COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA **37** (6) (2017) 1057–1072
DOI: 10.1007/s00493-016-3467-x

INDUCED SUBGRAPHS OF GRAPHS WITH LARGE CHROMATIC NUMBER.
III. LONG HOLES

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Received June 26, 2015
Online First November 28, 2016

We prove a 1985 conjecture of Gyárfás that for all k,ℓ, every graph with sufficiently large chromatic number contains either a clique of cardinality more than k or an induced cycle of length more than ℓ.

1. Introduction

All graphs in this paper are finite, and without loops or parallel edges. A *hole* in a graph G is an induced subgraph which is a cycle of length at least four, and an *odd hole* means a hole of odd length. (The *length* of a path or cycle is the number of edges in it, and we sometimes call a hole of length ℓ an *ℓ-hole.*) In 1985, A. Gyárfás [2] made three beautiful and well-known conjectures:

Conjecture 1.1. *For every integer $k>0$ there exists $n(k)$ such that every graph G with no clique of cardinality more than k and no odd hole has chromatic number at most $n(k)$.*

Conjecture 1.2. *For all integers $k,\ell>0$ there exists $n(k,\ell)$ such that every graph G with no clique of cardinality more than k and no hole of length more than ℓ has chromatic number at most $n(k,\ell)$.*

Conjecture 1.3. *For all integers $k,\ell>0$ there exists $n(k,\ell)$ such that every graph G with no clique of cardinality more than k and no odd hole of length more than ℓ has chromatic number at most $n(k,\ell)$.*

Mathematics Subject Classification (2000): 05C15, 05C17

[*] Supported by NSF grant DMS-1265803.
[†] Supported by ONR grant N00014-14-1-0084 and NSF grant DMS-1265563.

The third conjecture implies the first two, and remains open. Virtually no progress was made on any of them until 2013, when two of us proved the first conjecture [3]. On the second and third, there was no progress at all, until we proved [1] that the second and third are both true when $k=2$ and $\ell=6$. More recently two of us proved the third in full when $k=2$ [4]. (In fact we proved much more; that for all $\ell \geq 0$, in every graph with large enough chromatic number and no triangle, there is a sequence of holes of ℓ consecutive lengths). In this paper we prove the second; thus, our main result is:

1.4. *For all integers $k,\ell > 0$ there exists c such that every graph G with no clique of cardinality more than k and no hole of length more than ℓ has chromatic number at most c.*

We denote the chromatic number of a graph G by $\chi(G)$. If $X \subseteq V(G)$, the subgraph of G induced on X is denoted by $G[X]$, and we often write $\chi(X)$ for $\chi(G[X])$.

The proof of 1.4 is an extension of the method of [4]. In particular, we proceed by induction on k, and so we assume that 1.4 is true with k replaced by $k-1$. We will show in 5.1 that if G has no clique of cardinality more than k and no hole of length more than ℓ, and has large chromatic number, then there is a vertex v_1 such that the vertices within distance two of v_1 induce a subgraph with (not quite so) large chromatic number. The set of neighbours of v_1 induces a subgraph with bounded chromatic number (because it contains no clique of cardinality k), and so the vertices with distance exactly two from v_1 induce a subgraph, say G_2, with large chromatic number. By the same argument applied to G_2, there is a vertex v_2 of G_2 such that the subgraph of G_2 induced on the vertices with distance exactly two from v_2 has large chromatic number. And so on, many times; we obtain a sequence of "covers", each covering the next, and all covering some remaining subgraph which still has large chromatic number. The proof is by looking closely at a long such sequence of covers.

2. Multicovers

Given a long sequence of covers, each covering the next as just explained, we can clean up the relationship between each pair of them to make the relationship between them as simple as possible, in a way that we explain later. It turns out that after the cleanup, there are two ways each pair of the covers might be related, and an application of Ramsey's theorem will give us a long subsequence where all the pairs are related the same way. Thus, we need to extract something useful from a long sequence of covers pairwise related in the same way, where there are two possible cases for the "way". In this section we handle the first way; in that case we call the sequence of covers a "multicover".

If X, Y are disjoint subsets of the vertex set of a graph G, we say

- X is *complete* to Y if every vertex in X is adjacent to every vertex in Y;
- X is *anticomplete* to Y if every vertex in X is nonadjacent to every vertex in Y; and
- X *covers* Y if every vertex in Y has a neighbour in X.

(If $X = \{v\}$, we say v is complete to Y instead of $\{v\}$, and so on.) Let $x \in V(G)$, let N be some set of neighbours of x, and let $C \subseteq V(G)$ be disjoint from $N \cup \{x\}$, such that x is anticomplete to C and N covers C. In this situation we call (x, N) a *cover* of C in G. For $C, X \subseteq V(G)$, a *multicover of* C in G is a family $(N_x : x \in X)$ such that

- X is stable;
- for each $x \in X, (x, N_x)$ is a cover of C;
- for all distinct $x, x' \in X$, x' is anticomplete to N_x (and in particular all the sets $\{x\} \cup N_x$ are pairwise disjoint).

The multicover $(N_x : x \in X)$ is *stable* if each of the sets $N_x \, (x \in X)$ is stable. Let $(N_x : x \in X)$ be a multicover of C in G. If $X' \subseteq X$, and $N'_x \subseteq N_x$ for each $x \in X'$, we say that $(N'_x : x \in X')$ is *contained in* $(N_x : x \in X)$.

If $(N_x : x \in X)$ is a multicover of C, and F is a subgraph of G with $X \subseteq V(F)$ such that no vertex in $C \cup \bigcup_{x \in X} N_x$ belongs to or has a neighbour in $V(F) \setminus X$, we say that F is *tangent* to the multicover. We need to prove that if we are given a multicover $(N_x : x \in X)$ with $|X|$ large, of some set C with $\chi(C)$ large, then there is a multicover $(N'_x : x \in X')$ of some $C' \subseteq C$, contained in $(N_x : x \in X)$, with $|X'|$ and $\chi(C')$ still large (but much smaller than before), and with a certain desirable subgraph tangent, a "tick".

