

THE SUB-EXPONENTIAL TRANSITION FOR THE
CHROMATIC GENERALIZED RAMSEY NUMBERS

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A simple graph-product type construction shows that for all natural numbers $r \geq q$, there exists an edge-coloring of the complete graph on 2^r vertices using r colors where the graph consisting of the union of any q color classes has chromatic number 2^q . We show that for each fixed natural number q , if there exists an edge-coloring of the complete graph on n vertices using r colors where the graph consisting of the union of any q color classes has chromatic number at most $2^q - 1$, then n must be sub-exponential in r . This answers a question of Conlon, Fox, Lee, and Sudakov.

1. Introduction

The *Ramsey number* of a graph G is defined as the minimum integer n for which every edge two-coloring of K_n , the complete graph on n vertices, admits a monochromatic copy of G . Ramsey's theorem asserts that the Ramsey number of the complete graph K_k is finite for all natural numbers k . It is a fundamental result in combinatorics and its influence extends to various other fields of mathematics.

Several variants of the Ramsey number have been suggested since its introduction. Let p and q be positive integers satisfying $p \geq 2$ and $2 \leq q \leq \binom{p}{2}$. The *generalized Ramsey number* $F(r, p, q)$ is defined as the minimum n such that for every edge r -coloring of K_n there exists a set of p vertices having at most $q - 1$ distinct colors on the edges with both endpoints in the set. Note that $F(2, p, 2)$ is equivalent to the Ramsey number of K_p . Generalized

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Ramsey numbers encode several interesting problems in combinatorics in one function, and are closely connected to important problems such as the Hales-Jewett theorem, the $(6, 3)$ -problem, quasirandom graphs, and Ramsey-coloring of hypergraphs. It was introduced by Erdős and Shelah [4,5] around 40 years ago and then systematically studied by Erdős and Gyárfás [6].

Note that the definition trivially implies $F(r, p, q) \leq F(r, p, q')$ for $q' \leq q$. Erdős and Gyárfás proved a number of interesting results about the function $F(r, p, q)$, demonstrating how for fixed p , the function falls off from being at least exponential in r when $q = 2$ to being about $\sqrt{2r}$ when $q = \binom{p}{2}$ (for $p \geq 4$). In the process, they observed that the generalized Ramsey numbers satisfy the recurrence relation $F(r, p, q) \leq rF(r, p-1, q-1)$ for all $r \geq 2$ and $p, q \geq 3$. To prove the recurrence relation, suppose that $N \geq rF(r, p-1, q-1)$ for some $r \geq 2$, $p, q \geq 3$ and consider an edge r -coloring of K_N . Fix a vertex v , and note that by the pigeonhole principle, there exists a set X of size at least $\lceil \frac{N-1}{r} \rceil \geq F(r, p-1, q-1)$ for which all edges connecting v to X are of the same color. By definition, X contains a set of $p-1$ vertices having at most $q-2$ colors inside. Together with v , this set gives a set of p vertices having at most $q-1$ colors inside. Therefore $F(r, p, q) \leq N = rF(r, p-1, q-1)$. Note that since $F(r, 2, 2) = 2$, this relation implies $F(r, p, p) \leq 2r^{q-1}$ for all $p \geq 2$. Building on the work of Mubayi [8], and Eichhorn and Mubayi [3], recently Conlon, Fox, Lee, and Sudakov [2] showed that $F(r, p, q)$ is super-polynomial in r for all $q \leq p-1$. Hence a super-polynomial to polynomial transition of the generalized Ramsey numbers occurs at $p = q$. Understanding such transitions is connected to many interesting problems in Ramsey theory. See [1] for further information.

In another related work, Conlon, Fox, Lee, and Sudakov [1] introduced the following chromatic number version of generalized Ramsey numbers.

Definition 1.1. Let p and q be positive integers satisfying $p \geq 3$ and $2 \leq q \leq \binom{p}{2}$. For each positive integer r , define $F_\chi(r, p, q)$ as the minimum integer n for which every edge-coloring of K_n with r colors contains a p -chromatic subgraph receiving at most $q-1$ distinct colors on its edges.

An edge-coloring of the complete graph is a *chromatic- (p, q) -coloring* if the union of any $q-1$ color classes has chromatic number at most $p-1$. One can alternatively define $F_\chi(r, p, q)$ as the minimum n for which there does not exist a chromatic- (p, q) -coloring of K_n . While the former definition illuminates the Ramsey-type nature of the function and the connection between $F(r, p, q)$ and $F_\chi(r, p, q)$ more clearly (for example, it immediately implies $F_\chi(r, p, q) \leq F(r, p, q)$ for all r, p, q), the latter highlights the essence of the function and is arguably a more natural definition. Conlon, Fox, Lee, and

Sudakov realized the importance of chromatic- (p, q) -colorings while studying a question on generalized Ramsey numbers related to the Hales-Jewett theorem. They established some similarities and differences between $F(r, p, q)$ and $F_\chi(r, p, q)$, and suggested to study the two functions in more depth.

Suppose that there exists a chromatic- (p, q) -coloring of K_n using r colors. Partition the color set into $\lceil \frac{r}{q-1} \rceil$ sets each of size at most $q-1$ and note that each set induces a graph of chromatic number at most $p-1$. By the product formula of chromatic numbers, we see that $F_\chi(r, p, q) - 1 \leq (p-1)^{\lceil \frac{r}{q-1} \rceil}$. Hence we see that the function $F_\chi(r, p, q)$ has a natural exponential upper bound. Quite surprisingly, this simple bound turns out to be tight for some choice of parameters. Consider a complete graph on the vertex set $\{0, 1, \dots, 2^r - 1\}$ and color the edge $\{v, w\}$ with color $i \in [r]$, if the binary expansions of v and w first differ in the i -th digit. Since each color class induces a bipartite graph, we see that for all $q \leq r$, any union of q color classes induce a graph of chromatic number at most 2^q (one can in fact check that the chromatic number is exactly 2^q). Thus $F_\chi(r, 2^q + 1, q + 1) - 1 \geq 2^r$, and together with the upper bound established above, we see that $F_\chi(r, 2^q + 1, q + 1) = 2^r + 1$ whenever r is divisible by q . Conlon, Fox, Lee, and Sudakov asked whether these values of (p, q) are the ‘thresholds’ for $F_\chi(r, p, q)$ being exponential in r . More precisely, they asked whether $F_\chi(r, 2^q, q + 1) = 2^{o(r)}$ for all $q \geq 2$, and proved that this indeed is the case for $q = 2$. In this paper, we positively answer their question. That is, we prove that there is an $n = 2^{o(r)}$ for which any coloring of K_n with r colors contains a 2^q -chromatic subgraph with at most q colors.

Theorem 1.2. $F_\chi(r, 2^q, q + 1) = 2^{o(q)}$ for all $q \geq 2$.

In fact, we prove a slightly stronger statement asserting that for all $q \geq 2$, there exists a constant c_q such that $F_\chi(r, 2^q, q + 1) \leq 2^{c_q r^{1-1/q} (\log r)^q}$. As noted by Conlon, Fox, Lee, and Sudakov, Theorem 1.2 establishes for each fixed p , the maximum value of q for which $F_\chi(r, p, q)$ is exponential in r . To see this, suppose that $2^{d-1} < p \leq 2^d$ for some natural number d . As observed above, we have $F_\chi(r, p, d) \geq F_\chi(r, 2^{d-1} + 1, d) = 2^r$. On the other hand, Theorem 1.2 implies $F_\chi(r, p, d + 1) \leq F_\chi(r, 2^d, d + 1) = 2^{o(r)}$. Hence $F_\chi(r, p, q)$ is exponential in r if and only if $q \leq \lceil \log p \rceil$.

The rest of the paper is organized as follows. In Section 2 we prove some lemmas that will be repeatedly used throughout the paper. We first prove the $q = 3$ case of Theorem 1.2 in Section 3 to illustrate the main ideas of our proof, and then prove the remaining cases in Section 4. We conclude with some remarks in Section 5.

Notation. A graph $G = (V, E)$ is given by a pair of vertex set V and edge set E . The *density* of graph is defined as the fraction of pairs of distinct vertices that form an edge, i.e., it is $\frac{|E|}{\binom{|V|}{2}}$. For a family of sets \mathcal{W} , define

$\text{Vol}(\mathcal{W}) := \left| \bigcup_{W \in \mathcal{W}} W \right|$. Hence if \mathcal{W} consists of disjoint sets, then $\text{Vol}(\mathcal{W}) = \sum_{W \in \mathcal{W}} |W|$. We use \log to denote natural logarithm. We use subscripts such as $R_{2.2}$ to denote the constant R from Theorem/Lemma/Proposition 2.2.

2. Preliminaries

Fix a graph G . We say that a pair of vertex subsets (V_1, V_2) is *balanced* if $|V_1| = |V_2|$. For a positive real number ε , a pair of disjoint vertex subsets (V_1, V_2) is ε -*dense* if for every pair of subsets $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$ satisfying $|U_1| \geq \varepsilon|V_1|$ and $|U_2| \geq \varepsilon|V_2|$, we have $e(U_1, U_2) > 0$. The following lemma asserts that every graph of large density contains a dense pair. The logarithmic factor in the exponent can be removed by using a more detailed analysis such as that used by Peng, Rödl, and Ruciński [9], but we chose to provide a slightly weaker version for the sake of simplicity.

