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COLORFUL COVERINGS OF POLYTOPES AND PIERCING NUMBERS OF COLORFUL d-INTERVALS

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We prove a common strengthening of Bárány's colorful Carathéodory theorem and the KKMS theorem. In fact, our main result is a colorful polytopal KKMS theorem, which extends a colorful KKMS theorem due to Shih and Lee [*Math. Ann.* 296 (1993), no. 1, 35–61] as well as a polytopal KKMS theorem due to Komiya [*Econ. Theory* 4 (1994), no. 3, 463–466]. The (seemingly unrelated) colorful Carathéodory theorem is a special case as well. We apply our theorem to establish an upper bound on the piercing number of colorful d-interval hypergraphs, extending earlier results of Tardos [*Combinatorica* 15 (1995), no. 1, 123–134] and Kaiser [*Discrete Comput. Geom.* 18 (1997), no. 2, 195–203].

1. Introduction

The KKM theorem of Knaster, Kuratowski, and Mazurkiewicz [11] is a set covering variant of Brouwer's fixed point theorem. It states that for any covering of the k-simplex Δ_k on vertex set $[k+1]$ with closed sets $A_1,\dots,A_{k+1}$ such that the face spanned by vertices in S is contained in $\bigcup_{i\in S} A_i$ for every $S\subset[k+1]$, the intersection $\bigcap_{i\in[k+1]} A_i$ is nonempty.

The KKM theorem has inspired many extensions and variants, some of which we will briefly survey in Section 2. Important strengthenings include a colorful extension of the KKM theorem due to Gale [9] that deals with $k+1$ possibly distinct coverings of the k-simplex and the KKMS theorem of Shapley [16], where the sets in the covering are associated to faces of the k-simplex instead of its vertices. Further generalizations of the KKMS theorem are a polytopal version due to Komiya [12] and the colorful KKMS theorem of Shih and Lee [17].

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In this note we prove a colorful polytopal KKMS theorem, extending all results above. This result is finally sufficiently general to also specialize to Bárány's celebrated colorful Carathéodory theorem [5] from 1982, which asserts that if $X_1,\dots,X_{k+1}$ are subsets of $\mathbb{R}^k$ with $0 \in \operatorname{conv} X_i$ for every $i \in [k+1]$, then there exists a choice of points $x_1 \in X_1,\dots,x_{k+1} \in X_{k+1}$ such that $0 \in \operatorname{conv}\{x_1,\dots,x_{k+1}\}$. Carathéodory's classical result is the case $X_1 = X_2 = \dots = X_{k+1}$. We deduce the colorful Carathéodory theorem from our main result in Section 3.

For a set $\sigma \subset \mathbb{R}^k$ we denote by C_σ the *cone of* σ, that is, the union of all rays emanating from the origin that intersect σ. Our main result is the following:

Theorem 1.1. *Let P be a k-dimensional polytope with $0 \in P$. Suppose for every nonempty, proper face σ of P we are given $k+1$ points $y_\sigma^{(1)},\dots,y_\sigma^{(k+1)} \in C_\sigma$ and $k+1$ closed sets $A_\sigma^{(1)},\dots,A_\sigma^{(k+1)} \subset P$. If $\sigma \subset \bigcup_{\tau\subset\sigma} A_\tau^{(j)}$ for every face σ of P and every $j \in [k+1]$, then there exist faces $\sigma_1,\dots,\sigma_{k+1}$ of P such that $0 \in \operatorname{conv}\{y_{\sigma_1}^{(1)},\dots,y_{\sigma_{k+1}}^{(k+1)}\}$ and $\bigcap_{i=1}^{k+1} A_{\sigma_i}^{(i)} \neq \emptyset$.*

Our proof of this result relies on a topological mapping degree argument. As such, it is entirely different from Bárány's proof of the colorful Carathéodory theorem, and thus provides a new topological route to prove this theorem. Our argument is also less involved than the topological proof given recently by Meunier, Mulzer, Sarrabezolles, and Stein [14] to show that algorithmically finding the configuration whose existence is guaranteed by the colorful Carathéodory theorem is in PPAD (that is, informally speaking, it can be found by a path-following algorithm). Our method, however, involves a limiting argument and thus does not have immediate algorithmic consequences. Finally, our proof of Theorem 1.1 exhibits a surprisingly simple way to prove KKMS-type results and their polytopal and colorful extensions.

