

**SUBDIVISIONS, SHELLABILITY, AND
COLLAPSIBILITY OF PRODUCTS****KARIM A. ADIPRASITO*, BRUNO BENEDETTI†**

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We prove that the second derived subdivision of any rectilinear triangulation of any convex polytope is shellable. Also, we prove that the first derived subdivision of every rectilinear triangulation of any convex 3-dimensional polytope is shellable. This complements Mary Ellen Rudin’s classical example of a non-shellable rectilinear triangulation of the tetrahedron. Our main tool is a new relative notion of shellability that characterizes the behavior of shellable complexes under gluing.

As a corollary, we obtain a new characterization of the PL property in terms of shellability: A triangulation of a sphere or of a ball is PL if and only if it becomes shellable after sufficiently many derived subdivisions. This improves on PL approximation theorems by Whitehead, Zeeman and Glaser, and answers a question by Billera and Swartz.

We also show that any contractible complex can be made collapsible by repeatedly taking products with an interval. This strengthens results by Dierker and Lickorish, and resolves a conjecture of Oliver. Finally, we give an example that this behavior extends to non-evasiveness, thereby answering a question of Welker.

Shellability is one of the earliest notions in combinatorial topology. It provides a combinatorial analogue of the topological notion of the PL ball: Every shellable contractible manifold is a PL ball. However, the converse is false; some PL balls are not shellable [32]. One of the main goals of this pa-

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per is to show that a simple combinatorial manipulation, applied repeatedly, makes every PL ball shellable.

Our starting point is the famous *non-shellable* subdivision of the tetrahedron provided by Rudin [39], which gained much influence since its publication [9,37,33,40,41,42]. This example somewhat limited the appeal of the shellability property, because it demonstrated that even simple triangulations of a trivial geometry may fail to satisfy it. In our first main result, we will argue that Rudin's ball should not discourage us too much: while subdivisions of polytopes are not shellable in general, they are very close to being shellable.

Theorem A (Corollary I.2.2 and Theorem I.2.4). *If C is any subdivision of a convex polytope, the second derived subdivision of C is shellable. If $\dim C = 3$, already the first derived subdivision of C is shellable.*

The bound is best possible at least in dimension 3, as Rudin's ball is not shellable. For subdivisions of d -polytopes, $d \geq 4$, one needs either one or two derived subdivisions to guarantee shellability, and we conjecture the former to hold true. As for the case $d \leq 2$, it is not hard to see that no derived subdivision is needed: every triangulation of the d -disk, $d \leq 2$, is shellable.

Derived subdivisions preserve plenty of combinatorial properties, including collapsibility and shellability. Theorem A, which extends results in [1,2], shows that they even induce these properties after a relatively small number of steps. In contrast, PL balls can be arbitrarily nasty when $d \geq 3$: For each m and every $d \geq 3$ there is a PL d -ball whose m -th derived subdivision is still not shellable [25]. As a consequence of a relative variant of Theorem A, we obtain the following result:

Theorem B (Theorem I.3.9). *A simplicial complex is PL homeomorphic to a shellable complex if and only if it becomes shellable after finitely many derived subdivisions.*

In particular, a triangulated ball or sphere is PL if and only if it becomes shellable after a finite number of derived subdivisions, cf. Corollary I.3.11. This resolves a problem by Ed Swartz and Lou Billera (personal communication). A stronger version of the same conjecture has been resolved in parallel by the first author and Izmistiev [4], see also the remark after Corollary I.3.10.

A similar result to Theorem B holds if we replace “shellable” with “collapsible”:

Theorem C (Corollary II.0.2). *A simplicial complex is PL homeomorphic to a collapsible complex if and only if it becomes collapsible after finitely many derived subdivisions.*

Theorems B and C simplify and generalize some foundational results in PL topology: Whitehead [45, Thm. 7] and Glaser [24, pp. 58–69] proved that every PL ball admits a collapsible (iterated) stellar subdivision, and it is a fundamental result of Whitehead [45, Thm. 3] that this property characterizes PL balls among all PL manifolds. Theorem C shows that stellar subdivisions can be replaced by the one-parameter subclass of derived subdivisions. Similarly, one can use Theorem B to characterize the PL notion for manifolds: Theorem B shows that a (triangulated) homology manifold M is PL if and only if all vertex links of some iterated subdivision of M are shellable, cf. Corollary I.3.11.

Finally, we turn to a problem motivated by the Zeeman conjecture, which claims that the product of any contractible 2-dimensional complex C with the interval $I := [0, 1]$ is PL homeomorphic to a collapsible complex. This begs the question: Is it true that every contractible polytopal 2-complex, after multiplication with the interval $[0, 1]$, becomes collapsible? We provide the following answer.

Theorem D (Theorem II.1.5). *If two polytopal complexes C, D are simple homotopy equivalent, there exists an $n \geq 0$ such that $C \times I^n$ collapses to a polytopal complex combinatorially equivalent to a subdivision of D . In particular, for any contractible complex C there is an $n \geq 0$ such that $C \times I^n$ is collapsible.*

The analogous statement for shellability does not hold (cf. Remark II.1.7). The second statement in Theorem D was already conjectured by Bob Oliver in 1998, in a letter to Anders Björner. (We thank Björner for this communication). It also strengthens results by Dierker [21] and Lickorish [31, Lem. 1], who proved that for every contractible complex, there is an $n \geq 0$ such that $C \times I^n$ is PL homeomorphic to a collapsible complex. Using the intuition of Theorem D, we also give a concrete example of an evasive poset that becomes non-evasive when multiplied with the two-element poset $\{0, 1\}$, $0 < 1$, thereby solving a question by Welker [44, Open Problem 2], cf. Proposition II.1.9.

To this day the Zeeman conjecture remains open, although some related problems have been solved [16, 22]. Also, the Poincaré–Hamilton–Perelman Theorem implies the Zeeman conjecture for a wide class of complexes [23]. Via Theorem C, we can provide a rather concrete reformulation: The Zeeman conjecture is equivalent to the claim that for any contractible 2-complex C there is an $n \geq 0$ such that $\text{sd}^n(C \times I)$ is collapsible (Proposition II.2.1). However, for any $m, n \geq 0$, we can describe a contractible 2-complex $C_{m,n}$ such that the m -th derived subdivision of $C_{m,n} \times I^n$ is *not* collapsible (Corollary II.2.8).

Notation

For polytopes and polytopal complexes, we follow the standard notation in the literature [26,38,47]; for further details, see also [1]. We use $\text{conv } X$ to denote the *convex hull* of a set X . The abbreviations $\text{cl } X$, $\text{int } X$, $\text{relint } X$ and ∂X shall denote the *closure*, *interior*, *relative interior* and *boundary* of X , respectively.

By \mathbb{R}^d and S^d we denote the Euclidean d -space and the unit sphere in \mathbb{R}^{d+1} , respectively. A (*Euclidean*) *polytope* in \mathbb{R}^d is the convex hull of finitely many points in \mathbb{R}^d . Similarly, a *spherical polytope* in S^d is the convex hull of a finite number of points that all belong to some open hemisphere of S^d . Spherical polytopes are in natural one-to-one correspondence with Euclidean polytopes via radial projections. A *polytopal complex* in \mathbb{R}^d (resp. in S^d) is a finite collection of polytopes in \mathbb{R}^d (resp. S^d) that is closed under passing to faces of its elements and such that the intersection of any two polytopes is a face of both. We write “ d -complex” as an abbreviation of “ d -dimensional polytopal complex”.

A (polytopal) complex is *pure* if all its facets are of the same dimension. Two polytopal complexes C, D are *combinatorially equivalent*, denoted by $C \cong D$, if their face posets are isomorphic. A polytopal complex combinatorially equivalent to C is also called a *realization* of C . A polytopal complex is *simplicial* if all its faces are simplices. The set of k -dimensional faces of a polytopal complex C is denoted by $F_k(C)$, and the cardinality of this set is denoted by $f_k(C)$. A *principal d -complex* is a pure d -complex, or the empty complex. If C is a pure polytopal d -complex, then a *principal subcomplex* is a principal $(d-1)$ -complex in C . The *underlying space*, or *support*, $|C|$ of a polytopal complex C is the union of its faces. We will frequently abuse notation and identify a polytopal complex with its underlying space, as common in the literature. For instance, we do not distinguish between a polytope and the complex formed by its faces.

If C is a polytopal complex, and A is some set, we define the *restriction* $R(C, A)$ of C to A as the inclusion-maximal subcomplex D of C such that D lies in A . The *star* of σ in C , denoted by $\text{St}(\sigma, C)$, is the minimal subcomplex of C that contains all faces of C containing σ . The *deletion* $C - D$ of a subcomplex D from C is the subcomplex of C given by $R(C, C \setminus \text{relint } D)$.

We define links with a differential-geometric approach, cf. [1], [20, Sec. 2.2]. Let p be any point of a metric space X . By $T_p X$ we denote the tangent space of X at p . Let $T_p^1 X$ be the restriction of $T_p X$ to unit vectors. If Y is any subspace of X , then $N_{(p,Y)} X$ denotes the subspace of the tangent space spanned by vectors orthogonal to $T_p Y \subset T_p X$, and we define $N_{(p,Y)}^1 X := N_{(p,Y)} X \cap T_p^1 X$.