Lemma 2.1. *There exists ε_0 such that the following holds for all positive real numbers ε and d satisfying $0 < \varepsilon < \varepsilon_0$ and $0 < d < 1$. If G is an n -vertex graph of edge density at least d , then it contains a balanced ε -dense pair (X, Y) for which*

$$|X| = |Y| \geq nd^{3 \log(1/\varepsilon)/\varepsilon}.$$

Proof. For simplicity, assume that n is even (the odd case can be similarly handled). Let G be an n -vertex graph of density at least d . Let $V_1 \cup V_2$ be a bipartition of $V(G)$ satisfying $|V_1| = |V_2| = \frac{n}{2}$ chosen uniformly at random. Then the probability of a fixed edge crossing the partition is $\frac{2 \binom{n-2}{n/2-1}}{\binom{n}{n/2}} = \frac{n}{2(n-1)}$. Therefore by linearity of expectation, we have

$\mathbf{E}[e(V_1, V_2)] = \frac{n}{2(n-1)} \binom{n}{2} d = \frac{n^2}{4} d$. Hence there exists a particular partition $V_1 \cup V_2$ for which $e(V_1, V_2) \geq \frac{n^2}{4} d$. Fix the bipartite graph induced on the pair of sets (V_1, V_2) . As established above, it has density at least d .

Let ρ be the minimum positive real number for which there exists a pair $W_1 \subseteq V_1$ and $W_2 \subseteq V_2$ such that $m = |W_1| = |W_2| \geq \rho|V_1|$ and $e(W_1, W_2) \geq \rho^{-\varepsilon/(3 \log(1/\varepsilon))} d |W_1| |W_2|$. Note that we are taking a minimum over a non-empty set since the pair (V_1, V_2) satisfies the condition with $\rho = 1$. Further, since $e(W_1, W_2) \leq |W_1| |W_2|$, we have $\rho^{-\varepsilon/(3 \log(1/\varepsilon))} d \leq 1$, implying

$\rho \geq d^{3\log(1/\varepsilon)/\varepsilon}$. Hence the lemma immediately holds if the pair (W_1, W_2) is ε -dense. Otherwise, there exists a pair of subsets (U_1, U_2) such that $U_i \subseteq W_i$, $|U_i| \geq \varepsilon m$ for both $i = 1, 2$, and $e(U_1, U_2) = 0$. By abusing notation, we take subsets if necessary and assume that both U_1 and U_2 have sizes exactly $\lceil \varepsilon m \rceil$. Define $\alpha m = \lceil \varepsilon m \rceil$. If $\alpha m \geq \frac{m}{3}$, then $\frac{m}{3} \leq \alpha m \leq \varepsilon m + 1$ implies $m \leq 3$ if $\varepsilon < \frac{1}{12}$. Since (W_1, W_2) is not ε -dense, it is not complete. Thus we can take a dense pair (X, Y) consisting of a single edge satisfying $|X| = |Y| = 1 \geq \frac{m}{3} \geq \frac{\rho}{3}|V_1|$ and increase the density by a multiplicative factor at least $\frac{9}{8}$. However, this contradicts the minimality of ρ since $3^{\varepsilon/(3\log(1/\varepsilon))} < \frac{9}{8}$ for sufficiently small ε (say $\varepsilon < \frac{1}{4}$). Thus we may assume that $\alpha m < \frac{m}{3}$, i.e., $\alpha < \frac{1}{3}$. Define $U'_1 = W_1 \setminus U_1$ and $U'_2 = W_2 \setminus U_2$. By the minimality of ρ , we see that

$$e(U'_1, U'_2) \leq ((1 - \alpha)\rho)^{-\varepsilon/(3\log(1/\varepsilon))} d|U'_1||U'_2|$$

and

$$e(U_1, U'_2) \leq (\alpha\rho)^{-\varepsilon/(3\log(1/\varepsilon))} d|U_1||U'_2|,$$

where the second inequality can be obtained by taking the average over all balanced pairs (U_1, U''_2) with $U''_2 \subseteq U'_2$. Hence

$$\begin{aligned} e(W_1, W_2) &= e(U_1, U_2) + e(U'_1, U_2) + e(U_1, U'_2) + e(U'_1, U'_2) \\ &\leq 0 + 2\alpha(1 - \alpha)m^2 \cdot (\alpha\rho)^{-\varepsilon/(3\log(1/\varepsilon))} d \\ &\quad + (1 - \alpha)^2 m^2 \cdot ((1 - \alpha)\rho)^{-\varepsilon/(3\log(1/\varepsilon))} d. \end{aligned}$$

Since $\alpha < \frac{1}{3}$, we have $2\alpha(1 - \alpha) < (1 - \alpha)^2$. Further, $\alpha^{-\varepsilon/(3\log(1/\varepsilon))} + (1 - \alpha)^{-\varepsilon/(3\log(1/\varepsilon))}$ is decreasing in α in the range $\alpha < \frac{1}{2}$. Thus we may substitute $\alpha = \varepsilon$ to obtain an upper bound on $e(W_1, W_2)$. Since $e(W_1, W_2) \geq \rho^{-\varepsilon/(3\log(1/\varepsilon))} dm^2$, we see that

$$\begin{aligned} 1 &\leq 2\varepsilon(1 - \varepsilon)(\varepsilon)^{-\varepsilon/(3\log(1/\varepsilon))} + (1 - \varepsilon)^2(1 - \varepsilon)^{-\varepsilon/(3\log(1/\varepsilon))} \\ &= 2\varepsilon(1 - \varepsilon) \left(1 + \frac{\varepsilon}{3} + O(\varepsilon^2) \right) + (1 - 2\varepsilon + \varepsilon^2) \left(1 + \frac{\varepsilon^2}{3\log(1/\varepsilon)} + O(\varepsilon^{5/2}) \right) \\ &= \left(2\varepsilon - \frac{4}{3}\varepsilon^2 + O(\varepsilon^3) \right) + \left(1 - 2\varepsilon + \varepsilon^2 + \frac{\varepsilon^2}{3\log(1/\varepsilon)} + O(\varepsilon^{5/2}) \right) \\ &= 1 - \frac{1}{3}\varepsilon^2 + O(\varepsilon^2 \log^{-1}(1/\varepsilon)), \end{aligned}$$

where the asymptotics is taken as $\varepsilon \rightarrow 0$. The inequality above gives a contradiction for sufficiently small ε . ■

We also need a technical lemma that will be repeatedly used throughout the rest of this paper.

Lemma 2.2. *There exists a positive real R such that the following holds for all $r \geq R$ and $\varepsilon \leq \frac{1}{2}$. If X is a set of size n , and $X_1, \dots, X_r \subset X$ are subsets of size at least $(1-\varepsilon)n$, then there exists a set $I \subseteq [r]$ of size $|I| = \frac{r}{4}$ such that $|\bigcap_{i \in I} X_i| \geq (1-2\varepsilon)^{r/4}n$.*

Proof. For each $x \in X$, let $d(x)$ denote the number of subsets X_i containing x . Then

$$\sum_{x \in X} d(x) = \sum_i |X_i| \geq r(1-\varepsilon)n.$$

Now for each set $I \subseteq [r]$ of size $\frac{r}{4}$, let $X_I = \bigcap_{i \in I} X_i$. Then we have

$$\sum_{\substack{I \subseteq [r] \\ |I| = \frac{r}{4}}} |X_I| = \sum_{x \in X} \binom{d(x)}{r/4} \geq n \binom{r(1-\varepsilon)}{r/4} \geq (1-2\varepsilon)^{r/4}n \binom{r}{r/4},$$

where the first inequality holds by convexity since $(1-\varepsilon)r$ is sufficiently large, and the second inequality holds since $\binom{r(1-\varepsilon)}{r/4} / \binom{r}{r/4} \geq \left(\frac{r(1-\varepsilon)-r/4}{3r/4}\right)^{r/4} \geq (1-2\varepsilon)^{r/4}$. Thus, there exists some set I with $|X_I| \geq (1-2\varepsilon)^{r/4}n$, as desired. ■

3. Upper bound on $F_\chi(r, 8, 4)$

In this section, we prove the first case of our main theorem to illustrate the important ideas in a simpler form. The idea behind many results in Ramsey theory can be summarized as follows: given an r -coloring of the complete graph, find a subset of vertices in which less number of colors appear, and then repeat until no colors remain at which point the graph should be empty. For example, the upper bound on $F_\chi(r, 4, 3)$ of Conlon, Fox, Lee, and Sudakov was based on this idea. By utilizing the concept of ε -dense pairs, they were able to show that the recursion as above occurs. The straightforward generalization of their approach to chromatic- $(8, 4)$ -coloring fails because the structure one needs in order to force the recursion as above, a well-organized collection of ε -dense pairs, does not necessarily exist. Our key observation is that if such structure does not appear, then we can find a subset of vertices in which a large number of colors are ‘extremely’ sparse. Our proof is based on a modified version of the recursion given above with an extra flexibility that allows us to work with such sparse colors. The following definitions formalize and quantifies the notion of sparseness that we will use.

