COMPUTING THE PARTITION FUNCTION FOR GRAPH HOMOMORPHISMS

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We introduce the partition function of edge-colored graph homomorphisms, of which the usual partition function of graph homomorphisms is a specialization, and present an efficient algorithm to approximate it in a certain domain. Corollaries include efficient algorithms for computing weighted sums approximating the number of k-colorings and the number of independent sets in a graph, as well as an efficient procedure to distinguish pairs of edge-colored graphs with many color-preserving homomorphisms $G \to H$ from pairs of graphs that need to be substantially modified to acquire a color-preserving homomorphism $G \to H$.

1. Introduction and main results

(1.1) Graph homomorphism partition function. Let $G = (V, E)$ be an undirected graph with set V of vertices and set E of edges, without multiple edges or loops, and let $A = (a_{ij})$ be a $k \times k$ symmetric complex matrix. The *graph homomorphism partition function* is defined by

$$(1.1.1) \qquad P_G(A) = \sum_{\phi\colon V \to \{1,\dots,k\}} \prod_{\{u,v\} \in E} a_{\phi(u)\phi(v)}.$$

Here the sum is taken over all maps $\phi\colon V \to \{1,\dots,k\}$ and the product is taken over all edges in G.

The function $P_G(A)$ encodes many interesting properties of the graph G, and, not surprisingly, is provably hard to compute except in a few special

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cases, see [6], [7] and references therein. For example, if A is the adjacency matrix of an undirected graph H with vertices $1,\ldots,k$, that is, if

$$a_{ij} = \begin{cases} 1 & \text{if } \{i,j\} \text{ is an edge of } H \\ 0 & \text{otherwise,} \end{cases}$$

then $P_G(A)$ is the number of *homomorphisms* of G into H, that is, the number of maps $\phi: V \to \{1,\ldots,k\}$ such that $\{\phi(u),\phi(v)\}$ is an edge of H whenever $\{u,v\}$ is an edge of G.

Here are some examples of a particularly interesting choices of the matrix A, see also Section 5.3 of [14] for more.

(1.1.2) Colorings. If the $k \times k$ matrix A is defined by

$$a_{ij} = \begin{cases} 1 & \text{if } i \neq j \\ 0 & \text{if } i = j, \end{cases}$$

then $P_G(A)$ is the number of proper k-*colorings* of G, that is, the number of ways to color the vertices of G into k colors so that the endpoints of every edge of G have different colors. Indeed, each proper k-coloring of G contributes 1 to $P_G(A)$ in (1.1.1) via the map $\phi: V \to \{1,\ldots,k\}$ that maps the vertices colored in the i-th color into i. The smallest k for which a proper k-coloring of G exists is called the *chromatic number* of G. Approximating the chromatic number of a given graph within a factor $|V|^{1-\epsilon}$ is NP-hard for any fixed $\epsilon > 0$ [8], [17]. A graph G with the maximum degree $\Delta(G)$ of a vertex is obviously properly k-colorable for any $k > \Delta(G)$. A randomized polynomial time algorithm of [12] constructs a coloring with, up to lower terms in the logarithmic order, $\Delta(G)^{1-2/k}$ colors, provided the graph is k-colorable for some $k \geq 3$, see also [10] for some sharpening.

(1.1.3) Independent sets. Suppose that $k=2$ and that A is defined by

$$a_{ij} = \begin{cases} 0 & \text{if } i = j = 1 \\ 1 & \text{otherwise.} \end{cases}$$

Then $P_G(A)$ is the number of *independent sets* in G, that is, the number of subsets $U \subset V$ of vertices such that no two vertices of U span an edge of G. Indeed, each independent set U contributes 1 to $P_G(A)$ in (1.1.1) via the map $\phi: V \to \{1,2\}$ such that $\phi^{-1}(1) = U$. Approximating the size of the largest independent set of a given graph within a factor of $|V|^{1-\epsilon}$ is NP-hard for any fixed $\epsilon > 0$ [11], [17].

(1.1.4) Maximum cut. Suppose that $k=2$. For $0<\epsilon<1$, let us define $A=A_\epsilon$ by

$$a_{ij} = \begin{cases} \epsilon & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}$$

and let us consider the value of $\epsilon^{-|E|}P_G(A_\epsilon)$. Every map $\phi: V \to \{1,2\}$ in (1.1.1) is uniquely defined by the subset $S \subset V$ such that $S=\phi^{-1}(1)$. For a subset $S \subset V$ we define the *cut* associated with S by

$$\text{cut}_G(S) = |\{u, v\} \in E : u \in S, v \notin S|.$$

Then

$$\epsilon^{-|E|}P_G(A_\epsilon) = \sum_{S:\, S \subset V} \epsilon^{-\text{cut}_G(S)}.$$

Let

$$\mu(G) = \max_{S:\, S \subset V} \text{cut}_G(S)$$

be the maximum cut associated with a subset S of vertices. The polynomial time algorithm of [9] approximates the maximum cut within a factor of roughly 0.878.

We have

$$\epsilon^{-\mu(G)} \leq \epsilon^{-|E|}P_G(A_\epsilon) \leq 2^{|V|}\epsilon^{-\mu(G)}$$

and hence

$$\frac{\ln P_G(A_\epsilon)}{\ln(1/\epsilon)} + |E| - |V|\frac{\ln 2}{\ln(1/\epsilon)} \leq \mu(G) \leq \frac{\ln P_G(A_\epsilon)}{\ln(1/\epsilon)} + |E|.$$

In particular, computing $P_G(A_\epsilon)$ for a sufficiently small, yet fixed, $\epsilon > 0$, we can approximate $\mu(G)$ within an additive error of $\delta|V|$ for an arbitrarily small $\delta > 0$, fixed in advance.