If τ is any face of a polytopal complex C in $X^d = \mathbb{R}^d$ (or $X^d = S^d$) containing a nonempty face σ of C , and $p \in \text{relint } \sigma$, then $N_{(p,\sigma)}^1 \tau$ forms

a spherical polytope isometrically embedded in $N_{(p,\sigma)}^1|C| \subset N_{(p,\sigma)}^1X^d$. Here, the space $N_{(p,\sigma)}^1X^d$ is isometric to a sphere of dimension $d - \dim \sigma - 1$, and will be considered as such. The collection of all polytopes in $N_{(p,\sigma)}^1|C|$ obtained this way forms a polytopal complex denoted by $\text{Lk}_p(\sigma, C)$, the *link* of C at σ . Up to ambient isometry $\text{Lk}_p(\sigma, C)$ and $N_{(p,\sigma)}^1\tau$ in $N_{(p,\sigma)}^1|C| \subset N_{(p,\sigma)}^1X^d$ do not depend on p , thus, the base-point p will be omitted in notation whenever possible. By convention, we set $\text{Lk}(\emptyset, C) := C$, and $\text{Lk}(\tau, C) := \emptyset$ if τ is not in C . If C is simplicial, and v is a vertex of C , then $\text{Lk}(v, C) \cong (C - v) \cap \text{St}(v, C) = \text{St}(v, C) - v$.

Let C, D be polytopal complexes, and let C' and D' be realizations of C and D in skew affine subspaces of \mathbb{R}^d . A *join* $C * D$ of C and D is the complex consisting of faces $\text{conv}(\sigma \cup \tau)$, $\sigma \in C', \tau \in D'$. This is well-defined up to combinatorial equivalence. Caution: the join may not be realizable using C and D themselves: If C and D are complexes in \mathbb{R}^2 containing at least 3 vertices each, then $C * D$ contains a $K_{3,3}$ and is therefore not realizable in \mathbb{R}^2 . If C is a simplicial complex, and σ, τ are faces of C , then $\sigma * \tau$ is combinatorially equivalent to the minimal face of C containing both σ and τ (assuming it exists), hence they will be identified. If σ is a face of a simplicial complex C , and τ is a face of $\text{Lk}(\sigma, C)$, then $\sigma * \tau$ is combinatorially equivalent to the face χ of C with $\text{Lk}(\sigma, \chi) = \tau$. Again, we will simply denote χ by $\sigma * \tau$.

Analogously, if C and D are polytopal complexes, and C' and D' are realizations of C and D in orthogonal subspaces of \mathbb{R}^d , then a *product* $C \times D$ of C and D is given as the complex consisting of faces $\{(x, y) \in \mathbb{R}^d : x \in \tau, y \in \sigma\}$, $\sigma \in C', \tau \in D'$. Similar to the join, the product of two polytopal complexes is defined up to combinatorial equivalence only.

An *elementary collapse* is the deletion of a *free* face σ from a polytopal complex C , i.e. the deletion of a nonempty face σ of C that is strictly contained in only one other face of C . In this situation, we say that C (*elementarily*) *collapses* onto $C - \sigma$, and write $C \searrow_e C - \sigma$. More generally, we say that the complex C *collapses* to a subcomplex C' , and write $C \searrow C'$, if C can be reduced to C' by a sequence of elementary collapses. A *collapsible* complex is a complex that collapses onto a single vertex.

Let C be a pure polytopal complex with n facets. If $\dim C = 0$, a *shelling* of C is any ordering of its vertices. If $\dim C = d > 0$, a *shelling* for C is an ordering (F_1, \dots, F_n) , the *shelling order*, of its facets such that for each $i \in \{1, \dots, n-1\}$ the complex $B := F_i \cap \bigcup_{j=i+1}^n F_j$ is pure $(d-1)$ -dimensional and both B and $\partial F_i - B$ are shellable or empty¹. A complex C is *shellable* if it has a shelling.

¹ We will see in Corollary I.1.9 that this is equivalent to the definition of shellability classically used, although some authors reverse the shelling order, cf. [47, Sec. 8].

A *subdivision* of a polytopal complex C is a polytopal complex C' with the same underlying space as C , such that for every face F' of C' there is some face F of C for which $F' \subset F$. A *subdivision* of a set M is a polytopal complex whose underlying space is M . Now, let C denote any polytopal complex, and let τ denote any face of C . Let v_τ denote a point anywhere in the relative interior of τ . Define

$$\text{stlr}(\tau, C) := (C - \tau) \cup \{\text{conv}(\{v_\tau\} \cup \sigma) : \sigma \in \text{St}(\tau, C) - \tau\}.$$

The complex $\text{stlr}(\tau, C)$ is a *stellar subdivision* of C at τ . A *derived subdivision* $\text{sd}C$ of a polytopal complex C is any subdivision of C obtained by stellarly subdividing at all faces in order of decreasing dimension of the faces of C , cf. [29].

Finally, recall that two polytopal complexes C and D are called *PL equivalent* (or *PL homeomorphic*) if some subdivision C' of C is combinatorially equivalent to some subdivision D' of D . A *PL d -ball* is any polytopal complex that is PL homeomorphic to the d -simplex. A *PL d -sphere* is a polytopal complex that is PL homeomorphic to the boundary of the $(d+1)$ -simplex. A polytopal complex C is a *PL d -manifold* if for each vertex v of C the complex $\text{Lk}(v, C)$ is a PL $(d-1)$ -sphere or a PL $(d-1)$ -ball.

I. Derived subdivisions and shellability

We start with an introduction to shellability and the notion of relative shellings. Subdivisions of 3-polytopes are always collapsible [14], but not always shellable [39]. In the second part of this section, we reconcile this by showing that the second derived subdivision of any convex d -complex is shellable. In the third part of this section, we draw some consequences for PL topology from Theorem A, among them Theorem B. For some of the algebraic ramifications of relative shellability, we refer the reader to [5,41].

I.1. Basics in shellability

I.1.1. Shelling sequences and relative shellings Let C be any complex, and let τ be any face of C . We denote by $C \nabla \tau$ the complex $R(C, \text{cl}(|C| \setminus |\text{St}(\tau, C)|))$, the *removal* of τ from C . If $D \subset C$ is a subcomplex, we define

$$C \nabla D = \bigcap_{\tau \text{ facet of } D} C \nabla \tau.$$

Definition I.1.1. (Relative shellings, shelling sequences and shelling orders). Let C be a pure polytopal d -complex, and let D be a principal $(d-1)$ -dimensional complex such that $D \cup C$ is a polytopal complex. Assume that $d(f_d(C) - 1) > 0$, and let P be any facet of C . The removal $C \rightarrow C' := C \nabla P$ is a *shelling step relative to D* if $P \cap (C' \cup D)$ is pure of dimension $(d-1)$ and the complexes $P \cap (C' \cup D)$ and $\partial P \nabla (P \cap (C' \cup D))$ are shellable or empty (in the classical sense). Given two pure polytopal d -complexes C and C' then C *shells to C' relative to D* , and write $C \searrow_S^D C'$, if there is a sequence of shelling steps (relative to D) which deform C to C' .

The intermediate complexes between C and C' form a sequence of subcomplexes $(C = C_1, \dots, C_n = C')$, which we call *shelling sequence relative to D* . This is in contrast to the *shelling order relative to D* , which is a list (F_1, F_2, \dots, F_n) of facets of C in their order of removal from C .

If $D = \emptyset$, we abbreviate this to $C \searrow_S C'$, and simply say C *shells to C'* . A complex C is *shellable* (rsp. *shellable relative to D*) if and only if C shells (resp. shells relative to D) to some complex \tilde{C} with $d(f_d(\tilde{C}) - 1) = 0$ (i.e. if it can be reduced to a single facet) and, if D is nonempty, $D \cap \tilde{C}$ is shellable². We will write this simply as

$$C \searrow_S^{\emptyset} \emptyset \quad \text{or even simpler} \quad C \searrow_S \emptyset.$$

We stress that throughout this paper, shellability is only defined for *pure* complexes. However, there is also a notion of non-pure shellability in the literature [11], and most of the results in this preliminary sections (i.e. except those explicitly pertaining to manifolds) can in fact be extended to the non-pure case.

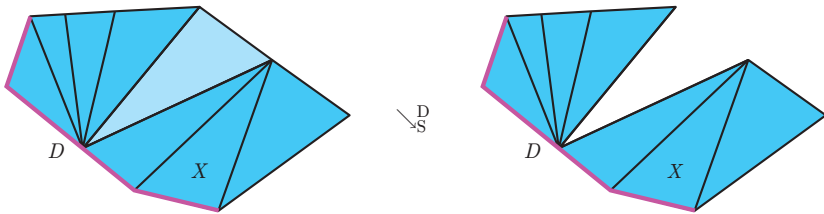


Figure 1. A relative shellable complex C (relative to a subcomplex D), and an elementary shelling step.

Shellings of simplicial complexes can be defined in a much simpler way, because if Δ is a simplex, *any* principal nonempty subcomplex C of Δ

² As usual, every 0-dimensional complex is shellable, both relative to the only principal subcomplexes $\{\emptyset\}$ and \emptyset .

is shellable. So many of the following consequences are much simpler to state/prove in the simplicial world.