Definition 3.1. Let $x, \varepsilon \in [0, 1]$ be real numbers and r_1, r be natural numbers. Given an edge-coloring of K_n , we say that a color c is (x, ε) -sparse if the subgraph consisting of edges of color c has no balanced ε -dense pairs with parts of size at least xn . An edge-coloring of K_n is (r_1, r, x, ε) -restricted if the edge-coloring uses r colors, out of which exactly $r - r_1$ colors are (x, ε) -sparse.

Note that a color is $(0, \varepsilon)$ -sparse only if there are no edges of that color. Define $G(r_1, r, x, \varepsilon)$ as the minimum n such that every (r_1, r, x, ε) -restricted edge-coloring of K_n contains an 8-chromatic subgraph receiving at most 3 distinct colors on its edges. Note that we can recast the problem of determining $F_\chi(r, 8, 4)$ using this notation since $F_\chi(r, 8, 4) = G(r, r, 0, \varepsilon)$ for any ε . We now present the key lemma needed to prove our result. It provides a recursive formula for the $G(r_1, r, x, \varepsilon)$ functions described above.

Lemma 3.2. *There exists a constant R such that the following holds. Suppose that r_1, r are non-negative integers satisfying $r_1 \geq R$, and x, ε are positive real numbers satisfying $x \leq 1$. Then, for every $(r_1, r, x, \varepsilon^2)$ -restricted chromatic- $(8, 4)$ -coloring of K_n , either (1) there exists a subset of $\beta\alpha_0 n$ vertices on which the coloring is $(\frac{31r_1}{32}, r, \beta^{-1}\alpha_0^{-1} \max(\alpha_1, x), \varepsilon^2)$ -restricted, or (2) a subset of $\beta\alpha_1\alpha_0 n$ vertices on which the coloring is $(\frac{31r}{32}, \frac{31r}{32}, 0, \varepsilon^2)$ -restricted, where $\alpha_1 = e^{-100 \log^2(r)r^{2/3}}$, $\alpha_0 = r^{-3 \log(1/\varepsilon^2)/\varepsilon^2}$, and $\beta = (1 - 32\varepsilon)^{\frac{7}{4}}$.*

Proof. Let $R = 8R_{2,2}$. Suppose that an $(r_1, r, x, \varepsilon^2)$ -restricted chromatic- $(8, 4)$ -coloring of K_n is given. Take a densest color, which we call red, and consider the graph \mathcal{G} induced by its edges. The graph has density d at least $\frac{1}{r}$ and so, by Lemma 2.1, we can obtain an ε^2 -dense pair $V_1 \cup V_{-1}$ with parts of size $m = |V_1| = |V_{-1}| \geq nr^{-3 \log(1/\varepsilon^2)/\varepsilon^2} = n\alpha_0$.

Define C_1 as the set of colors that are not (x, ε^2) -sparse and C_2 as the set of colors that are (x, ε^2) -sparse. Hence $|C_1| = r_1$ and $|C_2| = r - r_1$. Fix some color $c \in C_1$, and let \mathcal{G}_c be the graph consisting of edges of color c . Consider $\mathcal{G}_c[V_1]$, the subgraph of \mathcal{G}_c induced on V_1 . Let $W_{1,j} = V_{1,j} \cup V_{1,-j}$ for $j = 1, 2, \dots, k_1$, be a maximal collection of vertex-disjoint ε -dense pairs with parts of size at least $\alpha_1 m$. Define $\mathcal{L}_c(V_1) = \bigcup_{j \in [k_1]} \{V_{1,j}, V_{1,-j}\}$. Similarly, let $W_{-1,j} = V_{-1,j} \cup V_{-1,-j}$ for $j = 1, 2, \dots, k_{-1}$, be a maximal collection of vertex-disjoint balanced ε -dense pairs in $\mathcal{G}_c[V_{-1}]$ with parts of size at least $\alpha_1 m$. Define $\mathcal{L}_c(V_{-1}) = \bigcup_{j \in [k_{-1}]} \{V_{-1,j}, V_{-1,-j}\}$. We split up the remaining argument into two cases depending on whether there exists a color c such that $\text{Vol}(\mathcal{L}_c(V_i)) \geq 8\varepsilon m$ for both $i = \pm 1$.

Case 1. For every color $c \in C_1$, $\min\{\text{Vol}(\mathcal{L}_c(V_1)), \text{Vol}(\mathcal{L}_c(V_{-1}))\} < 8\varepsilon m$.

The condition of Case 1 and the maximality of the collection of sets $W_{i,j}$ imply that for each color $c \in C_1$, there exists a subset $S_c \subseteq V_1$ or $S_c \subseteq V_{-1}$ of size at least $(1 - 8\varepsilon)m$ in which there is no ε -dense pair with parts of size at least $\alpha_1 m$. Without loss of generality, we may assume that $S_c \subseteq V_1$ for at least $\frac{r_1}{2}$ colors c . By Lemma 2.2, there exists a set $\Gamma \subseteq C_1$ of $\frac{r_1}{8}$ colors such that $S = \bigcap_{c \in \Gamma} S_c$ is of size $|S| \geq (1 - 16\varepsilon)\frac{r_1}{8}m$. By abusing notation, we let Γ be an arbitrary subset of these colors of size $\frac{r_1}{32}$, and S be an arbitrary subset of the intersection of S_c for these $\frac{r_1}{32}$ colors of size exactly $(1 - 32\varepsilon)\frac{r}{4}m = \beta m$.

Consider the coloring of the subgraph $K_S = K_n[S]$ induced on S . Note that all colors in Γ have no ε -dense pair of size at least $\alpha_1 m$ in S . Since $|S| = \beta \alpha_0 n$, each color $c \in \Gamma$ is $(\beta^{-1}\alpha_1, \varepsilon)$ -sparse in K_S . We will use the weaker statement that they are $(\beta^{-1}\alpha_0^{-1}\alpha_1, \varepsilon^2)$ -sparse. Similarly, since the colors in C_2 were (x, ε^2) -sparse in K_n , they are $\beta^{-1}\alpha_0^{-1}x$ -sparse in K_S . Hence, the coloring induced on K_S is a $(\frac{31}{32}r_1, r, \beta^{-1}\alpha_0^{-1}\max(\alpha_1, x), \varepsilon^2)$ -restricted chromatic- $(8, 4)$ -coloring of the complete graph on $|S|$ vertices.

Case 2. There exists a color c_1 in C_1 , which we call blue, such that for $i = \pm 1$, $\text{Vol}(\mathcal{L}_{c_1}(V_i)) \geq 8\varepsilon m$.

Since the given coloring is a chromatic- $(8, 4)$ -coloring, for each color $c \in C_1 \cup C_2$, the graph consisting of edges of colors red, blue, and c is 7-colorable. Thus, we can partition the vertices into independent sets A_1, \dots, A_7 . If some set A_k intersects both V_1 and V_{-1} in more than $\varepsilon^2 m$ vertices, then by the definition of ε^2 -dense pairs, we would have a red edge between a vertex in $A_k \cap V_1$ and a vertex in $A_k \cap V_{-1}$, violating properness of the 7-coloring. Thus, either V_1 or V_{-1} , without loss of generality V_1 , intersects at least 4 of the A_i , say A_1, A_2, A_3, A_4 , in at most $\varepsilon^2 m$ vertices. Let I_c be the indices i for which $|W_{1,i} \cap A_k| \geq \varepsilon |W_{1,i}|$ for some $k \in \{1, 2, 3, 4\}$. Then, we have

$$\varepsilon \left| \bigcup_{i \in I_c} W_{1,i} \right| \leq \left| \bigcup_{i \in I_c} (W_{1,i} \cap \bigcup_{1 \leq k \leq 4} A_k) \right| \leq 4\varepsilon^2 m.$$

We can then deduce that

$$\left| \bigcup_{i \in I_c} W_{1,i} \right| \leq 4\varepsilon m.$$

For $i \notin I_c$, consider the ε -dense pair $W_{1,i} = V_{1,i} \cup V_{1,-i}$. For each $k = 5, 6, 7$, by the definition of dense pairs, we have $|V_{1,i} \cap A_k| \leq \varepsilon |V_{1,i}|$ or $|V_{1,-i} \cap A_k| \leq \varepsilon |V_{1,-i}|$ similarly as above, as otherwise there will be a blue edge within A_k . Moreover, for $k = 1, 2, 3, 4$, $|V_{1,i} \cap A_k| \leq 2\varepsilon |V_{1,i}|$ and $|V_{1,-i} \cap A_k| \leq 2\varepsilon |V_{1,-i}|$. Thus for at least one $k = 5, 6, 7$, $|V_{1,i} \cap A_k| \geq (1 - 10\varepsilon)|V_{1,i}|$ or $|V_{1,-i} \cap A_k| \geq$