Let $X \subseteq V(G)$ be stable. Let a and a_x $(x \in X)$ be distinct members of $V(G) \setminus X$, such that

- a is anticomplete to X;
- a_x is adjacent to a, x and is anticomplete to $X \setminus \{x\}$, for each $x \in X$;

We call the subgraph of G with vertex set $X \cup \{a\} \cup \{a_x : x \in X\}$ and edges x-a_x, a-a_x for each $x \in X$ a *tick* on X in G. This may not be an induced subgraph of G because the vertices a_x $(x \in X)$ may be adjacent to one another in G.

For a graph G, we denote by $\omega(G)$ the cardinality of the largest clique of G, and if $X \subseteq V(G)$ we sometimes write $\omega(X)$ for $\omega(G[X])$. We need:

2.1. *For all $j,k,m,c,\kappa \geq 0$ there exist $m_j, c_j \geq 0$ with the following property. Let G be a graph with $\omega(G) \leq k$, such that $\chi(H) \leq \kappa$ for every induced subgraph H of G with $\omega(H) < k$. Let $(N_x \colon x \in X)$ be a stable multicover in G of some set C, such that $|X| \geq m_j$, $\chi(C) \geq c_j$, and $\omega(\bigcup_{x \in X} N_x) \leq j$. Then there exist $X' \subseteq X$ with $|X'| \geq m$ and $C' \subseteq C$ with $\chi(C') \geq c$ and a stable multicover $(N'_x \colon x \in X')$ of C' contained in $(N_x \colon x \in X)$, such that there is a tick in G tangent to $(N'_x \colon x \in X')$.*

Proof. We may assume that $k \geq 2$, for otherwise the result is vacuous. We proceed by induction on j, keeping k,m,c,κ fixed. If $j=0$, then we may take $m_0 = c_0 = 1$ and the theorem holds vacuously; so we assume that $j>0$ and the result holds for $j-1$. Thus m_{j-1}, c_{j-1} exist. Let

$$\begin{aligned} m_j &= 2kmm_{j-1} \\ d_2 &= m_j 2^{m_j} c_{j-1} + 2^{m_j} c \\ d_1 &= d_2 + m_j \kappa \\ d_0 &= k 2^{m_j} d_1 \\ c_j &= d_0 + k\kappa. \end{aligned}$$

We claim that m_j, c_j satisfy the theorem. Let G, $(N_x \colon x \in X)$, and C be as in the theorem, with $|X| \geq m_j$ and $\chi(C) \geq c_j$, such that $\omega(\bigcup_{x \in X} N_x) \leq j$. We may assume that $|X| = m_j$. Since $c_j > \kappa$, there is a clique $A \subseteq C$ with $|A| = k$. Let C_0 be the set of vertices in $C \setminus A$ with no neighbour in A; then since every vertex in $C \setminus C_0$ has a neighbour in A, and for each $a \in A$ its set of neighbours has chromatic number at most κ (because it includes no k-clique), it follows that $\chi(C \setminus C_0) \leq k\kappa$, and so $\chi(C_0) \geq c_j - k\kappa = d_0$.

(1) *There exist $a \in A$, and $X_1 \subseteq X$ with $|X_1| \geq m_j/k$, and $C_1 \subseteq C_0$ with $\chi(C_1) \geq d_1$, such that for each $v \in C_1$ and each $x \in X_1$, there is a vertex in N_x adjacent to v and nonadjacent to a.*

For each $v \in C_0$ and each $x \in X$, v has a neighbour in N_x; and this neighbour is nonadjacent to some vertex in A, since $|A| = k = \omega(G)$. Thus there exists $a_{v,x} \in A$ such that some vertex in N_x is adjacent to v and nonadjacent to $a_{v,x}$. There are only k possible values for $a_{v,x}$ as x ranges over X, and so there exist $a_v \in A$ and $X_v \subseteq X$ with $|X_v| \geq |X|/k$, such that $a_{v,x} = a_v$ for all $x \in X_v$. There are only k possible values for a_v; so there exist $a \in A$ and $C' \subseteq C_0$ with $\chi(C') \geq \chi(C_0)/k \geq 2^{m_j} d_1$, such that $a_v = a$ for all $v \in C'$. Thus for each $v \in C'$ there exists $X_v \subseteq X$ with $|X_v| \geq |X|/k$, such that $a_{v,x} = a$ for all $x \in X_v$. There are at most 2^{m_j} possibilities for X_v; so there exists $C_1 \subseteq C'$ with $\chi(C_1) \geq d_1$, and $X_1 \subseteq X$ with $|X_1| \geq m_j/k$, such that $X_v = X_1$ for all $v \in C_1$. This proves (1).

Let a, X_1, C_1 be as in (1). For each $v \in C_1$ and each $x \in X_1$, let $n_{x,v} \in N_x$ be adjacent to v and nonadjacent to a. For each $x \in X_1$ choose $a_x \in N_x$ adjacent to a. Let C_2 be the set of all vertices in C_1 nonadjacent to each $a_x\,(x \in X_1)$; then $\chi(C_2) \geq \chi(C_1) - m_j\kappa \geq d_2$. For each $y \in X_1$, let C_y be the set of all $v \in C_2$ such that $n_{x,v}$ is adjacent to a_y, for at least m_{j-1} values of $x \in X_1 \setminus \{y\}$. Next, we show that we may assume that:

(2) $\chi(C_y) \leq c_{j-1}2^{m_j}$, *for each* $y \in X_1$.

