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THE 4/5 UPPER BOUND ON THE GAME TOTAL DOMINATION NUMBER

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The recently introduced total domination game is studied. This game is played on a graph G by two players, named Dominator and Staller. They alternately take turns choosing vertices of G such that each chosen vertex totally dominates at least one vertex not totally dominated by the vertices previously chosen. Dominator's goal is to totally dominate the graph as fast as possible, and Staller wishes to delay the process as much as possible. The game total domination number, $\gamma_{tg}(G)$, of G is the number of vertices chosen when Dominator starts the game and both players play optimally. The Staller-start game total domination number, $\gamma'_{tg}(G)$, of G is the number of vertices chosen when Staller starts the game and both players play optimally. In this paper it is proved that if G is a graph on n vertices in which every component contains at least three vertices, then $\gamma_{tg}(G) \le 4n/5$ and $\gamma'_{tg}(G) \le (4n+2)/5$. As a consequence of this result, we obtain upper bounds for both games played on any graph that has no isolated vertices.

1. Introduction

The domination game in graphs was introduced in [2] and extensively studied afterwards in [1,3,7,8,10,14,15,16] and elsewhere. A vertex u in a graph G *dominates* a vertex v if $u=v$ or u is adjacent to v in G. A *dominating set* of G is a set S of vertices of G such that every vertex in G is dominated by a vertex in S. The game played on a graph G consists of two players, *Dominator* and *Staller*, who take turns choosing a vertex from G. Each vertex chosen must dominate at least one vertex not dominated by the vertices previously chosen. The game ends when the set of vertices chosen becomes a dominating set in G. Dominator wishes to dominate the graph as fast as possible, and

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Staller wishes to delay the process as much as possible. The *game domination number* (resp. *Staller-start game domination number*), $\gamma_g(G)$ (resp. $\gamma_g'(G)$), of G is the number of vertices chosen when Dominator (resp. Staller) starts the game and both players play optimally.

Recently, the total version of the domination game was investigated in [11], where in particular it was demonstrated that these two versions differ significantly. A vertex u in a graph G *totally dominates* a vertex v if u is adjacent to v in G. A *total dominating set* of G is a set S of vertices of G such that every vertex of G is totally dominated by a vertex in S. The *total domination game*, played on a graph G again consists of two players called *Dominator* and *Staller* who take turns choosing a vertex from G. In this version of the game, each vertex chosen must totally dominate at least one vertex not totally dominated by the set of vertices previously chosen. At any particular point in the game some subset C of vertices has been chosen by the players, and it is the turn of one of the two players. We say that a vertex v of G is *playable* if v has a neighbor that is not totally dominated by C. If the player chooses v, then we say the player *played* v and we refer to this choice as the *move* of that player. For emphasis we may also say it was a *legal* move. The game ends when the set of vertices chosen is a total dominating set in G, or equivalently when G has no playable vertices. Dominator wishes to totally dominate the graph as fast as possible, and Staller wishes to delay the process as much as possible.

The *game total domination number*, $\gamma_{tg}(G)$, of G is the number of vertices chosen when Dominator starts the game and both players play optimally. The *Staller-start game total domination number*, $\gamma_{tg}'(G)$, of G is the number of vertices chosen when Staller starts the game and both players play optimally. For simplicity, we shall refer to the Dominator-start total domination game and the Staller-start total domination game as *Game 1* and *Game 2*, respectively.

A *partially total dominated graph* is a graph together with a declaration that some vertices are already totally dominated; that is, they need not be totally dominated in the rest of the game. In [11], the authors present a key lemma, named the *Total Continuation Principle*, which in particular implies that the number of moves in Game 1 and Game 2 when played optimally can differ by at most 1.

For notation and graph theory terminology not defined herein, we in general follow [9]. We denote the *degree* of a vertex v in a graph G by $d_G(v)$, or simply by $d(v)$ if the graph G is clear from the context. A *leaf* in a graph is a vertex of degree 1 in the graph. If X and Y are subsets of vertices in a graph G, then the set X *totally dominates* the set Y in G if every vertex of Y

is adjacent to at least one vertex of X. In particular, if X totally dominates the vertex set $V(G)$ of G, then X is a total dominating set in G. For more information on total domination in graphs see the recent book [12]. Since an isolated vertex in a graph cannot be totally dominated by definition, *all graphs considered will be without isolated vertices*. We use the standard notation $[k]=\{1,\ldots,k\}$.

Our aim in this paper is to establish upper bounds on the Dominator-start game total domination number and the Staller-start game total domination number played in a graph. More precisely, we shall prove the following result.

Theorem 1. *If G is a graph on n vertices in which every component contains at least three vertices, then*

$$\gamma_{tg}(G) \le \frac{4n}{5} \quad \textit{and} \quad \gamma'_{tg}(G) \le \frac{4n+2}{5}.$$

Our proof strategy is to modify an ingenious approach adopted by Csilla Bujtás [5] in order to attack the 3/5-*Conjecture* which asserts that $\gamma_g(G) \le 3|V(G)|/5$ holds for any isolate-free graph G. The conjecture was posed in [13] and proved to hold for forests in which each component is a caterpillar. In [5] Bujtás proved the conjecture for forests in which no two leaves are at distance 4 apart, while in [6] she further developed her method and proved upper bounds for $\gamma_g(G)$ which are better than $3|V(G)|/5$ as soon as the minimum degree of G is at least 3. On the other hand, large families of trees were constructed that attain the conjectured 3/5-bound and all extremal trees on up to 20 vertices were found in [4]. Further progress toward the 3/5-Conjecture was made in [10] where the conjecture is established over the class of graphs with minimum degree at least 2.

Bujtás's approach is to color the vertices of a forest with three colors that reflect three different types of vertices and to associate a weight with each vertex. In the total version of the domination game, we modify Bujtás's approach by coloring the vertices of a graph with four colors that reflect four different types of vertices. In the next section we introduce this coloring and deduce its basic properties. In Section 3 we assign weights to colored vertices and study the decrease of total weight of the graph as a consequence of playing vertices in the course of the game. In the subsequent section we define and study three phases of the game, while in Section 5 we describe and analyze a strategy of Dominator based on the three phases. The efforts of previous sections are culminated in Section 6 where a proof of Theorem 1 is given. As a consequence, we then prove that when G is a graph of order n with k components of order 2 and no isolated vertices, $\gamma_{tg}(G) \le \frac{4n+2k}{5}$ and $\gamma'_{tg}(G) \le \frac{4n+2k+2}{5}$. Finally, we pose a so-called 3/4-Conjecture.

2. A Residual Graph

We consider the total domination game played on a graph G in which every component contains at least three vertices. At any stage of the game, let D denote the set of vertices played to date where initially $D = \emptyset$. We define a *colored-graph* with respect to the played vertices in the set D as a graph in which every vertex is colored with one of four colors, namely white, green, blue, or red, according to the following rules.

- A vertex is colored *white* if it is not totally dominated by D and does not belong to D.
- A vertex is colored *green* if it is not totally dominated by D but belongs to D.
- A vertex is colored *blue* if it is totally dominated by D but has a neighbor not totally dominated by D.
- A vertex is colored *red* if it and all its neighbors are totally dominated by D.

Note that a vertex u is colored white if and only if $D \cap N[u] = \emptyset$ and is colored green if and only if $D \cap N[u] = \{u\}$, where $N[u]$ is the closed neighborhood of u.