$(1 - 10\varepsilon)|V_{1,-i}|$. Hence for all colors $i \notin I_c$, there exists a subset of $V_{1,i}$ of size at least $(1 - 10\varepsilon)|V_{1,i}|$ that contains no edge of color c , or a subset of $V_{1,-i}$ of size at least $(1 - 10\varepsilon)|V_{1,-i}|$ that contains no edge of color c . Let J_c be the indices i (positive or negative) for which $V_{1,i}$ contains a subset of size at least $(1 - 10\varepsilon)|V_{1,i}|$ having no edge of color c . For every color $c \in C_1 \cup C_2$, we have

$$\left| \bigcup_{i \in J_c} V_{1,i} \right| \geq \frac{1}{2} \left| \bigcup_{i \notin I_c} W_{1,i} \right| \geq \frac{1}{2} (\text{Vol}(\mathcal{L}_c(V_1)) - 4\varepsilon m)$$

or

$$\left| \bigcup_{i \in J_c} V_{-1,i} \right| \geq \frac{1}{2} (\text{Vol}(\mathcal{L}_c(V_{-1})) - 4\varepsilon m).$$

Without loss of generality, we may assume that at least half of the r colors satisfies the former. Let \mathcal{C} be the set of these colors (note that $|\mathcal{C}| \geq \frac{r}{2}$). We then have

$$\sum_{c \in \mathcal{C}} \sum_{i: i \in J_c} |V_{1,i}| \geq \frac{r}{2} \cdot \frac{1}{2} (\text{Vol}(\mathcal{L}_{c_1}(V_1)) - 4\varepsilon m).$$

Since

$$\sum_{c \in \mathcal{C}} \sum_{i: i \in J_c} |V_{1,i}| = \sum_i \sum_{c: i \in J_c} |V_{1,i}| \leq \sum_i |V_{1,i}| \cdot \max_i |\{c: i \in J_c\}|,$$

and $\sum_i |V_{1,i}| = \text{Vol}(\mathcal{L}_{c_1}(V_1)) \geq 8\varepsilon m$ we see that

$$\max_i |\{c: i \in J_c\}| \geq \frac{r}{4} \cdot \frac{\text{Vol}(\mathcal{L}_{c_1}(V_1)) - 4\varepsilon m}{\text{Vol}(\mathcal{L}_{c_1}(V_1))} \geq \frac{r}{8}.$$

In particular, there exists an index ι for which at least $\frac{r}{8}$ colors c satisfy $\iota \in J_c$. Recall that $\iota \in J_c$ implies that there exists a subset of $V_{1,\iota}$ of size at least $(1 - 10\varepsilon)|V_{1,\iota}|$ having no edge of color c . An application of Lemma 2.2 then gives a set S of size $(1 - 20\varepsilon)^{\frac{r}{32}}|V_{1,\iota}| \geq (1 - 32\varepsilon)^{\frac{r}{4}}|V_{1,\iota}|$ and a set of \mathcal{C}' of $\frac{r}{32}$ colors such that S contains no edge of color c for any $c \in \mathcal{C}'$. Thus S is a subset of size at least $\beta|V_{1,\iota}| \geq \beta\alpha_0\alpha_1 m$ on which the coloring is $(\frac{31r}{32}, \frac{31r}{32}, 0, \varepsilon^2)$ -restricted. ■

Remark. Note that in Case 2 above, instead of saying that the subgraph induced on S has $\frac{31r}{32}$ colors with no restriction, we could have kept track of the previously (x, ε^2) -sparse colors to conclude that the subgraph induced on S has some colors that are $\frac{x}{\beta\alpha_1\alpha_0}$ -sparse. However, we would not gain

anything from such refined analysis, since the lemma will later be used in the range $x \geq \alpha_1$. Since $\frac{x}{\beta\alpha_1\alpha_0} \geq 1$ for such values of x , there is no advantage in carrying out this more refined analysis.

Lastly, we need to take care of the base case of our recursion, for which we have the following lemma:

Lemma 3.3. *For every integer R , there exist constants γ and N_0 such that if r and x satisfy $x < e^{-3(\log(1/\varepsilon)/\varepsilon)\log((r-R)\gamma)}$, then*

$$G(R, r, x, \varepsilon) \leq N_0.$$

Proof. Define $\gamma = F_\chi(R, 8, 4)$. Let $N = \gamma(\gamma - 1)$, chosen appropriately for a later application of Turán’s theorem, and consider an (R, r, x, ε) -restricted chromatic- $(8, 4)$ -coloring of K_N . Call the $r - R$ colors that are (x, ε) -sparse as *restricted*, and the other R colors as *non-restricted*. If the subgraph H consisting of the edges of non-restricted colors contains a copy of K_γ , then by definition, we can find 3 colors whose union has chromatic number 8. Therefore, H does not contain a copy of K_γ , and thus by Turán’s theorem, H has density at most $1 - \frac{1}{\gamma}$. Then the complement of H in K_N has density at least $\frac{1}{\gamma}$. Since H is $(r - R)$ -colored with restricted colors, there exists a restricted color of density at least $\frac{1}{(r-R)\gamma}$ in K_N . By Lemma 2.1, we can find an ε -dense pair with parts of size at least $e^{-3(\log(1/\varepsilon)/\varepsilon)\log((r-R)\gamma)}N > xN$. However, this contradicts the fact that our color was a restricted color. ■

We can now combine our above results into an upper bound on $F_\chi(r, 8, 4)$.

Theorem 3.4. *There exists a constant C such that the following holds for all r :*

$$F_\chi(r, 8, 4) \leq e^{Cr^{2/3}\log^4(r)}.$$

Proof. Define $\varepsilon_0 = (\varepsilon_0)_{2,1}$ and let $\varepsilon = r^{-1/3}\log^{2/3}r$. Let $R > R_{3,2}$ be large enough so that for all $r \geq R$, we have $\varepsilon < \varepsilon_0$. Let γ and N_0 be the constants from Lemma 3.3 for this value of R . It suffices to prove the theorem for $r \geq R$, since then we can adjust the value of C so that the conclusion holds for all values of r . Let $\alpha_1 = e^{-100\log^2(r)r^{2/3}}$, $\alpha_0 = r^{-3\log(1/\varepsilon^2)/\varepsilon^2}$, and $\beta = (1 - 32\varepsilon)^{\frac{1}{4}}$.

Recall that $F_\chi(r, 8, 4) = G(r, r, 0, \varepsilon^2)$. Define $r_{1,0} = r_{2,0} = r$, $x_0 = 0$ and $n_0 = G(r, r, 0, \varepsilon^2) - 1$. We start with an $(r_{1,0}, r_{2,0}, x_0, \varepsilon^2)$ -restricted chromatic- $(8, 4)$ -coloring of a complete graph on n_0 vertices and repeatedly apply Lemma 3.2 to reduce the number of colors. Suppose that at the i -th step we are given an $(r_{1,i}, r_{2,i}, x_i, \varepsilon^2)$ -restricted coloring on n_i vertices. After applying Lemma 3.2, there are two possibilities depending on whether Case

1 or Case 2 of the lemma applies. If Case 1 applies, then we obtain a $(\frac{31}{32}r_{1,i}, r_{2,i}, \beta^{-1}\alpha_0^{-1}\max(\alpha_1, x), \varepsilon^2)$ -restricted coloring on $n_{i+1} = \beta\alpha_0 n$ vertices. If Case 2 applies, then we obtain a $(\frac{31}{32}r_{2,i}, \frac{31}{32}r_{2,i}, 0, \varepsilon^2)$ -restricted coloring on $n_{i+1} = \beta\alpha_1\alpha_0 n$ vertices.

Repeat the above until the first time T we have $r_{1,T} < R$ and let $x = x_T$. Note that there are at most $\log_{32/31}(r)$ iterations of Case 2, and at most $\log_{32/31}(r)$ consecutive iterations of Case 1. It suffices to prove that x is sufficiently small when the process ends, so that the condition of Lemma 3.3 is satisfied with $\varepsilon_{3,3} = \varepsilon^2$. When Case 2 applies, the density restriction factor x is reset to 0. So we only need to check the conditions when we have $\log_{32/31}(r)$ consecutive applications of Case 1. On the first iteration, the density factor is $\beta^{-1}\alpha_0^{-1}\alpha_1$ and each subsequent iteration adds a multiplicative factor of $\beta^{-1}\alpha_0^{-1}$. Thus for $c = \frac{1}{\log(32/31)}$ and some positive constant c' , we have (for small enough ε)

$$\begin{aligned} x &\leq \alpha_1(\beta^{-1}\alpha_0^{-1})^{c\log(r)} = \alpha_1 \left(r^{3\log(1/\varepsilon^2)/\varepsilon^2} \right)^{c\log(r)} (1 - 32\varepsilon)^{\frac{-rc\log(r)}{4}} \\ &\leq \alpha_1 \left(e^{3\log(r)\log(1/\varepsilon^2)/\varepsilon^2} \right)^{c\log(r)} e^{-64\varepsilon\frac{-rc\log(r)}{4}} \\ &< e^{-100\log^2(r)r^{2/3}} e^{c'r^{2/3}\log^{5/3}(r)} \\ &< e^{-\log(1/\varepsilon^2)/\varepsilon^2\log(r\gamma)}, \end{aligned}$$

as desired.

Suppose we applied Case 1 t times and Case 2 s times before reaching time T . We have seen above that $t < (\log_{32/31}(r))^2$ and $s < \log_{32/31}(r)$. Therefore, there exist positive constants C', C'', C such that

$$\begin{aligned} F_\chi(r, 8, 4) = G(r, r, 0, \varepsilon^2) &\leq N_0\alpha_0^{-t-s}\beta^{-t-s}\alpha_1^{-s} \\ &\leq N_0e^{C'r^{2/3}\log^{11/3}(r)}e^{C''\log^3(r)r^{2/3}} \\ &\leq e^{Cr^{2/3}\log^4(r)}. \end{aligned} \quad \blacksquare$$

4. Upper Bound on $F_\chi(r, 2^q, q + 1)$

Throughout this section, we fix a natural number $q \geq 2$.