We will show that if (2) is false, then there is a multicover $(N'_x\colon x \in X')$ contained in $(N_x\colon x \in X)$ with $\omega(\bigcup_{x\in X'} N'_x) \leq j-1$, to which we can apply inductive hypothesis on j. Suppose then that $\chi(C_y) > c_{j-1}2^{m_j}$ for some $y \in X_1$. For each $v \in C_y$, let $X_v \subseteq X_1 \setminus \{y\}$ with $|X_v| = m_{j-1}$, such that $n_{x,v}$ is adjacent to a_y for each $x \in X_v$. There are at most 2^{m_j} choices of X_v, and so there exist $C' \subseteq C_y$ and $X' \subseteq X_1 \setminus \{y\}$ with $\chi(C') \geq \chi(C_y)2^{-m_j} \geq c_{j-1}$ and $|X'| = m_{j-1}$, such that $X_v = X'$ for all $v \in C'$. Let N'_x be the set of neighbours of a_y in N_x, for each $x \in X'$; then $(N'_x\colon x \in X')$ is a multicover of C'. Moreover, since every vertex in $\bigcup_{x\in X'} N'_x$ is adjacent to a_y, it follows that $\omega(\bigcup_{x\in X'} N'_x) < j$. But then the result follows from the definition of m_{j-1}, c_{j-1}. This proves (2).

(3) *There exist* $C_3 \subseteq C_2$ *with* $\chi(C_3) \geq c$ *and* $X_3 \subseteq X_1$ *with* $|X_3| \geq m$, *such that* $n_{x,v}$ *is nonadjacent to* a_y *for all* $v \in C_3$ *and all distinct* $x, y \in X_3$.

Let C' be the set of all $v \in C_2$ that are not in any of the sets $C_y\,(y \in X_1)$, that is, such that for each $y \in X_1$, there are fewer than m_{j-1} values of $x \in X_1 \setminus \{y\}$ such that $n_{x,v}$ is adjacent to a_y. From (2), it follows that

$$\chi(C') \geq \chi(C_2) - m_j 2^{m_j} c_{j-1} \geq d_2 - m_j 2^{m_j} c_{j-1} = 2^{m_j} c.$$

Let $v \in C'$; and let G_v be the digraph with vertex set X_1 in which for distinct $x, y \in X_1$, y is adjacent from x in G_v if $n_{x,v}$ is adjacent to a_y. It follows from the definition of C_2 that every vertex of G_v has indegree at most $m_{j-1}-1$. Consequently, the undirected graph underlying G_v has degeneracy at most $2m_{j-1}-2$, and therefore is $2m_{j-1}$-colourable. Thus there exists $X_v \subseteq X_1$ with $|X_v| \geq |X_1|/(2m_{j-1})$ such that no two members of X_v are adjacent in G_v. There are at most 2^{m_j} choices of X_v, and so there exists $C_3 \subseteq C'$ with $\chi(C_3) \geq \chi(C')2^{-m_j} \geq c$ and $X_3 \subseteq X_1$ with

$$|X_3| \geq |X_1|/(2m_{j-1}) \geq m_j/(2km_{j-1}) = m,$$

such that $X_v = X_3$ for all $v \in C_3$. This proves (3).

For each $x \in X_3$, let N'_x be the set of vertices in N_x nonadjacent to each $a_y\,(y \in X_3)$. Thus $n_{x,v} \in N'_x$ for each $x \in X_3$ and $v \in C_3$. Hence $(N'_x\colon x \in X_3)$

is a multicover of C_3 contained in $(N_x\colon x \in X)$. Moreover, the subgraph consisting of a, the vertices $a_x\,(x\in X_3)$ and X_3, together with the edges a-a_x and a_x-x for each $x\in X_3$, form a tick which is tangent to this multicover. This proves 2.1. ∎

Let us say an *impression* of H in G is a map η with domain $V(H)\cup E(H)$, that maps $V(H)$ injectively into $V(G)$, and maps each edge $e=uv$ of H to a path of G of length at least two joining the vertices $\eta(u),\eta(v)$; such that the set $\{\eta(v)\colon v\in V(H)\}$ is stable, and for every two edges e,f of H with no common end, $V(\eta(e))$ is disjoint from and anticomplete to $V(\eta(f))$. Its *order* is the maximum length of the paths $\eta(e)(e\in E(H))$.

By repeated application of 2.1 with $j=|X|$, we can obtain many ticks on the same large subset X' of X, disjoint except for X' and with no edges joining them disjoint from X'. (Note that vertices in the same tick with degree two in that tick may be adjacent in G, but otherwise the subgraph formed by the union of the ticks is induced.) But such a "tick cluster" gives an impression of $K_{n,n}$ of order two, if we take n ticks clustered on a set X' with $|X'|=n$. We deduce:

2.2. *Let $k,\kappa,n\geq 0$ be integers. Then there exist m,c with the following property. Let G be a graph with $\omega(G)\leq k$, such that there is no impression of $K_{n,n}$ in G of order two, and such that $\chi(H)\leq\kappa$ for every induced subgraph H of G with $\omega(H)<k$. Then there is no stable multicover $(N_x\colon x\in X)$ in G of a set C, such that $|X|\geq m$ and $\chi(C)\geq c$.*

Let us eliminate the "stable" hypothesis from 2.2.

2.3. *Let $k,\kappa,n\geq 0$ be integers. Then there exist m,c with the following property. Let G be a graph with $\omega(G)\leq k$, such that there is no impression of $K_{n,n}$ in G of order two, and such that $\chi(H)\leq\kappa$ for every induced subgraph H of G with $\omega(H)<k$. Then there is no multicover $(N_x\colon x\in X)$ in G of a set C, such that $|X|\geq m$ and $\chi(C)\geq c$.*

