain three internally disjoint paths joining x_2 and three distinct vertices of H' . Two of these paths, say P''_1 and P''_2 , end at w and u_1 respectively, and the third path P''_3 ends at some w' . Again, by Jumper Operations, assume that $w' \notin \{u_1, u_2, u_3, w\}$. A straightforward check confirms that $w' \notin (\bigcup P_i) \cup (\bigcup P'_i) \cup C(x_1, w)$, for otherwise, a cycle of the type sought can be easily constructed. If $w' \in C(u_2, x_4]$, then

$$C' = C[x_1, w] \cup P'_3 \cup P'_2 \cup P_2 \cup P_1 \cup C^{-1}[u_1, x_2] \cup P''_3 \cup C[w', x_1]$$

is a cycle yielding a contradiction. If $w' \in C(x_4, u_3)$, then

$$C' = C[x_1, w] \cup P'_3 \cup P'_1 \cup C^{-1}[u_1, x_2] \cup P''_3 \cup C^{-1}[w', u_2] \cup P_2 \cup P_3 \cup C[u_3, x_1],$$

again yields a contradiction. So $w' \in C(u_3, x_1)$. But then

$$C' = C[x_1, w] \cup P'_3 \cup P'_2 \cup C[u_2, u_3] \cup P_3 \cup P_1 \cup C^{-1}[u_1, x_2] \cup P''_3 \cup C[w', x_1]$$

is a cycle which yields a contradiction yet again. This completes the proof of the Claim. ■

By the Claim and symmetry, both $C(x_1, x_2)$ and $C(x_4, x_1)$ contain one vertex from $\{u_1, u_2, u_3\}$, which implies that one of $C(x_2, x_3)$ and $C(x_3, x_4)$ does not contain a vertex from $\{u_1, u_2, u_3\}$. By symmetry again, we may assume that it is $C(x_2, x_3)$ which does not contain a vertex from $\{u_1, u_2, u_3\}$ and assume that $u_1 \in C(x_1, x_2)$, $u_2 \in C(x_3, x_4)$ and $u_3 \in C(x_4, x_1)$. Let $H = C[u_3, u_2] \cup (\bigcup P_i)$. By Lemma 3.6, x_4 and H are 3-linked in $G - z$. Now apply Perfect's Theorem to x_4 and H to obtain three internally disjoint paths from x_4 to H . Two of the paths, say P'_1, P'_2 , end at u_2, u_3 , respectively and the third path P'_3 ends at some vertex $w \notin \{u_1, u_2, u_3\}$ by Jumper Operations. First note that $w \notin P_i$ for any $i \in \{1, 2, 3\}$, for otherwise, x_5 can be inserted into the cycle $C' = C[u_3, u_2] \cup (P'_1 \cup P'_2)$ to generate a cycle which again yields a contradiction. Similarly, $w \notin C[x_3, u_2] \cup C[u_3, x_1]$ for otherwise, x_4 can be inserted into the cycle $C' = C[u_3, u_1] \cup P_2 \cup P_3$ to form a cycle which yields a contradiction. If $w \in C(x_1, u_1)$, then cycle

$$C'' = C[x_1, w] \cup P'_3 \cup P'_2 \cup C^{-1}[u_2, u_1] \cup P_1 \cup P_3 \cup C[u_3, x_1]$$

yields a contradiction. If $w \in C(u_1, x_2]$, then cycle

$$C'' = C[x_1, u_1] \cup P_1 \cup P_2 \cup C^{-1}[u_2, w] \cup P'_3 \cup P'_2 \cup C[u_3, x_1]$$

gives a contradiction. So $w \in C(x_2, x_3)$. Now consider the cycle $C' = C[u_3, u_2] \cup P_2 \cup P_3$ and the vertex x_4 , which contradicts Claim by interchanging the labels of x_4 and x_5 . This final contradiction completes the proof of Theorem 1.3. ■

Now, we show that Theorem 1.3 is sharp by providing infinitely many examples of 3-connected claw-free graphs which are *not* $C(6, 1)$.