4.1. Well-balanced colors

In this subsection, we generalize the definitions used in the previous section. For technical reasons, we need a slightly different definition of sparsity.

Definition 4.1. Let $x, \varepsilon \in [0, 1]$ be real numbers. Given an edge-coloring of K_n , we say that a color c is (x, ε) -sparse if the subgraph consisting of edges of color c has no balanced ε -dense pair with parts of size between xn and εn .

This definition is not much more restricted than the one given in the previous section.

Lemma 4.2. *Suppose that a color c is (x, ε) -sparse in an edge coloring of K_n . Then for every vertex-subset S of size $|S| \geq \gamma n$, the color c is $(\gamma^{-1}x, \varepsilon)$ -sparse in the induced subgraph $K_n[S]$.*

Proof. In the induced subgraph $K_n[S]$, there are no balanced ε -dense pairs in color c with parts of size between xn and εn . The conclusion follows since $[\gamma^{-1}x|S|, \varepsilon|S|] \subseteq [xn, \varepsilon n]$. ■

As in the previous section, we will bound F_χ by using a recursive formula. This time, we keep track of the number of sparse colors with various different sparsity conditions.

Definition 4.3. Let $\vec{r} = (r_1, \dots, r_{q-1})$ be a sequence of non-decreasing non-negative integers and $\vec{x} = (x_1, \dots, x_{q-1})$ be a sequence of non-negative real numbers. For a positive real number ε , an edge-coloring of K_n is $(\vec{r}, \vec{x}, \varepsilon)$ -restricted if there exist disjoint sets of colors $C_1 \cup \dots \cup C_{q-1}$ such that for each $i \in [q-1]$, $|C_i| = r_i - r_{i-1}$ (where $r_0 = 0$), and each color in C_i is (x_i, ε) -sparse.

We will always use $x_1 = 1$ so that there are r_1 colors that are not restricted. We define $G(\vec{r}, \vec{x}, \varepsilon)$ to be the minimum n such that every $(\vec{r}, \vec{x}, \varepsilon)$ -restricted coloring of K_n contains a 2^q -chromatic subgraph receiving at most q distinct colors on its edges. Then $F_\chi(r, 2^q, q + 1) = G((r, r, \dots, r), (1, 0, \dots, 0), \varepsilon)$ for any ε .

Consider a $(\vec{r}, \vec{x}, \varepsilon)$ -restricted coloring of K_n . As in the previous section, we take a densest color c_1 and a balanced ε -dense pair $W_1 = V_1 \cup V_{-1}$ in the color c_1 , where $|V_1| = |V_{-1}| \geq \alpha_0 n$ for some real number α_0 to be defined later. We refer to c_1 as the first level color, and define $\mathcal{L}_1 = \{V_1, V_{-1}\}$. Suppose that for some $k \in [q-2]$, we are given colors c_1, \dots, c_k with k -th level sets $\mathcal{L}_k = \bigcup_{a_1, \dots, a_k} \{V_{a_1, \dots, a_k}, V_{a_1, \dots, -a_k}\}$, where the sets are paired into $W_{a_1, \dots, a_k} = V_{a_1, \dots, a_k} \cup V_{a_1, \dots, -a_k}$ forming balanced ε -dense pairs in color c_k . For a color c_{k+1} , we construct the $(k+1)$ -th level sets by taking balanced ε -dense pairs of color c_{k+1} in each $V_{\vec{a}} \in \mathcal{L}_k$ as follows. Take a maximal collection of vertex-disjoint ε -dense pairs $W_{\vec{a}, a_{k+1}} = V_{\vec{a}, a_{k+1}} \cup V_{\vec{a}, -a_{k+1}}$ in $V_{\vec{a}}$ consisting of edges of color c_{k+1} and having sizes $\alpha_k |V_{\vec{a}}| \leq |V_{\vec{a}, a_{k+1}}| \leq \varepsilon |V_{\vec{a}}|$ for some parameter α_k to be defined later. Define $\mathcal{L}(V_{\vec{a}}) = \bigcup_{a_{k+1}} \{V_{\vec{a}, a_{k+1}}, V_{\vec{a}, -a_{k+1}}\}$

and $\mathcal{L}_{k+1} = \bigcup_{V \in \mathcal{L}_k} \mathcal{L}(V)$. Take a subfamily if necessary so that $\text{Vol}(\mathcal{L}(V_{\bar{a}})) \leq 2^{q+3}\varepsilon|V_{\bar{a}}|$ for each $V_{\bar{a}} \in \mathcal{L}_k$. Hence

$$(1) \quad \text{Vol}(\mathcal{L}_{k+1}) = \sum_{V_{\bar{a}} \in \mathcal{L}_k} \text{Vol}(\mathcal{L}(V_{\bar{a}})) \leq 2^{q+3}\varepsilon \text{Vol}(\mathcal{L}_k).$$

The following definition provides a threshold for a set containing ‘enough’ dense pairs in the next level.

Definition 4.4. For $k \geq 1$, we say that $V \in \mathcal{L}_k$ is *properly shattered* if $2^{q+2}\varepsilon|V| \leq \text{Vol}(\mathcal{L}(V)) \leq 2^{q+3}\varepsilon|V|$. Let $\mathcal{N}_k \subseteq \mathcal{L}_k$ be the family of sets that are not properly shattered.

The following lemma shows that a non properly shattered set contains a large subset on which c_{k+1} is sparse.

Lemma 4.5. *If $V \in \mathcal{L}_k$ is not properly shattered, then there exists $V' \subseteq V$ of size at least $|V'| \geq (1 - 2^{q+2}\varepsilon)|V|$ that contains no balanced ε -dense pair of size between $\alpha_k|V|$ and $\varepsilon|V|$.*

Proof. Let \mathcal{W} be the maximal family of vertex-disjoint balanced ε -dense pairs in V of color c_{k+1} with parts of size between $\alpha_k|V|$ and $\varepsilon|V|$. If $\text{Vol}(\mathcal{W}) < 2^{q+2}\varepsilon|V|$, then the conclusion holds since we can take $V' = V \setminus \bigcup_{W \in \mathcal{W}} W$. Otherwise, since V is not properly shattered, we must have $\text{Vol}(\mathcal{W}) > 2^{q+3}\varepsilon|V|$. In this case $\mathcal{L}(V)$ is obtained by repeatedly removing a balanced dense pairs from \mathcal{W} until the first time we reach $\text{Vol}(\mathcal{W}) < 2^{q+3}\varepsilon|V|$. Since each dense pair consists of parts of sizes at most $\varepsilon|V|$, we see that the final family has volume at least $2^{q+3}\varepsilon|V| - 2\varepsilon|V| > 2^{q+2}\varepsilon|V|$, showing that V is properly shattered. ■

Throughout the process, we will use a different analysis depending on whether there are enough properly shattered sets $V \in \mathcal{L}_k$. This can be considered as the analogue of Cases 1 and 2 of Lemma 3.2. We now wish to define the notion of a *well-balanced* sequence of colors. For the purposes of induction, we say that c_1 is *well-balanced*.

Definition 4.6. For $k \geq 1$, we say that (c_1, \dots, c_{k+1}) is well-balanced if (c_1, \dots, c_k) is well-balanced, and $\text{Vol}(\mathcal{N}_k) \leq 2^{-4(q-1)} \text{Vol}(\mathcal{L}_k)$. Also, we say that an edge-coloring of a complete graph is well-balanced up to the k -th level if there exists a sequence of colors c_1, \dots, c_k such that (c_1, \dots, c_k) is well-balanced.

4.2. Recursion

The proof of Theorem 1.2 uses a recursive formula obtained by considering the maximum level to which the given coloring is well-balanced. Let $\varepsilon = r^{-1/q} \log r$ and define

$$\delta = e^{-3(1/\varepsilon^{q-1}) \log r}, \quad z = \frac{2^{4q}}{2^{4q} + 1}, \quad y = \log_{1/z}(r), \quad \text{and} \quad \beta = (1 - 2^{q+3} \varepsilon)^{\frac{r}{2^{4q}}}.$$

For $i=0, 1, \dots, q-2$, define

$$\gamma_i = \beta \alpha_0 \cdots \alpha_i \quad \text{and} \quad \alpha_i = \beta^{-1} (\beta \delta)^{(3y)^i}.$$

Apply the framework of the previous subsection with this choice of $\{\alpha_i\}_{i=0}^{q-2}$ and ε^{q-1} instead of ε .