Proof. Let m,c' satisfy 2.2 (with c replaced by c'). Let $c=c'\kappa^m$. We claim that m,c satisfy the theorem. Let G be as in the theorem, and suppose that $(N_x\colon x\in X)$ is a multicover in G of a set C, such that $|X|\geq m$ and $\chi(C)\geq c$. We may assume that $|X|=m$. For each $x\in X$, the subgraph induced on N_x is κ-colourable; choose some such colouring, with colours $1,\dots,\kappa$, for each x. For each $v\in C$, let $f_v\colon X\to\{1,\dots,\kappa\}$ such that for each $x\in X$, some neighbour of v in N_x has colour $f_v(x)$. There are only $\kappa^{|X|}$ possibilities for f_v, so there is a function $f\colon X\to\{1,\dots,\kappa\}$ and a subset $C'\subseteq C$ with $\chi(C')\geq\chi(C)\kappa^{-|X|}\geq c'$, such that $f_v=f$ for all $v\in C'$. For each $x\in X$, let N'_x be the set of vertices in N_x with colour $f(x)$; then $(N'_x : x \in X)$ is a stable multicover of C', and the result follows from the choice of m, c'. This proves 2.3. ∎

If G admits an impression of $K_{n,n}$, then G has a hole of length at least $2n$. We deduce

2.4. *Let $k, \kappa, \ell \geq 0$ be integers. Then there exist m, c with the following property. Let G be a graph with no hole of length at least ℓ, such that $\omega(G) \leq k$, and $\chi(H) \leq \kappa$ for every induced subgraph H of G with $\omega(H) < k$. Then there is no multicover $(N_x : x \in X)$ in G of a set C, such that $|X| \geq m$ and $\chi(C) \geq c$.*

We remark that with a little more work, we can prove a version of 2.1, and of 2.4 below, which just assumes there is no odd hole of length at least ℓ, instead of assuming there is no hole of length at least ℓ. The proof is, roughly: use the argument above to get a large tick cluster, all tangent to a multicover $(N_x : x \in X)$ of some set C, with $|X|$ and $\chi(C)$ large. Use Ramsey's theorem repeatedly, to arrange that for each tick, its "knees" are stable (shrinking X to some smaller set); and then choose an odd path between two vertices $x, x' \in X$ via a vertex in N_x, a vertex in $N_{x'}$, and an $\omega(G)$-clique in C. We omit the details.

3. Cables

Now we return to the long sequence of covers mentioned at the start of the previous section. The goal of this section is just to introduce some terminology, describing precisely what results after the clean-up process (but before the application of Ramsey's theorem), and then to carry out the application of Ramsey's theorem.

Let $X \subseteq V(G)$ be a clique. If $|X| = k$, we call X a *k-clique.* We denote by $N^1_G(X)$ the set of all vertices in $V(G) \setminus X$ that are complete to X; and by $N^2_G(X)$ the set of all vertices in $V(G) \setminus X$ with a neighbour in $N^1(X)$ and with no neighbour in X. When $X = \{v\}$ we write $N^i_G(v)$ for $N^i_G(X)$ $(i = 1, 2)$.

Let G be a graph and let $t \geq 0$ and $h \geq 1$ be integers. An *h-cable* of *length* t in G consists of:

- t h-cliques $X_1, \ldots, X_t$, pairwise disjoint and anticomplete;
- for $1 \leq i \leq t$, a subset N_i of $N^1_G(X_i)$, such that the sets $N_1, \ldots, N_t$ are pairwise disjoint;
- for $1 \leq i \leq t$, disjoint subsets $Y_{i,t}$ and $Z_{i,i+1}, \ldots, Z_{i,t}$ of N_i; and
- a subset $C \subseteq V(G)$ disjoint from $X_1 \cup \cdots \cup X_t \cup N_1 \cup \cdots \cup N_t$

satisfying the following conditions.

- **(C1)** For $1 \le i \le t$, $Y_{i,t}$ covers C, and C is anticomplete to $Z_{i,j}$ for $i+1 \le j \le t$, and C is anticomplete to X_i.
- **(C2)** For $i < j \le t$, X_i is anticomplete to N_j.
- **(C3)** For all $i < j \le t$, every vertex in $Z_{i,j}$ has a non-neighbour in X_j.
- **(C4)** For $i < j < k \le t$, $Z_{i,j}$ is anticomplete to $X_k \cup N_k$.
- **(C5)** For all $i < j \le t$, either
  - some vertex in X_j is anticomplete to $Y_{i,t}$, and $Z_{i,j} = \emptyset$, or
  - X_j is complete to $Y_{i,t}$, and $Z_{i,j}$ covers N_j.

We call C the *base* of the h-cable, and say $\chi(C)$ is the *chromatic number* of the h-cable. Given an h-cable in this notation, let $I \subseteq \{1,\dots,t\}$; then the cliques $X_i\,(i \in I)$, the sets $N_i\,(i \in I)$, the sets $Z_{i,j}\,(i,j \in I)$, the sets $Y_{i,t}\,(i \in I)$ and C (after appropriate renumbering) define an h-cable of length $|I|$. We call this a *subcable.*

Thus there are two types of pair (i,j) with $i < j \le t$, and later we will apply Ramsey's theorem on these pairs to get a large subcable where all the pairs have the same type. Consequently, two special kinds of h-cables are of interest (t is the length in both cases):

- h-cables of *type 1*, where for all $i < j \le t$, some vertex in X_j has no neighbours in $Y_{i,t}$, and $Z_{i,j} = \emptyset$; and
- h-cables of *type 2*, where for all $i < j \le t$, X_j is complete to $Y_{i,t}$, and $Z_{i,j}$ covers N_j.

From 2.4 we deduce:

3.1. *For all $k, \kappa, n \ge 0$ and $h \ge 1$, there exist $t, c \ge 0$ with the following property. Let G be a graph with $\omega(G) \le k$, such that there is no impression of $K_{n,n}$ in G of order two, and $\chi(H) \le \kappa$ for every induced subgraph H of G with $\omega(H) < k$. Then G admits no h-cable of type 1 and length t with chromatic number more than c.*

Proof. Choose m, c to satisfy 2.3. By Ramsey's theorem there exists t such that for every partition of the edges of K_t into h sets, there is an m-clique of K_t for which all edges joining its vertices are in the same set. We claim that t, c satisfy the theorem.