We remark that in a partially total dominated graph the only playable vertices are those that have a white or green neighbor since a played vertex must totally dominate at least one new vertex. In particular, no red or green vertex is playable. Further, the status of a red vertex remains unchanged as the game progresses. Hence, once a vertex is colored red, it plays no role in the remainder of the game and can be deleted from the partially total dominated graph. Moreover, since blue vertices are already totally dominated, an edge joining two blue vertices plays no role in the game, and can be deleted. Therefore in what follows, we may assume a partially total dominated graph contains no red vertices and has no edge joining two blue vertices. Adopting the terminology in [13], we call the resulting graph a *residual graph*. We will also say that the original graph G, before any moves have been made in the game, is a residual graph. Note that at any stage of the game where the residual graph is H any white vertex that has degree 1 in H also had degree 1 in G.

Suppose that a residual graph contains a P_2-component, H. Since we are assuming that every component of the original graph G contains at least three vertices, at least one vertex, say v, of H has degree at least 2 in the original graph G. Let w be a neighbor of v that is not in H. When a move was made earlier in the game and the edge vw was removed it is the case that either the vertex w was red and was deleted or both v and w were blue. In both cases v is colored blue in H. Hence, every P_2-component in the residual graph consists of one blue vertex and either one white or one green vertex. The following additional properties of the residual graph follow readily from the properties of the coloring of the vertices.

Observation 2. *The following properties hold in a residual graph.*

(a) *Every playable vertex has a white or green neighbor.*
(b) *Every neighbor of a white vertex is a white or blue vertex.*
(c) *Every neighbor of a green vertex is a blue vertex.*
(d) *Every neighbor of a blue vertex is a white or green vertex.*
(e) *Every playable vertex is a white or blue vertex.*
(f) *There is no isolated vertex.*
(g) *Every P_2-component consists of one blue vertex and either one white or one green vertex.*

3. The Assignment of Vertex Weights

We next associate a weight with every vertex in the residual graph as follows:

Color of vertex	*Weight of vertex*
white	4
green	3
blue	2
red	0

Table 1. The weights of vertices according to their color.

Since the residual graph by definition contains no red vertex, the above assignment of weight 0 to red vertices seems redundant. However, in due course of the game new red vertices are created and at that moment we assign them weight 0.

We define the *weight* of the residual graph G as the sum of the weights of the vertices in G and denote this weight by $\omega(G)$. Hence, each white vertex contributes 4 to the sum $\omega(G)$, while each green and blue vertex contributes 3 and 2, respectively, to the sum $\omega(G)$. We observe that the weight of a vertex cannot increase during the game. We state this formally as follows.

Observation 3. *Let R be a residual component resulting from playing some, including the possibility of zero, moves in G. The following hold.*

(a) *Every white vertex in R is a white vertex in G.*
(b) *Every blue vertex in R is a white or blue vertex in G.*
(c) *Every green vertex in R is a white or green vertex in G.*
(d) *The degree of every white vertex in R is precisely its degree in G.*

3.1. Types of Playable Vertices

Let v be a playable vertex in G. As observed earlier, the vertex v is white or blue, and has a white or green neighbor. We distinguish eight different legal moves in the residual graph G according to the following types.

Type 1: v is colored white and has at least two white neighbors.
Type 2: v is colored white and has exactly one white neighbor.
Type 3: v is colored blue and has degree at least 3.
Type 4: v is colored blue and has degree 2 with two white neighbors.
Type 5: v is colored blue and has degree 2 with one white and one green neighbor.
Type 6: v is colored blue and has degree 2 with two green neighbors.
Type 7: v is a blue leaf with a green neighbor.
Type 8: v is a blue leaf with a white neighbor.

Suppose that the playable vertex v is colored white. When v is played its color status changes from white to green. Further, each white neighbor of v changes status from white to blue, while each blue neighbor of v retains its color status. Hence, if v is a type-1 vertex with k white neighbors, then when v is played the weight of G decreases by $1+2k$, which is at least 5. Suppose that v is a type-2 vertex. Let w denote the white neighbor of v. By Observation 2(g), we know that the component containing v contains at least three vertices. When v is played, its status changes from white to green, while the status of w changes from white to blue. Hence, when v is played the weight of G decreases by at least 3. Each blue neighbor of v retains its color status. The status of every blue neighbor of w, if any, remains unchanged (colored blue) or changes from blue to red, while the status of every white neighbor of w different from v remains unchanged. Therefore, when v is played the weight of G decreases by exactly 3 or by at least 5.

Suppose that v is colored blue. When v is played its color status changes from blue to red. Further, each white neighbor of v changes status from white to blue or red, while each green neighbor of v changes status from green to red. Hence, if v is a type-3 vertex with k white neighbors and ℓ green neighbors, where $k+\ell=d(v)\geq 3$, then when v is played the weight of G decreases by at least $2+2k+3\ell$ and $2+2k+3\ell\geq 2+2d(v)\geq 8$. If v is a type-4, type-5, or type-6 vertex, then when v is played the weight of G decreases by at least 6, 7 or 8, respectively. If v is a type-7 vertex and w denotes its green neighbor, then when v is played both v and w change status to red. Therefore, the weight of G decreases by at least 5 in this case. Suppose v is a type-8 vertex and w denotes its white neighbor. If w has no white neighbor (this includes the case when w is a leaf), then when v is played both v and

w are recolored red, implying that the weight of G decreases by at least 6. If w has a white neighbor, then when v is played v is recolored red and w is recolored blue, implying that the weight of G decreases by at least 4 in this case.

The decrease in the weight of G according to the different types of playable vertices is summarized in Table 2.

Type of playable vertex	*Decrease in weight of G*
Type-1	≥ 5
Type-2	3 or ≥ 5
Type-3	≥ 8
Type-4	≥ 6
Type-5	≥ 7
Type-6	≥ 8
Type-7	≥ 5
Type-8	≥ 4

Table 2. The decrease in weight when playing different types of vertices.

We therefore have the following observation.

Observation 4. *A played vertex in a residual graph decreases the weight by at least* 3.

The following lemmas will prove to be useful.

Lemma 5. *If the game is not yet over, then there exists a legal move that decreases the weight by at least* 5.

Proof. Suppose to the contrary that every legal move decreases the weight by at most 4. By Observation 4, every playable vertex in the residual graph G either decreases the weight by 3 or by 4. By Table 2, if a playable vertex decreases the weight by 3, then it is a type-2 vertex; if it decreases the weight by 4, then it is a type-8 vertex. Thus every white vertex has at most one white neighbor, and every blue vertex is a leaf with a white neighbor. By Observation 2(c), G has no green vertices and consequently G contains only white and blue vertices.

Suppose that G contains a blue vertex, v. As observed earlier, v is a leaf with a white neighbor, say w. If every neighbor of w is a blue vertex, then playing v decreases the weight by at least 6, a contradiction. Hence, the vertex w has a white neighbor x. This implies that playing x decreases the weight by at least 5, which is a contradiction. Hence, G contains only white vertices. By Observation 2(f) and 2(g), G contains at least three vertices. Hence there exists a white vertex with at least two white neighbors, a possibility ruled out in the first paragraph. ∎

Lemma 6. *If a played vertex in a residual graph is a white vertex, then the resulting total decrease in weight is odd.*

Proof. Let v be a white vertex in the residual graph G and suppose that v is playable. When v is played, its status changes from white to green; this contributes 1 to the decrease in weight of G. Every white neighbor of v changes status from white to blue and contributes 2 to the decrease in weight of G. Every blue neighbor of a white neighbor of v, if any, retains its blue color status or changes status from blue to red and therefore contributes either 0 or 2 to the decrease in weight of G. All other vertices retain their color status and contribute 0 to the decrease in weight of G. Therefore, when v is played the resulting total decrease in weight of G is odd. ∎

4. The Three Phases of the Game

Before any move of Dominator, the game is in one of the following three phases.