Lemma 4.7. *There exists $R \in \mathbb{N}$ such that the following holds for all $k < q - 1$. Let $\vec{r} = (r_1, \dots, r_{q-1})$ be a sequence of non-decreasing non-negative integers satisfying $R \leq r_k \leq r$ and let $\vec{x} = (x_1, \dots, x_{q-1})$ be a sequence of non-negative real numbers. If a $(\vec{r}, \vec{x}, \varepsilon^{q-1})$ -restricted coloring of K_n is well-balanced up to the k -th level, but not the $(k + 1)$ -th level, then there exists a subset of at least $\gamma_{k-1} n$ vertices on which the coloring is $(\vec{r}', \vec{x}', \varepsilon^{q-1})$ -restricted, where*

$$\begin{aligned} \vec{r}' &= (zr_k, \dots, zr_k, r_{k+1}, r_{k+2}, \dots, r_{q-1}) \\ \vec{x}' &= \left(1, 0, \dots, 0, \gamma_{k-1}^{-1} \max(\alpha_k, x_{k+1}), \gamma_{k-1}^{-1} x_{k+2}, \dots, \gamma_{k-1}^{-1} x_{q-1}\right). \end{aligned}$$

Proof. Let $R = 2^{4q} R_{2.2}$. Denote the set of colors as $C_1 \cup \dots \cup C_{q-1}$, where for each i , $|C_i| = r_i - r_{i-1}$ and each color in C_i is (x_i, ε^{q-1}) -sparse. By assumption, we have colors (c_1, \dots, c_k) that are well-balanced but for all colors c , the sequence of colors (c_1, \dots, c_k, c) is not well-balanced. Hence if we define \mathcal{N}_c as the family of non properly shattered sets $\mathcal{N}_k \subseteq \mathcal{L}_k$ obtained by considering the sequence of colors (c_1, \dots, c_k, c) , then $\text{Vol}(\mathcal{N}_c) > 2^{-4(q-1)} \text{Vol}(\mathcal{L}_k)$. Define $\mathcal{C} = C_1 \cup \dots \cup C_k$. Then

$$\sum_{c \in \mathcal{C}} \text{Vol}(\mathcal{N}_c) > |\mathcal{C}| \cdot 2^{-4(q-1)} \text{Vol}(\mathcal{L}_k).$$

On the other hand,

$$\begin{aligned} \sum_{c \in \mathcal{C}} \text{Vol}(\mathcal{N}_c) &= \sum_{V \in \mathcal{L}_k} |V| \cdot |\{c \in \mathcal{C} : V \in \mathcal{N}_c\}| \\ &\leq \text{Vol}(\mathcal{L}_k) \cdot \max_{V \in \mathcal{L}_k} |\{c \in \mathcal{C} : V \in \mathcal{N}_c\}|. \end{aligned}$$

Hence, there exists $V \in \mathcal{L}_k$ such that $V \in \mathcal{N}_c$ for at least $2^{-4(q-1)}|\mathcal{C}| = 2^{-4q+4}r_k$ colors $c \in \mathcal{C}$. Let \mathcal{C}' be an arbitrary subset of $2^{-4q+2}r_k$ of these colors. Since $V \in \mathcal{L}_k$, we have $|V| \geq \alpha_0 \cdots \alpha_{k-1}n$.

Note that by Lemma 4.5, for each $c \in \mathcal{C}'$, there exists a subset $V_c \subseteq V$ of size at least $|V_c| \geq (1 - 2^{q+2}\varepsilon^{q-1})|V|$ that does not contain an ε^{q-1} -dense pair of size between $\alpha_k|V|$ and $\varepsilon^{q-1}|V|$. By Lemma 2.2, there exists a set S of size $(1 - 2^{q+3}\varepsilon^{q-1})\frac{r_k}{2^{4q}}|V| \geq \beta|V| \geq \gamma_{k-1}n$ in which $\frac{r_k}{2^{4q}}$ colors do not contain an ε^{q-1} -dense pair of size between $\alpha_k|V|$ and $\varepsilon^{q-1}|V|$. For simplicity we take $\frac{r_k}{2^{4q+1}}$ of these colors, denoting the set as \mathcal{C}^* . Consider the coloring of the subgraph K_S induced on S . Define $C'_1 = C_1 \cup \dots \cup C_k$, $C'_2 = \dots = C'_k = \emptyset$, $C'_{k+1} = C_{k+1} \cup \mathcal{C}^*$, and $C'_j = C_j$ for all $j > k + 1$. Note that in K_S , by Lemma 4.2, the colors in C'_{k+1} are $(\gamma_{k-1}^{-1} \max(\alpha_k, x_{k+1}), \varepsilon^{q-1})$ -sparse, and for all $j > k + 1$, the colors in C'_j are $(\gamma_{k-1}^{-1}x_j, \varepsilon^{q-1})$ -sparse. Further, we have $|C'_1| = zr_k$, $|C'_{k+1}| = r_{k+1} - zr_k$, and $|C'_j| = |C_j|$ for all $j > k + 1$. Therefore, when restricted to K_S , the coloring is $(\vec{r}', \vec{x}', \varepsilon^{q-1})$ -restricted, proving the lemma (where the definitions of \vec{r}', \vec{x}' are given in the statement). ■

The following lemma handles the case when the given coloring is well-balanced up to the final, $(q - 1)$ -th, level.

Lemma 4.8. *There exists $R \in \mathbb{N}$ such that the following holds. Let $\vec{r} = (r_1, \dots, r_{q-1})$ be a sequence of non-decreasing non-negative integers satisfying $R \leq r_{q-1} \leq r$ and let $\vec{x} = (x_1, \dots, x_{q-1})$ be a sequence of non-negative real numbers. Suppose that a $(\vec{r}, \vec{x}, \varepsilon^{q-1})$ -restricted chromatic- $(2^q, q + 1)$ -coloring of K_n is well-balanced up to the $(q - 1)$ -th level. Then there exists a subset of at least $\gamma_{q-2}n$ vertices on which the coloring is $((zr_{q-1}, \dots, zr_{q-1}), (1, 0, \dots, 0), \varepsilon^{q-1})$ -restricted.*

Proof. Let $R = 2^{4q}R_{2.2}$. Denote the set of colors as \mathcal{C} , where $|\mathcal{C}| = r_{q-1}$. Suppose that (c_1, \dots, c_{q-1}) is well-balanced for $c_1, \dots, c_{q-1} \in \mathcal{C}$. Since the coloring is a chromatic- $(2^q, q + 1)$ -coloring, for each color c , the graph consisting of edges colored with c_1, \dots, c_{q-1} and c is $(2^q - 1)$ -colorable. Thus, we can partition the vertex set into sets A_1, \dots, A_{2^q-1} , each containing no edge of color c . Note that each A_k cannot satisfy $|A_k \cap V_i| \geq \varepsilon^{q-1}|V_i|$ for both $i = \pm 1$ since $V_1 \cup V_{-1}$ is ε^{q-1} -dense. Thus, at least half of them, without loss of generality A_1, \dots, A_{2^q-1} , intersect V_1 in smaller than ε^{q-1} fraction of its vertices. Define $\mathcal{S}_1 = \{V_1\}$. For $m \in [q - 1]$, we will iteratively construct families $\mathcal{S}_m \subseteq \mathcal{L}_m$ so that $\text{Vol}(\mathcal{S}_m) \geq 2^{-4m+2} \text{Vol}(\mathcal{L}_m)$ and each set $V \in \mathcal{S}_m$ satisfies $|V \cap A_k| \geq \varepsilon^{q-m}|V|$ for at most $2^{q-m} - 1$ sets A_k . Note that the condition holds for $m = 1$.

Suppose that we constructed the family \mathcal{S}_m for some $m \leq q-2$. For each $V \in \mathcal{S}_m$, consider the $(m+1)$ -th level sets $V_i, V_{-i} \in \mathcal{L}(V)$ and recall that these sets form ε^{q-1} -dense pairs $W_i = V_i \cup V_{-i}$. Let T be the set of indices k for which $|A_k \cap V| \geq \varepsilon^{q-m}|V|$, so that $|T| \leq 2^{q-m} - 1$ by construction. Let I_m be the set of indices i for which $|W_i \cap A_k| \geq \frac{1}{2}\varepsilon^{q-m-1}|W_i|$ for some $k \notin T$. Then, we have

$$\frac{1}{2}\varepsilon^{q-m-1} \left| \bigcup_{i \in I_m} W_i \right| \leq \left| \bigcup_{i \in I_m} (W_i \cap \bigcup_{k \notin T} A_k) \right| \leq (2^q - |T|)\varepsilon^{q-m}|V|$$

and deduce that

$$\left| \bigcup_{i \in I_m} W_i \right| \leq (2^{q+1} - 2^{q+1-m})\varepsilon|V|.$$

Fix $i \notin I_m$. Since the pair $W_i = V_i \cup V_{-i}$ is ε^{q-1} -dense, each $k \in T$ cannot satisfy both $|V_i \cap A_k| \geq \varepsilon^{q-m-1}|V_i|$ and $|V_{-i} \cap A_k| \geq \varepsilon^{q-m-1}|V_{-i}|$. Moreover, since $i \notin I_m$, each $k \notin T$ satisfies $|V_i \cap A_k| \leq |W_i \cap A_k| < \frac{1}{2}\varepsilon^{q-m-1}|W_i| = \varepsilon^{q-m-1}|V_i|$. Similarly, $|V_{-i} \cap A_k| < \varepsilon^{q-m-1}|V_{-i}|$ for each $k \notin T$. Therefore, at least one of the two sets V_i and V_{-i} intersects at most $\lfloor \frac{|T|}{2} \rfloor \leq 2^{q-m-1} - 1$ sets A_k in more than ε^{q-m-1} fraction of its vertices. Let J_m be the set of (positive or negative) indices i for which V_i satisfies $|V_i \cap A_k| \geq \varepsilon^{q-m-1}|V_i|$ for at most $2^{q-m-1} - 1$ sets A_k . Note that for each $i \notin I_m$, either $i \in J_m$ or $-i \in J_m$. Hence

$$\begin{aligned} \sum_{i \in J_m} |V_i| &\geq \frac{1}{2} \left(\text{Vol}(\mathcal{L}(V)) - \left| \bigcup_{i \in I_m} W_i \right| \right) \\ &\geq \frac{1}{2} \left(\text{Vol}(\mathcal{L}(V)) - (2^{q+1} - 2^{q+1-m})\varepsilon|V| \right). \end{aligned}$$