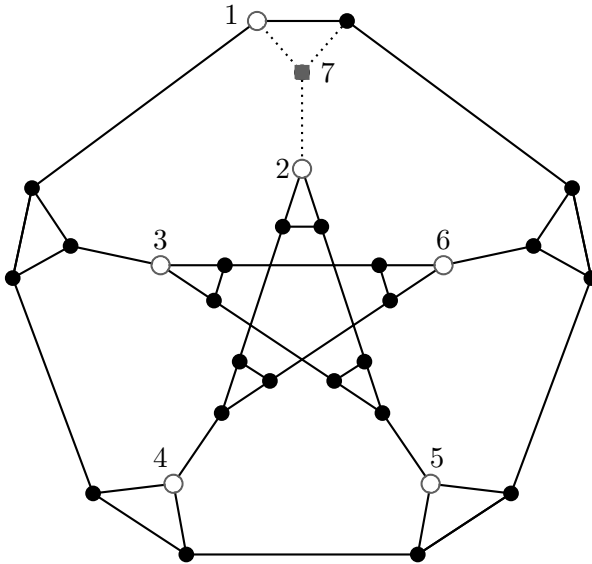


Figure 3. A cubic claw-free graph without a cycle through vertices 1, 2, ..., 6, which avoids 7

Let G be the 3-connected claw-free graph on thirty vertices obtained by replacing each of the ten vertices of the Petersen graph with a triangle. (See Figure 3.) It is easy to check that there is no cycle in this graph containing the six vertices numbered 1 through 6 which fails to contain the seventh vertex labeled 7. Hence, this graph does not possess the property $C(6, 1)$. To obtain infinitely many such counterexamples, one may simply replace any of the triangles with a larger complete graph.

It is also interesting to observe that the graph shown in Figure 3 is cubic as well. Hence 3-connected claw-free cubic graphs do not necessarily possess the property $C(6, 1)$ either. As we mentioned in the Introduction, a 3-connected cubic graph may not be $C(3, 1)$, an example being $K_{3,3}$.

6. Graphs on surfaces

Tutte [29] proved that every 4-connected plane graph G is Hamiltonian, and hence is $C(n,0)$ where $n = |V(G)|$. However, for 3-connected plane graphs, the maximum cyclability is only 5 and this bound is sharp in the sense that there exist 3-connected plane graphs which are not $C(6,0)$ [26,27]. The same fact holds true for 3-connected plane triangulations. There are infinitely many 3-connected plane triangulations which are not $C(6,0)$ and which are not $C(4,1)$. For example, in the eleven-vertex graph shown in Figure 4, there is no cycle through vertices 1, 2, \dots , 6, and neither is there a cycle through vertices 1, 2, 3 and 4, which avoids 7. Note that the example in Figure 4 can be extended to infinitely many examples by repeatedly adding a new vertex in the exterior face and connecting it to the three vertices of the exterior triangle.

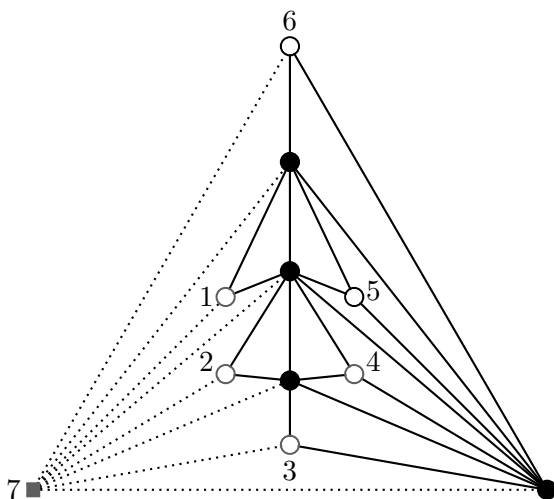


Figure 4. A plane triangulation without a cycle through vertices 1,2,3,4, which avoids 7

In fact, for any given closed surface Σ , there are infinitely many 3-connected graphs which are neither $C(6,0)$ nor $C(4,1)$, even for surface triangulations which have the maximal edge density among the graphs embedded in the surface. To construct a surface triangulation which is neither $C(6,0)$ nor $C(4,1)$: take any surface triangulation of Σ , then glue the exterior triangle of the graph in Figure 4 to any face-bounding triangle of the triangulation to generate a new triangulation. Then the new triangulation has no cycle through the six gray vertices, nor does it have a cycle through

1, 2, 3 and 4, which avoids 5. Note that a surface triangulation is always a polyhedral embedding. Hence, there are infinitely many graphs polyhedrally embedded in any closed surface which do not have the properties $C(6,0)$ or $C(4,1)$.