(1.2) Partition function of edge-colored graph homomorphisms. Let $G = (V, E)$ be a graph as above and let $B = \left(b_{ij}^{uv}\right)$ be a $|E| \times \frac{k(k+1)}{2}$ complex matrix with entries indexed by edges $\{u, v\} \in E$ and unordered pairs $1 \leq i, j \leq k$. Technically, we should have written $b_{\{i,j\}}^{\{u,v\}}$, but we write just b_{ij}^{uv}, assuming that

$$b_{ij}^{uv} = b_{ij}^{vu} = b_{ji}^{vu} = b_{ji}^{uv}.$$

We define the *edge-colored graph homomorphism partition function* by

(1.2.1)
$$Q_G(B) = \sum_{\phi:\, V \to \{1,\ldots,k\}} \prod_{\{u,v\}\in E} b_{\phi(u)\phi(v)}^{uv},$$

where, as in (1.1.1), the sum is taken over all maps $\phi\colon V \to \{1, \ldots, k\}$ and the product is taken over all edges of G. If $A = (a_{ij})$ is a $k \times k$ symmetric matrix and we define B by

$$b_{ij}^{uv} = a_{ij} \text{ for all } \{u, v\} \in E,$$

then

$$Q_G(B) = P_G(A),$$

so P_G defined by (1.1.1) is a specialization of Q_G defined by (1.2.1).

Let H be an undirected simple graph with k vertices and suppose that the edges of G and H are colored. Let us define

$$b_{ij}^{uv} = \begin{cases} 1 & \text{if } \{u, v\} \text{ and } \{i, j\} \text{ are edges of the same color} \\ & \text{of } G \text{ and } H, \text{ respectively} \\ 0 & \text{otherwise.} \end{cases}$$

Then $Q_G(B)$ is the number of *edge-colored homomorphisms* of G into H, that is, the number of maps $\phi\colon V \to \{1, \ldots, k\}$ such that for every edge $\{u, v\}$ of G, the pair $\{\phi(u), \phi(v)\}$ is an edge of H of the same color, cf., for example, [1].

(1.3) Our results. Let $\Delta(G)$ denote the largest degree of a vertex of G. We present a deterministic algorithm, which, given a graph $G = (V, E)$, an $\epsilon > 0$ and a (real or complex) $|E| \times \frac{k(k+1)}{2}$ matrix $B = \left(b_{ij}^{uv}\right)$ such that

$$\left|1 - b_{ij}^{uv}\right| \leq \frac{\gamma}{\Delta(G)} \text{ for all } \{u, v\} \in E \text{ and } 1 \leq i, j \leq k,$$

where $\gamma > 0$ is an absolute constant, computes the value of $Q_G(B)$ within relative error ϵ in $\left(|E|k\right)^{O(\ln|E| - \ln \epsilon)}$ time (this type of complexity is called *quasi-polynomial*). We can choose $\gamma = 0.34$, if $\Delta(G) \geq 3$ we can choose $\gamma = 0.45$, and for all sufficiently large $\Delta(G)$ we can choose $\gamma = 0.54$.

Consequently, we obtain an algorithm of $\left(|E|k\right)^{O(\ln|E| - \ln \epsilon)}$ complexity to approximate $P_G(A)$ for any $k \times k$ symmetric matrix $A = (a_{ij})$ which satisfies

$$\left|1 - a_{ij}\right| \leq \frac{\gamma}{\Delta(G)} \text{ for all } 1 \leq i, j \leq k.$$

This allows us to compute efficiently various "soft" relaxations of "hard" combinatorial quantities of interest. Here are the corresponding modification of Examples 1.1.2 and 1.1.3.

In Example 1.1.2, let us define the $k \times k$ matrix A by

$$a_{ij} = \begin{cases} 1 + \gamma/\Delta(G) & \text{if } i \neq j \\ 1 - \gamma/\Delta(G) & \text{if } i = j. \end{cases}$$

Then the value of

$$(1.3.1) \qquad \left(1 + \frac{\gamma}{\Delta(G)}\right)^{-|E|} P_G(A)$$

represents the weighted sum over all $k^{|V|}$ possible colorings of the vertices of G into k colors, where each proper coloring is counted with weight 1, whereas a coloring for which w edges are miscolored (that is, have their endpoints colored with the same color) is counted with weight

$$(1.3.2) \qquad \left(1 + \frac{\gamma}{\Delta(G)}\right)^{-w} \left(1 - \frac{\gamma}{\Delta(G)}\right)^{w} \leq \exp\left\{-\frac{2\gamma w}{\Delta(G)}\right\}.$$

Hence, we can compute in quasi-polynomial time the sum over all $k^{|V|}$ colorings of G, where each coloring is weighed down exponentially by the number w of miscolored edges. One can ask if there is a computationally efficient (quasi-polynomial) way to discount improper colorings more vigorously, for example, exponentially in $w^{1+\epsilon}$, where $\epsilon > 0$ is fixed in advance. The answer is "no" unless NP-complete problems admit quasi-polynomial algorithms. Indeed, given a graph G, for a positive integer m let us define G_m to be the union of m disjoint copies of G. Clearly, any proper k-coloring of G extends to a proper k-coloring of G_m. If G does not have a proper k-coloring, then each of the $k^{m|V|}$ colorings of G_m will have at least m miscolored edges. Consequently, if the penalty is exponential in $w^{1+\epsilon}$, by choosing m sufficiently large (but still polynomial in $|V|$ and and $\ln k$), we will be able to determine whether G has a proper k-coloring by comparing the sum of penalties for G_m with $1/2$.