- *Phase* 1, if there exists a legal move that decreases the weight by at least 7.
- *Phase* 2, if every legal move decreases the weight by at most 6, and there exists a legal move that decreases the weight by 6.
- *Phase* 3, if every legal move decreases the weight by at most 5.

By Lemma 5, if the game is in Phase 3, then there exists a legal move that decreases the weight by 5.

We remark that it is possible for a game to be in Phase 2 and later again in Phase 1. Further, it is possible for a game to be in Phase 3 and later in Phase 1 or Phase 2.

4.1. Phase 3

As a consequence of our discussion in Section 3 that is summarized in Table 2, we have the following observation.

Observation 7. *If the game is in Phase* 3*, then the following hold in the residual graph.*

(a) *Every white vertex has at most two white neighbors.*
(b) *Every blue vertex is a leaf.*
(c) *A green vertex, if it exists, is a leaf with a blue (leaf) neighbor.*

(d) *A white vertex with a blue (leaf) neighbor has at least one white neighbor.*
(e) *A white vertex has at most one blue (leaf) neighbor.*

We are now in a position to prove the following structural lemma.

Lemma 8. *If the game is in Phase* 3 *and* T *is any component of the residual graph, then one of the following holds.*

(a) $T = P_2$, *with one green vertex and one blue vertex.*
(b) $T = P_3$, *with a blue leaf and two white vertices.*
(c) $T = P_4$, *with two blue leaves and two white (central) vertices.*
(d) $T = K_{1,3}$, *with a blue leaf and three white vertices.*
(e) $T = P_k$, *where* $k \geq 3$, *consisting entirely of white vertices.*
(f) $T = C_k$, *where* $k \geq 3$, *consisting entirely of white vertices.*

Proof. Let T be a component of the residual graph as described in the statement of the lemma. If T contains a green vertex, then by Observation 7(c) we have that T satisfies condition (a) in the statement of the lemma. Hence we may assume that T contains no green vertex.

Suppose that T contains a blue vertex, v. By Observation 7(b), the vertex v is a leaf. Let w be the neighbor of v. By assumption, w is a white vertex. By Observation 7(b) and 7(e), the vertex v is the only blue neighbor of w. By Observation 7(d), the vertex w has a white neighbor, say x. If $T = P_3$, then T satisfies condition (b) in the statement of the lemma. Hence we may assume that $T \neq P_3$. If x has two white neighbors, then playing x decreases the weight of T by at least 7 (note that after playing x, the vertex v becomes red), contradicting the fact that we are in Phase 3. Hence, w is the only white neighbor of x. If $d(x) \geq 2$, then by Observation 7(e), $d(x) = 2$ and x has a blue leaf neighbor. Analogously in this case, $d(w) = 2$. Thus if $d(x) \geq 2$, then $T = P_4$ and T satisfies condition (c) in the statement of the lemma. Hence we may assume that x is a leaf. Since $T \neq P_3$, our earlier observations imply that $d(w) = 3$ and w has a white neighbor, say z, different from x. If z has a blue (leaf) neighbor, then playing w decreases the weight of T by at least 7, a contradiction. If z has a white neighbor different from w, then playing z decreases the weight of T by at least 7, a contradiction. Therefore, z is a leaf, implying that $T = K_{1,3}$ and T satisfies condition (d) in the statement of the lemma.

Hence we may assume that T contains no blue vertex, for otherwise T satisfies condition (a), (b), (c), or (d) in the statement of the lemma. However, in this case, by Observation 7(a) we have that T satisfies condition (e) or condition (f) in the statement of the lemma. ∎

By Lemma 8, if the game is in Phase 3, then a component of the residual graph can have one of six possible structures, which are illustrated in Figure 1.

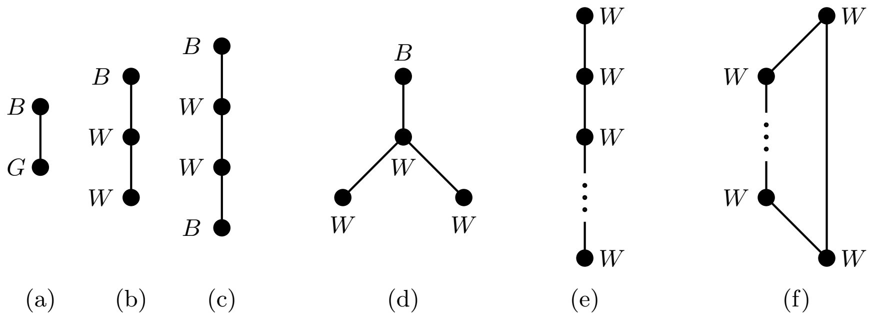

Figure 1. Six possible components of the residual graph in Phase 3.

4.2. Phase 2

In order to prove a structural lemma concerning properties of residual graphs in Phase 2, we first present the following lemma.

Lemma 9. *Suppose it is Dominator's turn and every legal move decreases the weight by at most* 6*. The following hold in the residual graph.*

(a) *Every white vertex has at most two white neighbors.*
(b) *Every blue vertex is a leaf or has degree* 2 *with two white neighbors.*
(c) *A green vertex, if it exists, is a leaf with a blue leaf neighbor.*
(d) *Every blue vertex of degree* 2 *is the central vertex of some path* P_5 *(not necessarily induced) that contains four white vertices.*

Proof. Properties (a) and (b) are a consequence of our discussion in Section 3 that is summarized in Table 2. To prove property (c), we remark that if a green vertex has degree at least 2, then by property (b) all of its neighbors are blue leaves. Playing one of them decreases the weight by at least 7, a contradiction.

Next, we prove property (d). Let v be a blue vertex of degree 2 in the residual graph. Let v_1 and v_2 be the two neighbors of v. By Lemma 9(b), both v_1 and v_2 are white vertices. Possibly, v_1 and v_2 are adjacent, but this component has order more than three since the game is in Phase 2. If every neighbor of v_1 different from v_2 is a blue vertex, then when v is played the resulting total decrease in weight is at least 8, a contradiction. Hence, v_1 has a white neighbor not adjacent to v and different from v_2; let u_1 be such a neighbor of v_1. Analogously, v_2 has a white neighbor not adjacent to v and different from v_1; let u_2 be such a neighbor of v_2. If u_1 is adjacent to v_2, then when u_1 is played the resulting total decrease in weight is at least 7, a contradiction. Hence, u_1 is not adjacent to v_2. Analogously, u_2 is not adjacent to v_1. In particular, $u_1 \neq u_2$. This establishes property (d). ∎

With the aid of Lemma 9, we are now in a position to prove the following structural lemma.

Lemma 10. *Suppose the game is in Phase* 2*. If T is a component of the residual graph R and contains a playable vertex that decreases the weight by* 6*, then T has the following properties.*

(a) *A playable vertex that decreases the weight by* 6 *is a blue vertex.*
(b) *T contains only blue and white vertices, with at least one blue vertex.*

Proof. Let T be as defined in the statement of the lemma. By Observation 2(e), every playable vertex is a white or blue vertex. By Lemma 6, a playable vertex in T that decreases the weight by 6 cannot be a white vertex, and is therefore a blue vertex. This establishes property (a). By Lemma 9(b), every blue vertex is a leaf or has degree 2 with two white neighbors.