For let G be as in the theorem, and suppose that G admits an h-cable of type 1 and length t with chromatic number more than c. In the usual notation for h-cables, fix an ordering of the members of X_i for each i; thus, we may speak of the rth member of X_i for $1 \le r \le h$. For each pair (i,j) with $i < j \le t$, let $f(i,j) = r$ where the rth member of X_j has no neighbours in $Y_{i,t}$. From the choice of t, there exist $I \subseteq \{1,\dots,t\}$ with $|I| = m$ and $r \in \{1,\dots,h\}$

such that $f(i,j)=r$ for all $i,j\in I$ with $i<j$. For each $j\in I$, let x_j be the rth member of X_j. Then the sets $(x_j,N_j)\,(j\in I)$ form a multicover of C, which is impossible by 2.3. This proves 3.1. ▮

We need an analogue for cables of type 2, but it needs an extra hypothesis. On the other hand, we only need to assume that there is no hole of length exactly ℓ.

3.2. *Let $\tau\geq 0$, $\ell\geq 5$ and $h\geq 1$, and let G be a graph with no ℓ-hole, such that $\chi(N^2(X))\leq\tau$ for every $(h{+}1)$-clique X of G. Then G admits no h-cable of type 2 and length $\ell-3$ with chromatic number more than $(\ell-3)\tau$.*

Proof. Let $t=\ell-3$, let G be as in the theorem, and suppose that G admits an h-cable of type 2 and length t with chromatic number more than $t\tau$. In the usual notation, choose $z_t\in Y_{t,t}$ (this is possible by **(C1)**), and choose $z_{t-1}\in Z_{t-1,t}$ adjacent to z_t (this is possible since the cable has type 2). Since $z_{t-1}\in Z_{t-1,t}$, it has a non-neighbour $x_t\in X_t$, by **(C3)**. Neither of x_t,z_t has a neighbour in $Z_{i,i+1}$ for $1\leq i\leq t-2$, by **(C4)**. Now z_{t-1} has a neighbour $z_{t-2}\in Z_{t-2,t-1}$, since the cable has type 1; and similarly for $i=t-3,\dots,1$ let $z_i\in Z_{i,i+1}$ be a neighbour of z_{i+1}. It follows that

$$z_1\text{-}z_2\text{-}\cdots\text{-}z_{t-1}\text{-}z_t\text{-}x_t$$

is an induced path.

For $1\leq i\leq t$, let C_i be the set of vertices $v\in C$ such that some vertex in $Y_{1,t}$ is adjacent to both v,z_i. Since X_i is complete to $Y_{1,t}$ (since the cable has type 2), it follows that $C_i\subseteq N^2_G(X_i\cup\{z_i\})$; and since $X_i\cup\{z_i\}$ is an $(h+1)$-clique, it follows from the hypothesis that $\chi(C_i)\leq\tau$. Thus the union $C_1\cup\cdots\cup C_t$ has chromatic number at most $t\tau$; and since $\chi(C)>t\tau$, there exists $u\in C$ not in any of the sets C_i $(1\leq i\leq t)$. Choose $v\in Y_{1,t}$ adjacent to u (this is possible by **(C1)**); then v is not adjacent to any of $z_1,\dots,z_t$, by definition of $C_1,\dots,C_t$. Choose $x_1\in X_1$; then

$$v\text{-}x_1\text{-}z_1\text{-}z_2\text{-}\cdots\text{-}z_{t-1}\text{-}z_t\text{-}x_t\text{-}v$$

is a hole of length $t+3=\ell$, a contradiction. This proves 3.2. ▮

From 3.1, 3.2 and Ramsey's theorem, we deduce:

3.3. *For all $k,\kappa,\tau,\ell\geq 0$ and $h\geq 1$, there exist $t,c\geq 0$ with the following property. Let G be a graph such that:*

- *G has no hole of length at least ℓ;*
- *$\omega(G)\leq k$;*
- *$\chi(H)\leq\kappa$ for every induced subgraph H of G with $\omega(H)<k$; and*
- *$\chi(N^2(X))\leq\tau$ for every $(h+1)$-clique X of G.*

Then every h-cable in G of length t has chromatic number at most c.

4. Clique control

In this section we explain the clean-up process that we plan to apply to the long sequence of covers; but before that, we need another concept, "clique control".

Let $\mathbb{N}$ denote the set of nonnegative integers, let $\phi\colon \mathbb{N} \to \mathbb{N}$ be a nondecreasing function, and let $h \geq 1$ be an integer. We say a graph G is (h,ϕ)*-clique-controlled* if for every induced subgraph H of G and every integer $n \geq 0$, if $\chi(H) > \phi(n)$, then there is an h-clique X of H such that $\chi(N^2_H(X)) > n$. Intuitively, this means that in every induced subgraph H of large chromatic number, there is an h-clique X with $N^2_H(X)$ of large chromatic number; the function ϕ is just a way of making "large" precise.

The following contains the clean-up process, somewhat disguised (rather than choose the whole sequence of covers and then clean up all the pairs of its terms separately, it is more convenient to grow the sequence term by term cleaning up all pairs involving the new term at each step).

4.1. *Let $t, c, \tau, \kappa \geq 0$ and $h > 0$, and let $\phi\colon \mathbb{N} \to \mathbb{N}$ be nondecreasing. Then there exists c' with the following property. Let G be a graph such that*

- *$\chi(N^1(v)) \leq \kappa$ for every $v \in V(G)$;*
- *G is (h,ϕ)-clique-controlled; and*
- *$\chi(N^2(X)) \leq \tau$ for every $(h+1)$-clique X of G.*

If $\chi(G) > c'$, then G admits an h-cable of length t with chromatic number more than c.

Proof. Let $\sigma_t = \max(c, \tau + h\kappa)$, and for $s = t-1, \dots, 0$ let

$$\sigma_s = \max(2^s \phi((h+1)^s \sigma_{s+1}), \tau + h\kappa).$$

Let $c' = \sigma_0$. We claim that c' satisfies the theorem.