On the other hand, Theorem 1.5 shows that a graph polyhedrally embedded in a closed surface must have the property $C(3,1)$. Note that the assumption that the embedding be polyhedral is necessary because $K_{3,3}$ has a closed 2-cell embedding in the projective plane but does not have the property $C(3,1)$. A graph G admitting a polyhedral embedding is 3-connected and the neighbors of any vertex $v \in V(G)$ belong to a cycle of $G-v$ (i.e., the symmetric difference of the face boundaries containing v) (see [23]).

Theorem 1.5 follows directly from the following more general result.

Theorem 6.1. *Let G be a 3-connected graph and $z \in V(G)$ such that $G-z$ has a cycle C_z containing all neighbors of z . Then, for any other three vertices x_1, x_2 and x_3 , G has a cycle passing through x_1, x_2, x_3 but avoiding z .*

Proof. Suppose to the contrary that G is a counterexample. Then G has vertices x_1, x_2, x_3 such that G does not have a cycle through x_1, x_2 and x_3 , which avoids z . Since G is 3-connected, for any pair of vertices x_i and x_j from $\{x_1, x_2, x_3\}$, there is a cycle C_{ij} containing x_i and x_j , but avoiding z . Because G is a counterexample, C_{ij} does not contain the third vertex x_k . On the other hand, G has three internally disjoint paths P_1, P_2 and P_3 from x_k to the cycle C_{ij} ending at three different vertices of C_{ij} by Menger’s Theorem. Again, since G is a counterexample, one of these three paths must contain the vertex z . Otherwise, two of P_1, P_2 and P_3 must end on the same segment of C_{ij} separated by x_i and x_j . These two paths then could be used to insert x_k into the cycle C_{ij} to generate a cycle of the type we seek, a contradiction. The following claim shows that all three paths from x_k to C_{ij} intersecting the cycle C_z .

Claim 3. *If x_k does not belong to C_z , then the three paths P_1, P_2 and P_3 intersect C_z at three different vertices.*

Proof of Claim 3. Since one of the paths P_1, P_2 and P_3 must contain the vertex z , it follows that at least one of them, say P_1 , intersects C_z . For a path P_t intersecting C_z , let y_t^1 and y_t^2 be the first and last vertices that P_t and C_z have in common when traversing P_t from x_k to C_{ij} for $t \in \{1, 2, 3\} \setminus \{i, j\}$.

If P_1 is the only path intersecting C_z , then $P'_1 = (P_1 - P_1(y_1^1, y_1^2)) \cup C_z[y_1^1, y_1^2]$ is a path from x_k to C_{ij} which avoids z and is internally disjoint from P_2 and P_3 . Then G has three internally disjoint paths from x_k to C_{ij} ending at three different vertices, a contradiction.