If T contains a green vertex, then by Lemma 9(c) it follows that T is a P_2 with one blue vertex and one green vertex. However, such a colored component does not have a playable vertex that decreases the weight by 6. Consequently, T only contains blue and white vertices, and by property (a) it has at least one blue vertex. This establishes property (b). ∎

5. Strategy of Dominator

We present in this section a strategy for Dominator playing in a residual graph G. The overall strategy of Dominator is to make sure that on average the weight decrease resulting from each played vertex in the game is at least 5. Let $m_1, \ldots, m_k$ denote a sequence of moves starting with Dominator's first move, m_1, and with moves alternating between Dominator and Staller, where either k is odd and the game is completed after move m_k or k is even (and the game may or may not be completed after move m_k). In addition, let ω_i denote the decrease in weight after move m_i is played where $i \in [k]$. At any stage of the game, Dominator's overall strategy is to play his moves so that for all possible moves of Staller the following holds:

$$\sum_{i=1}^{k} \omega_i \geq 5k. \tag{1}$$

Thus, if $k=1$, he finds some move m_1 such that $\omega_1 \geq 5$. If $k=2$, he finds some move m_1 such that for all possible moves m_2 of Staller, $\omega_1+\omega_2 \geq 10$. If $k=3$, he finds some move m_1 such that for all possible moves m_2 of Staller, there exists some move m_3 with $\omega_1+\omega_2+\omega_3 \geq 15$. If $k=4$, he finds some move m_1 such that for all possible moves m_2 of Staller, there exists some move m_3, such that for all possible moves m_4 of Staller, $\omega_1+\omega_2+\omega_3+\omega_4 \geq 20$, and so on. If Dominator can find such a sequence of moves, then he plays these moves which have the effect of decreasing the weight by an average of at least 5 on each played vertex in the sequence.

By *simplifying the game* we mean playing a sequence of k moves that reduces the current residual graph to a new residual graph whose weight is at least $5k$ less than that of the current residual graph, where either k is odd and the game is completed after move m_k or k is even.

Before we present the details of Dominator's strategy, we introduce some additional notation. Suppose the game enters Phase 2 or Phase 3 at some stage. The structure of the residual graph is then determined by Lemma 8 and Lemma 10. In particular, we note that if the residual graph contains a green vertex, then such a vertex belongs to a P_2-component with the other vertex colored blue. Further, we note that in Phase 2 or Phase 3 every vertex in a component of the residual graph that contains a white vertex contains only blue and white vertices.

We define a *white component* in a residual graph that is in Phase 2 or Phase 3 to be a component obtained from the residual graph by deleting all the blue vertices from a component that contains at least one white vertex. We note that every white component contains only white vertices. Further, by Observation 7(a) and Lemma 9(a), every white component is a path on at least one vertex or a cycle on at least three vertices. A white component isomorphic to a path P_n we call a *white P_n-component.*

We are now in a position to present a strategy for Dominator that will ensure that on average the weight decrease resulting from each played vertex in the game is at least 5. To achieve his objective, Dominator formulates an **opening-game** strategy, a **middle-game** strategy, and an **end-game** strategy. We remark that in each of these three strategies, it is possible for the game to oscillate back and forth between the three different phases, namely Phase 1, Phase 2 and Phase 3.

5.1. Opening-Game Strategy

In this section, we discuss Dominator's opening-game strategy. First, we define three types of moves or combinations of moves that Dominator can play.

5.1.1. Greedy Move The first type of move is the simplest to describe and is what we call a greedy move.

- A *greedy move* is a move that decreases the weight by as much as possible.

If the game is always in Phase 1, then by always playing greedy moves, Dominator can ensure that the weight decreases by an average of at least 5 per move. Each of Dominator's moves decreases the weight by at least 7, while each of Staller's moves decreases the weight by at least 3. Hence, since Dominator plays the first move, if $2k$ moves were played when the game is completed, then the total decrease in weight is at least $7k+3k$, while if $2k+1$ moves are played when the game is completed, then the total decrease in weight is at least $7(k+1)+3k$. In both cases, the average weight decrease for each move played is at least 5. We may therefore assume that the game is not always in Phase 1, and that the game enters Phase 2 or Phase 3 at some stage.

Suppose then, that Dominator has simplified the game so that it is not in Phase 1 and it is Dominator's turn. By Lemma 5 we may assume that the move of Dominator does not complete the game.

5.1.2. Path-Reduction Combination Suppose that the residual graph contains a white path component, say C, with at least four vertices and it is Dominator's turn. Let v be a leaf in C. There is a path P_v on four vertices emanating from v in C. Let this path be given by $P_v\colon vwxy$, where either y is a leaf in C or y has degree 2 in C. We note that the neighbors of v, w, and x, if any, in the residual graph that do not belong to the path P_v are all blue vertices. Dominator now plays the vertex x as his move m_1 in the residual graph, which results in $\omega_1 \ge 5$. Let T_v be the resulting component in the residual graph that contains v. We note that in T_v the colors of the vertices v, w, x, and y are white, blue, green, and blue, respectively. Every neighbor of v in T_v, if any, that is different from w is a blue vertex, while all neighbors of x in T_v are blue vertices. If Staller responds by playing her move m_2 on a neighbor of v or on a neighbor of x in T_v, then $\omega_2 \ge 5$ and Inequality (1) is satisfied where here $k=2$ (noting that $\omega_1+\omega_2 \ge 5+5=10=5k$). If, however, Staller plays her move m_2 on a vertex that is neither a neighbor of v nor a neighbor of x, then Dominator plays as his move m_3 the vertex w resulting in $\omega_3 \geq 9$. If move m_3 completes the game, then Inequality (1) is satisfied with $k=3$ (noting that $\omega_1+\omega_2+\omega_3 \geq 5+3+9=17>5k$). If move m_3 does not completes the game, then Inequality (1) is satisfied with $k=4$ (noting that $\omega_1+\omega_2+\omega_3+\omega_4 \geq 5+3+9+3=20=5k$).

- Using the notation described above, the move m_1 (which plays the vertex x) of Dominator, together with his move m_3 (which plays the vertex w if the move m_2 is neither a neighbor of v nor a neighbor of x), we call a *path-reduction combination.*

Thus, if Dominator plays a path-reduction combination, then Inequality (1) is satisfied for some k where $k \in \{2,3,4\}$, implying that in this case Dominator can simplify the game. Hence we may assume that Dominator has simplified the game so that every white component that is a path contains at most three vertices; that is, every such white component is a path P_1, a path P_2, or a path P_3. By our earlier assumptions, the game is currently in Phase 2 or Phase 3.

5.1.3. 2-Path-Structure Combination Next we define what we call a 2-path-structure combination. Suppose that there is a white P_2-component with both ends, v_4 and v_5 say, adjacent to blue vertices in G. Assume that the component of the residual graph that contains v_4 and v_5 is not a P_4 isomorphic to the graph in Figure 1(c) (with two blue leaves and two white central vertices). It follows from Lemma 8 that the game is in Phase 2. Let v_3 and v_6 be blue neighbors of v_4 and v_5, respectively. By assumption and Lemma 9(b), at least one of v_3 and v_6 has degree 2. Renaming vertices if necessary, we may assume that v_3 has degree 2. If $v_3 = v_6$, then playing the vertex v_3 results in a weight decrease of at least 10, implying that the game is in Phase 1, a contradiction. Hence, $v_3 \neq v_6$. By Lemma 9(d), there exists a path $v_1v_2v_3v_4v_5$, where v_3 is colored blue and the remaining vertices on the path are colored white. Further, if v_6 has degree 2, there exists a path $v_4v_5v_6v_7v_8$ in T where v_6 is colored blue and the remaining vertices on the path are colored white. In this case, we remark that possibly $\{v_1,v_2\} \cap \{v_7,v_8\} \neq \emptyset$. By our earlier assumptions, v_4 is the only white neighbor of v_5, and v_5 is the only white neighbor of v_4. Dominator now plays as his move m_1 the vertex v_3, and we see that $\omega_1 \geq 6$. Let T' be the resulting component in the residual graph that contains v_4, and note that v_4 is a blue leaf in T' with v_5 as its (white) neighbor. If Staller responds by playing her move m_2 on a neighbor of v_5, then $\omega_2 \geq 6$ and Inequality (1) is satisfied where here $k=2$ (noting that $\omega_1+\omega_2 \geq 6+6=12>5k$). If, however, Staller plays her move m_2 on a vertex that is not a neighbor of v_5 in T', then Dominator plays as his move m_3 the vertex v_6. This gives $\omega_3 \geq 8$. Analogously as before (see the discussion in Section 5.1.2), Inequality (1) is satisfied with $k=3$ (if move m_3 completes the game) or $k=4$.