Let G be a graph satisfying the hypotheses of the theorem, and therefore with $\chi(G) > c'$. Consequently G admits an h-cable of length 0 with chromatic number more than σ_0. We claim that for $s = 1, \dots, t$, G admits an h-cable of length s with chromatic number more than σ_s. For suppose the result holds for some $s < t$; we will prove it also holds for $s+1$.

Thus, G admits an h-cable of length s with chromatic number more than σ_s. In the usual notation, let C be the base of the h-cable. For each $v \in C$ and $1 \leq i \leq s$, let $C_{i,v}$ be the set of vertices $u \in C \setminus \{v\}$ nonadjacent to v, such that some vertex in $Y_{i,s}$ is adjacent to both u, v. Let $f_{i,v} = 1$ if $\chi(C_{i,v}) > \tau + h\kappa$, and $f_{i,v} = 0$ otherwise. There are only 2^s possibilities for the sequence $f_{1,v}, \dots, f_{s,v}$, so there is a subset $C_1 \subseteq C$ with $\chi(C_1) \geq 2^{-s}\chi(C) > 2^{-s}\sigma_s$ and a $0,1$-sequence $f_1,\dots,f_s$ such that $f_{i,v}=f_i$ for $1\le i\le s$ and all $v\in C_1$. For $0\le i\le s$ let $d_i=(h+1)^{s-i}\sigma_{s+1}$. Let $H=G[C_1]$; then since $2^{-s}\sigma_s\ge \phi(d_0)$, there is an h-clique X_{s+1} of H such that $\chi(D_0)>d_0$, where $D_0=N^2_H(X_{s+1})$. Let $N_{s+1}=Y_{s+1,s+1}=N^1_H(X_{s+1})$.

For $1\le i\le s$, we define $Y_{i,s+1},Z_{i,s+1}\subseteq Y_{i,s}$ and $D_i\subseteq D_{i-1}$ as follows. Assume that we have defined D_{i-1}, and $\chi(D_{i-1})>d_{i-1}$. Let W be the set of vertices in $Y_{i,s}$ that are complete to X_{s+1}, and for each $x\in X_{s+1}$, let U_x be the set of vertices in D_{i-1} with a neighbour in $Y_{i,s}$ that is nonadjacent to x. If $\chi(U_x)>d_i$ for some $x\in X_{s+1}$, let $D_i=U_x$, let $Y_{i,s+1}$ be the set of all vertices in $Y_{i,s}$ that are nonadjacent to x, and let $Z_{i,s+1}=\emptyset$. Let us call this "case 1".

Thus we assume that $\chi(U_x)\le d_i$ for each $x\in X_{s+1}$; and so $\bigcup_{x\in X_{s+1}}U_x$ has chromatic number at most hd_i. Let $D_i=D_{i-1}\setminus\bigcup_{x\in X_{s+1}}U_x$; then $\chi(D_i)>d_{i-1}-hd_i=d_i$. For each vertex in D_i, all its neighbours in $Y_{i,s}$ belong to W. In particular, let $x\in X_{s+1}$; then $C_{i,x}$ (defined earlier) has chromatic number more than

$$d_i\ge\sigma_{s+1}\ge\tau+h\kappa,$$

and so $f_{i,x}=1$. Since $x\in C_1$, it follows that $f_i=1$, and so $\chi(C_{i,v})>\tau+h\kappa$ for each $v\in C_1$.

Now let $v\in N_{s+1}$. If $u\in C$, and u has no neighbour in $X_{s+1}\cup\{v\}$, and some vertex in W is adjacent to both u,v, then $u\in N^2_G(X_{s+1}\cup\{v\})$; and so the set of all such u has chromatic number at most τ. On the other hand, the set of $u\in C$ with a neighbour in X_{s+1} has chromatic number at most $h\kappa$, since for each $x\in X_{s+1}$ its set of neighbours has chromatic number at most κ. Consequently, the set of vertices in C that are nonadjacent to v and adjacent to a neighbour of v in W has chromatic number at most $\tau+h\kappa$. Since $\chi(C_{i,v})>\tau+h\kappa$, it follows that there exists $u\in C_{i,v}$ such that no neighbour of v in W is adjacent to u. From the definition of $C_{i,v}$, it follows that v has a neighbour in $Y_{i,s}\setminus W$.

Since this is true for every vertex $v\in N_{s+1}$, we may define $Y_{i,s+1}=W$ and $Z_{i,s+1}=Y_{i,s}\setminus W$, and it follows that $Z_{i,s+1}$ covers N_{s+1}. This completes the definition of $Y_{i,s+1},Z_{i,s+1}$ and D_i. Let us call this "case 2".

In either case, $\chi(D_s)>d_s$, and we claim that $X_1,\dots,X_{s+1}$, the sets $N_1,\dots,N_{s+1}$, the sets $Z_{i,j}$ for $1\le i<j\le s+1$, the sets $Y_{i,s+1}$ for $1\le i\le s+1$, and D_s, define an h-cable of length $s+1$ and chromatic number more than d_s. To see this, we must verify **(C1)**–**(C5)**.

For **(C1)**, since D_s is anticomplete to $Z_{i,j}$ for $i+1\le j\le s$, and D_s is anticomplete to X_i for $1\le i\le s$, and D_s is anticomplete to X_{s+1} from its definition, it is enough to show that for $1\le i\le s+1$, $Y_{i,s+1}$ covers D_s, and if $i\le s$, then D_s is anticomplete to $Z_{i,s+1}$. Suppose first that $i=s+1$. Since