As an application of Theorem 1.1 we prove a bound on the piercing numbers of colorful d-interval hypergraphs. A *d-interval* is a union of at most d disjoint closed intervals on $\mathbb{R}$. A d-interval h is *separated* if it consists of d disjoint interval components $h = h^1 \cup \dots \cup h^d$ with $h^{i+1} \subset (i, i+1)$ for $i \in \{0,\dots,d-1\}$. A *hypergraph of (separated) d-intervals* is a hypergraph H whose vertex set is $\mathbb{R}$ and whose edge set is a finite family of (separated) d-intervals.

A *matching* in a hypergraph $H = (V, E)$ with vertex set V and edge set E is a set of disjoint edges. A *cover* is a subset of V intersecting all edges. The *matching number* $\nu(H)$ is the maximal size of a matching, and the *covering number* (or *piercing number*) $\tau(H)$ is the minimal size of a cover. Tardos

[19] and Kaiser [10] proved the following bound on the covering number in hypergraphs of d-intervals:

Theorem 1.2 (Tardos [19], Kaiser [10]). *In every hypergraph H of d-intervals we have $\tau(H) \leq (d^2 - d + 1)\nu(H)$. Moreover, if H is a hypergraph of separated d-intervals, then $\tau(H) \leq (d^2 - d)\nu(H)$.*

Matoušek [13] showed that this bound is not far from the truth: There are examples of hypergraphs of d-intervals in which $\tau = \Omega(\frac{d^2}{\log d}\nu)$. Aharoni, Kaiser and Zerbib [1] gave a proof of Theorem 1.2 that used the KKMS theorem and Komiya's polytopal extension, Theorem 2.1. Using Theorem 1.1 we prove here a colorful generalization of Theorem 1.2:

Theorem 1.3. *Let $\mathcal{F}_i$, $i \in [k+1]$, be $k+1$ hypergraphs of d-intervals and let $\mathcal{F} = \bigcup_{i=1}^{k+1} \mathcal{F}_i$.*

1. *If $\tau(\mathcal{F}_i) > k$ for all $i \in [k+1]$, then there exists a collection $\mathcal{M}$ of pairwise disjoint d-intervals in $\mathcal{F}$ of size $|\mathcal{M}| \geq \frac{k+1}{d^2-d+1}$, with $|\mathcal{M} \cap \mathcal{F}_i| \leq 1$.*
2. *If $\mathcal{F}_i$ consists of separated d-intervals and $\tau(\mathcal{F}_i) > kd$ for all $i \in [k+1]$, then there exists a collection $\mathcal{M}$ of pairwise disjoint separated d-intervals in $\mathcal{F}$ of size $|\mathcal{M}| \geq \frac{k+1}{d-1}$, with $|\mathcal{M} \cap \mathcal{F}_i| \leq 1$.*

Note that Theorem 1.2 is the case where all the hypergraphs $\mathcal{F}_i$ are the same. In Section 2 we introduce some notation and, as an introduction to our methods, provide a new simple proof of Komiya's theorem. Then, in Section 3, we prove Theorem 1.1 and use it to derive Bárány's colorful Carathéodory theorem. Section 4 is devoted to the proof of Theorem 1.3.

2. Coverings of polytopes and Komiya's theorem

Let Δ_k be the k-dimensional simplex with vertex set $[k+1]$ realized in $\mathbb{R}^{k+1}$ as $\{x \in \mathbb{R}^{k+1}_{\geq 0} : \sum_{i=1}^{k+1} x_i = 0\}$. For every $S \subset [k+1]$ let Δ^S be the face of Δ_k spanned by the vertices in S. Recall that the KKM theorem asserts that if $A_1, \ldots, A_{k+1}$ are closed sets covering Δ_k so that $\Delta^S \subset \bigcup_{i \in S} A_i$ for every $S \subset [k+1]$, then the intersection of all the sets A_i is non-empty. We will refer to covers $A_1, \ldots, A_{k+1}$ as above as *KKM covers*.