Lemma I.1.2. *Let B and C be two pure polytopal d -complexes that intersect in a principal common subcomplex D , and let $\widehat{D} \subset C$ be a principal $(d-1)$ -complex containing D . The following are equivalent:*

- (1) $C \searrow_{\mathbb{S}}^{\widehat{D}} C'$ via the shelling sequence (C_i) .
- (2) $B \cup C \searrow_{\mathbb{S}}^{\widehat{D} \nabla D} B \cup C'$ via the shelling sequence $(B \cup C_i)$. ■

Hence, the notion of relative shellings gives us a way to glue shellable complexes to obtain larger shellable complexes. This is not trivial, and not always possible: For example, Rudin's ball is the union of two shellable 3-balls B and C , glued together at a (shellable) 2-ball D in their boundary. However, neither B nor C is shellable relative to D , since Rudin's ball is not shellable.

Lemma I.1.3. *Let C be a pure simplicial complex with a shelling sequence (C_i) relative to a principal subcomplex D . Then $(v * C_i)$ is a shelling sequence relative to $v * D$. ■*

To generalize this, let us say that if (C_i) is a shelling sequence relative to a complex D , then the shelling *restricts* to a full-dimensional (or empty) subcomplex E of C_1 if for $E_i := C_i \cap E$, we have $E_i \searrow_{\mathbb{S}}^D E_{i+1}$ for all i .

Lemma I.1.4. *Let C be a pure simplicial complex with a shelling (C_i) relative to D that restricts to a subcomplex $E \subset C$. Then $(v * C_i)$ is a shelling sequence of $v * C$ relative to $E \cup (v * D)$.*

Proof. This can be checked directly from the definitions and using Lemma I.1.3. ■

If C is a simplicial complex, and v is one of its vertices, define $C \triangle v$ as the maximal subcomplex of C with $(C \triangle v) - v = C \nabla v$. Then $\text{Lk}(v, C \triangle v) \cong \text{St}(v, C) \cap (C \nabla v)$ (see also Figure 2 for an illustration for ∇, \triangle).

Corollary I.1.5. *If C is a pure simplicial complex, v is a vertex of C , and if $\text{Lk}(v, C)$ has a shelling relative to a principal subcomplex D that restricts to $\text{Lk}(v, C \triangle v)$, then $C \searrow_{\mathbb{S}}^{v * D} C \nabla v$.*

By Lemma I.1.2, it suffices to prove that $\text{St}(v, C)$ admits a shelling relative to $\text{Lk}(v, C \triangle v) \cup (v * D)$. But this follows directly from the assumption and Lemma I.1.4. ■

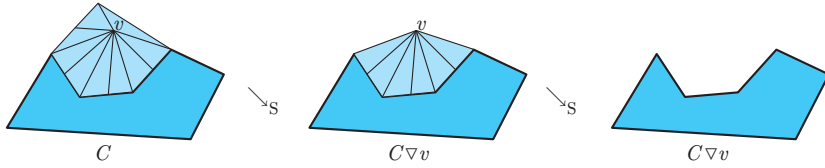


Figure 2. Illustration for Corollary I.1.5 (with $D = \emptyset$ for simplicity).

I.1.2. Reverse shellings and duality. To illustrate the notion of relative shellings further, we notice the following simple fact.

Proposition I.1.6. *Let C denote a simplicial d -ball with a shelling sequence that is simultaneously a shelling relative to \emptyset and ∂C . Then ∂C is shellable.*

Proof. Let (C_i) denote the shelling sequence as claimed. Then $(B_i := C_i \cap \partial C)$ is a sequence of subcomplexes of ∂C . Now, if $B_i \neq B_{i+1}$, then $D_i := B_i \nabla B_{i+1}$ is a nonempty principal subcomplex of the d -simplex $C_i \nabla C_{i+1}$.

It is not hard to see that if D is any principal nonempty subcomplex of a simplex, and E is any principal subcomplex of D , then $D \searrow_S^E \emptyset$ (cf. Lemma I.1.10). Hence, D_i is shellable relative to $D_i \cap B_{i+1}$, and Lemma I.1.2 yields that B_i shells to B_{i+1} for all i . Hence ∂C is shellable. ■

As a consequence we observe that a shelling of a PL ball is not automatically a shelling relative to its boundary (and vice versa).

Example I.1.7. Let C be any PL sphere that is not shellable, for instance one of the spheres of Lickorish [32]. Then C is the boundary of some shellable ball B by Pachner's Theorem [36]. By the previous proposition, any shelling of B is not a shelling relative to $\partial B = C$.

This does not mean that a shellable ball B does not admit a shelling relative to ∂B , it might just not be the same shelling.

Proposition I.1.8 (Alexander duality for relative shellings). *Let C be a polytopal d -ball with boundary ∂C , and consider a principal subcomplex D of C in ∂C . If C is shellable relative to D , then C is shellable relative to $D' := \partial C \nabla D$.*

Proof. Let $n := f_d(C)$ be the number of facets of C . If (F_1, F_2, \dots, F_n) is the shelling order for C relative to D , then we claim that *reversing* the shelling order to $(G_i) := (F_{n+1-i})$ gives the desired shelling order (G_1, \dots, G_n) of C relative to D' .

Let us explain why (G_1, \dots, G_n) is indeed associated to a shelling relative to D' . Let $G_i = F_{n+1-i}$ denote any facet of the shelling order (G_i) , and let

(C_i) denote the shelling sequence associated to the shelling order (F_i) , and let (C'_i) denote the shelling sequence associated to the shelling order (G_i) , where $C_{n+1} = C'_{n+1} = \emptyset$. With this notation,

$$\begin{aligned} G_i \cap (C'_{i+1} \cup D') \\ = \partial G_i - (G_i \cap (C_{n+2-i} \cup D)) = \partial F_{n+1-i} \nabla (F_{n+1-i} \cap (C_{n+2-i} \cup D)) \end{aligned}$$

since C is a manifold, and similarly

$$\partial G_i \nabla (G_i \cap (C'_{i+1} \cup D')) = F_{n+1-i} \cap (C_{n+2-i} \cup D).$$

Hence, the removal of G_i from C'_i is a valid shelling step relative to D' since the removal of $F_{n+1-i} = G_i$ from C_{n+1-i} is a valid shelling step relative to D . \blacksquare

Corollary I.1.9. *Let C denote a PL sphere, and let A and $B = C \nabla A$ denote shellable full-dimensional subcomplexes of C . Then A is the “beginning segment” of a shelling of C , i.e.*

$$C \searrow_S C \nabla A \searrow_S \emptyset. \quad \blacksquare$$

This shows the equivalence of our notion of shellability and the one classically used, cf. [10,47].

I.1.3. A few classical results on shellings, revisited Here we review some classical results on shellings. To justify that these results hold also for relative shellings, let us introduce the following rule of thumb.

Let C denote any d -complex, let C' denote any subcomplex and let D denote any principal $(d-1)$ -complex that together with C forms a polytopal complex. Then the following are equivalent:

- (1) $C \searrow_S^D C'$
- (2) $C \cup (v * D) \searrow_S C' \cup (v * D)$, where v is any additional vertex. \blacksquare

With this, the following results can be reduced to the classical setting shellability quite easily. (In each statement we reference the corresponding result in the classical setting.) If C is a polytopal complex, and τ is *not* an element of C , by convention we set $\text{stlr}(\tau, C) = C$.

Lemma I.1.10 (cf. Čukić–Delucchi [18]). *Assume C is a simplicial complex that shells to subcomplex C' relative to a complex D . Then any stellar subdivision $\text{stlr}(\tau, C)$ of C shells to the corresponding stellar subdivision $\text{stlr}(\tau, C')$ relative to $\text{stlr}(\tau, D)$. \blacksquare*

Proof. Iteratively remove the leaves of the path; since up to the last step there are always at least 2 leaves, we can choose to leave any chosen facet untouched by the shelling. \blacksquare

I.1.4. Derived orders and derived neighborhoods. We recall two notions related to derived subdivisions; the first is taken from [1], the second is classical.

Definition I.1.14 (Derived order). An *extension* of a partial order \prec on a set S is any partial order $\tilde{\prec}$ on any superset T of S such that $a \tilde{\prec} b$ whenever $a \prec b$ for $a, b \in S$.

Let now C be a polytopal complex, let S denote a subset of its faces, and let \prec denote any strict total order on S with the property that $a \prec b$ whenever $b \subsetneq a$. We extend this order to an irreflexive partial order $\tilde{\prec}$ on C as follows: Let σ be any face of C , and let $\tau \subsetneq \sigma$ be any strict face of σ .

- If τ is the minimal face (w.r.t. \prec) among the faces of σ in S , then $\tau \tilde{\prec} \sigma$.
- If τ is any other face of σ , then $\sigma \tilde{\prec} \tau$.

The transitive closure of the relation $\tilde{\prec}$ gives an irreflexive partial order on the faces of C , and by the correspondence of faces of C to the vertices of $\text{sd}C$, it gives an irreflexive partial order on $F_0(\text{sd}C)$. Any strict total order that extends the latter order is a *derived order* of $F_0(\text{sd}C)$ induced by \prec .

