

PROOF OF SCHUR'S CONJECTURE IN \mathbb{R}^D

ANDREY B. KUPAVSKII*, ALEXANDR POLYANSKII†

Received October 26, 2014

Revised December 28, 2015

Online First December 22, 2016

In this paper we prove Schur's conjecture in \mathbb{R}^d , which states that any diameter graph G in the Euclidean space \mathbb{R}^d on n vertices may have at most n cliques of size d . We obtain an analogous statement for diameter graphs with unit edge length on a sphere S_r^d of radius $r > 1/\sqrt{2}$. The proof rests on the following statement, conjectured by F. Morić and J. Pach: given two unit regular simplices Δ_1, Δ_2 on d vertices in \mathbb{R}^d , either they share $d-2$ vertices, or there are vertices $v_1 \in \Delta_1, v_2 \in \Delta_2$ such that $\|v_1 - v_2\| > 1$. The same holds for unit simplices on a d -dimensional sphere of radius greater than $1/\sqrt{2}$.

1. Introduction

One of the classical problems in discrete geometry, raised by P. Erdős in 1946 [7], is the following: given n points in the plane, how many unit distances they may determine? The key definition related to the question of P. Erdős is that of a *unit distance graph*. A graph G is a *unit distance graph* in \mathbb{R}^d if its set of vertices is a finite subset of \mathbb{R}^d and the edges are formed by the pairs of vertices which are at unit distance apart. In terms of distance graphs the question is to determine the maximal number of edges in a planar unit distance graph on n vertices. In this paper we focus on the questions of this type for *diameter graphs*. A graph $G = (V, E)$ is a *diameter graph* in \mathbb{R}^d , if

Mathematics Subject Classification (2000): 52C10

* Research supported in part by the Swiss National Science Foundation Grants 200021-137574 and 200020-14453 and by the grant N 15-01-03530 of the Russian Foundation for Basic Research.

† Research supported in part by the Presedent Grant MK-3138.2014.1.

$V \subset \mathbb{R}^d$ is a finite set of diameter 1, and edges of G are formed by the pairs of vertices that are at unit distance apart.

Diameter graphs arise naturally in the context of the finite version of the famous Borsuk's problem (see, e.g., [3,18] for the survey on Borsuk's problem), which is stated as follows: is it true that any (finite) set of unit diameter in \mathbb{R}^d can be partitioned into $d+1$ subsets of strictly smaller diameter? The finite version is equivalent to the following question concerning diameter graphs: is it true that any diameter graph G in \mathbb{R}^d satisfies $\chi(G) \leq d+1$?

A question about diameter graphs analogous to the question from the first paragraph has a simple answer: any set of n points in the plane generates at most n diameters, or any diameter graph on n vertices in the plane has at most n edges. This was proved by H. Hopf and E. Pannwitz in [9]. Interestingly, this result leads to a simple proof of the fact that Borsuk's question for finite sets in the plane have a positive answer. Indeed, it is easy to derive combinatorially that any graph G on n vertices with at most n edges and such that any of its subgraphs has at least as many vertices as edges satisfies $\chi(G) \leq 3$. A. Vázsonyi conjectured that any diameter graph in \mathbb{R}^3 on n vertices has at most $2n-2$ edges. Again, it is easy to see that Borsuk's conjecture for finite sets in \mathbb{R}^3 follows from this statement. Vázsonyi's conjecture was proved independently by B. Grünbaum [10], A. Heppes [11] and S. Straszewicz [20]. An interesting generalization of this result to the case of k -th diameters was obtained by F. Morić and J. Pach [17].

While the maximum number of edges in a diameter graph in $\mathbb{R}^2, \mathbb{R}^3$ is linear in the number of vertices, it becomes quadratic already in \mathbb{R}^4 . To put the discussion in a more general context, we introduce the following notations. Denote by $D_d(l, n)$ ($U_d(l, n)$) the maximum number of cliques of size l in a diameter (unit distance) graph on n vertices in \mathbb{R}^d . P. Erdős [7,8] studied $U_d(2, n)$ and $D_d(2, n)$ for different d . He showed that for $d \geq 4$ we have $U_d(2, n), D_d(2, n) = \frac{\lfloor d/2 \rfloor - 1}{2^{\lfloor d/2 \rfloor}} n^2 + o(n^2)$. K. Swanepoel [21] determined $U_d(2, n)$ for fixed even $d \geq 6$ and sufficiently large n depending on d and determined $D_d(2, n)$ for $d \geq 4$ and sufficiently large n .

Functions $D_d(l, n)$, $U_d(l, n)$ for $l > 2$ and similar functions were studied in several papers. In particular, the following conjecture was raised in [19]:

Conjecture 1 (Schur et al., [19]). *We have $D_d(d, n) = n$ for $n \geq d+1$.*

This was proved by H. Hopf and E. Pannwitz for $d=2$ in [9] and for $d=3$ by Z. Schur et al. in [19]. In the latter paper the authors also proved that $D_d(d+1, n) = 1$. In [16] P. Morić and J. Pach progressed towards resolving this conjecture. Namely, they showed that Schur's conjecture holds in the following special case:

Theorem 2 (Theorem 1 from [16]). *Given a diameter graph G on n vertices in \mathbb{R}^d , the number of d -cliques in G does not exceed n , provided that any two d -cliques share at least $d - 2$ vertices.*

As it turns out, Schur's conjecture and related questions are tightly connected with analogous questions for spherical sets. The spherical analogues were studied in a few papers. In particular, in the paper [4] V. Bulankina et al. noted that the statement of Theorem 2 holds for spheres of large radii: given a diameter graph G on n vertices in a d -dimensional sphere S_r^d with radius $r > 1/\sqrt{2}$, the number of d -cliques in G does not exceed n , provided that any two d -cliques share at least $d - 2$ vertices (Theorem 4 from [4]). Moreover, they showed that Schur's conjecture holds for S_r^3 for $r > 1/\sqrt{2}$. To be precise, we formulate Schur's conjecture for spheres separately:

Conjecture 3 (Schur's conjecture for spheres). *Any diameter graph G on n vertices (and with edges of unit Euclidean length) on a sphere S_r^d with $r > 1/\sqrt{2}$ has at most n d -cliques.*

In the paper [12] A. Kupavskii studied properties of diameter graphs in \mathbb{R}^4 , in particular proving the four-dimensional Schur's conjecture. The following theorem completes the description of the quantity $D_4(l, n)$ for different l :

Theorem 4 (Theorem 5 from [12]).

1. For $n \geq 52$ we have

$$D_4(2, n) = \begin{cases} \lfloor n/2 \rfloor \lfloor n/2 \rfloor + \lceil n/2 \rceil + 1, & \text{if } n \not\equiv 3 \pmod{4}, \\ \lfloor n/2 \rfloor \lfloor n/2 \rfloor + \lceil n/2 \rceil, & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

(In Corollary 3 from [21] the same was proved for sufficiently large n .)

2. For all sufficiently large n we have

$$D_4(3, n) = \begin{cases} (n - 1)^2/4 + n, & \text{if } n \equiv 1 \pmod{4}, \\ (n - 1)^2/4 + n - 1, & \text{if } n \equiv 3 \pmod{4}, \\ n(n - 2)/4 + n, & \text{if } n \equiv 0 \pmod{2}. \end{cases}$$

3. (Schur's conjecture in \mathbb{R}^4) For all $n \geq 5$ we have $D_4(4, n) = n$.

In [12] the first author also studied diameter graphs on S_r^3 with $r > 1/\sqrt{2}$. In particular, he showed that an analogue of Vázsonyi's conjecture holds for diameter graphs on spheres.

In the next section we present our main results and discuss related questions. In Section 3 we introduce some basic objects that are used in the proof. In Section 4 we present the proofs of the results.

2. New results and discussion

The main result of this paper is the proof of Schur's conjecture both in the Euclidean space and on the sphere in any dimension:

Theorem 5. *Schur's conjecture holds*

1. *In the space \mathbb{R}^d ,*
2. *On the sphere S_r^d of radius $r > 1/\sqrt{2}$.*

The proof of the first part actually relies heavily on the second part, so the questions for the Euclidean space and for the sphere are indeed interconnected.

Remark. Note that throughout the article by a k -simplex in \mathbb{R}^d we mean a set of $k+1$ vertices in \mathbb{R}^d in general position.