$D_s \subseteq D_0 = N^2_H(X_{s+1})$ and $Y_{s+1,s+1} = N^1_H(X_{s+1})$, it follows that $Y_{s+1,s+1}$ covers D_s, so the first claim holds; and the second holds vacuously. Thus we may assume that $1 \le i \le s$. Assume that case 1 applies, and let x be as in case 1. We recall that $D_i = U_x$, $Y_{i,s+1}$ is the set of all vertices in $Y_{i,s}$ that are nonadjacent to x, and $Z_{i,s+1} = \emptyset$. Consequently, $Y_{i,s+1}$ covers U_x (from the definition if U_x) and hence covers $D_s \subseteq D_i = U_x$, from the definition of U_x, so the first claim holds. Now D_s is anticomplete to $Z_{i,s+1}$ since the latter is empty; so the second claim holds. This proves **(C1)** in case 1. Now we assume that case 2 applies. With notation as in case 2, we recall that $D_i = D_{i-1} \setminus \bigcup_{x \in X_{s+1}} U_x$, $Y_{i,s+1} = W$ and $Z_{i,s+1} = Y_{i,s} \setminus W$. For every vertex in D_i, all its neighbours in $Y_{i,s}$ belong to W, and so $Z_{i,s+1}$ is anticomplete to $D_s \subseteq D_i$, and the second claim holds; and since every vertex in D_i has such a neighbour, it follows that $Y_{i,s+1}$ covers $D_s \subseteq D_i$, so the first claim holds. This completes the proof of **(C1)**.

For **(C2)**, it suffices to show that for $i < s+1$, X_i is anticomplete to N_{s+1}. But this is true since $N_{s+1} \subseteq C$ and X_i is anticomplete to C. This proves **(C2)**.

For **(C3)**, it suffices to show that for all $i < s+1$, every vertex in $Z_{i,s+1}$ has a non-neighbour in X_{s+1}. In case 1, this is true since $Z_{i,s+1} = \emptyset$, so we may assume that case 2 applies. In the notation of case 2, we recall that $Z_{i,s+1} = Y_{i,s} \setminus W$, and so every vertex in $Z_{i,s+1}$ has a non-neighbour in X_{s+1} as required. This proves **(C3)**.

For **(C4)**, it suffices to show that for $i < j < s+1$, $Z_{i,j}$ is anticomplete to $X_{s+1} \cup N_{s+1}$. But this is true since $Z_{i,j}$ is anticomplete to C and $X_{s+1} \cup N_{s+1} \subseteq C$. This proves **(C4)**.

For **(C5)**, we must show that for all $i < j \le s+1$, either

- some vertex in X_j is anticomplete to $Y_{i,s+1}$, and $Z_{i,j} = \emptyset$, or
- X_j is complete to $Y_{i,s+1}$, and $Z_{i,j}$ covers N_j.

If $j \le s$, then the claim holds since $Y_{i,s+1} \subseteq Y_{i,s}$ and either

- some vertex in X_j is anticomplete to $Y_{i,s}$, and $Z_{i,j} = \emptyset$, or
- X_j is complete to $Y_{i,s}$, and $Z_{i,j}$ covers N_j.

Consequently, we may assume that $j = s+1$. If case 1 applies, let x be as in case 1; then x is anticomplete to $Y_{i,s+1}$ from the definition of $Y_{i,s+1}$, and $Z_{i,s+1} = \emptyset$, so the claim holds. If case 2 applies, let W be as in case 2; then X_j is complete to $Y_{i,s}$ since $Y_{i,s+1} = W$, and $Z_{i,s+1}$ covers N_{s+1} since this was shown just before the definition of "case 2". This proves **(C5)**, and so completes the proof that G admits an h-cable of length $s+1$ with chromatic number more than σ_{s+1}.

We have shown then that for $s=0,\dots,t$, G admits an h-cable of length s with chromatic number more than σ_s. In particular, G admits an h-cable of length t with chromatic number more than $\sigma_t=c$. This proves 4.1. ∎

By combining 3.3 and 4.1, we deduce:

4.2. *Let $k,\kappa,\tau,\ell\geq 0$ and $h\geq 1$, and let $\phi\colon \mathbb{N}\to\mathbb{N}$ be nondecreasing. Then there exists $c(\tau)\geq 0$ with the following property. Let G be a graph such that:*

- *G has no hole of length at least ℓ;*
- *$\omega(G)\leq k$;*
- *$\chi(H)\leq\kappa$ for every induced subgraph H of G with $\omega(H)<k$;*
- *G is (h,ϕ)-clique-controlled; and*
- *$\chi(N^2(X))\leq\tau$ for every $(h+1)$-clique X of G.*

Then $\chi(G)\leq c(\tau)$.

Of course $c(\tau)$ depends on all of k,κ,τ,ℓ,h and ϕ, but this notation is convenient.

5. Proof of the main theorem

We need the following. A somewhat stronger version was proved in [1], but we give a proof here to make the paper self-contained.

5.1. *Let $\ell\geq 4$, $\kappa\geq 0$ and $\tau\geq 0$ be integers, and let G be a graph with no hole of length at least ℓ, such that $\chi(N^1(v))\leq\kappa$ and $\chi(N^2(v))\leq\tau$ for every vertex v. Then $\chi(G)\leq 2(\ell-3)(\kappa+\tau)+1$.*

Proof. Let G_1 be a component of G with $\chi(G_1)=\chi(G)$, let $z_0\in V(G_1)$, and for $i\geq 0$ let L_i be the set of vertices of G_1 with distance i from z_0. Choose k such that $\chi(L_k)\geq\chi(G_1)/2$. If $k=0$, then the theorem holds, so we may assume that $k\geq 1$. Let C_0 be the vertex set of a component of $G[L_k]$ with maximum chromatic number. Choose $v_0\in L_{k-1}$ with a neighbour in C_0. Let $t=\ell-3$, and suppose that $\chi(C_0)>t\kappa+t\tau$. We claim that:

(1) *For all i with $0\leq i\leq t$, there is an induced path v_0-v_1-$\cdots$-v_i where $v_1,\dots,v_i\in C_0$, and a subset C_i of C_0 such that $G[C_i]$ is connected, $\chi(C_i)>(t-i)\kappa+t\tau$, v_i has a neighbour in C_i, and $v_0,\dots,v_{i-1}$ have no neighbours in C_i.*

For this is true when $i=0$; suppose it is true for some value of $i<t$, and we prove it is also true for $i+1$. Let N be the set of neighbours of v_i in C_i. Thus

$$\chi(C_i\setminus N)\geq\chi(C_i)-\kappa>(t-i-1)\kappa+t\tau\geq 0,$$

and so $C_i \setminus N \neq \emptyset$; let C_{i+1} be the vertex set of a component of $G[C_i \setminus N]$ with maximum chromatic number. Thus $\chi(C_{i+1}) > (t-i-1)\kappa + t\tau$. Choose $v_{i+1} \in N$ with a neighbour in C_{i+1}. This completes the inductive definition of $v_1, \dots, v_i$ and C_i, and so proves (1).