Now assume there are exactly two paths from P_1, P_2 and P_3 intersecting C_z . Without loss of generality, assume these are P_1 and P_2 . Let $P_t^1 = P_t[x_k, y_t^1]$ and P_t^2 be the segment of P_t from y_t^2 to C_{ij} . If both y_2^1 and y_2^2 belong to the same segment $C_z[y_1^1, y_1^2]$ or $C_z[y_2^2, y_1^1]$, say $C_z[y_1^1, y_1^2]$, without loss of generality, we may assume that y_1^1, y_2^1, y_2^2 and y_1^2 appear in order along the clockwise direction of C_z . Let $P'_1 = P_1^1 \cup C_z^{-1}[y_1^1, y_1^2] \cup P_1^2$, and $P'_2 = P_2^1 \cup C_z[y_2^1, y_2^2] \cup P_2^2$. Then P'_1, P'_2 and P_3 are three internally disjoint paths from x_k to C_{ij} ending at three different vertices, a contradiction again. So y_2^1 and y_2^2 belong to different segments of C_z separated by y_1^1 and y_1^2 . Without loss of generality, assume that $y_2^1 \in C_z[y_1^1, y_1^2]$ and $y_2^2 \in C_z[y_2^2, y_1^1]$. Then let $P'_1 = P_1^1 \cup C_z^{-1}[y_1^1, y_2^2] \cup P_2^2$ and $P'_2 = P_2^1 \cup C_z[y_2^1, y_1^2] \cup P_1^2$. Then, P'_1, P'_2 and P_3 are three internally disjoint paths from x_k to C_{ij} ending at three different vertices, a contradiction again.

Hence, all three paths P_1, P_2 and P_3 intersect C_z . Because P_1, P_2 and P_3 are internally disjoint paths ending at three different vertices of C_{ij} , it follows that P_1, P_2 and P_3 intersect C_z at three different vertices. This completes the proof of Claim 3. ■

Again since G is a counterexample, C_z contains at most two vertices from $\{x_1, x_2, x_3\}$. If $|C_z \cap \{x_1, x_2, x_3\}| = 2$, then G has three internally disjoint paths from the vertex x_i not on C_z to z which intersects C_z at three different vertices. Further, the vertex x_i can be inserted into C_z using two of the three paths to generate a cycle of the type desired, a contradiction. Hence C_z contains at most one vertex from $\{x_1, x_2, x_3\}$. Without loss of generality, assume that $x_1, x_2 \notin C_z$. Since G is 3-connected, there are three internally disjoint paths joining each of x_1 and x_2 to z . Assume that the three paths from x_1 to z intersect C_z at v_1, v_2 and v_3 , and denote the segment from x_1 to v_i by $P[x_1, v_i]$ for $i \in \{1, 2, 3\}$. Similarly, there are three internally disjoint paths from x_2 to C_z ending at three different vertices u_1, u_2 and u_3 . Denote these paths by $P[x_2, u_i]$ for $i \in \{1, 2, 3\}$. Let $\Gamma_1 = \{v_1, v_2, v_3\}$ and $\Gamma_2 = \{u_1, u_2, u_3\}$.

Note that either $x_3 \in C_z$ or $x_3 \notin C_z$. First, we consider the case that $x_3 \in C_z$. Any two paths of the type $P[x_1, v_i]$, say $P[x_1, v_1]$ and $P[x_1, v_2]$, together with the segment of C_z separated by v_1 and v_2 which contains x_3 form a cycle. Let us denote such a cycle by C_{13} . Then by Claim 3, $P[x_2, u_i]$ does not internally intersect $P[x_1, v_j]$ (u_i and v_j may be the same), for $i, j \in \{1, 2, 3\}$. Otherwise, there are three internally disjoint paths from x_2 to C_{13} which do not intersect C_z at three different vertices, a contradiction to Claim 3. Since $x_3 \in C_z$, without loss of generality, assume that v_1, v_2 and v_3 appear in clockwise order on C_z and $x_3 \in C_z[v_1, v_2]$. By symmetry, assume that $|C_z(v_3, v_1) \cap \Gamma_2| \leq |C_z(v_2, v_3) \cap \Gamma_2|$. Then $|C_z(v_3, v_1) \cap \Gamma_2| \leq 1$