Similarly, there is no hope to raise the penalty to $\exp\left\{-\gamma w/\Delta^{1-\epsilon}(G)\right\}$ in computationally efficient way for any $0 < \epsilon \leq 1$, fixed in advance. To see that, given a graph G and a positive integer m, let us define the graph G_m as follows: for each vertex v of G we introduce m vertices $v_1, \ldots, v_m$ of G_m, which we call the *clones* of v. The edges of G_m are all pairs $\{v_i, u_j\}$ of clones such that $\{v, u\}$ is an edge of G. Then $\Delta(G_m) = m\Delta(G)$ and each proper k-coloring of G extends to a proper k-coloring of G_m. On the other hand, if G is not properly k-colorable, then any k-coloring of G_m will have at least m^2 miscolored edges. To see that, given a k-coloring of G_m, let us consider the following random coloring of G: independently for every vertex of G, we

choose uniformly at random its clone in G_m and replicate the color of the clone. Since G is not properly k-colorable, the expected number of miscolored edges in G is at least 1. On the other hand, the probability that any given edge in G_m is replicated in G is $1/m^2$ and hence the number of miscolored edges in G_m is at least m^2. If there were a quasi-polynomial algorithm to charge the penalty of $\exp\left\{-\gamma w/\Delta^{1-\epsilon}(G)\right\}$ for w miscolored edges, then by choosing a sufficiently large m (but still bounded by a polynomial in $|V|$ and $\ln k$), we would have been able to determine whether G is properly k-colorable by applying the algorithm to G_m and comparing the sum of penalties with $1/2$.

In Example 1.1.3, let us define the 2×2 matrix A by

$$a_{ij} = \begin{cases} 1 - \gamma/\Delta(G) & \text{if } i = j = 1 \\ 1 + \gamma/\Delta(G) & \text{otherwise.} \end{cases}$$

Then the value of (1.3.1) represents the weighted sum over all $2^{|V|}$ subsets of vertices of the graph G, where each independent set is counted with weight 1, whereas a set whose vertices span w edges of G is counted with weight (1.3.2). We note that the sum (1.3.1) differs from the partition function of independent sets (the "hard core" model) in which the sum is taken over all independent sets that are weighted exponentially in their cardinality, see [15].

Let us restrict ourselves to the class of graphs of bounded degree, with $\Delta(G) \le 3$, say. Then our result implies that the value of the partition function $P_G(A)$ can be efficiently approximated as long as $1 - \delta \le a_{ij} \le 1 + \delta$ for all i and j, where $0 < \delta < 1$ is an absolute constant (we can choose $\delta = 0.11$). It is tempting to conjecture that for any $0 < \delta < 1$, fixed in advance, the value of $P_G(A)$ can be efficiently approximated. This, however, cannot be so unless NP-hard problems can be solved by a quasi-polynomial algorithm. Indeed, approximating the maximum cut in G satisfying $\Delta(G) \le 3$ within a certain absolute constant factor $\beta_0 > 1$ is known to be NP-hard [4]. The problem remains NP-hard if we further restrict ourselves to connected graphs satisfying $\Delta(G) \le 3$. In this case the maximum cut is at least $|V| - 1$ and the construction of Section 1.1.4 shows that for some fixed $\epsilon > 0$ approximating $P_G(A_\epsilon)$ within some fixed factor $\beta_1 > 1$ is an NP-hard problem.

We note that for any positive A the problem of computing $P_G(A)$ exactly is #P-hard unless rank $A = 1$, in which case the problem admits a polynomial time algorithm [6].

Computing $Q_G(B)$ allows us to distinguish pairs of edge-colored graphs with many color-preserving homomorphisms $G \to H$ from pairs which are

sufficiently far from having a single color-preserving homomorphism. Indeed, given edge-colored graphs G and H, let us define $B = \left(b_{ij}^{uv}\right)$ by

$$
b_{ij}^{uv} = \begin{cases} 1 + \frac{\gamma}{\Delta(G)} & \text{if } \{u,v\} \text{ and } \{i,j\} \text{ are edges of the same color} \\ & \text{of } G \text{ and } H, \text{ respectively} \\ 1 - \frac{\gamma}{\Delta(G)} & \text{otherwise.} \end{cases}
$$

Then the value of

$$
(1.3.3) \qquad \left(1 + \frac{\gamma}{\Delta(G)}\right)^{-|E|} Q_G(B)
$$

represents the weighted sum over all $k^{|V|}$ maps $\phi \colon V \to \{1,\ldots,k\}$, where each color-preserving homomorphism is counted with weight 1 and a map ϕ which does not map some w edges of G onto the identically colored edges of H is counted with weight (1.3.2) at most.