- Using the notation described above, the move m_1 (which plays the vertex v_3) of Dominator, together with his move m_3 (which plays the vertex v_6) if the move m_2 is not a neighbor of v_5, we call a 2-*path-structure combination.*

Thus, if Dominator plays a 2-path-structure combination, then Inequality (1) is satisfied for some k where $k \in \{2,3,4\}$, implying that in this case Dominator can simplify the game. Hence we may assume that Dominator has simplified the game further so that for each white P_2-component C, one vertex of C is a leaf in G or C belongs to a component of G that has order 4 with two blue leaves.

5.1.4. Dominator's Opening-Game Strategy We are now in a position to formally state Dominator's opening-game strategy by providing sufficient, concrete instructions on how to play the game as Dominator.

Dominator's opening-game strategy:

1. *Whenever the game is in Phase* 1*, Dominator plays a greedy move. By playing greedy moves, Dominator simplifies the game until it reaches Phase* 2 *or Phase* 3*.*
2. *If the above does not apply and if the game contains a white P_n-component, where $n \geq 4$, then Dominator plays a path-reduction combination. By playing path-reduction combinations whenever the game is in Phase* 2 *or Phase* 3*, Dominator simplifies the game until it contains no white P_n-component, where $n \geq 4$.*
3. *If neither of the above applies and if the game contains a white P_2-component that does not contain a leaf in the original residual graph and is not contained in a component of the residual graph that is a path P_4 with two blue leaves, then Dominator plays a* 2*-path-structure combination. By playing* 2*-path-structure combinations whenever the game contains no white P_n-component, where $n \geq 4$, and the game is in Phase* 2 *or Phase* 3*, Dominator simplifies the game until every white P_2-component, if any, arises from a component of the residual graph that is isomorphic to a path P_4 with two blue leaves (and two white central vertices) or contains a leaf in the original residual graph G.*

4. *If none of the above applies, then Dominator proceeds to his middle-game strategy.*

Let $G_0 = G$ and let G_1 denote the simplified residual graph resulting from Dominator's opening-game strategy. The graph G_1 has the structure defined in Observation 11.

Observation 11. *After Dominator's opening-game strategy, if T is an arbitrary component in G_1, then T has one of the six possible structures described in Lemma 8 (and illustrated in Figure 1) or T has the properties defined in Lemma 10. Further, the following additional restrictions hold:*

- *Every white path component is a path P_1, a path P_2, or a path P_3.*
- *For every white P_2-component, C, of G_1 either one vertex of C is a leaf in G or the component of G_1 containing $V(C)$ has order 4 with two blue leaves.*

5.2. Middle-Game Strategy

Recall that the simplified residual graph, G_1, resulting from Dominator's opening-game strategy has the structure defined in Observation 11. In particular, the game is currently in Phase 2 or Phase 3, although as the game continues further it may possibly return to Phase 1 and oscillate back and forth between the three different phases. In order to present Dominator's middle-game strategy, we first define a blocking move.

- A *blocking move* is a move that plays a blue vertex of degree 2, which we call a *blocking vertex*, with two white neighbors at least one of which belongs to a white P_2-component that contains a leaf in the original residual graph G.

Dominator's middle-game strategy will consist of two parts which we call Part I and Part II. The middle-game strategy Part I is as follows.

Dominator's middle-game strategy: Part I

1. *Whenever the game is in Phase 1, Dominator plays a greedy move.*
2. *Whenever the game is in Phase 2 or Phase 3 and the game contains a white P_2-component that does not contain a leaf in the original residual graph and is not contained in a component of the residual graph that is a path P_4 with two blue leaves, then Dominator plays a 2-path-structure combination.*
3. *If neither of the above applies, then Dominator plays a blocking move if such a move exists.*

4. *If none of the above applies, then Dominator proceeds to Part II of his middle-game strategy.*

Part I of Dominator's middle-game strategy ends when the game is in Phase 2 or Phase 3 and there is no blocking move available for Dominator to play, and every white P_2-component arises from a component of the residual graph that is isomorphic to the graph illustrated in Figure 1(c) or contains a leaf in the original residual graph G. We remark that step 2 above in Part I of Dominator's middle-game strategy is the same strategy as employed in step 3 of his opening-game strategy. It is necessary for Dominator to repeat this opening-game strategy since new white P_2-components might arise in the course of steps 1 and 3 of his middle-game strategy.

We remark that whenever the game is in Phase 1, the greedy move played by Dominator decreases the weight by at least 7. We also remark when the game is in Phase 2 or Phase 3 and Dominator plays a blocking move, such a move decreases the weight by exactly 6 and creates at least one P_2-component that consists of one white vertex and one blue vertex. Associated with each blocking move, we select one such resulting P_2-component and call it a *surplus component*. Thus, a blocking move creates exactly one surplus component, even though possibly two new P_2-components with one white and one blue vertex may have been created when the blocking move is played. In the short term, a blocking move appears to be costly for Dominator since if Staller responds to a blocking move by playing a move that decreases the weight by exactly 3, then these two moves decrease the weight by 9 which is one short of the long term strategy of Dominator that on average the weight decrease resulting from each move be at least 5. In the long term, however, we show that this strategy of Dominator of playing blocking vertices allows him to recover his losses due to the creation of the surplus component.

Let G_2 denote the simplified residual graph resulting from Part I of Dominator's middle-game strategy applied to the residual graph G_1. Suppose that G_2 was obtained from G_1 after Dominator played $k_1 \geq 0$ blocking moves and $k_2 \geq 0$ greedy moves. Further, suppose that Staller plays $\ell_1 \geq 0$ moves from surplus components before the residual graph G_2 is created. The residual graph G_2 has the structure defined in Observation 12.

Observation 12. *After Part I of Dominator's middle-game strategy, the game is in Phase 2 or Phase* 3 *and there exists no blocking move. Further, every white path component is a path P_1, a path P_2 or a path P_3, and for every white P_2-component, C, either one vertex of C is a leaf in G or the component of G_2 containing $V(C)$ has order 4 with two blue leaves.*

Recall that by Observation 11 a vertex colored white or blue in G_2 is adjacent to at most two white vertices. By Observation 3(a), no new white vertices can be created that did not already exist in G_2. Further, by Observation 3(b), as the game progresses further any new blue vertex that is created and did not already exist in G_2 was originally colored white in G_2. Therefore, every vertex colored white or blue as the game progresses is adjacent to at most two white vertices. Further, every blue vertex has degree 1 or 2. We proceed further with the following key lemmas.