because $|\Gamma_2| = 3$. If $|C_z(v_3, v_1) \cap \Gamma_2| = 1$, say $u_3 \in C_z(v_3, v_1)$, then $C_z(v_2, v_3]$ has a vertex from Γ_2 , say u_2 . It follows that G has a cycle $C = P[x_1, v_1] \cup C[v_1, u_2] \cup P[x_2, u_2] \cup P[x_2, u_3] \cup C_z^{-1}[u_3, v_3] \cup P[x_1, v_3]$ which contains x_1, x_2 and x_3 , but not z , a contradiction. Hence, $C_z(v_3, v_1) \cap \Gamma_2 = \emptyset$. So $\Gamma_2 \subset C_z - C_z(v_3, v_1) = C_z[v_1, v_3]$. Then one of segments $C_z[v_1, x_3]$ and $C_z[x_3, v_3]$ contains two vertices from Γ_2 , say u_1 and u_2 . But then x_2 can be inserted into the cycle $P[x_1, v_1] \cup C_z[v_1, v_3] \cup P[x_1, v_3]$ using the two paths $P[x_2, u_1]$ and $P[x_2, u_2]$ to yield a cycle of the type we seek, a contradiction again. This contradiction implies that $x_3 \notin C_z$.

Let $H = C_z \cup (\bigcup P[x_1, v_i]) \cup (\bigcup P[x_2, u_j])$. It is easily seen that every path of the type $P[x_1, v_i]$ or $P[x_2, u_j]$ is contained in a cycle of H which contains both x_1 and x_2 , but not z . Since G is a counterexample and $x_3 \notin C_z$, it follows that $x_3 \notin H$. Since G is 3-connected, there are three internally disjoint paths from x_3 to H ending at three different vertices w_1, w_2 and w_3 by Menger's Theorem. Let $\Gamma_3 = \{w_1, w_2, w_3\}$. By Claim 3, $\Gamma_3 \subset C_z$.

Claim 4. For every two vertices $v, v' \in \Gamma_i$ with $i \in \{1, 2, 3\}$, both $C_z(v, v')$ and $C_z(v', v)$ contain a vertex $v'' \in \Gamma_j$ for some $j \in \{1, 2, 3\} \setminus \{i\}$.

Proof of Claim 4. Suppose that Claim 4 does not hold. Without loss of generality, assume that $C(v, v') \cap \Gamma_j = \emptyset$ for every $j \in \{1, 2, 3\} \setminus \{i\}$. Let $C'_z = (C_z - C_z(v, v')) \cup P[x_i, v] \cup P[x_i, v']$. Then $x_i \in C'_z$ and $\Gamma_j \subseteq C'_z$ for every $j \in \{1, 2, 3\} \setminus \{i\}$. Treat C'_z as C_z and x_i as x_3 in the case that $x_3 \in C_z$. An argument similar to the one used in the proof for the case $x_3 \in C_z$ shows that G must have a cycle containing x_1, x_2 and x_3 , but not z , which contradicts the assumption that G is a counterexample. This completes the proof of Claim 4. ■

Note that either $C_z(v_1, v_2]$ or $C_z(v_2, v_1]$ contains two vertices from Γ_2 . Without loss of generality, assume that Γ_2 has two vertices in $C_z(v_2, v_1]$, namely u_1 and u_2 (relabeling if necessary). Assume that v_1, v_2, u_1 and u_2 appear in clockwise order on C_z . Consider the four vertices v_1, v_2, u_1 and u_2 on C_z , and observe that the segment $C_z(v_1, u_1]$ is symmetric to the segment $C_z(u_1, v_1]$. Then either $C_z(v_1, u_1]$ or $C_z(u_1, v_1]$ contains two vertices from Γ_3 . By the symmetry of $C(v_1, u_1]$ and $C(u_1, v_1]$, assume that $|\Gamma_3 \cap C_z(v_1, u_1)] \geq 2$, and $w_1, w_2 \in C_z(v_1, u_1]$ are such that v_1, w_1 and w_2 appear in clockwise order on C_z .

If $w_2 \in C_z(v_2, u_1]$, then x_3 can be inserted into the cycle

$$C_z[u_2, v_1] \cup P(x_1, v_1) \cup P(x_1, v_2) \cup C_z[v_2, u_1] \cup P(x_2, u_1) \cup P(x_2, u_2)$$

using the two paths $P(x_3, w_1)$ and $P(x_3, w_2)$ if $w_1 \in C_z[v_2, u_1]$ (or the two paths $P(x_3, w_1) \cup C_z[w_1, v_2]$ and $P(x_3, w_2)$ if $w_1 \in C_z(v_1, v_2)$) to give a cycle

of the type we seek, and again we have a contradiction to the assumption that G is a counterexample. This contradiction implies that both w_1 and $w_2 \in C_z(v_1, v_2)$.