If V is properly shattered, then $\text{Vol}(\mathcal{L}(V)) \geq 2^{q+2}\varepsilon|V|$ and thus

$$\sum_{i \in J_m} |V_i| \geq \frac{1}{4} \text{Vol}(\mathcal{L}(V)).$$

Define \mathcal{S}_{m+1} as the union of the family of sets V_i for $i \in J_m$ over all $V \in \mathcal{S}_m$ (here we are abusing notation since the set J_m differs for each V). Recall that $\mathcal{N}_m \subseteq \mathcal{L}_m$ is the family of sets that are not properly shattered. We have

$$\text{Vol}(\mathcal{S}_{m+1}) \geq \sum_{V \in \mathcal{S}_m \setminus \mathcal{N}_m} \frac{1}{4} \text{Vol}(\mathcal{L}(V)).$$

By the definition of properly shattered sets, well-balanced colors, and our hypothesis $\text{Vol}(\mathcal{S}_m) \geq 2^{-4m+2} \text{Vol}(\mathcal{L}_m)$,

$$\begin{aligned} \sum_{V \in \mathcal{S}_m \setminus \mathcal{N}_m} \text{Vol}(\mathcal{L}(V)) &\geq \sum_{V \in \mathcal{S}_m \setminus \mathcal{N}_m} 2^{q+2} \varepsilon |V| = 2^{q+2} \varepsilon (\text{Vol}(\mathcal{S}_m) - \text{Vol}(\mathcal{N}_m)) \\ &\geq 2^{q+2} \varepsilon (2^{-4m+2} \text{Vol}(\mathcal{L}_m) - 2^{-4q+4} \text{Vol}(\mathcal{L}_m)) \\ &\geq 2^{-4m+1} \cdot 2^{q+2} \varepsilon \text{Vol}(\mathcal{L}_m). \end{aligned}$$

Hence

$$\text{Vol}(\mathcal{S}_{m+1}) \geq 2^{-4m-1} \cdot 2^{q+2} \varepsilon \text{Vol}(\mathcal{L}_m) \geq 2^{-4m-2} \text{Vol}(\mathcal{L}_{m+1}),$$

where the second inequality follows from (1). Note that \mathcal{S}_{q-1} is the family of sets $V \in \mathcal{L}_{q-1}$ that intersect at most one A_k in more than ε fraction of its vertices. This implies that there is a subset of size $(1 - (2^q - 2)\varepsilon)|V|$ in V containing no edge of color c , obtained by taking $A_k \cap V$.

Since the analysis above was for a fixed color $c \in \mathcal{C}$, in order to distinguish between different choices of c , we abuse notation and write \mathcal{S}_c for the set \mathcal{S}_{q-1} obtained by considering color c . Since $\text{Vol}(\mathcal{S}_c) \geq 2^{-4q+6} \text{Vol}(\mathcal{L}_{q-1})$ for each $c \in \mathcal{C}$, we have

$$\sum_{c \in \mathcal{C}} \text{Vol}(\mathcal{S}_c) \geq |\mathcal{C}| \cdot 2^{-4q+6} \text{Vol}(\mathcal{L}_{q-1}).$$

On the other hand,

$$\begin{aligned} \sum_{c \in \mathcal{C}} \text{Vol}(\mathcal{S}_c) &= \sum_{V \in \mathcal{L}_{q-1}} |V| \cdot |\{c \in \mathcal{C} : V \in \mathcal{S}_c\}| \\ &\leq \text{Vol}(\mathcal{L}_{q-1}) \cdot \max_{V \in \mathcal{L}_{q-1}} |\{c \in \mathcal{C} : V \in \mathcal{S}_c\}|. \end{aligned}$$

Hence there exists $V \in \mathcal{L}_{q-1}$ such that $V \in \mathcal{S}_c$ for at least $2^{-4q+6} |\mathcal{C}|$ colors $c \in \mathcal{C}$. Let \mathcal{C}' be an arbitrary subset of $2^{-4q+2} |\mathcal{C}| = 2^{-4q+2} r_{q-1}$ of these colors. For each $c \in \mathcal{C}'$, there exists a subset $V_c \subseteq V$ of size at least $(1 - (2^q - 2)\varepsilon)|V|$ that contains no edge of color c . Therefore, an application of Lemma 2.2 gives a set $S \subseteq V$ of size $(1 - (2^{q+1} - 4)\varepsilon)^{\frac{r_{q-1}}{2^{4q}}} |V|$ in V and a set \mathcal{C}'' of $\frac{r_{q-1}}{2^{4q}}$ colors such that S contains no edge of color c for any $c \in \mathcal{C}''$. For simplicity, we will take an arbitrary subcollection of $\frac{r_{q-1}}{2^{4q+1}}$ of these colors. Since S has size at least $(1 - 2^{q+3}\varepsilon)^{\frac{r_{q-1}}{2^{4q}}} |V| \geq \alpha_0 \cdots \alpha_{q-2} \beta n = \gamma_{q-2}^{-1} n$ and the subgraph induced on S is colored by at most $\frac{2^{4q}}{2^{4q+1}} r_{q-1}$ colors, this proves the lemma. ■

The next lemma takes care of the cases when one of the coordinates of $\vec{r} = (r_1, \dots, r_{q-1})$ is small, and will be used as the base cases of our recursion.

The proof is almost identical to that of Lemma 3.3, but needs to be slightly modified due to the fact that our definition of sparse colors has changed.

Lemma 4.9. *For every pair of natural numbers R and $k < q - 1$, there exist constants γ and N_0 such that if ε, r and x_2, \dots, x_{q-1} satisfy $\max(x_{k+1}, \dots, x_{q-1}) < \varepsilon e^{-3(\log(1/\varepsilon)/\varepsilon)\log((r-R)\gamma)}$, then*

$$G((R, \dots, R, r_{k+1}, \dots, r_{q-2}, r), \vec{x}, \varepsilon) \leq \varepsilon^{-1} N_0.$$

Proof. Define $\gamma = F_\chi(R, 2^q, q + 1)$. Let $N_0 = \gamma(\gamma - 1)$ and N be a natural number satisfying $N \geq \varepsilon^{-1} N_0$. Consider a chromatic- $(2^q, q + 1)$ -coloring of K_N that is $((R, \dots, R, r_{k+1}, \dots, r_{q-2}, r), \vec{x}, \varepsilon)$ -restricted. We may view this coloring as a coloring with r colors where R colors have no restriction, and $r - R$ colors are (x, ε) -sparse for $x = \max(x_{k+1}, \dots, x_{q-1})$. We refer to the former R colors as *non-restricted*, and the latter $r - R$ colors as *restricted*. Consider an arbitrary subset of vertices of size $\varepsilon N \geq N_0$ and let K be the subgraph induced on these vertices.

If the subgraph H of K consisting of the edges of non-restricted colors contains a copy of K_γ , then by definition, we can find q colors whose union has chromatic number 2^q . Therefore, H does not contain a copy of K_γ , and thus by Turán’s theorem, H has density at most $1 - \frac{1}{\gamma}$. Then the complement of H in K has density at least $\frac{1}{\gamma}$. Since H is $(r - R)$ -colored with restricted colors, there exists a restricted color of density at least $\frac{1}{(r-R)\gamma}$ in K_N . By Lemma 2.1, we can find an ε -dense pair with parts of size at least $e^{-3(\log(1/\varepsilon)/\varepsilon)\log((r-R)\gamma)} \varepsilon N > xN$ and at most $N_0 \leq \varepsilon N$. However, this contradicts the fact that our color was a restricted color. ■

We now combine the results into an upper bound on $F_\chi(r, 2^q, q + 1)$.

Theorem 4.10.

$$F_\chi(r, 2^q, q + 1) \leq e^{Cr^{1-1/q}(\log r)^q}.$$

Proof. Define $\varepsilon_0 = (\varepsilon_0)_{2.1}$ and recall that $\varepsilon = r^{-1/q} \log r$. Let $R > \max\{R_{4.7}, R_{4.8}\}$ be large enough so that for all $r \geq R$, we have $\varepsilon < \varepsilon_0$. Let γ and N_0 be the constants from Lemma 4.9 for this value of R . It suffices to prove the theorem for $r \geq R$, since then we can adjust the value of C so that the conclusion holds for all values of r .