Next we discuss several questions mentioned in the paper [16]. Since the authors of [16] proved Theorem 2, they naturally raised the following problem:

Conjecture 6 (F. Morić and J. Pach, Problem 1 from [16]). *Any two unit regular simplices on d vertices in \mathbb{R}^d must share at least $d-2$ vertices, provided the diameter of their union is 1.*

We confirm this conjecture (and its spherical version) in our paper, which together with Theorem 2 and its spherical analogue from [4], mentioned in the previous section, gives us the proof of Schur's conjecture both in the space and on the sphere. Another problem the authors of [16] raised deals with irregular simplices.

Conjecture 7 (Conjecture 3 from [16]). *Let a_1, \dots, a_d and b_1, \dots, b_d be two simplices on d vertices in \mathbb{R}^d with $d \geq 3$, such that all their edges have length at least 1. Then there exist $i, j \in \{1, \dots, d\}$ such that $\|a_i - b_j\| \geq 1$.*

By slightly modifying the proof of Theorem 5 it is not difficult to obtain the following theorem:

Theorem 8. *Consider a regular unit simplex $\{a_1, \dots, a_d\}$ and a simplex $\{b_1, \dots, b_d\}$ in \mathbb{R}^d (or on S_r^d with $r > 1/\sqrt{2}$), where the second simplex satisfies the property $\|b_i - b_j\| \geq 1$ for $i \neq j$. Then either these two simplices share $d-2$ vertices, or $\|a_i - b_j\| > 1$ for some $i, j \in \{1, \dots, d\}$.*

This theorem solves Conjecture 7 in a stronger form in the case where one of the two simplices is regular. We omit the proof, but the main additional ingredient needed is that the radius of the smallest ball that contains the simplex b_1, \dots, b_d is at least as big as for a regular unit $(d-1)$ -simplex, provided that $\|b_i - b_j\| \geq 1$ for $i \neq j$. This, in turn, is an easy application of Kirszbraun's theorem (see [1] for a short and nice proof):

Theorem 9 (Kirschbraun's theorem). *Let U be a subset of X , where X is \mathbb{R}^d , S^d or H^d (a d -dimensional hyperbolic space). Then any nonexpansive map $f: U \rightarrow X$ can be extended to a nonexpansive map $f': X \rightarrow X$. A nonexpansive map $f: Y \rightarrow X$ is a map which satisfies $\|f(a) - f(b)\| \leq \|a - b\|$ for any $a, b \in Y$.*

After having prepared the first version of this paper, we came across a paper by H. Maehara [15], in which the author studies a seemingly unrelated concept of *sphericity* of a graph: given a graph G , the *sphericity* of G is the minimum dimension in which the vertices of the graph can be represented as unit spheres in such a way that two spheres intersect (or touch) iff the corresponding vertices are connected by an edge. In [15] the author discusses the sphericity of complete bipartite graphs. And, as it turned out, the main result of the paper is, in fact, the proof of Conjecture 7, which was given 20 years before the conjecture was formulated! For a bit more on Maehara's result in the context of Schur's conjecture see Section 4.1.

Finally, in the paper [16] the authors raised the following general problem:

Problem 10 (Problem 6 from [16]). *For a given d , characterize all pairs k, l of integers such that for any set of k red and l blue points in \mathbb{R}^d we can choose a red point r and a blue point b such that $\|r - b\|$ is at least as large as the smallest distance between two points of the same color.*

For $k = d + 1$ and $l = \lfloor \frac{d+1}{2} \rfloor$ it is not difficult to construct an example of two regular unit simplices in \mathbb{R}^d on k and l vertices respectively, such that the distance between any two vertices from different simplices is smaller than 1, which we describe at the end of the next section. (An analogous, but somewhat different, example appeared in the latter version of the paper [16].) We think that this is the extremal example, thus, we conjecture the following.

Conjecture 11. *Given two unit simplices in \mathbb{R}^d , one on $d + 1$ vertices, the other on $\lfloor \frac{d+1}{2} \rfloor + 1$ vertices, either they share a vertex, or the diameter of their union is strictly larger than 1.*

In [13] the authors proved Conjecture 11 for $d = 4$.

3. Preliminaries

Given a hyperplane π , we denote by π^+ and π^- two closed half-spaces (half-spheres in the spherical case) that are determined by π .

The following object is very important for understanding diameter graphs:

Definition 1. A Reuleaux simplex Δ in \mathbb{R}^d is a set formed by the intersection of the balls $B_i = B_1^d(v_i)$ of unit radius with centers in $v_i, i = 1, \dots, d+1$, where v_i 's are the vertices of a unit simplex in \mathbb{R}^d . In the case $d = 3$ we call this object a Reuleaux tetrahedron, and in the case $d = 2$ we call it a Reuleaux triangle.

We denote the $(d - 1)$ -dimensional spheres of unit radii with centers in v_1, \dots, v_{d+1} (the boundary spheres of B_1, \dots, B_{d+1}) by S_1, \dots, S_{d+1} . A Reuleaux simplex is a spherical polytope, so one can naturally partition the boundary of a Reuleaux simplex into spherical faces of different dimensions: the vertices of the underlying simplex are the zero-dimensional faces, the arcs that connect the vertices are the one-dimensional faces and so on. We discuss it in more details a bit later in this section. The analogous definition could be given in the case of $S_r^d, r > 1/\sqrt{2}$. In this case we call the body a *spherical Reuleaux simplex*. The only thing one has to keep in mind is that on a d -dimensional sphere we still consider spherical Reuleaux simplices on $d + 1$ vertices. Note that, by Jung's theorem, on a d -dimensional sphere of radius $r = \sqrt{(d + 1)/(2d + 4)}$ one can have a regular unit $(d + 2)$ -simplex, which is, however, impossible for other radii (and, in particular, impossible for $r > 1/\sqrt{2}$).

For a given set W we denote its interior by $\text{int } W$. In the paper we use several times the following simple observation.

Observation 12. Consider two d -balls B, B' in \mathbb{R}^d of radii r, r' , correspondingly. Denote their boundary spheres by S, S' . Assume that $r > r'$ and that S and S' intersect in a $(d - 2)$ -dimensional sphere. Denote the hyperplane that contains $S \cap S'$ by σ and assume that the centers of S, S' are in the same closed halfspace σ^+ with respect to σ . The other halfspace we denote by σ^- . Then $B \cap \sigma^+ \supset B' \cap \sigma^+$ and $B \cap \sigma^- \subset B' \cap \sigma^-$. Moreover, $\text{int}(B \cap \sigma^+) \supset B' \cap \text{int}(\sigma^+)$ and $B \cap \text{int}(\sigma^-) \subset \text{int}(B' \cap \sigma^-)$.

See Figure 1, illustrating the observation. We apply the observation above to deduce the following two lemmas.

Lemma 13. Consider a Reuleaux simplex $\Delta \subset \mathbb{R}^d$ with the set of vertices $v_i, i = 1, \dots, d + 1$ and a ball B with a boundary sphere S , circumscribed around the d -simplex $\{v_1, \dots, v_{d+1}\}$. Then Δ lies inside B , moreover, $\Delta \cap S = \{v_1, \dots, v_{d+1}\}$.

Proof. Denote by π_i the hyperplane that passes through the vertices $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{d+1}$. We denote the closed halfspace defined by π_i and that contains v_i by π_i^+ . This halfspace contains the convex hull of the vertices

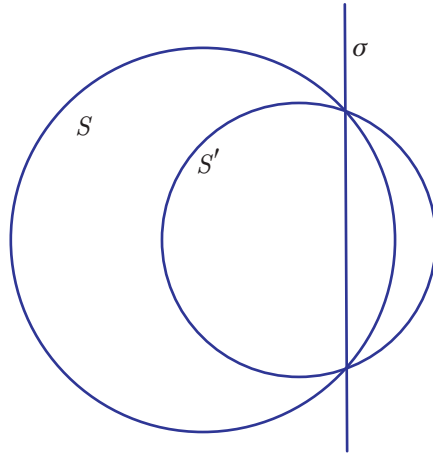


Figure 1. Ball B contains the part of B' to the left from σ

$\{v_1, \dots, v_{d+1}\}$. By π_i^- we denote the other halfspace. Applying Observation 12 to the balls B_i, B , we get that $\text{int } B \supset \text{int}(B \cap \pi_i^-) \supset B_i \cap \text{int}(\pi_i^-)$. After going through all possible values of i , we get that

$$\text{int } B \supset \cup_{i=1}^{d+1} (B_i \setminus \pi_i^+) \supset (\cap_{i=1}^{d+1} B_i) \setminus (\cap_{i=1}^{d+1} \pi_i^+) = \Delta \setminus (\cap_{i=1}^{d+1} \pi_i^+).$$

On the other hand, $\cap_{i=1}^{d+1} \pi_i^+$ is just a convex hull of the points $\{v_1, \dots, v_{d+1}\}$, and it is for sure contained in B , moreover, it intersects S only in its vertices v_1, \dots, v_{d+1} . ■

Lemma 14. Consider a Reuleaux simplex $\Delta \subset \mathbb{R}^d$ with the set of vertices $v_i, i = 1, \dots, d + 1$. Then the intersection of Δ with the hyperplane π that passes through v_1, \dots, v_d is a Reuleaux simplex Δ' with vertices $v_i, i = 1, \dots, d$.