A generalization of this result, known as the KKMS theorem, was proven by Shapley [16] in 1973. Now we have a cover of Δ_k by closed sets A_T, $T \subset [k+1]$, so that $\Delta^S \subset \bigcup_{T \subset S} A_T$ for every $S \subset [k+1]$. Such a collection of sets A_T is called a *KKMS cover*. The conclusion of the KKMS theorem is that there exists a balanced collection $T_1, \ldots, T_m$ of subsets of $[k+1]$ for which

$\bigcap_{i=1}^{m} A_{T_i} \neq \emptyset$. Here $T_1, \dots, T_m$ form a balanced collection if the barycenters of the corresponding faces $\Delta_{T_1}, \dots, \Delta_{T_m}$ contain the barycenter of Δ_k in their convex hull.

A different generalization of the KKM theorem is a colorful version due to Gale [9]. It states that given $k+1$ KKM covers $A_1^{(i)}, \dots, A_{k+1}^{(i)}$, $i \in [k+1]$, of the k-simplex Δ_k, there is a permutation π of $[k+1]$ such that $\bigcap_{i\in[k+1]} A_{\pi(i)}^{(i)}$ is nonempty. This theorem is colorful in the sense that we think of each KKM cover as having a different color; the theorem then asserts that there is an intersection of $k+1$ sets of pairwise distinct colors associated to pairwise distinct vertices. Asada et al. [2] showed that one can additionally prescribe $\pi(1)$.

In 1993 Shih and Lee [17] proved a common generalization of the KKMS theorem and Gale's colorful KKM theorem: Given $k+1$ KKMS covers A_T^i, $T \subset [k+1]$, $i \in [k+1]$, of Δ_k, there exists a balanced collection $T_1, \dots, T_{k+1}$ of subsets of $[k+1]$ for which we have $\bigcap_{i=1}^{m} A_{T_i}^i \neq \emptyset$.

Another far reaching extension of the KKMS theorem to general polytopes is due to Komiya [12] from 1994. Komiya proved that the simplex Δ_k in the KKMS theorem can be replaced by any k-dimensional polytope P, and that the barycenters of the faces can be replaced by any points y_σ in the face σ:

Theorem 2.1 (Komiya's theorem [12]). *Let P be a polytope, and for every nonempty face σ of P choose a point $y_\sigma \in \sigma$ and a closed set $A_\sigma \subset P$. If $\sigma \subset \bigcup_{\tau\subset\sigma} A_\tau$ for every face σ of P, then there are faces $\sigma_1, \dots, \sigma_m$ of P such that $y_P \in \operatorname{conv}\{y_{\sigma_1}, \dots, y_{\sigma_m}\}$ and $\bigcap_{i=1}^{m} A_{\sigma_i} \neq \emptyset$.*

This specializes to the KKMS theorem if P is the simplex and each point y_σ is the barycenter of the face σ. Moreover, there are quantitative versions of the KKM theorem due to De Loera, Peterson, and Su [6] as well as Asada et al. [2] and KKM theorems for general pairs of spaces due to Musin [15].

To set the stage we will first present a simple proof of Komiya's theorem. Recall that the KKM theorem can be easily deduced from Sperner's lemma on vertex labelings of triangulations of a simplex. Our proof of Komiya's theorem – just as Shapley's original proof of the KKMS theorem – first establishes an equivalent Sperner-type version. A *Sperner–Shapley labeling* of a triangulation T of a polytope P is a map $f\colon V(T) \longrightarrow \{\sigma\colon \sigma \text{ a nonempty face of } P\}$ from the vertex set $V(T)$ of T to the set of nonempty faces of P such that $f(v) \subset \operatorname{supp}(v)$, where $\operatorname{supp}(v)$ is the minimal face of P containing v. We prove the following polytopal Sperner–Shapley theorem that will imply Theorem 2.1 by a limiting and compactness argument:

Theorem 2.2. *Let T be a triangulation of the polytope $P \subset \mathbb{R}^k$, and let $f\colon V(T) \longrightarrow \{\sigma\colon \sigma$ a nonempty face of $P\}$ be a Sperner–Shapley labeling of T. For every nonempty face σ of P choose a point $y_\sigma \in \sigma$. Then there is a face τ of T such that $y_P \in \operatorname{conv}\{y_{f(v)}\colon v$ vertex of $\tau\}$.*

Proof. The Sperner–Shapley labeling f maps vertices of the triangulation T of P to faces of P; thus mapping a vertex v to the chosen point $y_{f(v)}$ in the face $f(v)$ and extending linearly onto faces of T defines a continuous map $F\colon P \longrightarrow P$. By the Sperner–Shapley condition for every face σ of P we have that $F(\sigma) \subset \sigma$. This implies that F is homotopic to the identity on ∂P, and thus $F|_{\partial P}$ has degree one. Then F is surjective and we can find a point $x \in P$ such that $F(x) = y_P$. Let τ be the smallest face of T containing x. By the definition of F the image $F(\tau)$ is equal to the convex hull $\operatorname{conv}\{y_{f(v)}\colon v$ vertex of $\tau\}$. ∎