Lemma 13. *If R is a residual component resulting by playing some, including the possibility of zero, moves in G_2, then there exists no blocking move in R.*

Proof. Suppose, to the contrary, that R contains a blocking vertex w. Then, w is a blue vertex in R with two white neighbors at least one of which belongs to a white P_2-component that contains a leaf in the original residual graph G. Let v and x be the two white neighbors of w, where v belongs to a white P_2-component that contains one vertex, u say, that is a (white) leaf in the original residual graph G. Since u is a white leaf in G, it is also a white leaf in R. We note that v has degree at least 2 in R and every neighbor of v in R different from u is colored blue. Further, we note that v and x are the only neighbors of w in R. By Observation 3(a) x is also colored white in G_2. By Observation 3(b), w is colored white or blue in G_2. If w is colored white in G_2, then $uvwx$ is a white path in G_2, implying that G_2 has a white path component of order at least 4, contradicting Observation 11. If w is colored blue in G_2, then the vertex w is blocking vertex in G_2, a contradiction. Therefore, there is no blocking vertex in R. ∎

Lemma 14. *Let R be a residual component resulting by playing some, including the possibility of zero, moves in G_2. If Staller plays a move in R that decreases the weight by exactly 3, then immediately after the move is played, Dominator can play a vertex in R that decreases the weight by at least 7.*

Proof. Suppose that there is a vertex u in the residual component R which when played by Staller decreases the weight by exactly 3.

By Table 2 it follows that u is colored white and has exactly one white neighbor, say v. Suppose that v has a white neighbor different from u. When u is played, its color changes to green and the color of v changes to blue. If the vertex v is now played immediately after u is played, then since v is currently a blue vertex with one green neighbor and one white neighbor, this results in a decrease in weight of at least 7, as desired. Hence we may assume that u is the only neighbor of v colored white. Thus, u and v belong to a common white P_2-component in R. By Observation 11, one of u and v is a white leaf in G, or R is a path of order 4, say $xuvy$ where x and y are blue vertices. Suppose that R is the path $xuvy$. When Staller plays u the weight decreases by exactly 5, a contradiction. Suppose that u is a white leaf in G and therefore also a white leaf in R. In this case, v has degree at least 2 in R. Further, every neighbor of v in R different from u is colored blue. Let w be an arbitrary blue neighbor of v in R. If w has degree 1 in R, then playing u decreases the weight by at least 5, a contradiction. Hence, w has degree 2 in R. Let x be the neighbor of w different from v. If x is colored white in R, then w is a blocking vertex in R, contradicting Lemma 13. Hence, x is colored green in R. By Observation 3(c) and Lemma 9(c), x must have been colored white in G_2. As in the proof of Lemma 13, either $uvwx$ is a white path in G_2 or w is blocking vertex in G_2. Both cases produce a contradiction. Hence, v is a white leaf in G.

Since v is a leaf in G, it is a leaf in R. In this case, the vertex u has degree at least 2 in R and each neighbor of u in R different from v is colored blue. Let z be an arbitrary blue neighbor of u in R. When u is played, its color changes to green and the color of v changes to blue. If Dominator now plays the (blue) vertex z immediately after u is played, then this results in a decrease in weight of at least 7, as desired. ∎

Recall that the simplified residual graph, G_2, resulting from Dominator's middle-game strategy Part I has the structure defined in Observation 12. In particular, the game is currently in Phase 2 or Phase 3, although as the game continues further it may possibly return to Phase 1 and oscillate back and forth between the three different phases. Dominator's middle-game strategy Part II is to simplify the game until it reaches Phase 3. This he achieves readily by repeated applications of Lemma 14. Abusing notation, for $i \in \{1,2,3\}$ we say that a component of a residual graph is in Phase i if the game played on that component is in Phase i.

Dominator's middle-game strategy: Part II

1. *If the residual graph G_2 contains a component in Phase 1 or Phase 2 that is not a surplus component, then Dominator plays a greedy move in this component.*
2. *Thereafter, Dominator applies the following strategy.*
   2.1. *Whenever Staller responds to a move of Dominator by playing a vertex that decreases the weight by exactly 3, Dominator responds by playing a move that decreases the weight by at least 7. (We note that such a move exists by Lemma 14.)*

2.2. *Whenever Staller responds to a move of Dominator by playing a vertex that decreases the weight by at least* 4 *and the resulting game contains a component in Phase* 1 *or Phase* 2 *that is not a surplus component, Dominator responds by playing a greedy move from that component. (We note that such a move decreases the weight by at least* 6*).*

2.3. *If every component is in Phase* 3 *or is a surplus component, then Dominator proceeds to his end-game strategy.*

Let G_3 denote the simplified residual graph resulting from Dominator's middle-game strategy Part II applied to the residual graph G_2. Suppose that G_3 was obtained from G_2 after Dominator played $k_3 \geq 0$ moves. Dominator's middle-game strategy guarantees that the residual graph G_3 has the structure defined in Observation 15.

Observation 15. *Every component in the residual graph G_3 that is not a surplus component is in Phase* 3*.*

5.3. End-Game Strategy

In this section, we discuss Dominator's end-game strategy. We define a *special vertex* to be a blue vertex that belongs to a residual component that is isomorphic to a path P_5 both of whose ends are colored blue and with all three internal vertices colored white. We define a *special component* to be a component obtained from playing a special vertex. We note that a special component is a path P_4 with two blue leaves and two white (central) vertices.

Before presenting Dominator's end-game strategy, we shall need the following key lemma about properties of a residual graph in Phase 3. Recall that the simplified residual graph G_3 has the structure defined in Observation 15.

Lemma 16. *Let R be a residual component resulting by playing some, including the possibility of zero, moves in G_3. If Staller plays a move in R that is not a special vertex and decreases the weight by exactly* 4*, then immediately after the move is played, Dominator can play a vertex in R that decreases the weight by at least* 6*.*

Proof. Let R be as defined in the statement of the lemma. Suppose that there is a vertex u in R which when played by Staller decreases the weight by exactly 4 and that u is not a special vertex. By Table 2, u is a blue vertex with exactly one white neighbor, say v. Since playing u decreases the weight by exactly 4, we note that v has at least one white neighbor, say w. Thus, R

contains the path uvw, where u is a blue leaf and where v and w are colored white. In particular, we note that R is not a surplus component.

Suppose the residual component R resulted from playing some, including the possibility of zero, moves in a component, C, of G_3. Since R is not a surplus component, by Observation 15 the component C is in Phase 3 and therefore satisfies one of the conditions (a)–(f) in the statement of Lemma 8 (and illustrated in Figure 1) with the additional restriction that if C satisfies condition (e), then $C = P_3$. Since R contains the path uvw and since playing u decreases the weight by exactly 4, we deduce that C satisfies one of the conditions (b), (c), (d) or (f) in the statement of Lemma 8.

If C satisfies condition (b), then $R = C$ and R is the path uvw. When u is played, the resulting component is a P_2-component with one blue vertex, v, and one white vertex, w. Dominator now plays on the vertex v which decreases the weight by 6.

If C satisfies condition (c), then $R = C$. Let R be the path $uvwx$, where we note that x is colored blue. When u is played, the resulting component is a path vwx with v and x colored blue and with w colored white. Dominator now plays on the vertex v which decreases the weight by 8.

If C satisfies condition (d), then $R = C$. Let x be the neighbor of v different from u and w. We note that x is colored white. When u is played, the resulting component is a path wvx with w and x colored white and with v colored blue. Dominator now plays on the vertex v which decreases the weight by 10.

If C satisfies condition (f), then $C = C_n$ for some $n \ge 3$. In this case, R is necessarily a path with both ends colored blue, implying that w is not an end of R. Let x be the neighbor of w on R different from v. If x is colored blue, then when u is played, the resulting path contains the subpath vwx where v and x are colored blue and w is colored white. Dominator now plays on the vertex x which decreases the weight by at least 8. Hence we may assume that x is colored white. Let y be the neighbor of x on R different from w. If y is colored white, then when u is played, the resulting path contains the subpath $vwxy$ where w, x and y are colored white and v is colored blue. Dominator now plays on the vertex x which decreases the weight by at least 7. Hence we may assume that y is colored blue. Suppose that y is not an end of R and let z be the neighbor of y on R different from x. Since the color of y changed from white to blue, we note that z is colored green in R. Dominator now plays on the vertex y which decreases the weight by at least 7. Hence we may assume that R is the path $uvwxy$, where u and y are colored blue and v, w and x are colored white. Thus, R is a special path component and the vertex u is a special vertex in R, contrary to assumption. ∎

We are now in a position to present Dominator's end-game strategy.