Proof. We have $\Delta' = \pi \cap \bigcap_{i=1}^{d+1} B_i$ and we have to prove that $\Delta' = \pi \cap \bigcap_{i=1}^d B_i$. Thus, it is enough to show that $\pi \cap \bigcap_{i=1}^d B_i \subset B_{d+1} \cap \pi$. Denote the circumscribed ball of the Reuleaux simplex $\pi \cap \bigcap_{i=1}^d B_i$ by B' and its boundary sphere by S' . Then $B_{d+1} \cap \pi = B'$, and the statement of the lemma follows from the first claim of Lemma 13. ■

We need some knowledge about the structure of a Reuleaux simplex Δ as a spherical polytope. Let the vertices of Δ be v_1, \dots, v_{d+1} and their convex hull be denoted by T (see Figure 2, where T is the shaded triangle and Δ is the set bounded by the arcs). The boundary of the Reuleaux simplex Δ can be partitioned into relatively open spherical regions of different dimensions in the following way. Consider all spheres U that are formed as intersections

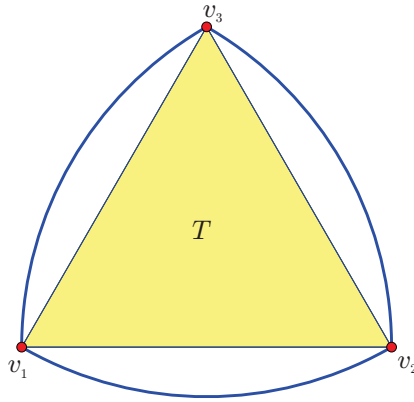


Figure 2. A simplex, its convex hull T and its Reuleaux simplex Δ

of several spheres out of S_1, \dots, S_{d+1} . For example, the intersection of the first d spheres is a two-point set, with one of its points being v_{d+1} . We only exclude the intersection of all the spheres, which is empty. We denote the set of all such spheres U by \mathcal{S} . For each point on the boundary of Δ we may find the sphere from \mathcal{S} of minimal dimension that contains it. The set of all points from the boundary of Δ that correspond to a given $U \in \mathcal{S}$ we call a *face*. It is defined by the set of strict quadratic inequalities and, thus, is naturally a relatively open set.

The center of each sphere $U \in \mathcal{S}$ coincides with the center of some of the faces of T . Namely, if $U = \bigcap_{j=1}^l S_{i_j}$, where $1 \leq i_1 < i_2 < \dots < i_l \leq d+1$, then the center of U is the centerpoint of the $(l-1)$ -dimensional face with the vertices v_{i_1}, \dots, v_{i_l} . On the other hand, U contains all the vertices from $\{v_1, \dots, v_{d+1}\} \setminus \{v_{i_1}, \dots, v_{i_l}\}$ and, since U is a $(d-l)$ -dimensional sphere and points $\{v_1, \dots, v_{d+1}\}$ are in general position, these points determine U . We call these points the *vertices of the face*.

Each face of Δ is a connected set, moreover, it is obtained from the face of T with the same set of vertices via projection from the center O of T on the boundary of Δ . We verify this property in what follows. Fix a vertex set of the face F of Δ . W.l.o.g., it is $\{v_1, \dots, v_i\}$. The points of F on the boundary of Δ are the ones that, first, are at distance 1 from v_{i+1}, \dots, v_{d+1} and, second, are at distance strictly less than 1 from v_1, \dots, v_i . From the first condition we have $F \subset S_{i+1} \cap \dots \cap S_{d+1} \subset \alpha$, where the flat α is an affine hull of the points O, v_1, \dots, v_i (we may replace O by the centerpoint of the face $\{v_{i+1}, \dots, v_{d+1}\}$ of T). Since Ov_j has the same length for all j , the second condition is equivalent to the fact that each point w from F satisfies the angular inequality $\angle wOv_j \leq \angle wOv_{i+1}$ for all $j = 1, \dots, i$. Each of these

inequalities is satisfied in a halfspace, defined by the hyperplane that passes through O and vertices $\{v_1, \dots, v_{d+1}\} \setminus \{v_j, v_{i+1}\}$. These halfspaces bound the face $\{v_1, \dots, v_i\}$ in $T \cap \alpha$ and, as we have showed above, they also bound the face with the same set of vertices in $\Delta \cap \alpha$. Moreover, they pass through O , which concludes the proof of the statement about the central projection. We formulate the findings of the last three paragraphs in a lemma:

Lemma 15. *In the notations introduced above, consider a Reuleaux simplex $\Delta \subset \mathbb{R}^d$. Then its boundary may be split into relatively open connected spherical regions (faces), each of which corresponds to an intersection of several spheres out of S_1, \dots, S_{d+1} . The face F of Δ that corresponds to the intersection of the spheres S_{i+1}, \dots, S_{d+1} lies on the sphere with the center in the centerpoint C of the face F' of T with vertices v_{i+1}, \dots, v_{d+1} : $C = (v_{i+1} + \dots + v_{d+1}) / (d - i + 1)$. The flat of minimal dimension that contains F is an affine hull of points O, v_1, \dots, v_i . Moreover, the face F is equal to the central projection of the convex hull of $\{v_1, \dots, v_i\}$ from the center of Δ to the boundary of Δ .*

Next we define the object which is of a particular importance for the paper:

Definition 2. A rugby ball Θ in \mathbb{R}^d is a set formed by the intersection of the balls $B_i = B_1^d(v_i)$ of unit radius with centers in $v_i, i = 1, \dots, d$, where v_i 's are the vertices of a unit $(d - 1)$ -simplex in \mathbb{R}^d .

We omit the analogous definition of a *spherical rugby ball*. Note the difference between a Reuleaux simplex and a rugby ball. The latter is an intersection of d balls instead of $d + 1$ for the former. The intersection of the hyperplane π that passes through v_1, \dots, v_d and the corresponding rugby ball is a Reuleaux simplex of codimension 1. The rugby ball is symmetric with respect to π .

Consider a Reuleaux simplex Δ on the vertices v_1, \dots, v_{d+1} , the rugby ball Θ on the vertices v_1, \dots, v_d , and the hyperplane π containing vertices v_1, \dots, v_d . Suppose that $v_{d+1} \in \pi^+$.

Lemma 16. *In the notations introduced above, we have $\Delta^+ := \Delta \cap \pi^+ = \Theta \cap \pi^+$.*

Proof. Since $\Delta \cap \pi^+ = \Theta \cap \pi^+ \cap B_{d+1}$, it is obviously sufficient to show that $B_{d+1} \cap \pi^+ \supset \Theta \cap \pi^+$. Consider a ball B circumscribed around Δ . By using the same argument as in Lemma 13, we get that $B \cap \pi^+ \supset \Theta \cap \pi^+$. On the other hand, applying Observation 12 to B, B_{d+1} , we get that $B_{d+1} \cap \pi^+ \supset B \cap \pi^+$. ■

Now we describe the construction mentioned in the end of the previous section. Take a regular simplex on $d + 1$ vertices in \mathbb{R}^d as the set of red points.

Next, construct the Reuleaux simplex on the the red points and choose $l = \lfloor \frac{d+1}{2} \rfloor$ midpoints y_1, \dots, y_l of some l pairwise disjoint arcs (1-dimensional faces) that connect the vertices of the Reuleaux simplex. It could be checked that the distance between the midpoints of two such arcs is strictly bigger than 1. To see this, one have to consider a coordinate representation of the simplex $\{v_1, \dots, v_{d+1}\}$ in the hyperplane $x_1 + \dots + x_{d+1} = 1$ in \mathbb{R}^{d+1} and calculate the coordinates of such a midpoint. Thus, if we consider the simplex on y_1, \dots, y_l and contract it a little, we will get a simplex on vertices x_1, \dots, x_l with all vertices inside the Reuleaux simplex and with all sides greater than 1. We take $\{x_i\}$ as the set of blue points, which together with the red points gives us the desired example.