Let us choose some positive integer w. Hence, if every map ϕ does not map some w edges of G onto the identically colored edges of H, the value of (1.3.3) does not exceed $k^{|V|}e^{-2\gamma w/\Delta(G)}$ (in this case, we say that G and H are sufficiently far from having a color-preserving homomorphism $G \to H$). If, however, the probability that a random map ϕ is a color-preserving homomorphism is at least $2e^{-2\gamma w/\Delta(G)}$, then the sum (1.3.3) is at least $2k^{|V|}e^{-2\gamma w/\Delta(G)}$ (in this case we say that there are sufficiently many color-preserving homomorphisms). Computing the value of $Q_G(B)$ within relative error 0.1, say, we can tell apart these two cases. The most interesting situation is when G is not far from regular, so $|E| \geq \delta|V|\Delta(G)$ for some constant $\delta > 0$, fixed in advance, and $w \approx \epsilon|E|$ for some fixed $\epsilon > 0$, in which case "many" may still mean that the probability to hit a color-preserving homomorphism at random is exponentially small.

(1.4) The idea of the algorithm. Let J denote the $|E| \times \frac{k(k+1)}{2}$ matrix filled with 1s. Given a $|E| \times \frac{k(k+1)}{2}$ matrix $B = \left(b_{ij}^{uv}\right)$, where $\{u,v\} \in E$ and $1 \leq i, j \leq k$, we consider the univariate function

$$
(1.4.1) \qquad f(t) = \ln Q_G\big(J + t(B - J)\big),
$$

so that

$$
f(0) = \ln Q_G(J) = |V| \ln k \quad \text{and} \quad f(1) = \ln Q_G(B).
$$

Hence our goal is to approximate $f(1)$ and we do it by using the Taylor polynomial expansion of f at $t = 0$:

$$(1.4.2) \qquad f(1) \approx f(0) + \sum_{m=1}^{n} \frac{1}{m!} \frac{d^m}{dt^m} f(t)\Big|_{t=0}.$$

It turns out that the approximation (1.4.2) can be computed in $(|E|k)^{O(n)}$ time. We present the algorithm in Section 2. The quality of approximation (1.4.2) depends on the location of *complex* zeros of Q_G.

(1.5) Lemma. *Suppose that there is a real $\beta > 1$ such that*

$$Q_G\big(J + z(B - J)\big) \neq 0 \text{ for all } z \in \mathbb{C} \text{ satisfying } |z| \leq \beta.$$

Then the right hand side of (1.4.2) approximates $f(1)$ within an additive error of

$$\frac{|E|}{(n+1)\beta^n (\beta - 1)}.$$

In particular, for a fixed $\beta > 1$, to ensure an additive error of $0 < \epsilon < 1$, we can choose $n = O(\ln|E| - \ln\epsilon)$, which would result in the algorithm for approximating $Q_G(B)$ within relative error ϵ in $(|E|k)^{O(\ln|E| - \ln\epsilon)}$ time. We prove Lemma 1.5 in Section 2.

It remains to identify a class of matrices B for which the number $\beta > 1$ of Lemma 1.5 exists. We prove the following result.

(1.6) Theorem. *There exists an absolute constant $\alpha > 0$ such that for any undirected graph G and any complex $|E| \times \frac{k(k+1)}{2}$ matrix $B = \big(b_{ij}^{uv}\big)$ satisfying*

$$\left|1 - b_{ij}^{uv}\right| \leq \frac{\alpha}{\Delta(G)} \text{ for all } \{u, v\} \in E \text{ and } 1 \leq i, j \leq k,$$

where $\Delta(G)$ is the largest degree of a vertex of G, one has

$$Q_G(B) \neq 0.$$

One can choose $\alpha = 0.35$, if $\Delta(G) \geq 3$ one can choose $\alpha = 0.46$ and if $\Delta(G)$ is sufficiently large, one can choose $\alpha = 0.55$.

We prove Theorem 1.6 in Section 3. Theorem 1.6 implies that if

$$\left|1 - b_{ij}^{uv}\right| \leq \frac{0.34}{\Delta(G)} \text{ for all } \{u, v\} \in E \text{ and } 1 \leq i, j \leq k,$$

we can choose $\beta = 35/34$ in Lemma 1.5 and hence obtain an algorithm which computes $Q_G(B)$ within relative error ϵ in $(|E|k)^{O(\ln|E|-\ln\epsilon)}$ time. Similarly, if $\Delta(G) \geq 3$ and

$$\left|1 - b_{ij}^{uv}\right| \leq \frac{0.45}{\Delta(G)} \text{ for all } \{u, v\} \in E \text{ and } 1 \leq i, j \leq k,$$

we can choose $\beta = 46/45$ and if

$$\left|1 - b_{ij}^{uv}\right| \leq \frac{0.54}{\Delta(G)} \text{ for all } \{u, v\} \in E \text{ and } 1 \leq i, j \leq k,$$

and $\Delta(G)$ is sufficiently large, (namely, if $\Delta(G) \geq 30$) we can choose $\beta = 55/54$.

A similar approach was used earlier to compute the permanent of a matrix [2] and the partition function for cliques of a given size in a graph [3]. In terms of statistical physics, one can interpret the method as the approximation of the logarithm of the partition function at a lower temperature by a low degree Taylor polynomial computed at a higher (in fact, infinitely high) temperature (the role of the temperature is played by $\Delta(G)/\gamma$ in (1.3.2)). As long as the partition function has no complex zeros, the interpolation is very efficient. As is known since [16] and [13], complex zeros of the partition function are responsible for phase transitions. In short, the intuition underlying the method is that the logarithm of the partition function is well-approximated by a low-degree Taylor polynomial at the temperatures higher than the phase transition threshold. We note that our method is different from the "correlation decay" approach, see [15], although the latter also links computability and the absence of phase transitions.