In particular, such a path $v_0\text{-}\cdots\text{-}v_t$ and subset C_t exist. Since $\chi(C_t) > t\tau$, there is a vertex $v \in C_t$ in none of the sets $N_G^2(v_i)$ $(0 \leq i \leq t-1)$, and therefore with distance at least three from all of $v_0, \dots, v_{t-1}$, since $t \geq 1$. Choose $u \in L_{k-1}$ adjacent to v; then u has distance at least two from all of $v_0, \dots, v_{t-1}$. Let P be an induced path of $G[C_t \cup \{u, v_t\}]$ between u, v_t; thus P has length at least one. Let Q be an induced path of G between u, v_0 with all internal vertices in $L_0 \cup \cdots \cup L_{k-2}$; then Q has length at least two. The union of P, Q and $v_0\text{-}v_1\text{-}\cdots\text{-}v_t$ is a hole of length at least $t+3 = \ell$, which is impossible.

This proves that $\chi(C_0) \leq t\kappa + t\tau$. Consequently $\chi(L_k) \leq t(\kappa + \tau)$, and so $\chi(G) \leq 2t(\kappa+\tau)$. This proves 5.1. ∎

From 5.1 we deduce:

5.2. *Let $\ell \geq 4$, and let $k \geq 1$ and $\kappa \geq 0$ be such that $\chi(H) \leq \kappa$ for every graph H with no hole of length at least ℓ and $\omega(H) < k$. For $x \geq 0$ let $\phi_1(x) = 2(\ell-3)(\kappa+x)+1$. Then every graph G with no hole of length at least ℓ and with $\omega(G) \leq k$ is $(1, \phi_1)$-clique-controlled.*

Proof. Let G be a graph with no hole of length at least ℓ and with $\omega(G) \leq k$. Let $n \geq 0$, and let H be an induced subgraph of G with $\chi(H) > \phi_1(n)$. Consequently $V(H) \neq \emptyset$; choose $v \in V(H)$ with $\chi(N_H^2(v))$ maximum, $\chi(N_H^2(v)) = \tau$ say. Since H has no hole of length at least ℓ, and $\chi(N_H(u)) \leq \kappa$ and $\chi(N_H^2(u)) \leq \tau$ for every vertex u of H, 5.1 implies that $\chi(H) \leq 2(\ell - 3)(\kappa + \tau) + 1$, and so $\phi_1(n) < \chi(H) \leq \phi_1(\chi(N_H^2(v)))$. Consequently $\chi(N_H^2(v)) > n$. This proves 5.2. ∎

We claim:

5.3. *Let $\ell \geq 4$, and let $k \geq 1$ and $\kappa \geq 0$ be such that $\chi(H) \leq \kappa$ for every graph H with no hole of length at least ℓ and $\omega(H) < k$. For all h with $1 \leq h \leq k$ there is a nondecreasing function $\phi_h \colon \mathbb{N} \to \mathbb{N}$ such that every graph G with no hole of length at least ℓ and with $\omega(G) \leq k$ is (h, ϕ_h)-clique-controlled.*

Proof. We proceed by induction on h. In view of 5.2, the result holds for $h = 1$, so we may assume that $h < k$ and the result holds for h, and we will prove it holds for $h+1$. Since the result holds for h, ϕ_h exists as in the theorem. By 4.2, for each $\tau \geq 0$, there exists $c(\tau)$ as in 4.2 with ϕ replaced by ϕ_h. For each $n \geq 0$, let $\phi_{h+1}(n) = \max_{0 \leq \tau \leq n} c(\tau)$; we claim that ϕ_{h+1} satisfies the theorem. For let G be a graph with no hole of length at least ℓ, and $\omega(G) \leq k$. It follows that G is (h, ϕ_h)-clique-controlled. We must show that G is $(h+1, \phi_{h+1})$-clique-controlled. Thus, let H be an induced subgraph of G, and let $\chi(H) > \phi_{h+1}(n)$ for some $n \geq 0$; we must show that there is an $(h+1)$-clique X of H such that $\chi(N_H^2(X)) > n$. Let τ be the maximum of $\chi(N_H^2(X))$ over all $(h+1)$-cliques X of H, or 0 if there is no such X. By 4.2, $\chi(H) \leq c(\tau)$, and so $c(\tau) > \phi_{h+1}(n)$. It follows that $\tau > n$, and so there is an $(h+1)$-clique X of H such that $\chi(N_H^2(X)) > n$. This proves 5.3. ▮

Proof of 1.4. By induction on k, we may assume that there exists $\kappa \geq 0$ such that $\chi(H) \leq \kappa$ for every graph H with no hole of length at least ℓ and $\omega(H) < k$. Given k, ℓ, let ϕ_k be as in 5.3, and let $c = \phi_k(0)$. We claim that c satisfies 1.4. For let G be a graph with no hole of length at least ℓ and with $\omega(G) \leq k$; then G is (k, ϕ_k)-clique-controlled, by 5.3 with $h = k$. If $\chi(G) > \phi_k(0)$, then there is a k-clique X of G such that $\chi(N_G^2(X)) > 0$, which is impossible since $\omega(G) \leq k$ and so $\chi(N_G^2(X)) = 0$ for every k-clique X. This proves that $\chi(G) \leq \phi_k(0) = c$, and so proves 1.4. ▮

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