4. Proof

4.1. Reduction to an auxiliary theorem

The proof of Theorem 5 involves an inductive argument based on the following auxiliary theorem, which is of interest by itself:

Theorem 17. *Given a diameter graph G*

1. *In the space \mathbb{R}^d , $d \geq 3$;*
2. *On the sphere S_r^d of radius $r > 1/\sqrt{2}$, $d \geq 3$, any two d -cliques in G must share a vertex.*

As we came across the paper [15] we were thinking whether or not to try to give a proof of Theorem 17 using Maehara's result. The Euclidean case of Theorem 17 is almost equivalent to [15, Theorem 2] (which is equivalent to Conjecture 7). However, to get part 1 of Theorem 17 from Conjecture 7, one has to replace the non-strict inequality on distances by a strict one. It turns out that this seemingly technical detail is not easy to overcome. If applied directly, [15, Theorem 2] gives only that, if there are two regular unit $(d-1)$ -simplices in a diameter graph in \mathbb{R}^d that do not share a vertex, then there must be at least one edge between them. This is clearly not sufficient (and it is fairly easy to reduce Theorem 17 to this case). The proof of Maehara, however, may be modified to give a proof of Theorem 17 in the Euclidean case, but becomes significantly more complicated. Maehara's technique relies heavily on linear algebra, thus we do not even know if it is possible to apply his techniques in the spherical case. We believe that in sum it would not simplify the proof of Theorem 17. Therefore, we decided to leave the proof as it is and not to utilize Maehara's ideas.

In this subsection we describe how to derive Theorem 5 from Theorem 17. Consider two d -cliques in a diameter graph G in \mathbb{R}^d (or on S_r^d with $r > 1/\sqrt{2}$). Then, by Theorem 17, these two cliques must share a vertex. All the remaining vertices of the two simplices must lie on the $(d-1)$ -dimensional unit sphere S with the center in the common vertex of the two simplices. The vertices on S form two $(d-2)$ -dimensional unit simplices, and, since the subgraph that lies on S is a diameter graph, we can again apply Theorem 17 and obtain that the two $(d-2)$ -dimensional simplices on S must share a vertex, which gives the second common vertex for the d -cliques.

Finally, we obtain that any two d -cliques must share $d-2$ common vertices and apply a spherical analogue of Theorem 2, which was proved in [4]. This completes the proof of Schur's conjecture. We only have to verify the following: the spheres that we work with during this process always have radius greater than $1/\sqrt{2}$. This was shown to be true in [4, Lemma 4]. We state this fairly easy lemma and present its proof for completeness.

Lemma 18. *Consider a d -dimensional sphere $S = S_r^d$ of radius $r > 1/\sqrt{2}$ and a unit simplex Δ on k vertices v_1, \dots, v_k with all its vertices on S . Then the intersection Ω of the sphere S and the k unit spheres with centers in v_1, \dots, v_k is a sphere of radius $r_\Omega > 1/\sqrt{2}$.*

Proof. We assume that the sphere is embedded into a Euclidean space, and we work in that space. Denote by $v = \frac{1}{k} \sum_{i=1}^k v_i$ the center of the sphere S' , circumscribed around Δ . By Jung's theorem, the radius r' of S' is equal to $\sqrt{\frac{k-1}{2k}}$. So, the radius r'' of the sphere S'' , which is the intersection of k unit spheres with centers in v_1, \dots, v_k is $\sqrt{1 - \frac{k-1}{2k}} = \sqrt{\frac{k+1}{2k}}$. Note that the center of S'' is also v . Denote the center of S by O . Then the center w of Ω lies on the segment Ov of length b . Since v_1, \dots, v_k lie on S , we have $b^2 = r^2 - (r')^2 = r^2 - \frac{k-1}{2k}$. Suppose w splits the segment Ov into the parts of length $b-a, a$, respectively. Then, since $\Omega \subset S$, we get $r_\Omega^2 = r^2 - (b-a)^2$. We also have $\Omega \subset S''$ so we get $r_\Omega^2 = \frac{k+1}{2k} - a^2$. Therefore,

$$2r_\Omega^2 = r^2 - b^2 + \frac{k+1}{2k} + 2ab - 2a^2 = \frac{k-1}{2k} + \frac{k+1}{2k} + 2a(b-a) > 1,$$

because it is easy to see that $a, b-a > 0$. ■

4.2. Sketch of the proof of Theorem 17

Our main goal is to prove Theorem 17. The proof of this theorem also goes by induction. The base case $d=3$ is known to be true (it was verified for \mathbb{R}^3 in [6] and for S_r^3 with $r > 1/\sqrt{2}$ in [12]). We reduce the problem for the d -dimensional space or for the d -dimensional sphere to the analogous problem for the $(d-1)$ -dimensional sphere. Since the base case is already verified, this concludes the proof of the theorem.

To justify the induction step, we proceed as follows. We consider two unit simplices on d vertices K_1, K_2 and build a rugby ball Θ around K_1 . We analyze the possible positions of the vertices of K_2 with respect to Θ , in particular, with respect to the plane π that passes through K_1 .

The first step is to prove Lemma 19, which, roughly speaking, tells us that if we have two vertices in one of the halfspaces π^+, π^- , one of which projects inside the convex hull $T \subset \pi$ of K_1 , then K_1 and K_2 have common vertices. This already reduces a lot the work to be done.

The second step, which is considered in (i) of the next subsection, is to consider the case when none of the vertices of K_2 are projected strictly inside T . In that case we use the fact that then all these vertices lie inside a full-dimensional ball B , circumscribed around K_1 and with the center in the center of K_1 . This is due to the fact, stated in Observation 20, that Θ is contained in the union of B and the set of points of Θ that project inside T . Once we know that K_2 is contained in B , we conclude that it is contained in the boundary S of B , which leads to a conclusion that all the vertices are actually projected inside T . This leads to the same conclusion as in the previous step.

Finally, if none of the two are true, we have just one vertex w_1 in π^+ , which projects inside T , and the rest in π^- . This corresponds to the case (ii) of the next subsection. We describe a procedure that rotates K_1 and creates more and more unit distances between w_1 and the vertices of K_1 . At the end this results in having a full-dimensional regular unit simplex $K_1 \cup \{w_1\}$. Then all the vertices of $K_1 \cap K_2$ apart from w_1 lie on a unit sphere centered at w_1 and we can apply induction.

It is tempting to give a unified proof of Theorem 17, in which the Euclidean and the spherical cases are both treated at the same time. But, on the other hand, it makes the proof more difficult to understand. So we chose an intermediate option. We give two separate proofs, first for the Euclidean case and then we describe the differences and peculiarities of the spherical case in a separate subsection. The key lemma (Lemma 19), however, has a unified proof.

4.3. Proof of Theorem 17. Euclidean case

We begin with the following important lemma:

Lemma 19. *Take a Reuleaux simplex Δ in \mathbb{R}^d and the hyperplane π containing the vertices v_1, \dots, v_d of Δ . Consider the body Δ^+ . Suppose $v, w \in \Delta^+$, and suppose that the projection v' of v on the hyperplane π lies inside the convex hull T of v_1, \dots, v_d . Then $\|v-w\| \leq 1$, with the equality possible in the following two cases: 1. One of the vertices v, w coincides with one of v_1, \dots, v_d . 2. The vertex w lies in the hyperplane π on the border of a Reuleaux simplex Δ_π , constructed on the vertices v_1, \dots, v_d . At the same time the projection v' of the vertex v on the hyperplane π must lie on ∂T .*

We defer the proof of this lemma until Section 4.5, where we give a unified proof in both Euclidean and spherical cases.

Consider a diameter graph G and two d -cliques K_1, K_2 in G . Denote by v_1, \dots, v_d the vertices of K_1 . Form a rugby ball Θ on K_1 and denote the hyperplane containing K_1 by π . The following step is essential for the proof. Consider a d -dimensional ball B , circumscribed around the clique K_1 . It has a center in the center O of the clique K_1 and radius $\sqrt{(d-1)/2d}$, but has one dimension more than a normal circumscribed ball in the hyperplane π . Denote the boundary sphere of B by S . As usually, denote the $(d-1)$ -dimensional spheres of unit radii with centers in v_1, \dots, v_d (the boundary spheres of B_1, \dots, B_d) by S_1, \dots, S_d .

The set $S \cap S_i$ for any $i = 1, \dots, d$ is a sphere that lies in the hyperplane π_i orthogonal to π . Indeed, it is true due to the fact that both centers (O and v_i) lie in π . This together with Observation 12 gives us the following crucial observation.