While the algorithm of Section 2 and Lemma 1.5 are pretty straightforward modifications of the corresponding results of [2] and [3], the proof of Theorem 1.6 required new ideas.

2. The algorithm

(2.1) The algorithm to approximate the partition function. We present an algorithm, which, given a $|E| \times \frac{k(k+1)}{2}$ matrix $B = \left(b_{ij}^{uv}\right)$, computes the approximation (1.4.2) for the function f defined by (1.4.1). Let

(2.1.1) $$g(t) = Q_G\big(J + t(B - J)\big),$$

so $f(t) = \ln g(t)$. Hence

$$f'(t) = \frac{g'(t)}{g(t)} \text{ and } g'(t) = g(t)f'(t).$$

Therefore, for $m \geq 1$, we have

$$(2.1.2) \qquad \frac{d^m}{dt^m}g(t)\Big|_{t=0} = \sum_{j=0}^{m-1} \binom{m-1}{j} \left(\frac{d^j}{dt^j}g(t)\Big|_{t=0} \right) \left(\frac{d^{m-j}}{dt^{m-j}}f(t)\Big|_{t=0} \right)$$

(we agree that the 0-th derivative of g is g). We note that $g(0) = k^{|V|}$. If we compute the values of

$$(2.1.3) \qquad \frac{d^m}{dt^m}g(t)\Big|_{t=0} \quad \text{for } m = 1, \ldots, n,$$

then the formulas (2.1.2) for $m = 1, \ldots, n$ provide a non-degenerate triangular system of linear equations that allows us to compute

$$\frac{d^m}{dt^m}f(t)\Big|_{t=0} \quad \text{for } m = 1, \ldots, n.$$

Hence our goal is to compute the values (2.1.3). We have

$$\frac{d^m}{dt^m}g(t)\Big|_{t=0} = \sum_{\phi:\, V \to \{1,\ldots,k\}} \sum_{\substack{I = \left(\{u_1,v_1\}, \\ \ldots,\{u_m,v_m\} \right)}} \left(b^{u_1 v_1}_{\phi(u_1)\phi(v_1)} - 1 \right) \cdots \left(b^{u_m v_m}_{\phi(u_m)\phi(v_m)} - 1 \right),$$

where the inner sum is taken over all ordered sets I of m distinct edges $\{u_1,v_1\}, \ldots, \{u_m,v_m\}$ of G. Let $S(I)$ be the set of all distinct vertices among $u_1, v_1, \ldots, u_m, v_m$. Then

$$\frac{d^m}{dt^m}g(t)\Big|_{t=0}$$
$$= \sum_I k^{|V|-|S(I)|} \sum_{\phi:\, S(I) \to \{1,\ldots,k\}} \left(b^{u_1 v_1}_{\phi(u_1)\phi(v_1)} - 1 \right) \cdots \left(b^{u_m v_m}_{\phi(u_m)\phi(v_m)} - 1 \right),$$

where the outer sum is taken over not more than $|E|^m$ ordered sets I of m distinct edges $\{u_1,v_1\}, \ldots, \{u_m,v_m\}$ of G and the inner sum is taken over not more than k^{2m} maps $\phi: S(I) \to \{1,\ldots,k\}$. Hence, the complexity of computing the approximation (1.4.2) is $(|E|k)^{O(n)}$ as claimed.

(2.2) *Proof of Lemma 1.5.*   The function $g(t)$ defined by (2.1.1) is a polynomial of degree $d \leq |E|$ and $g(0) = k^{|V|} \neq 0$, so we factor

$$g(z) = g(0) \prod_{i=1}^{d} \left(1 - \frac{z}{\alpha_i}\right),$$

where $\alpha_1, \ldots, \alpha_d \in \mathbb{C}$ are the roots of $g(z)$. By the condition of Lemma 1.5, we have

$$|\alpha_i| \geq \beta > 1 \text{ for } i = 1, \ldots, d.$$

Therefore,

$$(2.2.1) \qquad f(z) = \ln g(z) = \ln g(0) + \sum_{i=1}^{d} \ln\left(1 - \frac{z}{\alpha_i}\right) \text{ for } |z| \leq 1,$$

where we choose the branch of $\ln g(z)$ that is real at $z = 0$. Using the standard Taylor expansion, we obtain

$$\ln\left(1 - \frac{1}{\alpha_i}\right) = -\sum_{m=1}^{n} \frac{1}{m}\left(\frac{1}{\alpha_i}\right)^m + \zeta_n,$$

where

$$|\zeta_n| = \left| \sum_{m=n+1}^{+\infty} \frac{1}{m}\left(\frac{1}{\alpha_i}\right)^m \right| \leq \frac{1}{(n+1)\beta^n(\beta-1)}.$$

Therefore, from (2.2.1) we obtain

$$f(1) = f(0) + \sum_{m=1}^{n} \left(-\frac{1}{m}\sum_{i=1}^{d}\left(\frac{1}{\alpha_i}\right)^m\right) + \eta_n,$$

where

$$|\eta_n| \leq \frac{|E|}{(n+1)\beta^n(\beta-1)}.$$

It remains to notice that

$$-\frac{1}{m}\sum_{i=1}^{d}\left(\frac{1}{\alpha_i}\right)^m = \frac{1}{m!}\frac{d^m}{dt^m}f(t)\Big|_{t=0}. \qquad \blacksquare$$