Proof of Theorem 2.1 Let $\varepsilon > 0$, and let T be a triangulation of P such that every face of T has diameter at most ε. Given a cover $\{A_\sigma\colon \sigma$ a nonempty face of $P\}$ that satisfies the covering condition of the theorem we define a Sperner–Shapley labeling in the following way: For a vertex v of T, label v by a face $\sigma \subset \operatorname{supp}(v)$ such that $v \in A_\sigma$. Such a face σ exists since $v \in \operatorname{supp}(v) \subset \bigcup_{\sigma \subset \operatorname{supp}(v)} A_\sigma$. Thus by Theorem 2.2 there is a face τ of T whose vertices are labeled by faces $\sigma_1, \dots, \sigma_m$ of P such that $y_P \in \operatorname{conv}\{y_{\sigma_1}, \dots, y_{\sigma_m}\}$. In particular, the ε-neighborhoods of the sets A_{σ_i}, $i \in [m]$, intersect. Now let ε tend to zero. As there are only finitely many collections of faces of P, one collection $\sigma_1, \dots, \sigma_m$ must appear infinitely many times. By compactness of P the sets A_{σ_i}, $i \in [m]$, then all intersect since they are closed. ∎

Note that Theorem 2.1 is true also if all the sets A_σ are open in P. Indeed, given an open cover $\{A_\sigma\colon \sigma$ a nonempty face of $P\}$ of P as in Theorem 2.1, we can find closed sets $B_\sigma \subset A_\sigma$ that have the same nerve as A_σ (namely, any collection of sets $\{B_{\sigma_i}\colon i \in I\}$ intersects if and only if the corresponding collection $\{A_{\sigma_i}\colon i \in I\}$ intersects) and still satisfy $\sigma \subset \bigcup_{\tau \subset \sigma} B_\tau$ for every face σ of P.

3. A colorful Komiya theorem

Recall that the colorful KKMS theorem of Shih and Lee [17] states the following: If for every $i \in [k+1]$ the collection $\{A^i_\sigma\colon \sigma$ a nonempty face of $\Delta_k\}$ forms a KKMS cover of Δ_k, then there exists a balanced collection of faces $\sigma_1, \dots, \sigma_{k+1}$ so that $\bigcap_{i=1}^{k+1} A^i_{\sigma_i} \neq \emptyset$. Theorem 1.1, proved in this section, is a colorful extension of Theorem 2.1, and thus generalizes the colorful KKMS theorem to any polytope.

Let P be a k-dimensional polytope. Suppose that for every nonempty face σ of P we choose $k+1$ points $y_\sigma^{(1)},\dots,y_\sigma^{(k+1)} \in \sigma$ and $k+1$ closed sets $A_\sigma^{(1)},\dots,A_\sigma^{(k+1)} \subset P$, so that $\sigma \subset \bigcup_{\tau\subset\sigma} A_\tau^{(j)}$ for every face σ of P and every $j \in [k+1]$. Theorem 2.1 now guarantees that for every fixed $j \in [k+1]$ there are faces $\sigma_1^{(j)},\dots,\sigma_{m_j}^{(j)}$ of P such that $y_P^{(j)} \in \operatorname{conv}\{y_{\sigma_1}^{(j)},\dots,y_{\sigma_{m_j}}^{(j)}\}$ and $\bigcap_{i=1}^{m_j} A_{\sigma_i}^{(j)}$ is nonempty. Now let us choose $y_P^{(1)} = y_P^{(2)} = \cdots = y_P^{(k+1)}$ and denote this point by y_P. The colorful Carathéodory theorem implies the existence of points $z_j \in \{y_{\sigma_1}^{(j)},\dots,y_{\sigma_{m_j}}^{(j)}\}$, $j \in [k+1]$, such that $y_P \in \operatorname{conv}\{z_1,\dots,z_{k+1}\}$. Theorem 1.1 shows that this implication can be realized simultaneously with the existence of sets $B_j \in \{A_{\sigma_1}^{(j)},\dots,A_{\sigma_{m_j}}^{(j)}\}$, $j \in [k+1]$, such that $\bigcap_{j=1}^{k+1} B_j$ is nonempty. We prove Theorem 1.1 by applying the Sperner–Shapley version of Komiya's theorem – Theorem 2.2 – to a labeling of the barycentric subdivision of a triangulation of P. The same idea was used by Su [18] to prove a colorful Sperner's lemma. For related Sperner-type results for multiple Sperner labelings see Babson [3], Bapat [4], and Frick, Houston-Edwards, and Meunier [7].