Observation 20. *In the notations introduced above, whenever a point lies in $\Theta \setminus \text{int } B$, its projection on the hyperplane π falls inside the convex hull T of v_1, \dots, v_d . If a point lies in $\Theta \setminus B$, then its projection falls in $\text{int } T$.*

Suppose that there are at least two vertices w_1, w_2 of K_2 in $\pi^+ \cap \Theta$. If one of them, say w_1 , does not lie in B , then, by Observation 20, its projection on π falls strictly inside T , and we are done by Lemma 19. Indeed, checking the conditions in Lemma 19 that allow $\|w_1 - w_2\| = 1$ to hold, one sees that condition 2 does not take place since the projection of w_1 falls strictly inside T , so the first condition must hold and, consequently, one of the vertices w_1, w_2 must coincide with one of the vertices of K_1 . The same reasoning apply for any two points w_1, w_2 of K_2 lying in $\pi^- \cap \Theta$.

Now we are left with two possibilities.

(i) On both sides of the hyperplane π we have at least two points of K_2 , or all vertices of K_2 lie on one side. This case, which seems to be essential, actually has a short resolution. In this case all points from K_2 lie inside the ball B , and we are able to use some of its properties. Namely, we know that, since K_2 is a clique of size d , then the radius of the minimal ball that contains the clique equals the radius of B (even though it has a smaller dimension). This means that the center of that minimal ball must coincide with O , and all the points of K_2 must in fact lie on S (otherwise the minimal ball will have a smaller radius). This, by Observation 20, gives us that all the points of K_2 are projected inside T . Since there are at least 3 vertices in K_2 , then in one of the halfspaces, say, π^+ , there are at least two vertices w_1, w_2 of K_2 . We may then apply Lemma 19. The next step is to check the two conditions from the lemma that allow the equality $\|w_1 - w_2\| = 1$ to hold.

Condition 1 gives that one of w_1, w_2 coincides with one of the vertices of K_1 . Condition 2 gives that w_1 lies on π , therefore, $w_1 \in \Theta \cap \pi \cap S$. We note that that $\Theta \cap \pi$ is a Reuleaux simplex in the hyperplane π and $S \cap \pi$ is its circumscribed sphere. Thus, by Lemma 13, we conclude that $\Theta \cap \pi \cap S = \{v_1, \dots, v_d\}$. Therefore, in any case one of w_1, w_2 coincide with some of v_1, \dots, v_d and we are done in Case (i).

(ii) The other possibility is that exactly one vertex, say, w_1 , lies in π^+ , while the others lie in π^- . Moreover, we may assume that $w_1 \notin B$, otherwise, all the vertices of K_2 lie inside B and we argue as in the previous case.

We treat this case as follows. We prove that the two cliques have a common vertex by contradiction. Assuming the contrary, we start with a configuration of two d -cliques K_1, K_2 without common vertices of the type described above. We perturb the first clique, obtaining a valid configuration of two simplices K'_1, K_2 without common vertices, provided that the initial configuration was valid. Thus, if we obtain a contradiction at some point, it means that the initial configuration was as well impossible.

We try to perturb the simplex K_1 so that w_1 will get to the top of the rugby ball Θ , constructed on the perturbed K_1 , or, in other words, that w_1 will form a regular unit simplex with the (possibly perturbed) vertices v_1, \dots, v_d . Note that we do not modify K_2 .

Here is the procedure. Suppose the distance between w_1 and v_1 is strictly less than 1. We rotate v_1 around the vertices v_2, \dots, v_d , which are fixed. The possible trajectory of v_1 is a circle, and we push v_1 towards π^- . Denote the image of v_1 by v' . We stop the rotation if one of the following two events happen:

Event 1. The distance between v' and w_1 is equal to 1.

Event 2. Some of w_2, \dots, w_d fall on the hyperplane π' that passes through v', v_2, \dots, v_d .

Before analyzing these two possibilities, we have to make some preparations. We start with the following simple observation.

Observation 21. Consider two points x, y in \mathbb{R}^d and a hyperplane τ that passes through the middle of the segment xy and is orthogonal to it. Denote by τ^+ the closed halfspace bounded by τ and that contains x . Then for any point $z \in \tau^+$ we have $\|z - x\| \leq \|z - y\|$.

Next we formulate in a lemma that, while having valid configurations all the time, we eventually arrive at one of the two events above. Define B' as the image of B under the rotation, and similarly define Θ', S', S'_1 , and K'_1 .

- Lemma 22.**
1. The point w_1 stays inside Θ' , moreover, $w_1 \notin B'$.
 2. No vertex among w_2, \dots, w_d can escape from $\Theta' \cap (\pi')^-$ without falling onto π' first.
 3. No vertex among w_2, \dots, w_d can coincide with some of the vertices of K'_1 .

Proof. Part 1. The point w_1 stays inside Θ' since we do not move v_2, \dots, v_d and because of Event 1. In what follows we prove that $w_1 \notin B'$.

Consider the hyperplane γ , which contains the intersection of S and S' . It passes through the vertices v_2, \dots, v_d and through the middle of the segment vv' . Denote by γ^+ the halfspace that contains v . We apply Observation 21 for the hyperplane γ and the centers of balls B, B' , which are obviously symmetric with respect to γ . We get that $B \cap \gamma^+ \supset B' \cap \gamma^+$ and $B \cap \gamma^- \subset B' \cap \gamma^-$.

Next, we show that $w_1 \in \gamma^+$. This is due to the fact that any point from $\pi^+ \cap \Theta \setminus B$ projects from above inside the convex hull T of the vertices of the Reuleaux simplex $\Theta \cap \pi$ (see Observation 20). We have $T \subset \gamma^+$. Since the rotation is made continuously, we may for sure assume that the angular distance between v and v' is less than 90° . Therefore, any point that is projected on π inside T from above, lies in γ^+ (see Figure 3, where points that are projected inside T from above, lie in the shaded rectangle). From $\pi^+ \cap \Theta \setminus B \subset \gamma^+$ it follows that $\pi^+ \cap \Theta \setminus B \subset \pi^+ \cap \Theta \setminus B'$, and, consequently, $w_1 \notin B'$.

Part 2. This fact is proved in a similar fashion. Consider the hyperplane that contains the intersection of S_1 and S'_1 . It is again γ , moreover, due to Observation 21, we have $\Theta' \cap \gamma^- \supset \Theta \cap \gamma^-$. Indeed, since the spheres S_2, \dots, S_d do not change, we only have to look at the intersection of B_1 and B'_1 , and we fall into a situation which is similar to the one considered in the previous part. Therefore, the only way for a point w_i to escape Θ' is to fall onto γ

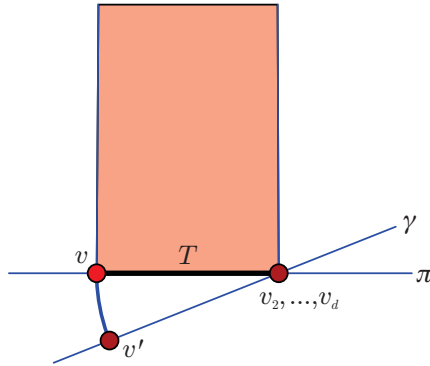


Figure 3. $\pi^+ \cap \Theta \setminus B$ projects inside T and thus is inside γ^+

first. But this is not possible, because any position of γ was a position of π' at the earlier stage of rotation, so the point w_i has to fall onto π' first.

Part 3. Clearly, w_i cannot coincide with v_2, \dots, v_d since none of them is moved during the rotation. Assume $w_i = v'$. But it means that before the rotation w_i lay on the same arc as v' . However, the projection of this arc on the hyperplane π is a straight segment connecting the vertex v_1 and the center O . Therefore, the projection of w_i onto π falls inside T , and, since we have more than 1 vertex in π^- , we have a common vertex between K_1 and K_2 by Lemma 19. This contradicts our assumptions. ■

The only thing that is left to do is to analyze the two events given by the procedure. Suppose that Event 2 happens, and the point w_2 from K_2 falls onto π' . Then we have two vertices of K_2 in $(\pi')^+$, and, since $w_2 \neq \{v', v_2, \dots, v_d\}$, we have a common vertex of K'_1 and K_2 using Lemma 19, a contradiction.

If Event 1 happens, then we take another vertex of K_1 and proceed in the same way. Finally, assume that we cannot continue the procedure and there is no common vertex of K'_1 and K_2 . Denote by v'_1, \dots, v'_d the images of the vertices of K_1 after all the rotations. Then w_1 forms a unit d -simplex with v'_1, \dots, v'_d . In this case all the vertices $v'_1, \dots, v'_d, w_2, \dots, w_d$ lie on the unit sphere with center in w_1 .