Definition I.1.15 (Derived neighborhoods, cf. [46,29]). Let C be a polytopal complex. Let D be a set of faces of C (for instance a subcomplex of C). The (*first*) *derived neighborhood* $N(D, C)$ of D in C is the polytopal complex

$$N(D, C) := \bigcup_{\substack{v_\sigma \text{ vertices of } \text{sd}C \\ \text{corresponding to faces } \sigma \text{ of } D}} \text{St}(v_\sigma, \text{sd}C).$$

Proposition I.1.16. *Let K and Λ denote simplicial complexes, and let V denote a subset of the vertex set of Λ . Assume $\text{sd}K$ is shellable relative to a subcomplex $\text{sd}D$, $D \subset K$, and that*

$$\text{sd}\Lambda \searrow_S N(V, \Lambda) \searrow_S \emptyset.$$

*Then $\text{sd}(K * \Lambda)$ has a shelling relative to $\text{sd}(D * \Lambda)$ that restricts to $N(K * V, K * \Lambda)$.*

Proof. We abbreviate $X := \text{sd}K * \text{sd}\Lambda$, $Y := \text{sd}(K * \Lambda)$, $\chi := \text{sd}D * \text{sd}\Lambda$ and $v := \text{sd}(D * \Lambda)$. First, notice that by Lemma I.1.12, applied twice, we have

$$\text{sd}K * \text{sd}\Lambda = X \searrow_S^\chi \text{sd}K * N(V, \Lambda) \searrow_S^\chi \emptyset.$$

Let e be any edge of X . If the vertices of e correspond to faces of dimension k of K and λ of Λ , respectively, then let us define the *height* of e as $k + \lambda$. Perform stellar subdivisions at the edges of $\text{sd}K * \text{sd}\Lambda$ not in $\text{sd}K \cup \text{sd}\Lambda$ in order of decreasing height of the edges. Since this operation introduces to $\text{sd}(K) * \text{sd}(\Lambda)$ the chains between K and Λ in the join, the resulting complex is combinatorially equivalent to $\text{sd}(K * \Lambda)$, and is therefore identified with the latter. Since stellar subdivisions preserve shellability (Lemma I.1.10) we obtain that whenever

$$(X = A_1, A_2, \dots, A_j, \text{sd}K * N(V, \Lambda) = B_1, B_2, \dots)$$

is a shelling sequence (relative to χ) for X , then

$$(\mathcal{R}(\mathcal{Y}, A_1), \mathcal{R}(\mathcal{Y}, A_2), \dots, \mathcal{R}(\mathcal{Y}, A_j), \mathcal{R}(\mathcal{Y}, B_1), \mathcal{R}(\mathcal{Y}, B_2), \dots)$$

can be refined to a shelling sequence (relative to v) of \mathcal{Y} . It remains to describe such a shelling that restricts to $N(K * V, K * \Lambda)$. This can be done as follows: Set $\Delta_i := A_i \nabla A_{i+1}$. Then $\mathcal{R}(\mathcal{Y}, \Delta_i)$ is a path of $m := \binom{\dim K + \dim \Lambda + 2}{\dim K + 1}$ facets of \mathcal{Y} . Using Lemma I.1.13, shell $\mathcal{R}(\mathcal{Y}, \Delta_i)$ to the unique facet of $\mathcal{R}(\mathcal{Y}, \Delta_i)$ containing $\Delta_i \cap \text{sd}K$, and let $\Delta_{i,1}, \Delta_{i,2}, \dots, \Delta_{i,m}$ denote the associated shelling sequence. Then we shell \mathcal{Y} relative to v using the shelling sequence

$$\begin{aligned} (\mathcal{Y} = & \\ \mathcal{R}(\mathcal{Y}, A_1) = & \mathcal{R}(\mathcal{Y}, A_2) \cup \Delta_{1,1}, \mathcal{R}(\mathcal{Y}, A_2) \cup \Delta_{1,2}, \dots, \mathcal{R}(\mathcal{Y}, A_2) \cup \Delta_{1,m}, \\ \mathcal{R}(\mathcal{Y}, A_2) = & \mathcal{R}(\mathcal{Y}, A_3) \cup \Delta_{2,1}, \mathcal{R}(\mathcal{Y}, A_3) \cup \Delta_{2,2}, \dots, \mathcal{R}(\mathcal{Y}, A_3) \cup \Delta_{2,m}, \\ & \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \\ \mathcal{R}(\mathcal{Y}, A_j) = & \mathcal{R}(\mathcal{Y}, B_1) \cup \Delta_{j,1}, \mathcal{R}(\mathcal{Y}, B_1) \cup \Delta_{j,2}, \dots, \\ & \qquad \qquad \qquad \mathcal{R}(\mathcal{Y}, B_1) \cup \Delta_{j,m}, \mathcal{R}(\mathcal{Y}, B_1)) \end{aligned}$$

This shells \mathcal{Y} to $\mathcal{R}(\mathcal{Y}, B_1)$ relative to v and the shelling clearly restricts to $N(K * V, K * \Lambda)$. It remains to show that $\mathcal{R}(\mathcal{Y}, B_1)$ is shellable relative to v ; whatever shelling we choose for this will automatically restrict to $N(K * V, K * \Lambda)$ since $\mathcal{R}(\mathcal{Y}, B_1) \subset N(K * V, K * \Lambda)$. Hence, we can finish the proof by applying Lemma I.1.10, since B_1 is shellable relative to χ . ■

I.2. Shellability of convex balls

We can now prove Theorem A, building on ideas from [1, Ch. 2], see also [2]. We first prove that any subdivision of a convex polytope (of arbitrary dimension) is shellable after 2 derived subdivisions; afterwards, we argue

that 1 derived subdivision suffices for $d \leq 3$. The essence of the proofs can be summarized as follows:

- For general d , we start with a derived subdivision in order to be able to order the vertices nicely; for $d \leq 3$ this step turns out not to be necessary. Let us call the resulting complex X .
- We now order the vertices v_1, \dots, v_n of X by an order coming from the geometry of the complex. Then, we define the complexes $\Sigma_i := \text{sd} X \nabla \{v_0, v_1, \dots, v_{i-1}\}$ and prove the shellability of $\text{sd} X$ by proving

$$\Sigma_i \searrow_S^D \Sigma_i \nabla v_i = \Sigma_{i+1}$$

for every i , compare Figure 2.

Arbitrary dimension. A hemisphere is in *general position* w.r.t. a complex C in S^d if its boundary contains no vertex of C . Recall that a *polyhedron* in S^d is a subset of S^d that is obtained as the intersection of a finite number of closed halfspaces in the sphere.

Theorem I.2.1. *Let C be any subdivision of a polyhedron P in S^d . Let H_+ denote a general position hemisphere in S^d that intersects C non-trivially. Let H_- denote the hemisphere complementary to H_+ . Define $D = \emptyset$ or $D = \text{sd}^2 \partial C$. Then*

- (I) $\text{sd} N(\mathbb{R}(C, H_+), C)$ is shellable relative to D , and
- (II) $\text{sd}^2 C \searrow_S^D \text{sd} N(\mathbb{R}(C, H_+), C) \searrow_S^D \emptyset$.

Corollary I.2.2. *Let C be any subdivision of a convex polytope in \mathbb{R}^d . Then $\text{sd}^2 C$ is shellable.*

Before we begin with the proof, let us state a variant of Theorem I.2.1 that will become useful later. The proof is analogous to the proof of Theorem I.2.1, but requires a more involved notation due to an additional variable.

Theorem I.2.3. *Let C be any subdivision of a polyhedron P in S^d . Let p denote any facet of P and let H_+ denote a general position hemisphere in S^d that intersects P and p non-trivially. Let H_- denote the hemisphere complementary to H_+ . Define $D := \mathbb{R}(\text{sd}^2 C, p)$ or $D := \partial \text{sd}^2 C \nabla \mathbb{R}(\text{sd}^2 C, p)$. Then*

- (I) $\text{sd} N(\mathbb{R}(C, H_+), C)$ is shellable relative to D , and
- (II) $\text{sd}^2 C \searrow_S^D \text{sd} N(\mathbb{R}(C, H_+), C) \searrow_S^D \emptyset$.

Proof of Theorem I.2.1. The proof is by induction on the dimension; the case $d=1$ is obvious. As for the case $d > 1$, we will see that for a fixed dimension d , (I) implies (II) straightforwardly. Hence, we first prove (I) for dimension $d+1$, and we may assume that (I) and (II) have been already proven for all dimensions at most d .

Proof of (I). Let x denote the center of mass of H_+ , and let $d(y)$ denote the distance of a point $y \in S^d$ to x with respect to the intrinsic metric on S^d . Let $M(C, H_+)$ denote the set of faces σ of $R(C, H_+)$ for which the function $\min_{y \in \sigma} d(y)$ attains its minimum in the relative interior of σ . With this, we order the elements of $M(C, H_+)$ strictly by defining $\sigma < \sigma'$ whenever $\min_{y \in \sigma} d(y) < \min_{y \in \sigma'} d(y)$. Since H_+ is in general position, we may perturb it slightly without changing $N(R(C, H_+), C)$, and therefore we may assume that $<$ is a strict total order.