Proof of Theorem 1.1 Let $\varepsilon > 0$, and let T be a triangulation of P such that every face of T has diameter at most ε. We will also assume that the chosen points $y_\sigma^{(1)},\dots,y_\sigma^{(k+1)}$ are contained in σ. This assumption does not restrict the generality of our proof since $0 \in \operatorname{conv}\{x_1,\dots,x_{k+1}\}$ for vectors $x_1,\dots,x_{k+1} \in \mathbb{R}^k$ if and only if $0 \in \operatorname{conv}\{\alpha_1 x_1,\dots,\alpha_{k+1}x_{k+1}\}$ with arbitrary coefficients $\alpha_i > 0$. Denote by T' the barycentric subdivision of T. We now define a Sperner–Shapley labeling of the vertices of T': For $v \in V(T')$ let σ_v be the face of T so that v lies at the barycenter of σ_v, let $\ell = \dim \sigma_v$, and let σ be the minimal supporting face of P containing σ_v. By the conditions of the theorem, v is contained in a set $A_\tau^{(\ell+1)}$ where $\tau \subset \sigma$. We label v by τ. Thus by Theorem 2.2 there exists a face τ of T' (without loss of generality τ is a facet) whose vertices are labeled by faces $\sigma_1,\dots,\sigma_{k+1}$ of P such that $0 \in \operatorname{conv}\{y_{\sigma_1}^{(1)},\dots,y_{\sigma_{k+1}}^{(k+1)}\}$. In particular, the ε-neighborhoods of the sets $A_{\sigma_i}^{(i)}$, $i \in [k+1]$, intersect. Now use a limiting argument as before. ∎

Note that by the same argument as before, Theorem 1.1 is true also if all the sets $A_\sigma^{(i)}$ are open.

For a point $x \neq 0$ in $\mathbb{R}^k$ let $H(x) = \{y \in \mathbb{R}^k : \langle x,y\rangle = 0\}$ be the hyperplane perpendicular to x and let $H^+(x) = \{y \in \mathbb{R}^k : \langle x,y\rangle \geq 0\}$ be the closed halfspace with boundary $H(x)$ containing x. Let us now show that Bárány's colorful Carathéodory theorem is a special case of Theorem 1.1.

Theorem 3.1 (Colorful Carathéodory theorem, Bárány [5]). *Let $X_1,\dots,X_{k+1}$ be finite subsets of $\mathbb{R}^k$ with $0\in\operatorname{conv}X_i$ for every $i\in[k+1]$. Then there are $x_1\in X_1,\dots,x_{k+1}\in X_{k+1}$ such that $0\in\operatorname{conv}\{x_1,\dots,x_{k+1}\}$.*

Proof. We will assume that 0 is not contained in any of the sets $X_1,\dots,X_{k+1}$, for otherwise we are done. Let $P\subset\mathbb{R}^k$ be a polytope containing 0 in its interior, such that if points x and y belong to the same face of P, then $\langle x,y\rangle\geq 0$. For example, a sufficiently fine subdivision of any polytope that contains 0 in its interior (slightly perturbed to be a strictly convex polytope) satisfies this condition. We can assume that any ray emanating from the origin intersects each X_i in at most one point by arbitrarily deleting any additional points from X_i. This will not affect the property that $0\in\operatorname{conv}X_i$. Furthermore, we can choose P in such a way that for each face σ and $i\in[k+1]$ the intersection $C_\sigma\cap X_i$ contains at most one point.

Now for each nonempty, proper face σ of P choose points $y_\sigma^{(i)}$ and sets $A_\sigma^{(i)}$ in the following way: If there exists $x\in C_\sigma\cap X_i$, then let $y_\sigma^{(i)}=x$ and $A_\sigma^{(i)}=\{y\in P\colon \langle y,x\rangle\geq 0\}=P\cap H^+(x)$; otherwise let $y_\sigma^{(i)}$ be some point in σ and let $A_\sigma^{(i)}=\sigma$.