By induction, a unit $(d - 1)$ -simplex and a unit $(d - 2)$ -simplex on a $(d - 1)$ -dimensional sphere of radius greater than $1/\sqrt{2}$ must share a common vertex. Therefore, we get a contradiction in any case and K_1, K_2 must have a common vertex in the first place.

4.4. Proof of Theorem 17. Spherical case

4.4.1. Preliminaries on spherical geometry In Section 4.4 we work on a d -dimensional sphere Γ of radius greater than $1/\sqrt{2}$.

Spherical geometry is very similar to Euclidean. To make the proof work in this case, one should, more or less, only change the notation: planes are changed to diametral (or great) spheres, halfspaces to hemispheres, balls to spherical caps. We will often use the Euclidean names for the spherical objects, e.g., say “a plane” instead of “a diametral sphere”. This should not cause confusion, since we will mostly work in terms of internal spherical geometry. However, when it is convenient, we will think of the sphere as a subset of a Euclidean space, and interpret points of Γ as vectors. In the next several paragraphs we will list the facts from spherical geometry that we will use in the proof. For an introduction to elementary spherical geometry we refer the reader to Chapter 1 of the book due to L. Fejes Tóth [14]. For a systematic treatment of spherical geometry, that by far covers all the material used in this paper we refer to [2].

1. There is a natural way to assign dimensions to spherical planes, such that the definition will work the same way as in the Euclidean case. Namely, the dimension of a diametral sphere is equal to the dimension of the minimal Euclidean plane that contains it. Note that a spherical line consists of two points.

2. For a flat (diametral sphere) γ denote by γ^* the maximal flat such that any vector from γ^* is orthogonal to any vector in γ . If γ is a hyperplane (diametral sphere of codimension 1), by γ^+, γ^- we denote the closed halfspaces bounded by γ .

3. For a given hyperplane γ and an arbitrary point $\Gamma \setminus \gamma^*$ we can define the projection of a point v to γ . Consider the two-point set γ^* . Then the projection v' of v on γ is the closest intersection point to v of the great circle that goes through γ^* and v with the plane γ .

4. We define the reflection R_γ with respect to a given hyperplane γ . For any given point v in $\Gamma \setminus \gamma^*$ we consider the great circle that contains γ^* and v , and find a point $R_\gamma(v)$ on that circle, which is symmetric to v with respect to the projection of v on γ . As for the γ^* , the reflection interchanges the two points in γ^* .

5. Using reflections, it is easy to introduce the notion of orthogonality to a hyperplane, which would be convenient for us. Namely, a plane σ is orthogonal to a hyperplane γ , if $R_\gamma(\sigma) = \sigma$.

6. Suppose that we have a k -sphere Ω on Γ , which is not diametral. It is easy to show that any such sphere is contained in a (spherical) plane γ of dimension $k+2$. Indeed, taking a Euclidean point of view, for any k -sphere

there is a $(k+2)$ -dimensional plane that passes through the center of Γ and contains Ω . Its intersection with Γ is the desired (spherical) plane. Note that this is the minimal plane that contains Ω .

7. For points in an open hemisphere Γ^+ of Γ one can easily define the distance between two points as the shorter angle between the corresponding vectors. In particular, the distance between any point in γ and any point in γ^* is $\pi/2$. We denote the spherical distance between $u_1, u_2 \in \Gamma^+$ by $\rho(u_1, u_2)$.

8. We define an angle between the two intersecting arcs as the dihedral angle between the corresponding vector planes. For three distinct points $u_1, u_2, u_3 \in \Gamma^+$ we denote by $A(u_1, u_2, u_3)$ the angle between the arcs u_1u_2 and u_2u_3 .

9. There is a version of Pythagoras' theorem for spherical triangles. Namely, given a right spherical triangle u_1, u_2, u_3 in Γ^+ with $A(u_1, u_2, u_3) = \pi/2$, one have $\cos(\rho(u_1, u_3)) = \cos(\rho(u_1, u_2))\cos(\rho(u_2, u_3))$. Moreover, Pythagoras' theorem is a corollary of the spherical cosine law:

$$(1) \quad \cos(\rho(u_1, u_3)) = \cos(\rho(u_1, u_2)) \cos(\rho(u_2, u_3)) + \sin(\rho(u_1, u_2)) \sin(\rho(u_2, u_3)) \cos(A(u_1, u_2, u_3)).$$

One can deduce the following statement out (1): suppose we are given three distinct points $u_1, u_2, u_3 \in \Gamma^+$ and the angle $A(u_1, u_2, u_3)$ between the arcs u_1u_2 and u_2u_3 is at least $\pi/2$. Assume moreover, that $\rho(u_1, u_2), \rho(u_2, u_3)$ are less than $\pi/2$. Then $\rho(u_1, u_3) > \max\{\rho(u_1, u_2), \rho(u_2, u_3)\}$.

10. We need the notion of a convex hull of a set of points $\{u_1, \dots, u_k\}$. The straightforward way to define it is by using the Euclidean interpretation. It is simply the intersection of the cone formed by vectors corresponding to u_1, \dots, u_k and Γ . Note that the boundary of such a convex hull is formed by planes (diametral spheres).

4.4.2. The proof The distance in Γ , that corresponds to Euclidean distance 1, we denote by ϕ . Note that, since in our case Γ has radius greater than $1/\sqrt{2}$, we have $\phi < \pi/2$. Suppose we are given a diameter graph on Γ , which contains two simplices K_1 and K_2 on d vertices. We consider the spherical rugby ball Θ , formed by vertices of K_1 , and the diametral sphere π that contains K_1 .

The proof stays almost the same as in the Euclidean case. We describe all the differences in what follows. All the notations are translated to this case from the Euclidean case.

First, we show that the spherical rugby ball Θ is contained in one of the open hemispheres of Γ . We denote such a hemisphere by Γ^+ . Consider the unit ball B_1 with the center in v_1 (one of the vertices of K_1) . On one

hand, since the radius of Γ is bigger than $1/\sqrt{2}$, B_1 is contained in the open hemisphere Γ^+ with the center in v_1 . On the other hand, surely, $B_1 \supset \Theta$.

Consider a spherical Reuleaux simplex Δ and a point x inside the (spherical) convex hull of its vertices. Then an open halfsphere with the center in x contains Δ . This is due to the fact that Δ is contained in the intersection of open halfspheres with centers in the vertices of Δ , therefore, if we think about the vector interpretation of the situation, any vector y from Δ has positive scalar products with any of the vectors representing the vertices. Since x is a convex combination of the vectors representing the vertices, x and y have positive scalar product. We formulate it as the first part of the following observation:

Observation 23. *1. Consider a spherical Reuleaux simplex Δ on Γ and a point x that lies in the spherical convex hull of the vertices of Δ . Then for any $y \in \Delta$ we have $\rho(x, y) < \pi/2$.
 2. Consider any diametral hypersphere γ on Γ and any point x , $x \notin \gamma^*$. Let x' be the projection of x to γ . Then $\rho(x, x') < \pi/2$.*

The proof of Theorem 17 starts with Lemma 19. Its proof is deferred till the next subsection, and for now we assume that it holds in the spherical case as well. In what follows, we go through the changes in the remaining part of the proof.

Observation 24. *Spheres $S \cap S_i$ lie in the hyperplane π_i , which is orthogonal to π .*

Proof. The first thing we note is that $S \cap S_i$ is a $(d - 2)$ -sphere that is not diametral. It would be diametral only if both S and S_i are diametral, which is not the case. Thus, by point 6 from the previous subsection, the minimal plane that contains $S \cap S_i$ is of codimension 1. Next, note that $R_\pi(S \cap S_i) = S \cap S_i$. This is due to the fact that both centers of S and S_i lie in π . Hence, the same should hold for π_i , and by point 5 from the previous subsection we obtain the desired orthogonality. ■

To conclude the proof in the case when there are either none or at least two vertices in each halfspace π^+, π^- (corresponding to case (i) of the Euclidean proof), we use the same proof as in the Euclidean case. The ingredients that we add are the spherical version of Lemma 19, Observation 24 and the following observation, which shows that the circumscribed ball considerations still work in the spherical case.

Observation 25. *Suppose we are given two unit simplices K_1, K_2 on d vertices. Suppose K_2 lies inside the d -dimensional ball B of diameter f ,*

which is a ball of minimal diameter that contains K_1 . Denote by B_2 the circumscribed ball for K_2 . Then, if B and B_2 do not coincide, the intersection $B \cap B_2$ is contained in a ball of diameter strictly smaller than f .