This allows us to induce an associated derived order on the vertices of $\text{sd}C$, which we restrict to the vertices of $N(R(C, H_+), C)$. Let $v_0, v_1, v_2, \dots, v_n$, $n = f_0(N(R(C, H_+), C)) - 1$, denote the vertices of $N(R(C, H_+), C)$ labeled according to the latter order, starting with the maximal element v_0 . Let Σ_i denote the complex

$$\text{sd} N(R(C, H_+), C) \nabla \{v_0, v_1, \dots, v_{i-1}\}.$$

We will prove that $\Sigma_i \searrow_S^D \Sigma_{i+1}$ for all i , $0 \leq i \leq n-1$. To finish the proof, we then only have to observe that Σ_n is a join of v_n and $\text{Lk}(v_n, \text{sd}^2 C)$, which is shellable relative to $\text{Lk}(v_n, D)$ by inductive assumption. Hence, by Lemma I.1.12, Σ_n is shellable relative to D . To prove $\Sigma_i \searrow_S^D \Sigma_{i+1} = \Sigma_i \nabla v_i$, there are two cases to consider.

- (1) v_i corresponds to an element of $M(C, H_+)$.
- (2) v_i corresponds to a face of C not in $M(C, H_+)$.

We need some notation to prove these two cases. Recall that we can define N , N^1 and Lk with respect to a base-point; we shall need this notation in case (1). We shall abbreviate $v := v_i$ for the duration of the proof. Let us denote by τ the face of C corresponding to v in $\text{sd}C$, and let m denote the point in τ with $d(m) = \min_{y \in \tau} d(y)$. Finally, define the ball B_m as the set of points y in S^d with $d(y) \leq d(m)$, and define $D' := \text{Lk}(v, D)$.

Case (1): In this case, $\text{Lk}(v, \Sigma_i) \cong \text{sd}(\text{sd} \partial \tau * \text{sd} \text{Lk}(\tau, C))$ and $\text{Lk}(v, \Sigma_i \Delta v) \subset \text{Lk}(v, \Sigma_i)$ is combinatorially equivalent to

$$\text{sd} \left(\text{sd} \partial \tau * N(\text{LLk}_m(\tau, C), \text{Lk}(\tau, C)) \right),$$

where $\text{LLk}_m(\tau, \sigma) := R(\text{Lk}(\tau, C), N_{(m, \tau)}^1 B_m)$.

The complexes $\text{sd}(\text{sd} \partial \tau * \text{sd} \text{Lk}(\tau, C))$ and $\text{sd}(\text{sd} \partial \tau * N(\text{LLk}_m(\tau, C), \text{Lk}(\tau, C)))$ are obtained as stellar subdivisions of the complexes

$$\begin{aligned} & \text{sd}^2 \partial \tau * \text{sd}^2 \text{Lk}(\tau, C) \quad \text{and} \\ & \text{sd}^2 \partial \tau * \text{sd} N(\text{LLk}_m(\tau, C), \text{Lk}(\tau, C)) \subset \text{sd}^2 \partial \tau * \text{sd}^2 \text{Lk}(\tau, C), \end{aligned}$$

respectively (compare also the proof of Proposition I.1.16). Hence, by inductive assumption (II), Lemma I.1.10 and Lemma I.1.13, $\text{Lk}(v, \Sigma_i) \searrow_S^{D'} \text{Lk}(v, \Sigma_i \Delta v) \searrow_S^{D'} \emptyset$. Hence, Corollary I.1.5 proves

$$\Sigma_i \searrow_S^D \Sigma_i \nabla v = \Sigma_{i+1}.$$

Case (2): If τ is not an element of $M(C, H_+)$, let σ denote the face of τ containing m in its relative interior. Then

$$\text{Lk}(v, \Sigma_i) \cong \text{sd}(\text{sd} \partial \tau * \text{sd} \text{Lk}(\tau, C))$$

and

$$\text{Lk}(v, \Sigma_i \Delta v) \cong N(v_\sigma * \text{sd} \text{Lk}(\tau, C), \text{sd} \partial \tau * \text{sd} \text{Lk}(\tau, C)),$$

where v_σ is the vertex of $\text{sd} \partial \tau$ corresponding to σ . By the inductive assumption (II), $\text{sd}^2 \text{Lk}(\tau, C) = \text{sd}(\text{sd} \text{Lk}(\tau, C))$ is shellable relative to $\text{sd}^2 \text{Lk}(\tau, D)$. Furthermore, $\text{sd}^2 \partial \tau$ can be realized as the boundary of a polytope since τ is a polytope, and therefore $\text{sd}^2 \partial \tau$ shells to the shellable complex $\text{St}(v_\sigma, \text{sd}^2 \partial \tau) = N(v_\sigma, \text{sd} \partial \tau)$ by the Bruggesser-Mani Theorem [12]. Hence, Proposition I.1.16 shows that $\text{Lk}(v, \Sigma_i)$ is shellable relative to $\text{sd}(\text{sd} \partial \tau * \text{sd} \text{Lk}(\tau, D))$, and this shelling can be chosen to restrict to $\text{Lk}(v, \Sigma_i \Delta v)$. Corollary I.1.5 and Lemma I.1.13 now proves

$$\Sigma_i \searrow_S^D \Sigma_i \nabla v = \Sigma_{i+1}.$$

This finishes the proof of (I).

Proof of (II). It suffices to prove the case where $D = \emptyset$, the case $D = \partial \text{sd}^2 C$ is then obtained by reversing the shelling (Proposition I.1.8). Observe that

$$A \cup B = \text{sd}^2 C \quad \text{and} \quad A \cap B = \partial A \nabla \partial \text{sd}^2 C = \partial B \nabla \partial \text{sd}^2 C,$$

where $A := \text{sd} N(\text{R}(C, H_+), C)$ and $B := \text{sd} N(\text{R}(C, H_-), C)$. By assumption on (I) the complex B is shellable relative to $\partial \text{sd}^2 C$. Then B is shellable relative to $\partial B \nabla \partial \text{sd}^2 C = A \cap B$ by Proposition I.1.8. Hence, by Lemma I.1.2 we see that $\text{sd}^2 C \searrow_S A$, and A is shellable by (I). \blacksquare

Dimension $d \leq 3$. To finish the proof of Theorem A, it remains to show that for $d \leq 3$, one derived subdivision is enough.

Theorem I.2.4. *Let C be any subdivision of a polytope in \mathbb{R}^d , $d \leq 3$. Then $\text{sd}C$ is shellable.*

The proof relies on the following analogue of Theorem I.2.1 for $d \leq 2$:

Proposition I.2.5. *Let C be any subdivision of a polyhedron in S^d , $d \leq 2$. Let H_+ denote a general position hemisphere in S^d that intersects C non-trivially. Let H_- denote the hemisphere complementary to H_+ . Define $D = \emptyset$ or $D = \text{sd}\partial C$. Then*

- (I) $N(\text{R}(C, H_+), C)$ is shellable relative to D , and
- (II) $\text{sd}C \searrow_S^D N(\text{R}(C, H_+), C) \searrow_S^D \emptyset$.

Proof. The case $d = 1$ is trivial, so assume $d = 2$. We already saw that (I) straightforwardly implies (II) in the proof of Theorem I.2.1. Hence, it remains to show that $X := N(\text{R}(C, H_+), C) \searrow_S^D \emptyset$. The case $D = \emptyset$ (and therefore also the case $D = \partial X$) is folklore. To prove the case $\emptyset \subsetneq D \subsetneq \partial X$, pick any facet Δ of X for which $\Delta \cap (\partial X \nabla D)$ is 1-dimensional.

Then $X \rightarrow X \nabla \Delta$ is a shelling step relative to D , cf. Figure 1. Proceeding this way, all facets of X can be removed with shelling steps relative to D . ■

Proof of Theorem I.2.4. Let μ be a generic vector in \mathbb{R}^d , and order the vertices of $C \subset \mathbb{R}^d$ according to their value under $\langle \mu, \cdot \rangle$. Since μ is generic, this gives a total order on the vertices of C . Let $v_0, v_1, v_2, \dots, v_n$, $n = f_0(C) - 1$, denote the vertices of C labeled according to that order, starting with the vertex v_0 maximizing the inner product $\langle \mu, \cdot \rangle$. Let Σ_i denote the complex $\text{sd}C \nabla \{v_0, v_1, \dots, v_{i-1}\}$. We will prove that $\Sigma_i \searrow_S \Sigma_{i+1}$ for all i , $0 \leq i \leq n - 1$. Set $v := v_i$ and define H_v as the set of points y in S^d with $\langle \mu, y \rangle \leq \langle \mu, v \rangle$. Then $\text{Lk}(v, \Sigma_i) \cong \text{sd} \text{Lk}(\tau, C)$ (where $v \in \text{relint } \tau$), and

$$\begin{aligned} \text{Lk}(v, \Sigma_i \Delta v) &\cong N(\text{LLk}_v(\tau, C), \text{Lk}(\tau, C)), \text{ where} \\ \text{LLk}_v(\tau, \sigma) &:= \text{R}(\text{Lk}(\tau, C), \text{T}_v^1 H_v). \end{aligned}$$

Hence, by Proposition I.2.5,

$$\text{Lk}(v, \Sigma_i) \searrow_S \text{Lk}(v, \Sigma_i \Delta v) \searrow_S \emptyset.$$

Thus, Corollary I.1.5 proves

$$\Sigma_i \searrow_S \Sigma_i \nabla v = \Sigma_{i+1}.$$

To finish the proof, observe that Σ_n is a join of v_n and $\text{Lk}(v_n, \text{sd}C)$, which is shellable by Proposition I.2.5. Hence, by Lemma I.1.12, Σ_n is shellable. ■

I.3. Consequences in combinatorial topology

We derive Theorem B and some further consequences in PL topology.