3. Proof of Theorem 1.6

For a $0 < \delta < 1$, we define the polydisc $\mathcal{U}(\delta) \subset \mathbb{C}^{k(k+1)|E|/2}$ by

$$\mathcal{U}(\delta) = \left\{ Z = \left(z_{ij}^{uv} \right) : \left| 1 - z_{ij}^{uv} \right| \leq \delta \text{ for all } \{u,v\} \in E \text{ and } 1 \leq i,j \leq k \right\}.$$

Thus we have to prove that for $\delta = \alpha/\Delta(G)$, where $\alpha > 0$ is an absolute constant, we have $Q_G(Z) \neq 0$ for all $Z \in \mathcal{U}(\delta)$.

(3.1) Recursion. For a sequence of distinct vertices $W = (v_1, \ldots, v_m)$ of the graph G and a sequence $L = (l_1, \ldots, l_m)$ of not necessarily distinct numbers $1 \leq l_1, \ldots, l_m \leq k$, we define

$$Q_L^W(Z) = \sum_{\substack{\phi:\, V \to \{1,\ldots,k\} \\ \phi(v_1)=l_1,\ldots,\phi(v_m)=l_m}} \prod_{\{u,v\} \in E} z_{\phi(u)\phi(v)}^{uv}$$

(we suppress the graph G in the notation). In words: we restrict the sum (1.2.1) defining $Q_G(Z)$ onto the maps $\phi: V \to \{1, \ldots, k\}$ that map selected vertices $v_1, \ldots, v_m$ of G into preassigned indices $l_1, \ldots, l_m$. We denote by $|W|$ the number of vertices in W and by $|L|$ the number of indices in L (hence we have $|W| = |L|$).

  We denote by (W, u) a sequence W appended by u (distinct from all previous vertices in W) and by (L, l) a sequence L appended by l (not necessarily distinct from all previous indices in L). Then for any sequence W of distinct vertices, for any u distinct from all vertices in W and for any sequence L of indices such that $|L| = |W|$, we have

$$(3.1.1) \qquad\qquad Q_L^W(Z) = \sum_{l=1}^{k} Q_{(L,l)}^{(W,u)}(Z).$$

When W and L are both empty, then $Q_L^W(Z) = Q_G(Z)$.

  We start with a geometric inequality.

(3.2) Lemma. *Let $x_1, \ldots, x_n \in \mathbb{R}^2$ be non-zero vectors such that for some $0 \leq \alpha < 2\pi/3$ the angle between any two vectors x_i and x_j does not exceed α. Let $x = x_1 + \ldots + x_n$. Then*

$$\|x\| \geq \left(\cos \frac{\alpha}{2} \right) \sum_{i=1}^{n} \|x_i\|.$$

Proof. We note that 0 is not in the convex hull of any three vectors x_i, x_j, x_k, since otherwise the angle between some two of those three vectors would have been at least $2\pi/3$. The Carathéodory Theorem implies that 0 is not in the convex hull of $x_1, \ldots, x_n$ and hence the vectors lie in an angle of at most α with vertex at the origin. Let us consider the bisector of that angle and the orthogonal projections of each x_i onto the bisector. The length of the orthogonal projection of each x_i is at least $\|x_i\| \cos(\alpha/2)$ and hence the length of the orthogonal projection of $x_1 + \ldots + x_n$ is at least $(\|x_1\| + \ldots + \|x_n\|) \cos(\alpha/2)$. Since the vector $x_1 + \ldots + x_n$ is at least as long as its orthogonal projection, the proof follows. ∎

Lemma 3.2 was suggested by Boris Bukh [5] and by an anonymous referee. It replaces a weaker bound of $\sqrt{\cos\alpha}\,(\|x_1\| + \ldots + \|x_n\|)$, assuming that $\alpha \leq \pi/2$, of an earlier version of the paper.

Our proof of Theorem 1.6 is based on the following two lemmas.

(3.3) Lemma. *Let $\tau > 0$ be real, let W be a sequence of distinct vertices of G, let u be a vertex distinct from the vertices in W and let L be a sequence of not necessarily distinct numbers from the set $\{1, \ldots, k\}$ such that $|L| = |W|$. Suppose that for all $Z \in \mathcal{U}(\delta)$ and for all $1 \leq l \leq k$, we have*

$$Q^{(W,u)}_{(L,l)}(Z) \neq 0$$

and, moreover,

$$\left| Q^{(W,u)}_{(L,l)}(Z) \right| \geq \frac{\tau}{\Delta(G)} \sum_{\substack{v:\,\{u,v\}\in E \\ j:\,1\leq j\leq k}} |z^{uv}_{lj}| \left| \frac{\partial}{\partial z^{uv}_{lj}} Q^{(W,u)}_{(L,l)}(Z) \right|.$$