By Claim 4, $C_z(u_1, u_2)$ contains a vertex from $\Gamma_1 \cup \Gamma_3$. If $w_3 \in C_z(u_1, u_2)$, then x_3 can be inserted into the cycle

$$C' = C_z[u_2, v_1] \cup P(x_1, v_1) \cup P(x_1, v_2) \cup C_z[v_2, u_1] \cup P(x_2, u_1) \cup P(x_2, u_2)$$

using the two paths $P(x_3, w_2) \cup C_z[w_2, v_2]$ and $P(x_3, w_3) \cup C_z^{-1}[w_3, u_1]$, a contradiction. Hence, $v_3 \in C_z(u_1, u_2)$. But then the cycle

$$\begin{aligned} P[x_1, v_1] \cup C_z[v_1, w_1] \cup P[x_3, w_1] \cup P[x_3, w_2] \cup C[w_2, u_1] \\ \cup P[x_2, u_1] \cup P[x_2, u_2] \cup C_z^{-1}[u_2, v_3] \cup P[x_1, v_3] \end{aligned}$$

is a cycle of the type desired, a contradiction. This final contradiction completes the proof. \blacksquare

7. Concluding remarks

It is worth mentioning that Kelmans and Lomonosov [19] characterized k -connected graphs without the property $C(k+2, 0)$ for integer $k \geq 2$, and Watkins and Mesner [30] characterized 2-connected graphs without the property $C(3, 0)$. It might be possible to use these characterizations to give alternative proofs of Theorem 4.2 (using Kelmans and Lomonosov's characterization for $k=2$) and Theorem 6.1 (using Watkins and Mesner's characterization). However, these characterizations are not simple, and the applications of them are not straightforward.

Let $f_{\mathcal{F}}(k, t)$ be the largest integer m such that every k -connected graph in the family \mathcal{F} is $C(m, t)$ where t, k are integers with $k \geq t + 2$. If every k -connected graph in \mathcal{F} is Hamiltonian, define $f_{\mathcal{F}}(k, 0) = \infty$. Since a k -connected graph has a cycle through any given k vertices (cf. [5]), it follows immediately that $f_{\mathcal{F}}(k, t) \geq k - t$. But if \mathcal{F} contains $K_{k,k}$, then $f_{\mathcal{F}}(k, t) = k - t$. For some interesting families of graphs, $f_{\mathcal{F}}(k, t)$ could be bigger than $k - t$. For example, if we let \mathcal{F} be the family of claw-free graphs, then $f_{\mathcal{F}}(3, 1) = 5$ (Theorem 1.3); if we let \mathcal{F} be the family of polyhedral maps, then $f_{\mathcal{F}}(3, 1) = 3$ (Theorem 1.5). Particularly, if let \mathcal{F} be the family of plane graphs, then $f_{\mathcal{F}}(3, 0) = 5$ ([26, 27]), and $f_{\mathcal{F}}(4, 0) = f_{\mathcal{F}}(4, 1) = f_{\mathcal{F}}(4, 2) = \infty$ ([29, 24, 28]) which implies $f_{\mathcal{F}}(5, t) = \infty$ for any $t \in \{0, 1, 2, 3\}$. Note that the connectivity of plane graphs is at most 5. Therefore, the exact value of $f_{\mathcal{F}}(k, t)$ for plane graphs has been determined. It would be interesting to study $f_{\mathcal{F}}(k, t)$ for other families of graphs as well.

The cyclability for graphs embedded in surfaces also deserves to be further explored. It would be interesting to determine $f_{\mathcal{F}}(k, t)$ when \mathcal{F} is the family of graphs embedded in a closed surface Σ , or the family of triangulations of a closed surface Σ . Note that, with only finitely many exceptions, a graph embedded in a surface has average degree less than 7. Therefore, if \mathcal{F} is an infinite family of k -connected graphs embedded in a surface, then k could only be a small integer between 2 and 6.

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