Proof. Choose an arbitrary point u on the open segment connecting the centers O, O_2 of B, B_2 , respectively. By the last property in point 9 from the previous subsection, combined with the fact that $\rho(x, u), \rho(u, O), \rho(u, O_2) < \pi/2$ for any $x \in B \cap B_2$, we have $\rho(x, u) < \max\{\rho(x, O), \rho(x, O_2)\}$, since $A(O, u, x)$ or $A(O_2, u, x)$ is at least $\pi/2$. We obtain that K_2 is contained in a ball of radius strictly smaller than f , which is impossible. Thus, the centers of B and B_2 coincide. ■

We are left to modify Case (ii) of the Euclidean proof, in which we have one vertex of K_2 in $\pi^+ \cap \Theta$, while the rest lie in $\pi^- \cap \Theta$.

Lemma 22 works exactly as in the Euclidean case. To prove it, we note that the plane γ satisfies the following equation: $R_\gamma(B') = B$, which is why $B \cap \gamma^+ \supset B' \cap \gamma^+$. Indeed, $B' \cap \gamma^+ = R_\gamma(B \cap \gamma^-)$ and $B \cap \gamma^-$ is less than a halfball, while $B \cap \gamma^+$ is bigger than a halfball. Similar reasoning applies for the inclusion $\Theta' \cap \gamma^- \supset \Theta \cap \gamma^-$.

As for the third part of Lemma 22, consider the case when w_2 falls into π' and, moreover, w_2 coincides with v' . We need to check that the arc on which v', v_1 lie projects inside the spherical convex hull T of vertices v_1, \dots, v_d . It is clear that the circle $S_2 \cap \dots \cap S_d$ and the sphere S touch at v_1 (in the plane π). Thus the circle lies in the exterior of the ball B and the point w_2 lies in $\Theta \setminus B$. This by Observation 20 gives that the arc projects inside T . Therefore, we can apply the spherical analogue of Lemma 19 to conclude the proof of Lemma 22 in the spherical case.

The analysis of the possibilities work the same, and the proof of Theorem 17 in the spherical case is complete.

4.5. Proof of Lemma 19

We give a unified proof of Lemma 19 in both Euclidean and spherical cases. We use a unified terminology, in particular, $\rho(x, y)$ for both Euclidean and spherical distance and hyperplanes for both hyperplanes and hyperspheres. The distance between points forming edges in a diameter graph, as in the spherical case, we denote by ϕ .

It is enough to consider the case when none of v, w coincide with the vertices of Δ . Consider the projections v', w' of v, w on the hyperplane π (see Figure 4). We have two possibilities:

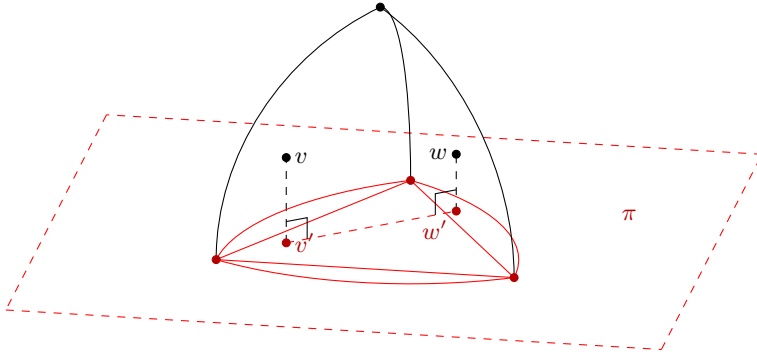


Figure 4.

1. $\rho(w', w) \geq \rho(v', v)$. Since v' lies inside T and the maximum of the distances from a fixed point to the points of a polytope is attained on the vertices of the polytope, there exists a vertex of T , say, v_1 , such that $\rho(v_1, w') > \rho(v', w')$. Using Pythagoras theorem (in the spherical case, point 9 from Section 4.4.1 and Observation 23), we get that $\phi \geq \rho(v_1, w) > \rho(v', w)$.

To complete the proof of Lemma 19 in this case we have to show that $\rho(v', w) \geq \rho(v, w)$. In the Euclidean case it follows right away from the fact that v is closer to π than w . In the spherical case we also have the same property, since $\pi/2 > \rho(w', w) \geq \rho(v', v)$. It implies the inequality $A(w, v, v') \geq \pi/2$, and by point 9 from Section 4.4.1 we get that $\rho(w, v') \geq \max\{\rho(w, v), \rho(v, v')\}$.

2. $\rho(w', w) < \rho(v', v)$. Assume for a moment that we obtained an inequality similar to the one in the previous case: there exists a vertex of T , say, v_1 , such that $\rho(v_1, v') \geq \rho(v', w')$. Note that the inequality is not strict in this case. Then, arguing in a similar way, we first get that

$$\phi \geq \rho(v_1, v) \stackrel{(1)}{\geq} \rho(v, w') \stackrel{(2)}{\geq} \rho(v, w).$$

Unfortunately, we do not have any strict inequality in this chain. However, it is clear that the inequality (2) is strict unless $w = w'$. In the Euclidean case it is obvious, and in the spherical case it again follows from point 9 of Section 4.4.1. Moreover, it is easy to show that the inequality (1) is strict unless w is on the border of Δ_π . Indeed, if w is in Δ_π , but not on the border, then change w to the point of intersection of the ray $v'w$ with the border of Δ_π . The distance between v and w will increase, which makes the inequality strict. This proves that $\rho(v, w) = \phi$ may only occur if the first part of condition 2 from the lemma is satisfied.

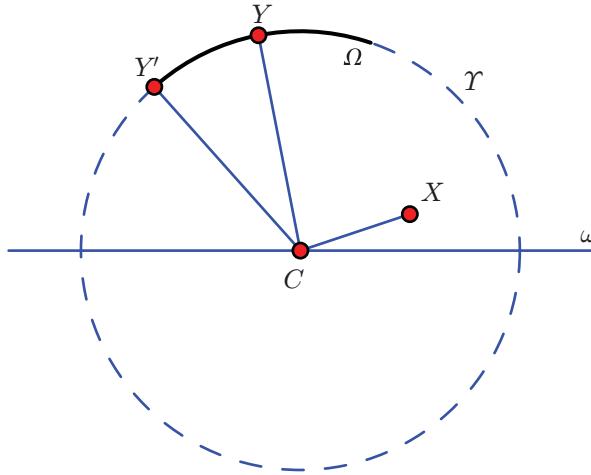


Figure 5. Y' is farther away from X than Y

Therefore, to conclude the proof of Lemma 19 it is sufficient to show that such a vertex v_1 exists (and to verify that in the case $\rho(v, w) = \phi$ the other part of condition 2 from the lemma holds as well). We formulate it as a separate lemma:

Lemma 26. Consider a unit Reuleaux simplex Δ_π in \mathbb{R}^{d-1} (or S_r^{d-1} , $r > 1/\sqrt{2}$) with the set of vertices v_1, \dots, v_d and two points $v', w \in \Delta_\pi$, different from the vertices of Δ_π , so that v' lies in the convex hull T of v_1, \dots, v_d . Denote by w' the projection of w on π . Then there exists i so that $\rho(v_i, v') \geq \rho(w', v')$, with the equality possible only in case if v' lies on the boundary of T .

Lemma 26 is proved via a repetitive application of the proposition below.

Proposition 27. Consider a closed half-space ω^+ in \mathbb{R}^d or S^d bounded by a hyperplane ω . Let \mathcal{Y} be a sphere with center in C , where $C \in \omega$. Let Ω be an open region on \mathcal{Y} , $\Omega \subset \mathcal{Y} \cap \omega^+$. Consider two points $X \in \omega^+$, $Y \in \Omega$. Then one can find a point $Y' \in \partial\Omega$ such that

- $\rho(X, Y) < \rho(X, Y')$, if $X \neq C$;
- $\rho(X, Y) = \rho(X, Y')$, if $X = C$.