Then, for any two $1 \leq l, m \leq k$ and any $A \in \mathcal{U}(\delta)$, the angle between two complex numbers

$$Q^{(W,u)}_{(L,l)}(A) \text{ and } Q^{(W,u)}_{(L,m)}(A),$$

interpreted as vectors in $\mathbb{R}^2 = \mathbb{C}$, does not exceed

$$\theta = \frac{2\delta\Delta(G)}{\tau(1-\delta)}.$$

Proof. Since $Q^{(W,u)}_{(L,l)}(Z) \neq 0$ for all $Z \in \mathcal{U}(\delta)$, we can and will consider a branch of $\ln Q^{(W,u)}_{(L,l)}(Z)$ for $Z \in \mathcal{U}(\delta)$. Then

$$\frac{\partial}{\partial z^{uv}_{lj}} \ln Q^{(W,u)}_{(L,l)}(Z) = \frac{\partial}{\partial z^{uv}_{lj}} Q^{(W,u)}_{(L,l)}(Z) \Big/ Q^{(W,u)}_{(L,l)}(Z)$$

and since
$$\left|z_{ij}^{xy}\right| \geq 1 - \delta \text{ for all } x, y, i, j$$

we conclude that
$$\sum_{\substack{v:\,\{u,v\}\in E \\ j:\,1\leq j\leq k}} \left|\frac{\partial}{\partial z_{lj}^{uv}} \ln Q_{(L,l)}^{(W,u)}(Z)\right| \leq \frac{\Delta(G)}{\tau(1-\delta)} \text{ for all } Z \in \mathcal{U}(\delta).$$

Given $A \in \mathcal{U}(\delta)$, $A = \left(a_{ij}^{xy}\right)$, and $1 \leq l, m \leq k$, we define $B \in \mathcal{U}(\delta)$, $B = \left(b_{ij}^{xy}\right)$, by
$$b_{lj}^{uv} = a_{mj}^{uv} \text{ for all } v \in V \text{ such that } \{u,v\} \in E \text{ and all } 1 \leq j \leq k$$

and
$$b_{ij}^{xy} = a_{ij}^{xy} \text{ in all other cases.}$$

Then
$$Q_{(L,l)}^{(W,u)}(B) = Q_{(L,m)}^{(W,u)}(A)$$

and hence
$$\left|\ln Q_{(L,l)}^{(W,u)}(A) - \ln Q_{(L,m)}^{(W,u)}(A)\right| = \left|\ln Q_{(L,l)}^{(W,u)}(A) - \ln Q_{(L,l)}^{(W,u)}(B)\right|$$
$$\leq \max_{Z\in\mathcal{U}(\delta)} \sum_{\substack{v:\,\{u,v\}\in E \\ j:\,1\leq j\leq k}} \left|\frac{\partial}{\partial z_{lj}^{uv}} \ln Q_{(L,l)}^{(W,u)}(Z)\right| \times \max_{\substack{v\in V:\,\{u,v\}\in E \\ j:\,1\leq j\leq k}} \left|a_{lj}^{uv} - b_{lj}^{uv}\right| \leq \frac{2\delta\Delta(G)}{\tau(1-\delta)},$$

where the last inequality follows since $\left|a_{ij}^{xy} - b_{ij}^{xy}\right| \leq 2\delta$ for all $A, B \in \mathcal{U}(\delta)$. The proof now follows. ∎

(3.4) Lemma. *Let $0 \leq \theta < 2\pi/3$ be a real number, let W be a sequence of distinct vertices and let L be a sequence of not necessarily distinct indices from the set $\{1,\ldots,k\}$ such that $|L| = |W|$. Suppose that for any $Z \in \mathcal{U}(\delta)$, for every $v \in V$ distinct from the vertices of W, and for every $1 \leq i, j \leq k$ we have*
$$Q_{(L,i)}^{(W,v)}(Z), \quad Q_{(L,j)}^{(W,v)}(Z) \neq 0$$

and that the angle between
$$Q_{(L,i)}^{(W,v)}(Z) \text{ and } Q_{(L,j)}^{(W,v)}(Z),$$

considered as vectors in $\mathbb{R}^2 = \mathbb{C}$, does not exceed θ.

Let $W=(W',u)$ and $L=(L',l)$. Then for all $Z \in \mathcal{U}(\delta)$ we have

$$\left| Q_L^W(Z) \right| \geq \frac{\tau}{\Delta(G)} \sum_{\substack{v:\, \{u,v\} \in E \\ j:\, 1 \leq j \leq k}} |z_{lj}^{uv}| \left| \frac{\partial}{\partial z_{lj}^{uv}} Q_L^W(Z) \right|,$$

where

$$\tau = \cos \frac{\theta}{2}.$$

Proof. Let v be a vertex of G such that $\{u,v\} \in E$. If v is an element of W', then

$$\frac{\partial}{\partial z_{lj}^{uv}} Q_L^W(Z) = \frac{1}{z_{lj}^{uv}} Q_L^W(Z),$$

provided j is the element in the L' sequence which corresponds to v and

$$\frac{\partial}{\partial z_{lj}^{uv}} Q_L^W(Z) = 0$$

if the element in the L' sequence corresponding to v is not j.