Suppose the statement of the theorem was incorrect; then in particular, we can slightly perturb the vertices of P and those points $y_\sigma^{(i)}$ that were chosen arbitrarily in σ, to make sure that for any collection of points $y_{\sigma_1}^{(1)},\dots,y_{\sigma_{k+1}}^{(k+1)}$ and any subset S of this collection of size at most k, $0\notin\operatorname{conv}S$.

Let us now check that with these definitions the conditions of Theorem 1.1 hold. Clearly, all the sets $A_\sigma^{(i)}$ are closed. The fact that P is covered by the sets $A_\sigma^{(i)}$ for every fixed i follows from the condition $0\in\operatorname{conv}X_i$. Indeed, this condition implies that for every $p\in P$ there exists a point $x\in X_i$ with $\langle p,x\rangle\geq 0$, and therefore, for the face σ of P for which $x\in C_\sigma$ we have $p\in A_\sigma^{(i)}$.

Now fix a proper face σ of P. We claim that $\sigma\subset A_\sigma^{(i)}$ for every i. Indeed, either $X_i\cap C_\sigma=\emptyset$ in which case $A_\sigma^{(i)}=\sigma$, or otherwise, pick $x\in X_i\cap C_\sigma$ and let $\lambda>0$ such that $\lambda x\in\sigma$; then for every $p\in\sigma$ we have $\langle p,\lambda x\rangle\geq 0$ by our assumption on P, and thus $\langle p,x\rangle\geq 0$, or equivalently $p\in A_\sigma^{(i)}$.

Thus by Theorem 1.1 there exist faces $\sigma_1,\dots,\sigma_{k+1}$ of P such that $\bigcap_{i=1}^{k+1}A_{\sigma_i}^{(i)}\neq\emptyset$ and $0\in\operatorname{conv}\{y_{\sigma_1}^{(1)},\dots,y_{\sigma_{k+1}}^{(k+1)}\}$. We claim that $\bigcap_{i=1}^{k+1}A_{\sigma_i}^{(i)}$ can contain only the origin. Indeed, suppose that $0\neq x_0\in\bigcap_{i=1}^{k+1}A_{\sigma_i}^{(i)}$. Fix $i\in[k{+}1]$. If $y_{\sigma_i}^{(i)}\in C_{\sigma_i}\cap X_i$, then since $x_0\in A_{\sigma_i}^{(i)}$ we have $y_{\sigma_i}^{(i)}\in H^+(x_0)$ by definition. Otherwise $x_0\in A_{\sigma_i}^{(i)}=\sigma_i$ and $y_{\sigma_i}^{(i)}\in\sigma_i$, so by our choice of P we obtain again that $y_{\sigma_i}^{(i)}\in H^+(x_0)$. Thus all the points $y_{\sigma_1}^{(1)},\dots,y_{\sigma_{k+1}}^{(k+1)}$ are in $H^+(x_0)$. But since $0\in\operatorname{conv}\{y^{(1)}_{\sigma_1},\dots,y^{(k+1)}_{\sigma_{k+1}}\}$ this implies that the convex hull of the points in $\{y^{(1)}_{\sigma_1},\dots,y^{(k+1)}_{\sigma_{k+1}}\}\cap H(x_0)$ contains the origin. Now, the dimension of $H(x_0)$ is $k-1$, and thus by Carathéodory's theorem there exists a set S of at most k of the points in $y^{(1)}_{\sigma_1},\dots,y^{(k+1)}_{\sigma_{k+1}}$ with $0\in\operatorname{conv}S$, in contradiction to our general position assumption.

We have shown that $\bigcap_{i=1}^{k+1}A^{(i)}_{\sigma_i}=\{0\}$, and thus in particular, $A^{(i)}_{\sigma_i}\neq\sigma_i$ for all i. By our definitions, this implies $y^{(i)}_{\sigma_i}\in X_i$ for all i, concluding the proof of the theorem. ∎

Remark 3.2. Note that we could have avoided the usage of Carathéodory's theorem in the proof of Theorem 3.1 by taking a more restrictive assumption on the polytope P, namely, that $\langle x,y\rangle>0$ whenever the points x and y belong to the same face of P. Therefore, in particular, Theorem 3.1 specializes to Carathéodory's theorem in the case where all the sets X_i are the same.