Dominator's end-game strategy:

1. *Dominator starts by playing a greedy move.*
2. *Thereafter, Dominator applies the following strategy.*
   - 2.1. *Whenever Staller responds to a move of Dominator by playing a vertex that decreases the weight by exactly* 3*, Dominator responds by playing a move that decreases the weight by at least* 7*. (We note that such a move exists by Lemma 14.)*
   - 2.2. *Whenever Staller responds to a move of Dominator by playing a vertex that is not a special vertex and decreases the weight by exactly* 4*, Dominator responds by playing a move that decreases the weight by at least* 6 *and belongs to the residual component that contains Staller's move. (We note that such a move exists by Lemma 16.)*
   - 2.3. *Whenever Staller responds to a move of Dominator by playing a special vertex or a vertex that decreases the weight by at least* 5*, Dominator responds by playing a greedy move. (We note that such a move decreases the weight by at least* 5*).*

Suppose that when Dominator implements his end-game strategy a total of s special vertices were played by Staller.

5.4. Analysis of Dominator's Strategy

It remains for us to show that Dominator's Strategy does indeed guarantee that on average the weight decrease resulting from each played vertex in the game is at least 5. Recall that G_0 denotes the original residual graph G. The residual graph G_1 denotes the simplified residual graph resulting from Dominator's opening-game strategy. The residual graph G_2 denotes the simplified residual graph resulting from Dominator's middle-game strategy Part I applied to G_1. The residual graph G_3 denotes the simplified residual graph resulting from Dominator's middle-game strategy Part II applied to G_2.

Suppose that the residual graph G_1 was obtained from the residual graph G_0 after Dominator played k_0 moves. Recall that the residual graph G_2 was obtained from G_1 after Dominator played k_1 blocking moves and k_2 greedy moves. Further, recall that Staller plays ℓ_1 moves from surplus components during the process of creating G_2 from G_1. Since a total of k_1 surplus components were created and since Staller plays ℓ_1 moves from surplus components before the residual graph G_2 is created, we note that $\ell_1 \le k_1$ and that G_2 contains exactly $k_1 - \ell_1$ surplus components. Finally, recall that the residual graph G_3 was obtained from G_2 after Dominator played k_3 moves. Suppose that Staller plays ℓ_2 moves from surplus components during the process of creating G_3 from G_2. Since G_2 contains exactly $k_1-\ell_1$ surplus components, we note that $\ell_2 \le k_1-\ell_1$ and that G_3 contains exactly $k_1-\ell_1-\ell_2$ surplus components. Recall finally that a total of s special vertices were played by Staller when the game played on G_3 is completed.

Suppose that Dominator's strategy results in a total of m_0, m_1, m_2 and m_3 moves played in G_0, G_1, G_2 and G_3, respectively. In particular, we note that m_0 moves were played in the game. We proceed further with the following series of claims.

Claim A. The following holds.

(a) $m_0 = m_1 + 2k_0$.
(b) $m_1 = m_2 + 2(k_1 + k_2)$.
(c) $m_2 = m_3 + 2k_3$.

Proof. Since G_1 was obtained from G_0 after Dominator played k_0 moves, $2k_0$ moves were played in obtaining G_1 from G_0, implying that $m_0 = m_1 + 2k_0$. Since G_2 was obtained from G_1 after Dominator played k_1 blocking moves and k_2 greedy moves, $2(k_1+k_2)$ moves were played in obtaining G_2 from G_1, implying that $m_1 = m_2 + 2(k_1 + k_2)$. Since G_3 was obtained from G_2 after Dominator played k_3 moves, $2k_3$ moves were played in obtaining G_3 from G_2, implying that $m_2 = m_3 + 2k_3$. ∎

Claim B. $\omega(G_1) \le \omega(G_0) - 10k_0$.

Proof. The opening-game strategy of Dominator discussed in Section 5.1 guarantees that on average the weight decrease resulting from each move played during the process of creating G_1 from G_0 is at least 5. Hence after the $2k_0$ moves played by Dominator and Staller in order to obtain G_1 from G_0, we note that $\omega(G_0) \ge \omega(G_1) + 10k_0$. ∎

Claim C. $\omega(G_2) \le \omega(G_1) - 9k_1 - 10k_2 - 3\ell_1$.

Proof. Each blocking move played by Dominator in the middle-game strategy Part I decreases the weight by at least 6 while each greedy move decreases the weight by at least 7. Every move of Staller decreases the weight by at least 3. If, however, a move of Staller is played in a surplus component, then such a move decreases the weight by 6. After the $2(k_1+k_2)$ moves played by Dominator and Staller in order to obtain G_2 from G_1, we therefore note that $\omega(G_2) \le \omega(G_1) - 9k_1 - 10k_2 - 3\ell_1$. ∎

Claim D. $\omega(G_3) \le \omega(G_2) - 10k_3 - 2\ell_2$.

Proof. We note that the first move played by Dominator in G_2 is a greedy move that decreases the weight by at least 6. Thereafter, whenever Staller plays a move that decreases the weight by exactly 3, Dominator responds with a move that decreases the weight by at least 7. All other moves of Staller decrease the weight by at least 4 and can be uniquely associated with a move by Dominator that decreases the weight by at least 6. If, however, a move of Staller is played in a surplus component, then such a move decreases the weight by 6. After the $2k_3$ moves played by Dominator and Staller in order to obtain G_3 from G_2, we therefore note that Dominator's middle-game strategy Part II implies that $\omega(G_3) \le \omega(G_2) - 10k_3 - 2\ell_2$. ∎

Claim E. $\omega(G_3) \ge 5m_3 + s + k_1 - \ell_1 - \ell_2$.

Proof. The end-game strategy of Dominator discussed in Section 5.3 guarantees that on average the weight decrease resulting from each move that is not one of the s special vertices played by Staller and that does not belong to a surplus component or one of the s special components is at least 5. Each of the s special vertices played by Staller decrease the weight by exactly 4. Exactly two vertices are played in each of the s special components (created by playing the s special vertices) and these two vertices decrease the weight by 12. Hence, the s special vertices, together with the associated s special components, result in a total of $3s$ moves and these moves decrease the weight by $4s + 12s = 16s = 15s + s$. Each move played in one of the $k_1 - \ell_1 - \ell_2$ surplus components decreases the weight by 6. Hence the $k_1 - \ell_1 - \ell_2$ moves played in surplus components decrease the weight by $5(k_1 - \ell_1 - \ell_2) + k_1 - \ell_1 - \ell_2$. Dominator's end-game strategy therefore implies that $\omega(G_3) \ge 5m_3 + s + k_1 - \ell_1 - \ell_2$. ∎

We are now in a position to show that the average weight decrease resulting from these m_0 moves is at least 5. Equivalently, we wish to prove the following result.

Theorem 17. $\omega(G_0) \ge 5m_0$.