Some basic facts in simplicial approximation. Let us start by recalling a fundamental approximation Lemma of PL topology:

Lemma I.3.1 (cf. [46, Ch. 1, Lem. 4]). *Let C be a polytopal complex, and let C' be any subdivision of C . Then there is an $n \geq 0$ such that some n -th derived subdivision $\text{sd}^n C$ of C is a subdivision of both C and C' .*

Corollary I.3.2. *Let C and D be two PL homeomorphic polytopal complexes. Then there is an $n \geq 0$ and a subdivision D' of D such that $\text{sd}^n C \cong D'$.*

Furthermore, we need a fundamental result of PL topology going back to Alexander [6] and Newman [35].

Theorem I.3.3 ([46, Ch. 3, Thm. 2]). *Let C_1 and C_2 denote two PL d -balls, and let D_1, D_2 denote PL $(d-1)$ -balls in their respective boundaries. Then every PL homeomorphism $\phi: D_1 \rightarrow D_2$ extends to a PL homeomorphism $\phi: C_1 \rightarrow C_2$.*

If we combine this with Lemma I.3.1, we obtain:

Lemma I.3.4. *Let C be any PL d -ball, and let D denote any PL $(d-1)$ -ball in ∂C . Let Δ be the d -simplex, and let δ be any one of its facets. Then there is an $n \geq 0$ and a subdivision Δ' of Δ such that $\text{sd}^n D = \text{R}(\text{sd}^n C, D) \cong \delta' := \text{R}(\Delta', \delta)$, and the combinatorial isomorphism extends to an isomorphism $\text{sd}^n C \rightarrow \Delta'$.*

Finally, we state a result that allows us to replace PL homeomorphisms with subdivisions when dealing with shellability.

Proposition I.3.5. *Let C and $D \subset C$ denote polytopal complexes PL homeomorphic to complexes $\Gamma = \phi(C)$ and $\Delta = \phi(D) = \text{R}(\Gamma, \phi(D))$. Assume that Γ shells to Δ via ϕ . Then there exist subdivisions C' of C and $D' = \text{R}(C', D)$ of C and D such that C' shells to D' .*

Proof. By Lemma I.3.1, there is an $n \geq 0$ and a subdivision C' of C such that $\text{sd}^n C' \cong C'$, and such that $\text{sd}^n \Delta$ gets taken to $D' := \text{R}(C', D)$ under this combinatorial isomorphism. Since barycentric subdivisions preserve shellability (Lemma I.1.10), we see that $\text{sd}^n C' \searrow_S \text{sd}^n \Delta$ and therefore $C' \searrow_S D'$. ■

Characterization of shellability. As a direct consequence of Lemma I.3.4 and Theorems I.2.1 and I.2.3, we note the following:

Theorem I.3.6. *Let C be any PL ball, and let $D = \partial C$, $D = \emptyset$ or $D \subset \partial C$ a PL $(d-1)$ -ball. Then there is an $n \geq 0$ so that $\text{sd}^n C \searrow_{\mathcal{S}}^{\text{sd}^n D} \emptyset$.*

We conclude that shellability is almost preserved under subdivisions.

Theorem I.3.7. *Let C and $D \subset C$ be polytopal complexes such that C shells to D . Let C' be any subdivision of C . Then there is an $n \geq 0$ so that $\text{sd}^n C'$ shells to $\text{sd}^n D'$ where $D' = R(C', D)$.*

Remark I.3.8. By using a variant of Theorem I.2.3 one can show that in the notation of Theorem I.3.7, it suffices to take $n=2$.

Proof. We may assume that $C \searrow_{\mathcal{S}} D$ is only a single shelling step that consists of the removal of a single facet Δ . Set $\Delta' := R(C', \Delta)$. Now, $\Delta \cap D$ is either a PL $(d-1)$ -ball or coincides with $\partial\Delta$. Hence, the same is true for $\Delta' \cap D'$ which is a subdivision of $\Delta \cap D$. By Theorem I.3.6, there is an $n \geq 0$ such that $\text{sd}^n \Delta'$ is shellable relative to $\text{sd}^n \Delta' \cap \text{sd}^n D' = \text{sd}^n (\Delta' \cap D')$. Hence, by Lemma I.1.2,

$$\text{sd}^n C' = \text{sd}^n \Delta' \cup \text{sd}^n D' = \text{sd}^n (\Delta' \cup D') \searrow_{\mathcal{S}} \text{sd}^n D'. \quad \blacksquare$$

Finally, we can finish the proof of Theorem B; by Proposition I.3.5, it is equivalent to the following theorem, which we will prove instead.

Theorem I.3.9. *Let C and $D \subset C$ be polytopal complexes such that $C \searrow_{\mathcal{S}} D$. Suppose that some subdivision C' of C shells onto some subdivision D' of D . Then for n large enough, $\text{sd}^n C \searrow_{\mathcal{S}} \text{sd}^n D$.*

Proof. By Lemma I.3.1, there is an $\ell \geq 0$ so that $\text{sd}^{\ell} C$ is a subdivision of C' , and such that $\text{sd}^{\ell} D$ is a subdivision of $D' = R(C', D)$. Moreover, by Theorem I.3.7, there is an $m \geq 0$ so that

$$\text{sd}^{m+\ell} C = \text{sd}^m \text{sd}^{\ell} C \searrow_{\mathcal{S}} \text{sd}^m \text{sd}^{\ell} D = \text{sd}^{m+\ell} D. \quad \blacksquare$$

Now, a *triangulated manifold* is a simplicial complex whose underlying space is a topological manifold.

Corollary I.3.10. *For a triangulated manifold M , the following are equivalent:*

- (1) *For some $m \geq 0$, the m -th derived subdivision of M is shellable.*
- (2) *M is a PL ball or a PL sphere.*

Proof.

- (1) \Rightarrow (2) is well known: One can prove by induction on the dimension, combined with elementary cellular homology and Poincaré duality, that all shellable d -pseudomanifolds, for $d > 0$, are either PL balls or PL spheres, cf. [19].
- (2) \Rightarrow (1) follows from Theorem I.3.9, and the fact that every PL d -ball has some subdivision which is also some iterated derived subdivision of the d -simplex (hence shellable). ■

Corollary I.3.10 answers a problem of Billera and Swartz.

Update added in the course of publication: Corollary I.3.10 also follows from the following stronger result, very recently proven by the first author and Izmistiev [4]: For any PL sphere M , there is some $m \geq 0$ such that the m -th derived subdivision of M is combinatorially equivalent to the boundary of a polytope. (This had also been conjectured by Lou Billera.)

We conclude by expanding Corollary I.3.10 to a characterization of PL manifolds:

Corollary I.3.11. *For any triangulated manifold M , the following are equivalent:*

- (1) M is PL.
- (2) Every vertex link in M becomes shellable after suitably many derived subdivisions.
- (3) For some $m \geq 0$, the m -th derived subdivision of any vertex link in M is shellable.
- (4) For some $m \geq 0$, all vertex links in $\text{sd}^m M$ are shellable.

II. Collapsibility, products, and the Zeeman conjecture

The work of the previous chapter shows that derived subdivisions induce shellability. An analogue of Theorem I.3.9 can be proven for collapsibility, either by invoking Theorem I.3.7, or by using the more powerful result proven in [1,2]:

Theorem II.0.1 ([1, Thm. 2.3.B]). *Let C, D be polytopal complexes such that C collapses to D . Let C' be any subdivision of C . Then $\text{sd} C'$ collapses to $\text{sd} D'$, where $D' = R(C', D)$.*

This follows from the fact that a convex polytope is collapsible after one derived subdivision; the proof of this statement, in turn, uses geometry to give an order on the faces of C in a similar way to Theorem A. This proves Theorem C:

Corollary II.0.2. *Let C, D be polytopal complexes such that $D \subset C$. Suppose that some subdivision C' of C collapses onto some subdivision $D' = R(C', D)$ of D . Then for n large enough, $\text{sd}^n C \searrow \text{sd}^n D$.*

Proof. With Lemma I.3.1, let us choose n large enough, so that $\text{sd}^{n-1} C$ is a subdivision of C' . Then $\text{sd}^n C \searrow \text{sd}^n D$ by Theorem II.0.1. ■

So, derived subdivisions induce collapsibility as well. We will now see that instead of subdividing, one could take repeated product with intervals: The effect is similar, and that is not a coincidence.

II.1. Collapsibility of products

In this section, we show that any contractible complex can be made collapsible by taking products with the n -dimensional cube, for n suitably large.

In order to study products with cubes, we start by introducing a special subdivision, which is in some sense a cubical analogue of the classical operation of stellar subdivision.

Definition II.1.1. Let C be a polytopal complex in \mathbb{R}^d , and let τ be any face of C . Let x_τ denote a point anywhere in the relative interior of τ , and let λ be any number in the interval $(0, 1)$. Define

$$\begin{aligned} \text{c-stlr}(\tau, C) := & (C - \tau) \cup \\ & \{\text{conv}(\sigma \cup (\lambda\sigma + (1 - \lambda)x_\tau)) : \sigma \in \text{St}(\tau, C) - \tau\} \cup \\ & (\lambda \text{St}(\tau, C) + (1 - \lambda)x_\tau). \end{aligned}$$

The complex $\text{c-stlr}(\tau, C)$ is the *cubical stellar subdivision*, or *c-stellar subdivision* of C at τ .