4. A colorful d-interval theorem

Recall that a *fractional matching* in a hypergraph $H=(V,E)$ is a function $f\colon E\longrightarrow\mathbb{R}_{\geq 0}$ satisfying $\sum_{e\colon e\ni v}f(e)\leq 1$ for all $v\in V$. A *fractional cover* is a function $g\colon V\longrightarrow\mathbb{R}_{\geq 0}$ satisfying $\sum_{v\colon v\in e}g(v)\geq 1$ for all $e\in E$. The *fractional matching number* $\nu^*(H)$ is the maximum of $\sum_{e\in E}f(e)$ over all fractional matchings f of H, and the *fractional covering number* $\tau^*(H)$ is the minimum of $\sum_{v\in V}g(v)$ over all fractional covers g. By linear programming duality, $\nu\leq\nu^*=\tau^*\leq\tau$. A *perfect fractional matching* in H is a fractional matching f in which $\sum_{e\colon v\in e}f(e)=1$ for every $v\in V$. It is a simple observation that a collection of sets $\mathcal{I}\subset 2^{[k+1]}$ is balanced if and only if the hypergraph $H=([k+1],\mathcal{I})$ has a perfect fractional matching (see e.g., [1]). The *rank* of a hypergraph $H=(V,E)$ is the maximal size of an edge in H. H is d-partite if there exists a partition $V_1,\dots,V_d$ of V such that $|e\cap V_i|=1$ for every $e\in E$ and $i\in[d]$.

For the proof of Theorem 1.3 we will use the following theorem by Füredi.

Theorem 4.1 (Füredi [8]). *If H is a hypergraph of rank $d\geq 2$, then $\nu(H)\geq\frac{\nu^*(H)}{d-1+\frac{1}{d}}$. If, in addition, H is d-partite, then $\nu(H)\geq\frac{\nu^*(H)}{d-1}$.*

We will also need the following simple counting argument.

Lemma 4.2. *If a hypergraph $H=(V,E)$ of rank d has a perfect fractional matching, then $\nu^*(H)\geq\frac{|V|}{d}$.*

Proof. Let $f\colon E \longrightarrow \mathbb{R}_{\geq 0}$ be a perfect fractional matching of H. Then $\sum_{v\in V}\sum_{e\colon v\in e} f(e) = \sum_{v\in V} 1 = |V|$. Since $f(e)$ was counted $|e|\leq d$ times in this equation for every edge $e\in E$, we have that $\nu^*(H)\geq \sum_{e\in E} f(e)\geq \frac{|V|}{d}$. ∎

We are now ready to prove Theorem 1.3. The proof is an adaption of the methods in [1]. For the first part we need the simplex version of Theorem 1.1, which was already proven by Shih and Lee [17], while the second part requires our more general polytopal extension.

Proof of Theorem 1.3. For a point $\vec{x} = (x_1,\dots,x_{k+1}) \in \Delta_k$ let $p_{\vec{x}}(j) = \sum_{t=1}^{j} x_t \in [0,1]$. Since $\mathcal{F}$ is finite, by rescaling $\mathbb{R}$ we may assume that $\mathcal{F} \subset (0,1)$. For every $T \subset [k+1]$ let A^i_T be the set consisting of all $\vec{x} \in \Delta_k$ for which there exists a d-interval $f\in\mathcal{F}_i$ satisfying:

(a) $f \subset \bigcup_{j\in T}(p_{\vec{x}}(j-1), p_{\vec{x}}(j))$, and
(b) $f \cap (p_{\vec{x}}(j-1), p_{\vec{x}}(j)) \neq \emptyset$ for each $j\in T$.

Note that $A^i_T = \emptyset$ whenever $|T|>d$.

Clearly, the sets A^i_T are open. The assumption $\tau(\mathcal{F}_i)>k$ implies that for every $\vec{x}=(x_1,\dots,x_{k+1})\in\Delta_k$, the set $P(\vec{x})=\{p_{\vec{x}}(j)\colon j\in[k]\}$ is not a cover of $\mathcal{F}_i$, meaning that there exists $f\in\mathcal{F}_i$ not containing any $p_{\vec{x}}(j)$. This, in turn, means that $\vec{x}\in A^i_T$ for some $T\subseteq[k+1]$, and thus the sets A^i_T form a cover of Δ_k for every $i\in[k+1]$.