If v is not an element of W', then

$$\frac{\partial}{\partial z_{lj}^{uv}} Q_L^W(Z) = \frac{\partial}{\partial z_{lj}^{uv}} Q_{(L,j)}^{(W,v)}(Z) = \frac{1}{z_{lj}^{uv}} Q_{(L,j)}^{(W,v)}(Z).$$

Denoting by d_0 the number of vertices v in the sequence W' such that $\{u,v\} \in E$, we obtain

(3.4.1)

$$\sum_{\substack{v:\, \{u,v\} \in E \\ j:\, 1 \leq j \leq k}} |z_{lj}^{uv}| \left| \frac{\partial}{\partial z_{lj}^{uv}} Q_L^W(Z) \right| = d_0 \left| Q_L^W(Z) \right| + \sum_{\substack{v \text{ not in } W' \\ \{u,v\} \in E \\ 1 \leq j \leq k}} \left| Q_{(L,j)}^{(W,v)}(Z) \right|.$$

On the other hand, from (3.1.1) and Lemma 3.2, we conclude that for each v not in the sequence W', we have

(3.4.2) $$\left| Q_L^W(Z) \right| \geq \left(\cos \frac{\theta}{2} \right) \sum_{j=1}^{k} \left| Q_{(L,j)}^{(W,v)}(Z) \right|.$$

Denoting by d_1 the number of vertices v not in the sequence W' such that $\{u, v\} \in E$, we deduce from (3.4.1) and (3.4.2) that

$$\sum_{\substack{v:\ \{u,v\}\in E \\ j:\ 1\leq j\leq k}} |z_{lj}^{uv}| \left| \frac{\partial}{\partial z_{lj}^{uv}} Q_L^W(Z) \right| \leq d_0 \left| Q_L^W(Z) \right| + \frac{d_1}{\cos \frac{\theta}{2}} \left| Q_L^W(Z) \right|,$$

from which the proof follows. ∎

(3.5) Proof of Theorem 1.6. One can see that for all sufficiently small $\alpha > 0$, the equation

$$\theta = \frac{2\alpha}{(1-\alpha)\cos\frac{\theta}{2}}$$

has a solution $0 \leq \theta < 2\pi/3$. Numerical computations show that one can choose

$$\alpha = 0.35 \text{ and } \theta \approx 1.420166551.$$

Let

$$\tau = \cos \frac{\theta}{2} \approx 0.7583075916.$$

Given a graph $G = (V, E)$ we define

$$\delta = \frac{\alpha}{\Delta(G)}$$

and prove by descending induction on $n = |V|, \ldots, 1$ the following three statements (3.5.1)–(3.5.3).

(3.5.1) For any sequence W of n distinct vertices of G, for every sequence L of not necessarily distinct indices $1 \leq l \leq k$ such that $|W| = |L|$, for any $Z \in \mathcal{U}(\delta)$, we have $Q_L^W(Z) \neq 0$;

(3.5.2) Let W be a sequence of n distinct vertices of G such that $W = (W', v)$ and let L' be a sequence of not necessarily distinct indices $1 \leq l \leq k$ such that $|L'| = |W'|$. Then for every $1 \leq i, j \leq k$ and every $Z \in \mathcal{U}(\delta)$, the angle between $Q_{(L',i)}^{(W',v)}(Z)$ and $Q_{(L',j)}^{(W',v)}(Z)$, interpreted as vectors in $\mathbb{R}^2 = \mathbb{C}$, does not exceed θ;

(3.5.3) Let W be a sequence of n distinct vertices of G such that $W = (W', u)$ and let L be a sequence of not necessarily distinct indices $1 \leq l \leq k$ such that $L = (L', l)$ and $|W| = |L|$. Then for all $Z \in \mathcal{U}(\delta)$, we have

$$\left| Q_L^W(Z) \right| \geq \frac{\tau}{\Delta(G)} \sum_{\substack{v:\ \{u,v\}\in E \\ j:\ 1\leq j\leq k}} |z_{lj}^{uv}| \left| \frac{\partial}{\partial z_{lj}^{uv}} Q_L^W(Z) \right|.$$

Suppose that $n = |V|$. If $W = (v_1, \ldots, v_n)$ and $L = (l_1, \ldots, l_n)$, then

$$Q_L^W(Z) = \prod_{\{v_i, v_j\} \in E} z_{l_i j_j}^{v_i v_j} \neq 0,$$

so (3.5.1) holds. Moreover, denoting $\deg(v_n)$ the degree of v_n, we obtain

$$\sum_{\substack{v: \, \{v_n, v\} \in E \\ j: \, 1 \leq j \leq k}} \left| z_{l_n j}^{v_n v} \right| \left| \frac{\partial}{\partial z_{l_n j}^{v_n v}} Q_L^W(Z) \right| = \deg(v_n) \left| Q_L^W(Z) \right|,$$

so (3.5.3) holds as well.

Statements (3.5.1) and (3.5.3) for sequences W of length n and Lemma 3.3 imply statement (3.5.2) for sequences W and L of length n.

Formula (3.1.1), Lemma 3.2 and statement (3.5.2) for sequences W of length n imply statement (3.5.1) for sequences W of length $n - 1$.

Statements (3.5.1) and (3.5.2) for sequences W of length n and Lemma 3.4 imply statement (3.5.3) for sequences W of length $n - 1$.

This proves that (3.5.1)–(3.5.3) hold for sequences W of length 1. Formula (3.1.1), Lemma 3.2 and statement (3.5.2) for $n = 1$ imply that $Q_G(Z) \neq 0$ for all $Z \in \mathcal{U}(\delta)$.

We can improve the value of the constant α by defining θ as a solution to the equation

$$\theta = \frac{2\alpha}{\left(1 - \frac{\alpha}{\Delta(G)}\right) \cos \frac{\theta}{2}}.$$

Numerical computations show that one can choose $\alpha = 0.55$ provided $\Delta(G) \geq 30$ and that one can choose $\alpha = 0.46$ provided $\Delta(G) \geq 3$. ∎

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