Lemma II.1.2. *Let C be an arbitrary complex, and let τ be any face of C . Then $C \times [0, 1]$ collapses to (a complex combinatorially equivalent to) $\text{c-stlr}(\tau, C)$.*

Proof. Let σ be a face of C . We perform collapses on $C \times [0, 1]$ as follows:

- If σ contains τ , we delete $\sigma \times \{0\}$ from the complex.
- If σ is not in $\text{St}(\tau, C)$, we delete $\sigma \times \{1\}$ from the complex.

The deletions are conducted in order of decreasing dimension, in order to ensure that faces are free when deleted. The result is combinatorially equivalent to $\text{c-stlr}(\tau, C)$, compare Figure 3. ■

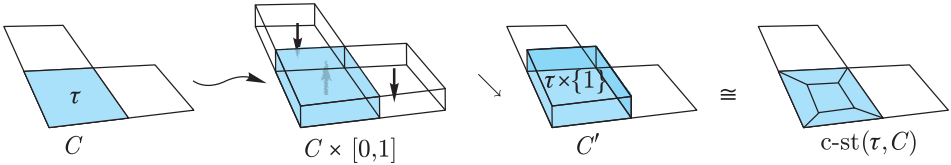


Figure 3. $C \times [0,1]$ collapses vertically to C' , which is combinatorially equivalent to the cubical stellar subdivision $\text{c-st}(\tau, C)$ of C .

Definition II.1.3 (Cubical derived subdivision). Let C denote any polytopal d -complex, with faces ordered by decreasing dimension- A *cubical derived subdivision*, or *c-derived subdivision*, is any subdivision C obtained by c-stellar subdivision, subdividing first all d -faces of C , then perform cubical stellar subdivisions on resulting complex at all $(d-1)$ -faces of C , then all $(d-2)$ -faces, and so on up to the faces of dimension 0.

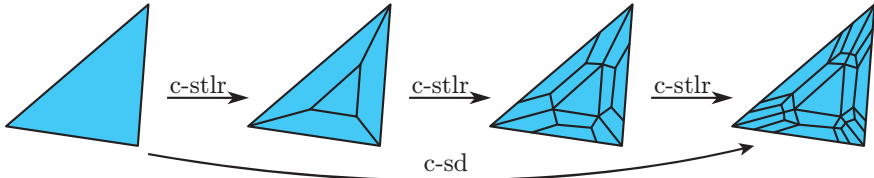


Figure 4. A cubical derived subdivision (here, of a triangle) performs cubical stellar subdivisions first at the faces of maximal dimension, then at the codimension one faces of the original complex etc.

c-stellar and c-derived subdivisions mimic the classical notions of stellar and derived subdivisions, respectively. Notice that in contrast to other forms of “cubical subdivisions”, the cubical stellar subdivision of a polytopal complex is a cubical complex only if the original complex was also a cubical complex. In contrast, other types of cubical subdivision, like [13, Construction 4.5] or the construction in [28], always yield cubical complexes (if one starts with a polytopal complexes consisting of simple polytopes).

Recall that a derived subdivision of a d -complex can be obtained by iterated stellar subdivision, beginning with the d -faces of C , then all $(d-1)$ -faces of C , and so on. In fact, c-derived subdivisions enjoy several analogous properties, among them the following theorem:

Theorem II.1.4. *Let C, D be polytopal complexes such that $D \subset C$. Suppose that some subdivision C' of C collapses onto some subdivision $D' = \text{R}(C', D)$ of D . Then for n large enough, $\text{c-sd}^n C \searrow \text{c-sd}^n D$. \blacksquare*

The proof is analogous to the proof of Corollary II.0.2. We are now ready to show that contractible complexes can be made collapsible by taking products with high-dimensional cubes. Recall that two complexes are called *simple homotopy equivalent* if there exists a sequence of expansions (inverses of collapses) and collapses (on the level of CW complexes) that transforms one complex into the other, cf. [15].

Theorem II.1.5. *If two polytopal complexes C, D are simple homotopy equivalent, there exists an $n \geq 0$ such that $C \times I^n$ collapses to a polytopal complex combinatorially equivalent to a subdivision of D .*

Proof. Let C and D be two complexes that are simple homotopy equivalent. By a classical result of Dierker, Lickorish and Cohen [21,31,16], there is an $\ell \geq 0$ such that the complex $E := C \times I^\ell$ is PL equivalent to a complex that collapses to a complex F that is PL homeomorphic to D .

By Theorem II.1.4 there exists a c-derived subdivision $\text{c-sd}^m E$ of E that collapses to $\text{c-sd}^m F = R(\text{c-sd}^m E, F)$. Lemma II.1.2 now provides an $n \geq 0$ so that $E \times I^n$ collapses to a complex combinatorially equivalent to $\text{c-sd}^m F$. Thus, $C \times I^{n+\ell} = E \times I^n$ collapses to a complex PL homeomorphic to D . ■

Corollary II.1.6. *For any contractible complex C , there is an $n \geq 0$ for which $C \times I^n$ is collapsible.*

Proof. Every contractible complex is simple homotopy equivalent to a point, cf. [45]. ■

Remark II.1.7. The analogous statement does not hold with respect to shellability. Indeed, not even the results of Dierker, Lickorish and Cohen extend to shellability. For instance, let $C := v * T$ be the join of any triangulation T of the torus $S^1 \times S^1$ with a vertex, and let n be any nonnegative integer. Let $\Sigma = \varphi(C \times I^n)$ be any complex PL homeomorphic to $C \times I^n$, and let σ be any facet of $R(\Sigma, \varphi(v \times I^n))$. Then $\text{Lk}(\sigma, \Sigma)$ is PL homeomorphic to T , and therefore not shellable (Corollary I.3.10). Hence $\Sigma \cong C \times I^n$ is *not* PL homeomorphic to a shellable complex.

Non-evasiveness. A stronger notion than collapsibility, called non-evasiveness, was introduced in [30] in connection with the Aanderaa–Karp–Rosenberg conjecture. A simplicial d -complex C is called *non-evasive* if either $d=0$ and C is a single point, or $d>0$ and there is a vertex v of C so that both $\text{Lk}(v, C)$ and $C - v$ are non-evasive. Every non-evasive complex is collapsible, and therefore contractible. To make sense of non-evasiveness with respect to products, recall that an *order complex* of a poset \mathcal{R} is any

simplicial complex $\Omega(\mathcal{R})$ whose face poset is combinatorially equivalent to the poset of chains in \mathcal{R} , ordered by inclusion. A poset is *non-evasive* if its order complex is non-evasive, and the product of posets is defined as follows: If \mathcal{A} and \mathcal{B} are two posets on sets A and B , respectively, then the product $\mathcal{A} \times \mathcal{B}$ is a poset on the set $A \times B$, where $(a, b) \prec (a', b')$ if and only if $a \prec a'$ and $b \prec b'$.

Proposition II.1.8 (Welker [44]). *The product of any two non-evasive posets is non-evasive.*

Welker [44, Open Problem 2] asked whether the converse of Proposition II.1.8 holds; more precisely, he asked whether the non-evasiveness of the product $\mathcal{A} \times \mathcal{B}$ of two order complexes \mathcal{A}, \mathcal{B} implies that both \mathcal{A} and \mathcal{B} are non-evasive.

To see the connection to products with intervals, let us denote by \mathcal{I} the poset on two elements 0, 1, ordered by $0 < 1$. Clearly, $\Omega(\mathcal{I})$ is combinatorially equivalent to I . Moreover, products of posets with \mathcal{I} behave like products of complexes with I : Indeed $\Omega(\mathcal{P} \times \mathcal{I})$ is a subdivision of $\Omega(\mathcal{P}) \times \Omega(\mathcal{I})$ that introduces no new vertices. This connection allows us to answer Welker's problem.

Proposition II.1.9. *There exists an evasive poset \mathcal{P} such that $\mathcal{P} \times \mathcal{I}$ is non-evasive.*

The example we present here is closely modeled after a classical example of a simplicial complex that is not non-evasive, but that is non-evasive after one derived subdivision [7, Example 5.4].

Proof. Let \mathcal{R} denote the face poset of the polytopal complex given Figure 5(1). Let \mathcal{P} denote the poset given by adding an element v to \mathcal{R} , that is defined to be larger than all faces of the highlighted subcomplex in Figure 5(1). Then the order complex $C := \Omega(\mathcal{P})$ can be pictured as in Figure 5(2).

The complex C is collapsible (starting from the bottom edge), but the links of all vertices deformation retract to wedges of 1-spheres, hence they are not contractible and cannot be non-evasive. Consequently, C and in particular \mathcal{P} are not non-evasive; compare [7, Example 5.4].