Proof. By Claim E, $\omega(G_3) \ge 5m_3 + s + k_1 - \ell_1 - \ell_2$. By Claim A(c), $m_3 = m_2 - 2k_3$. Therefore, by Claim D this implies that

$$\begin{aligned}
\omega(G_2) &\ge \omega(G_3) + 10k_3 + 2\ell_2 \\
&\ge (5m_3 + s + k_1 - \ell_1 - \ell_2) + 10k_3 + 2\ell_2 \\
&= 5m_3 + 10k_3 + s + k_1 - \ell_1 + \ell_2 \\
&= 5(m_2 - 2k_3) + 10k_3 + s + k_1 - \ell_1 + \ell_2 \\
&= 5m_2 + s + k_1 - \ell_1 + \ell_2.
\end{aligned}$$

By Claim A(b), $m_2 = m_1 - 2(k_1 + k_2)$. Therefore, by Claim C this implies that

$$\begin{aligned}\omega(G_1) &\geq \omega(G_2) + 9k_1 + 10k_2 + 3\ell_1 \\ &\geq (5m_2 + s + k_1 - \ell_1 + \ell_2) + 9k_1 + 10k_2 + 3\ell_1 \\ &= 5m_2 + 10k_1 + 10k_2 + 2\ell_1 + \ell_2 + s \\ &= 5(m_1 - 2(k_1 + k_2)) + 10k_1 + 10k_2 + 2\ell_1 + \ell_2 + s \\ &= 5m_1 + 2\ell_1 + \ell_2 + s.\end{aligned}$$

By Claim A(a), $m_1 = m_0 - 2k_0$. Therefore, by Claim B this implies that

$$\begin{aligned}\omega(G_0) &\geq \omega(G_1) + 10k_0 \\ &\geq (5m_1 + 2\ell_1 + \ell_2 + s) + 10k_0 \\ &= 5(m_0 - 2k_0) + 2\ell_1 + \ell_2 + s + 10k_0 \\ &= 5m_0 + 2\ell_1 + \ell_2 + s \\ &\geq 5m_0.\end{aligned}$$

This completes the proof of Theorem 17. ∎

The strategy of Dominator implemented in this section, together with our detailed analysis of Dominator's Strategy that culminated in Theorem 17, shows that on average the weight decrease resulting from each played vertex in the graph G is at least 5. We state this formally as follows.

Theorem 18. *In a residual graph G, we have that $\gamma_{tg}(G) \leq \omega(G)/5$.*

6. Proof of Theorem 1 and a General Bound

In this section we present a proof of our main result, namely, Theorem 1. Recall its statement.

Theorem 1. *Let G be a graph on n vertices in which every component contains at least three vertices. Then,*

$$\gamma_{tg}(G) \leq \frac{4n}{5} \quad \textit{and} \quad \gamma'_{tg}(G) \leq \frac{4n+2}{5}.$$

Proof. Coloring the vertices of G with the color white we produce a colored-graph in which every vertex is colored white and every component contains at least three vertices. In particular, we note that G has n white vertices and has weight $\omega(G) = 4n$. Applying Theorem 18 to the resulting colored-graph G, we have that $\gamma_{tg}(G) \le 4n/5$. To prove the upper bound on the Staller-start game total domination number, we note that the first move of Staller is on a white vertex with at least one white neighbor and decreases the weight by at least 3. If R_G denotes the resulting residual colored-graph, then $\omega(R_G) \le \omega(G) - 3$ and $\gamma'_{tg}(G) = 1 + \gamma_{tg}(R_G)$. Applying Theorem 18 to the residual graph R_G, we have that

$$\gamma'_{tg}(G) = 1 + \gamma_{tg}(R_G) \le 1 + \frac{\omega(R_G)}{5} \le 1 + \frac{\omega(G) - 3}{5} = \frac{\omega(G) + 2}{5} = \frac{4n + 2}{5}.$$

This completes the proof of Theorem 1. ∎

When either Game 1 or Game 2 is played on a P_2 exactly two moves are required. Using the bounds in Theorem 1 we are now able to prove general upper bounds for the total domination games played on any graph that has no isolated vertices. As observed earlier, the game total domination number is not well-defined on graphs having isolated vertices.

Corollary 19. *If G is a graph of order n with k components of order 2 and no isolated vertices, then*

$$\gamma_{tg}(G) \le \frac{4n + 2k}{5} \quad \textit{and} \quad \gamma'_{tg}(G) \le \frac{4n + 2k + 2}{5}.$$

Proof. If every component of G has order at least 3, then the bounds are the same as in Theorem 1. If all the components of G have order 2, then every vertex of G will be played in both Game 1 and Game 2, and the upper bounds follow immediately. Finally, suppose G has at least one component of order 2 and at least one component of larger order. Let G_1 denote the union of all the components of G that have order at least 3, and let G_2 be the union of all k components of order 2. Note that G_2 has order $2k$, and G_1 has order $n - 2k$. The graph G is the disjoint union of these two graphs.

When Game 1 is played on G, Dominator's strategy is to make an optimal move in Game 1 played on G_1. If on any turn Staller plays a vertex v in G_2, then Dominator plays the neighbor of v. If Staller finishes the game on G_1, but G_2 is not empty, then all remaining components have order 2 and every vertex will be played. Otherwise, Dominator always responds to Staller's move with an optimal move in the subgraph G_1. Following this strategy it follows from Theorem 1 that

$$\gamma_{tg}(G) \le \gamma_{tg}(G_1) + 2k \le \frac{4(n - 2k)}{5} + 2k = \frac{4n + 2k}{5}.$$

When Game 2 is played on G the strategy of Dominator is similar to that in Game 1. In particular, when Staller plays a vertex in G_1, Dominator responds with an optimal move in Game 1 restricted to G_1. When Staller plays a vertex v in G_2, Dominator plays the neighbor of v. In this way Dominator can limit the number of moves in G_1 to $\gamma'_{tg}(G_1)$. Since all $2k$ vertices of G_2 must be played, Theorem 1 implies

$$\gamma'_{tg}(G) \le \gamma'_{tg}(G_1) + 2k \le \frac{4(n-2k)+2}{5} + 2k = \frac{4n+2k+2}{5}. \qquad \blacksquare$$

We believe that the upper bounds of Theorem 1 cannot be achieved and pose the following conjecture, which we call the $\frac{3}{4}$-*Game Total Domination Conjecture* or, simply, the 3/4-*Conjecture.*

3/4-Conjecture: Let G be a graph on n vertices in which every component contains at least three vertices. Then,

$$\gamma_{tg}(G) \le \frac{3n}{4} \quad \textit{and} \quad \gamma'_{tg}(G) \le \frac{3n+1}{4}.$$

We remark that if the 3/4-Conjecture is true, then the upper bound on the Dominator-start game total domination number is tight as may be seen by taking, for example, $G = k_1P_4 \cup k_2P_8$ where $k_1, k_2 \ge 0$ and $k_1 + k_2 \ge 1$. Since $\gamma_{tg}(P_4) = \gamma'_{tg}(P_4) = 3$ and $\gamma_{tg}(P_8) = \gamma'_{tg}(P_8) = 6$, the optimal strategy of Staller is whenever Dominator starts playing on a component of G, she plays on that component and adopts her optimal strategy on the component. This shows that $\gamma_{tg}(G) = 3k_1 + 6k_2 = 3n/4$, where $n = 4k_1 + 8k_2$ is the number of vertices in G.

That the upper bound on the Staller-start game total domination number given the 3/4-Game Total Domination Conjecture is tight may be seen by taking, for example, $G = P_5 \cup k_1P_4 \cup k_2P_8$ where $k_1 + k_2 \ge 1$. Since $\gamma'_{tg}(P_5) = 4$, the optimal strategy of Staller is to play her first move optimally on the P_5-component (by playing a leaf of the P_5) and thereafter whenever Dominator starts playing on a component of G, she plays on that component and adopts her optimal strategy on the component. This shows that $\gamma'_{tg}(G) = 4 + 3k_1 + 6k_2 = (3n+1)/4$, where $n = 5 + 4k_1 + 8k_2$ is the number of vertices in G.

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