To show that this is a KKMS cover, let Δ^S be a face of Δ_k for some $S\subset[k+1]$. If $\vec{x}\in\Delta^S$ then $(p_{\vec{x}}(j-1), p_{\vec{x}}(j)) = \emptyset$ for $j\notin S$, and hence it is impossible to have $f\cap(p_{\vec{x}}(j-1), p_{\vec{x}}(j))\neq\emptyset$. Thus $\vec{x}\in A^i_T$ for some $T\subseteq S$. This proves that $\Delta^S\subseteq\bigcup_{T\subseteq S} A^i_T$ for all $i\in[k+1]$.

By Theorem 1.1 there exists a balanced collection of sets $\mathcal{T} = \{T_1,\dots,T_{k+1}\}$ of subsets of $[k+1]$, satisfying $\bigcap_{i=1}^{k+1} A^i_{T_i}\neq\emptyset$. In particular, $|T_i|\leq d$ for all i. (Recall that we think of a collection of sets $\mathcal{I}\subset 2^{[k+1]}$ as faces of the k-dimensional simplex to apply the earlier geometric definition of balancedness.) Then by the observation mentioned above, the hypergraph $H=([k+1],\mathcal{T})$ of rank d has a perfect fractional matching, and thus by Lemma 4.2 we have $\nu^*(H)\geq\frac{k+1}{d}$. Therefore, by Theorem 4.1, $\nu(H)\geq\frac{\nu^*(H)}{d-1+\frac{1}{d}}\geq\frac{k+1}{d^2-d+1}$.

Let M be a matching in H of size $m\geq\frac{k+1}{d^2-d+1}$. Let $\vec{x}\in\bigcap_{i=1}^{k+1}A^i_{T_i}$. For every $i\in[k+1]$ let $f(T_i)$ be the d-interval of $\mathcal{F}_i$ witnessing the fact that $\vec{x}\in A^i_{T_i}$. Then the set $\mathcal{M}=\{f(T_i)\,|\,T_i\in M\}$ is a matching of size m in $\mathcal{F}$ with $|\mathcal{M}\cap\mathcal{F}_i|\leq 1$. This proves the first assertion of the theorem.

Now suppose that $\mathcal{F}_i$ is a hypergraph of separated d-intervals for all $i\in[k+1]$. For $f\in\mathcal{F}$ let $f^t\subset(t-1,t)$ be the t-th interval component of f. We can assume without loss of generality that f^t is nonempty. Let $P=(\Delta_k)^d$. For a d-tuple $T=(j_i,\dots,j_d)\subset[k+1]^d$ let A^i_T consist of all $\vec{X}=\vec{x}^1\times\dots\times\vec{x}^d\in P$ for which there exists $f\in\mathcal{F}_i$ satisfying $f^t\subset(t-1+p_{\vec{x}^t}(j_t-1),t-1+p_{\vec{x}^t}(j_t))$ for all $t\in[d]$.

Since $\tau(\mathcal{F})>kd$, the points $t-1+p_{\vec{x}^t}(j), t\in[d], j\in[k]$, do not form a cover of $\mathcal{F}$. Therefore, as before, the sets A^i_T are open and satisfy the covering condition of Theorem 1.1. Thus, by Theorem 1.1, there exists a set $\mathcal{T}=\{T_1,\dots,T_{k+1}\}$ of d-tuples in $[k+1]^d$ containing the point $(\frac{1}{k+1},\dots,\frac{1}{k+1})\times\dots\times(\frac{1}{k+1},\dots,\frac{1}{k+1})\in P$ in its convex hull and satisfying $\bigcap_{i\in[k+1]}A^i_{T_i}\neq\emptyset$. Then the d-partite hypergraph $H=(\bigcup_{i=1}^d V_i,\mathcal{T})$, where $V_i=[k+1]$ for all i, has a perfect fractional matching, and hence by Lemma 4.2 we have $\nu^*(H)\geq k+1$. By Theorem 4.1, this implies $\nu(H)\geq\frac{\nu^*(H)}{d-1}\geq\frac{k+1}{d-1}$. Now, by the same argument as before, by taking $\vec{X}\in\bigcap_{i\in[k+1]}A^i_{T_i}$ we obtain a matching in $\mathcal{F}$ of the same size as a maximal matching in H, concluding the proof of the theorem. ∎

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