Consider now the complex $\Omega(\mathcal{P} \times \mathcal{I})$, in which every vertex τ of C corresponds to two copies $(\tau, 0)$ and $(\tau, 1)$ of $\Omega(\mathcal{P} \times \mathcal{I})$. Moreover, we call the vertices of C marked in Figure 5(2) by a cross the “interior vertices of the first kind”, and the vertices marked by a circle the “interior vertices of the second kind”. Their union are the interior vertices, and the remaining

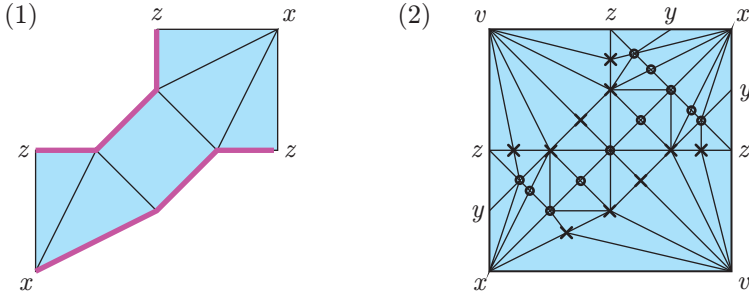


Figure 5. (1) A construction step for the poset \mathcal{P} . (2) The order complex $C = \Omega(\mathcal{P})$

vertices are the “boundary vertices”. These notions are naturally inherited when passing from \mathcal{P} to products. Note that without the identifications on the boundary, C is only a 2-disk, and non-evasive by elementary arguments (see [3]). A *non-evasive deletion* is the deletion of a vertex whose link is non-evasive. Clearly, a complex is non-evasive if it can be deformed to a point by non-evasive deletions, which we will show for $\Omega(\mathcal{P} \times \mathcal{I})$.

- (1) Delete $(y, 1)$ and all the interior vertices of $\Omega(\mathcal{P} \times \mathcal{I})$ of type $(\cdot, 1)$ of the second kind in an order given by the poset, i.e., first delete those vertices $(\cdot, 1)$ that are maximal in \mathcal{P} , then the highest remaining ones and so on.
- (2) From the resulting complex, delete the vertices $(v, 0)$ and $(x, 1)$, in that order.
- (3) From the resulting complex, delete all remaining interior vertices, which can be done by deleting the vertices in an order that gives the deletion order on the disk \mathcal{P} without the boundary identifications.
- (4) From the resulting complex, delete the vertices $(v, 1)$, $(z, 1)$, $(z, 0)$ and $(y, 0)$ in that order.

Only the vertex $(x, 0)$ remains, and one can easily check the deletions described above are non-evasive deletions. ■

II.2. A simplicial approach to the Zeeman conjecture

The *Zeeman conjecture* claims that the product of any contractible 2-dimensional complex C with the interval $I = [0, 1]$ is PL homeomorphic to a collapsible complex. Its importance is highlighted by the fact that it implies, quite straightforwardly, the three-dimensional Poincaré conjecture, recently proven by Hamilton–Perelman (cf. [24, pp. 78–79]). In turn, the work of Hamilton–Perelman straightforwardly shows the validity of the Zeeman conjecture for all “special” contractible 2-complexes that embed in \mathbb{R}^3

(or equivalently 3-manifolds), cf. [23,34]. Using Theorem C, this (still open) conjecture can be rephrased in terms of derived subdivisions.

Proposition II.2.1. *The Zeeman conjecture is equivalent to the following conjecture: For any contractible 2-complex C , there is an $m \geq 0$ such that $\text{sd}^m(C \times I)$ is collapsible. One can prove the following weaker statement: For any contractible 2-complex C , there is an $m \geq 0$ such that $\text{sd}^m(C \times I^6)$ is collapsible.*

Proof. The first claim follows from Corollary II.0.2. As for the second claim, Cohen showed for any contractible 2-complex C , the product $C \times I^6$ is PL equivalent to a collapsible complex [16]. The conclusion follows then from Corollary II.0.2. ■

Our principal goal in this section is to show that the number of subdivisions needed, in order to achieve collapsibility, can be arbitrarily high.

Lemma II.2.2. *Let B be any polytopal 3-ball. Let m, n be two non-negative integers. The link of every proper face in $\text{sd}^m(B \times I^n)$ is shellable.*

Proof. Since B is three-dimensional, each combinatorial link in B is a sphere or a ball of dimension ≤ 2 , and therefore shellable by Proposition I.2.5. Hence, for every face τ of B , the complex $\text{St}(\tau, B)$ is shellable. For brevity, set $A := B \times I^n$ and $A' := \text{sd}^m A$.

For any face τ of A , $\text{St}(\tau, A)$ is shellable, since shellability is preserved under products with I . Proposition I.1.11 yields the shellability of $\text{Lk}(\tau, A)$.

Now, for any face τ' of A' , there is a face τ in A such that $\text{St}(\tau', A') \subset \text{St}(\tau, A)$. By Lemma I.1.10, the complex $\text{R}(A', \text{St}(\tau, A)) = \text{sd}^m \text{St}(\tau, A)$ is shellable. Consequently, $\text{Lk}(\tau', A')$ is shellable by Proposition I.1.11. ■

Definition II.2.3 (cf. [8]). A triangulation C of a d -manifold with non-empty (resp. empty) boundary is called *endo-collapsible* if C minus a d -face collapses onto ∂C (resp. onto a vertex).

Lemma II.2.4 ([8, Corollary 3.21]). *Let B be a collapsible PL d -ball. If $\text{sd} \text{Lk}(\sigma, B)$ is endo-collapsible for every proper face σ , then $\text{sd} B$ is also endo-collapsible.*

Lemma II.2.5 ([8, Theorem 4.1]). *All shellable manifolds are endo-collapsible.*

Lemma II.2.6 ([8, Theorem 3.15]). *Let M be an endo-collapsible PL d -manifold. Let L be a subcomplex of M , with $\dim L = \ell \leq d - 2$, such that all facets of L lie in the interior of M . Then the homotopy group $\pi_{d-\ell-1}(|M| - |L|)$ has a presentation with (at most) $f_\ell(L)$ generators.*

Theorem II.2.7. *For any $m, n \geq 0$, there is a 3-ball B such that $\text{sd}^m(B \times I^n)$ is not collapsible.*

Proof. We prove the claim in two parts:

- (1) we prove that if $\text{sd}^m(B \times I^n)$ is collapsible, then $\text{sd}^{m+1}(B \times I^n)$ is endo-collapsible;
- (2) we construct a 3-ball B such that $\text{sd}^{m+1}(B \times I^n)$ is not endo-collapsible.

The conclusion follows immediately by combining (1) and (2). So, let us prove these claims.

- (1) Via Lemma II.2.4, it suffices to show that the link of every proper face in $\text{sd}^{m+1}(B \times I^n)$ is endo-collapsible. In Lemma II.2.2 we showed that any such link is shellable; by Lemma II.2.5, shellability implies endo-collapsibility.
- (2) Set $\ell := f_{n+1}(\text{sd}^{m+1}(I^{n+1}))$, and let B be a simplicial 3-ball with a 3-edge subcomplex K , isotopic to the connected sum of 3ℓ trefoil knots. (For how to construct it, see [27].) Then any presentation of the group $\pi_1(|B| - |K|)$ must have at least $3\ell + 1$ generators, cf. Goodrick [25]. Now, set $M := \text{sd}^{m+1}(B \times I^n)$ and $L := \text{sd}^{m+1}(K \times I^n)$. This L is $(n+1)$ -dimensional, and has exactly 3ℓ facets, while M is $(n+3)$ -dimensional. Clearly $|B| - |K|$ is a deformation retract of $|M| - |L|$, so the homotopy groups of these two spaces are the same. In particular, any presentation of $\pi_1(|M| - |L|)$ must have at least $3\ell + 1$ generators. Were M endo-collapsible, by Lemma II.2.6 we would obtain a presentation of $\pi_1(|M| - |L|)$ with (at most) 3ℓ generators, a contradiction. ■

Corollary II.2.8. *For any pair of non-negative integers m, n , there is a contractible 2-dimensional simplicial complex C such that $\text{sd}^m(C \times I^n)$ is not collapsible.*

Proof. Let B be any 3-ball for which $\text{sd}^m(B \times I^n)$ is not collapsible; for example, the one constructed in Theorem II.2.7. Every polytopal d -ball collapses to a $(d-1)$ -dimensional subcomplex, so that B collapses to some 2-complex C . The complex C is a deformation retract of B and therefore contractible. Moreover, since B collapses onto C , $B \times I^n$ collapses onto $C \times I^n$ and $\text{sd}^m(B \times I^n)$ collapses onto $\text{sd}^m(C \times I^n)$. Were the latter complex collapsible, $\text{sd}^m(B \times I^n)$ would be collapsible as well, a contradiction. ■

Remark II.2.9. While the above Theorems use knot theory to provide complexes that are far from collapsible, one can also use the algorithmic undecidability. Indeed, were there universal constants m, n such that for every

contractible simplicial complex C the complex $\text{sd}^m(C \times I^n)$ is collapsible, then by checking collapsibility of $\text{sd}^m(C \times I^n)$, which can be done by checking all possible collapsing sequences, we would have an algorithm to decide the contractibility of simplicial complexes, and in particular of triangulated manifolds.

Recall now that Novikov [43] proved that it is not decidable whether a given manifold with the homology of the 5-sphere is actually the 5-sphere. Now, a triangulated manifold M with the homology of S^5 is homeomorphic to S^5 if and only if for any 5-ball B in M , the manifold with boundary $M \setminus B$ is contractible. Hence, deciding whether a given 5-manifold is contractible is not decidable by any algorithm. This contradicts the soundness of the proposed algorithm above, and it follows that no universal constants m , n exist such that contractibility of C implies the collapsibility of $\text{sd}^m(C \times I^n)$.

Notice that this approach does not give us a 2-complex, but simplicial complexes of higher dimension. It is not known to the authors whether the contractibility of 2-complexes is decidable.

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