Proof. The equality from the statement of the lemma is obvious since Ω and $\partial\Omega$ both lie on \mathcal{Y} . As for the inequality, consider the two-dimensional plane γ that contains the points C, X, Y (See Figure 5). The line CY splits

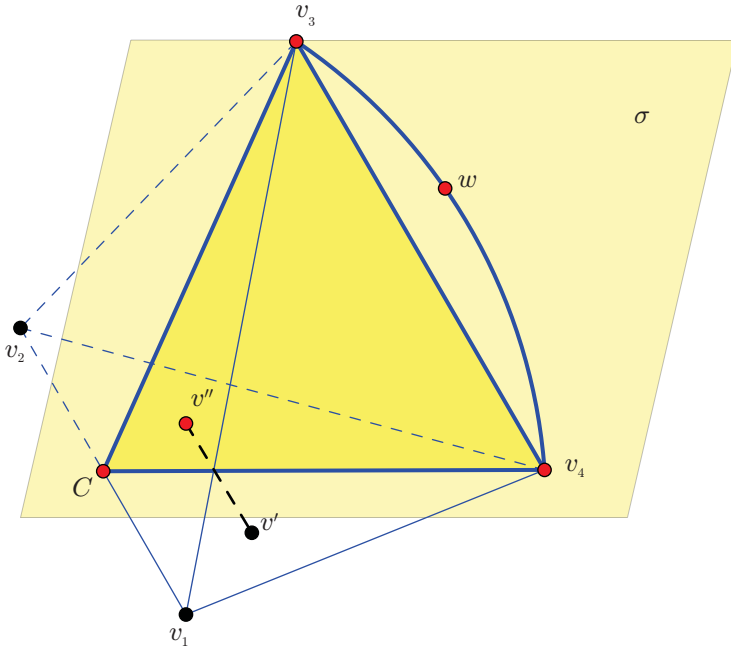


Figure 6. v' projects inside the convex hull of C, v_3, v_4

the plane into two closed halfplanes γ^+, γ^- . Let $X \in \gamma^+$. There is at least one point $Y' \in \partial\Omega$ in γ^- , which is different from Y . Then we have the inequality for the angles $\angle XCY < \angle XCY'$ and, thus, by the law of cosines, $\rho(X, Y) < \rho(X, Y')$. In the spherical case it holds because the first summand in (1) stays the same in both cases, while in the second one the only change is that the last multiple gets smaller. Note that we do not need to put any restrictions on $\rho(X, C), \rho(C, Y)$, as it is done in the end of point 9 of the previous subsection. ■

Proof of Lemma 26. As we have already said, we apply Lemma 27 repeatedly. We may assume that $w \in \partial\Delta_\pi$. The boundary of a Reuleaux simplex can be partitioned into the open spherical regions of different dimensions, as it is stated in Lemma 15. We remark that considerations of Section 3 transfer with some obvious changes to the spherical case. Thus, we have a spherical analogue of Lemma 15.

We first find an open spherical region Ω that contains w and the sphere U of minimal dimension that contains Ω . We denote its center by C . Then we project T on the flat σ of minimum dimension that contains U . Assume that σ is an affine hull of the points O, v_{k+1}, \dots, v_d (see Lemma 15). Then

$C = \nu(v_1 + \dots + v_k)/k$ with $\nu = 1$ in the Euclidean case. The projection is fairly simply arranged (see Figure 6, in which we show T , the face Ω of Δ_π that contains w and the plane σ ; v_1, v_2 are projected into C , while $v_3, v_4 \in \sigma$; all points lying in σ are marked by red). Any of the points v_1, \dots, v_k project into C . This is true due to the fact that for any $i = 1, \dots, k$ we have $S_i \cap \sigma = U$. To the contrary, all the points v_{k+1}, \dots, v_d project to themselves, since they lie in σ . It is clear that the projection v'' of v' falls into the projection of T .

In σ consider a hyperplane ω which passes through C and such that the vertices w, v'' and the whole projection of T lie in the halfspace ω^+ (recall that $w \in \Delta_\pi$). Any hyperplane in σ that touches the set $T \cap \sigma$ in a single point C would fit, and we obviously have such hyperplanes.

It is possible to apply Lemma 27 and find a point w'' on an open spherical region of a smaller dimension such that $\rho(v'', w'') \geq \rho(v'', w)$. Moreover, if $v' \notin \partial T$, then v'' cannot coincide with C , which means that the inequality is strict. By Pythagoras' theorem we get $\rho(v', w'') \geq \rho(v', w)$ (and a strict inequality in the case when $v' \notin \partial T$). Reducing the dimension of the spherical region which contains the current image of w step by step, we eventually arrive at a vertex of T , which concludes the proof of both Lemma 26 and 19. ■

Acknowledgement. We are grateful to Filip Morić and János Pach for several interesting and stimulating discussions on the problem, Eyal Ackerman for pointing out the paper [15], Ido Shahaf for his valuable remarks on the presentation of the paper and for providing us with Figure 4, to Arseniy Akopyan and Roman Karasev for valuable discussions and for bringing Kirszbraun's theorem to our attention, and anonymous referees for numerous valuable comments that helped to significantly improve the presentation of the paper.

References

- [1] A. V. AKOPYAN and A. S. TARASOV: A Constructive Proof of Kirszbraun's Theorem, *Mathematical Notes* **84** (2008), 725–728.
- [2] D. V. ALEKSEEVSKIJ, E. B. VINBERG and A. S. SOLODOVNIKOV: Geometry of spaces of constant curvature Geometry II: Spaces of constant curvature, *Encycl. Math. Sci.* **29** (1993), 1–138. Translation from Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., *Fundam. Napravleniya* **29** (1988), 5–146.
- [3] P. BRASS, W. MOSER and J. PACH: *Research problems in discrete geometry*, Springer, Berlin, 2005.
- [4] V. V. BULANKINA, A. B. KUPAVSKII and A. A. POLYANSKII: Note on Schur's conjecture in \mathbb{R}^4 , *Doklady Mathematics* **89** (2014), 88–91.
- [5] V. V. BULANKINA, A. B. KUPAVSKII and A. A. POLYANSKIY: On Schur's conjecture in \mathbb{R}^4 , *Mathematical notes* **97** (2015), 21–29.

- [6] V. L. DOL'NIKOV: Some properties of graphs of diameters, *Discrete Comput. Geom.* **24** (2000), 293–299.
- [7] P. ERDŐS: On a set of distances of n points, *Amer. Math. Monthly* **53** (1946), 248–250.
- [8] P. ERDŐS: On sets of distances of n points in Euclidean space, *Magyar Tud. Akad. Mat. Kut. Int. Közl.* **5** (1960), 165–169.
- [9] H. HOPF and E. PANNWITZ: Aufgabe Nr. 167, *Jahresbericht Deutsch. Math.-Verein.* **43** (1934), 114.
- [10] B. GRÜNBAUM: A proof of Vázsonyi's conjecture, *Bull. Res. Council Israel, Sect. A* **6** (1956), 77–78.
- [11] A. HEPPES: Beweis einer Vermutung von A. Vázsonyi, *Acta Math. Acad. Sci. Hungar.* **7** (1957), 463–466.
- [12] A. KUPAVSKII: Diameter graphs in \mathbb{R}^d , *Discrete and Computational Geometry* **51** (2014), 842–858.
- [13] A. KUPAVSKII and A. POLYANSKII: On simplices in diameter graphs in \mathbb{R}^d , *Mathematical Notes*, accepted.
- [14] L. FEJES TÓTH: *Lagerungen in der Ebene, auf der Kugel und im Raum*, Springer-Verlag, 1972.
- [15] H. MAEHARA: Dispersed points and geometric embedding of complete bipartite graphs, *Discrete and Computational Geometry* **6** (1991), 57–67.
- [16] F. MORIĆ and J. PACH: Remarks on Schur's conjecture, *Comput. Geom.* **48** (2015), 520–527.
- [17] F. MORIĆ and J. PACH: Large simplices determined by finite point sets, *Beitr. Algebra Geom.* **54** (2013), 45–57.
- [18] A. M. RAIGORODSKII: Around Borsuk's conjecture, *J. of Math. Sci.* **154** (2008), 604–623.
- [19] Z. SCHUR, M. A. PERLES, H. MARTINI and Y. S. KUPITZ: On the number of maximal regular simplices determined by n points in \mathbb{R}^d , in: *Discrete and Computational Geometry, The Goodman-Pollack Festschrift*, B. Aronov, S. Basu, J. Pach, M. Sharir eds., Springer, 2003.
- [20] S. STRASZEWICZ: Sur un problème géométrique de P. Erdős, *Bull. Acad. Polon. Sci. Cl. III.* **5** (1957), 39–40.
- [21] K. J. SWANEPOEL: Unit distances and diameters in Euclidean spaces, *Discrete and Computational Geometry* **41** (2009), 1–27.

Andrey B. Kupavskii

Ecole Polytechnique Fédérale de Lausanne
Moscow Institute of Physics and Technology
kupavskii@yandex.ru

Alexandr Polyanskii

Moscow Institute of Physics and Technology
Technion – Israel Institute of Technology
alexander.polyanskii@yandex.ru