Recall that $F_\chi(r, 2^q, q + 1) = G((r, \dots, r), (1, 0, \dots, 0), \varepsilon^{q-1})$. Define $\vec{r}_0 = (r, \dots, r)$, $\vec{x}_0 = (1, 0, \dots, 0)$, and $n_0 = G(\vec{r}_0, \vec{x}_0, \varepsilon^{q-1}) - 1$ so that there exists a $(\vec{r}_0, \vec{x}_0, \varepsilon^{q-1})$ -restricted chromatic- $(2^q, q + 1)$ -coloring of K_{n_0} . We obtain a bound on n_0 by recursively using Lemmas 4.7 and 4.8. At each step, we

take as input a $(\vec{r}_i, \vec{x}_i, \varepsilon^{q-1})$ -restricted coloring of the complete graph on n_i vertices and find a $(\vec{r}_{i+1}, \vec{x}_{i+1}, \varepsilon^{q-1})$ -restricted coloring of a complete graph on n_{i+1} vertices, using Lemma 4.7 or Lemma 4.8.

Given a $(\vec{r}_i, \vec{x}_i, \varepsilon^{q-1})$ -restricted coloring of the complete graph on n_i vertices, suppose that it is well-balanced up to the k -th level. If $k = q - 1$, then by Lemma 4.8, we may take

$$\vec{r}_{i+1} = (zr_{q-1}, \dots, zr_{q-1}), \quad \vec{x}_{i+1} = (1, 0, \dots, 0), \quad \text{and} \quad n_{i+1} \geq \gamma_{q-2}n_i.$$

For $k < q - 1$, by Lemma 4.7, we may take

$$\begin{aligned} \vec{r}_{i+1} &= (zr_{i,k}, \dots, zr_{i,k}, r_{i,k+1}, \dots, r_{i,q-1}), \\ \vec{x}_{i+1} &= (1, 0, \dots, 0, \gamma_{k-1}^{-1} \max(\alpha_k, x_{i,k+1}), \gamma_{k-1}^{-1} x_{i,k+2}, \dots, \gamma_{k-1}^{-1} x_{i,q-1}), \quad \text{and} \\ n_{i+1} &\geq \gamma_{k-1}n_i. \end{aligned}$$

Repeat the process above as long as $r_{i,k} > R$ for all k . We say that an iteration ran the k -th level process if the coloring was well-balanced up to the k -th level.

Let T be the time of termination. Recall that $y = \log_{1/z}(r)$ and note that $y \leq 2^{4q+1} \log r$. The termination condition immediately implies that there can be at most y occurrences of the $(q - 1)$ -th level process since the $(q - 1)$ -th coordinate of \vec{r}_i shrinks by a factor of z at each such iteration. Similarly, for $k < q - 1$, the process terminates if the k -th level process occurs more than y times without any occurrence of the j -th level process for $j > k$ in-between since the k -th co-ordinate of \vec{r}_i increases only if a j -th level process for $j > k$ occurs and shrinks by a factor of z at each k -th level process. We claim that the conditions of Lemma 4.9 are satisfied with $\varepsilon_{4.9} = \varepsilon^{q-1}$ when the process terminates. Fix an index $k \in [q - 2]$. Let T_0 be the last iteration before T on which the j -th level process for some $j \geq k + 1$ occurred (if there were no such occurrences, then let $T_0 = 0$). Since we have $x_{T_0, k+1} = 0$, at the first time $t > T_0$ at which $x_{t, k+1}$ becomes non-zero (which is when a k -th level process occurs), we have $x_{t, k+1} = \gamma_{k-1}^{-1} \alpha_k$. Let T_1 be this time. The observation above implies that for each $j \leq k$, there are at most y^{k-j+1} occurrences of the j -th level process from time T_1 to T . Therefore

$$\begin{aligned} x_{T, k+1} &\leq \alpha_k \gamma_0^{-y^k} \gamma_1^{-y^{k-1}} \dots \gamma_{k-1}^{-y} \\ &= \alpha_k \alpha_0^{-y^k - \dots - y} \alpha_1^{-y^{k-1} - \dots - y} \dots \alpha_{k-1}^{-y} \beta^{-y^k - \dots - y - 1} \\ &\leq \alpha_k \alpha_0^{-2y^k} \alpha_1^{-2y^{k-1}} \dots \alpha_{k-1}^{-2y} \beta^{-2y^k} \\ &= (\beta\delta)^{(3y)^k} (\beta\delta)^{-2y^k} (\beta\delta)^{-2 \cdot 3y^k} \dots (\beta\delta)^{-2 \cdot 3^{k-1} y^k} \beta^{-2y^k - 1 + 2y^k + 2y^{k-1} + \dots + 2y} \end{aligned}$$

$$\begin{aligned}
 &\leq (\beta\delta)^{y^k} \\
 &\leq \left((1 - 2^{q+3}\varepsilon)^{\frac{r}{2^{4q}}} \right)^{y^k} (e^{-\varepsilon^{1-q} \log(r)})^{y^k} \\
 &\leq e^{-r^{(q-1)/q}(\log r)} 2^{3-3q} y^k e^{-r^{(q-1)/q}(\log r)^{2-q} y^k} \\
 &\leq \varepsilon^{q-1} e^{-(\log(1/\varepsilon^{q-1})/\varepsilon^{q-1}) \log(r\gamma)}.
 \end{aligned}$$

Hence, the conditions of Lemma 4.9 are satisfied at time T and we have $n_T \leq \varepsilon^{-(q-1)} N_0$. Suppose for each k , we applied the k -th level process a_k times before reaching time T . The discussion above implies $a_k \leq y^{q-k}$ for all $k \in [q-1]$. Therefore

$$\begin{aligned}
 n_0 &\leq \varepsilon^{-(q-1)} N_0 \gamma_0^{-a_1} \gamma_1^{-a_2} \dots \gamma_{q-2}^{-a_{q-1}} \\
 &\leq \varepsilon^{-(q-1)} N_0 \gamma_0^{-y^{q-1}} \gamma_1^{-y^{q-2}} \dots \gamma_{q-2}^{-y} \\
 &= \varepsilon^{-(q-1)} N_0 \alpha_0^{-y^{q-1}-\dots-y} \alpha_1^{-y^{q-2}-\dots-y} \dots \alpha_{q-2}^{-y} \beta^{-y^{q-1}-\dots-y} \\
 &\leq \varepsilon^{-(q-1)} N_0 \alpha_0^{-2y^{q-1}} \dots \alpha_{q-2}^{-2y} \beta^{-2y^{q-1}} \\
 &= \varepsilon^{-(q-1)} N_0 (\beta\delta)^{-2y^{q-1}} (\beta\delta)^{-2 \cdot 3y^{q-1}} \\
 &\quad \dots (\beta\delta)^{-2 \cdot 3^{q-2} y^{q-1}} \beta^{-2y^{q-1}+2y^{q-1}+2y^{q-2}+\dots+2y} \\
 &\leq \varepsilon^{-(q-1)} N_0 (\beta\delta)^{-3^{q-1} y^{q-1}} \\
 &\leq \varepsilon^{-(q-1)} N_0 (\beta\delta)^{-3^{q-1} 2^{(4q+1)(q-1)} (\log r)^{q-1}} \\
 &< \varepsilon^{-(q-1)} N_0 e^{C_1 \varepsilon r (\log r)^{q-1}} e^{C_2 / \varepsilon^{q-1} (\log r)^q},
 \end{aligned}$$

where C_1, C_2 are positive constants. Since $\varepsilon = r^{-1/q} \log r$, we have

$$F_\chi(r, 2^q, q + 1) \leq e^{C r^{1-1/q} (\log r)^q},$$

for some constant C , as desired. ■

5. Remarks

The study of chromatic generalized Ramsey numbers raises interesting questions regarding the structure of edge-colorings of complete graphs. These questions seem to be new types of questions that have not been asked before. Establishing lower bounds on $F_\chi(r, p, q)$ for various choices of parameters (p, q) seems especially interesting since these questions ask to find an edge-coloring of the complete graph where the union of color classes have small chromatic number. For example, Conlon, Fox, Lee, and Sudakov [1] found a chromatic- $(4, 3)$ -coloring proving $F_\chi(r, 4, 3) \geq 2^{\Omega(\log^2 r)}$, and used

it to study a problem of Graham, Rothschild, and Spencer related to the Hales-Jewett theorem. One can see, using the product formula for chromatic number of union of graphs, that a chromatic- $(4, 3)$ -coloring is a chromatic $(2^q, q + 1)$ -coloring for all $q \geq 2$ (for $q = 3$ we need the additional condition that each color class induces a bipartite graph). Hence their result implies

$$F_\chi(r, 2^q, q + 1) \geq F_\chi(r, 4, 3) \geq 2^{\Omega(\log^2 r)}.$$

It would be interesting to improve this bound for $q > 2$. The following question posed by Conlon, Fox, Lee, and Sudakov is closely related, and a positive answer to it will establish for each fixed p , the maximum value of q for which $F_\chi(r, p, q)$ is super-polynomial.

Question 5.1. *Is $F_\chi(r, p, p - 1)$ super-polynomial in r for all $p > 4$?*

As mentioned in the introduction, the corresponding question for generalized Ramsey numbers has been answered by Conlon, Fox, Lee, and Sudakov [2] who provided an explicit edge-coloring of the complete graph in which all subgraphs induced on p -vertex subsets contain at least $p - 1$ distinct colors. It is not clear whether the coloring (or some modification of it) can be used to answer the question above.

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