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FINITE FORMS OF GOWERS' THEOREM ON THE OSCILLATION STABILITY OF C_0*

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We give a constructive proof of the finite version of Gowers' FIN_k Theorem for both the positive and the general case and analyse the corresponding upper bounds provided by the proofs.

1. Introduction

It was observed by Milman (see [7, p.6]) that given a real-valued Lipschitz function defined on the unit sphere of an infinite dimensional Banach space, one can always find a finite dimensional subspace of any given dimension on the unit sphere of which the function is almost constant. This motivated the question of whether in this setting one could also pass to an infinite dimensional subspace with the same property.

It was only in 1992 when W.T. Gowers proved in [2] that c_0, the classical Banach space of real sequences converging to 0 endowed with the supremum norm, has this property. Gowers associated a discrete structure to a net for the sphere of c_0, and proved a partition theorem for the structure that we shall refer to as the $\mathrm{FIN}_k^{\pm}$ Theorem.

For a fixed $k \in \mathbb{N}$, $\mathrm{FIN}_k^{\pm}$ is the set of all functions $f\colon \mathbb{N} \to \{-k,\dots,-1,0,1,\dots,k\}$ that attain one of the values k or $-k$ at least once

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and whose support $\mathrm{supp}(f)=\{n\in\mathbb{N}\colon f(n)\neq 0\}$ is finite. Given $f,g\in\mathrm{FIN}_k^{\pm}$ we say that $f<g$ if the support of f occurs before the support of g. We consider two operations in $\mathrm{FIN}_k^{\pm}$ defined pointwise as follows:

(i) *Sum*: $(f+g)(n)=f(n)+g(n)$ for $f<g$,
(ii) *Tetris*: $T\colon \mathrm{FIN}_k^{\pm}\to\mathrm{FIN}_{k-1}^{\pm}$. For $f\in\mathrm{FIN}_k^{\pm}$,

$$(Tf)(n)=\begin{cases} f(n)-1 & \text{if } f(n)>0\\ f(n)+1 & \text{if } f(n)<0\\ 0 & \text{otherwise.}\end{cases}$$

Note that if $f_0<\ldots<f_{n-1}\in\mathrm{FIN}_k^{\pm}$ and $\delta_i=\pm 1$ for $i<n$, then $\delta_0 T^{l_0}(f_0)+\ldots+\delta_{n-1}T^{l_{n-1}}(f_{n-1})\in\mathrm{FIN}_k^{\pm}$ as long as one of $l_0,\ldots,l_{n-1}$ is zero. A sequence $(f_i)_{i\in I}$ of elements of $\mathrm{FIN}_k^{\pm}$ with $I=\mathbb{N}$ or $I=n$ for some $n\in\mathbb{N}$ such that $f_i<f_j$ for all $i<j\in I$ is called a *block sequence*.

We will also consider the structure FIN_k consisting of finitely supported functions $f\colon\mathbb{N}\to\{0,1,\ldots,k\}$ that attain the value k at least once, with the operations $+$ and T as defined above. We shall refer to the combinatorial theorem associated to this structure as the FIN_k Theorem.

The FIN_k Theorem is a generalization of Hindman's Theorem. It states that for any finite coloring $c\colon\mathrm{FIN}_k\to\{0,1,\ldots,r-1\}$ there exists an infinite block sequence $(f_i)_{i\in\mathbb{N}}$ such that the combinatorial space $\langle f_i\rangle_{i\in\mathbb{N}}$ generated by the sequence $(f_i)_{i\in\mathbb{N}}$,

$$\langle f_i\rangle_{i\in\mathbb{N}}=\{T^{l_0}(f_{i_0})+\ldots+T^{l_n}(f_{i_n})\colon i_0<\ldots<i_n,\\ \min\{l_1,\ldots,l_n\}=0, n\in\mathbb{N}\},$$

is monochromatic.

In the structure $\mathrm{FIN}_k^{\pm}$ we consider the metric defined by $||f-g||_\infty=\max\{|f(n)-g(n)|\colon n\in\mathbb{N}\}$. For $A\subset\mathrm{FIN}_k^{\pm}$ let

$$(A)_1=\{g\colon ||f-g||_\infty\leq 1 \text{ for some } f\in A\}.$$

The $\mathrm{FIN}_k^{\pm}$ Theorem states that for any finite coloring $c\colon\mathrm{FIN}_k^{\pm}\to\{0,\ldots,r-1\}$ there exists an infinite block sequence $(f_i)_{i\in\mathbb{N}}$ such that the combinatorial space $\langle f_i\rangle_{i\in\mathbb{N}}^{\pm}$ generated by the sequence $(f_i)_{i\in\mathbb{N}}$,

$$\langle f_i\rangle_{i\in\mathbb{N}}^{\pm}=\{\delta_0T^{l_0}(f_{i_0})+\ldots+\delta_nT^{l_n}(f_{i_n})\colon i_0<\ldots<i_n,\\ \min\{l_0,\ldots,l_n\}=0,\delta_i=\pm1, n\in\mathbb{N}\},$$

is almost monochromatic; in the sense that there exists $i<r$ such that $\langle f_i\rangle_{i\in\mathbb{N}}^{\pm}\subseteq(c^{-1}(i))_1$.

The proofs of these theorems use Galvin-Glazer methods of ultrafilter dynamics. While the standard modern proof of Hindman's Theorem uses ultrafilter dynamics (see [10, Ch. 2]), Hindman's theorem was originally proved by constructive methods [5] (see also [1]). Until now there is no constructive proof of the $\mathrm{FIN}_k^{\pm}$ Theorem or of the FIN_k Theorem for $k>1$. In this paper we provide a constructive proof of the finite version of these theorems, namely we prove the following:

Theorem 1. *For all natural numbers m,k,r there exists a natural number n such that for every r-coloring of $\mathrm{FIN}_k(n)$, the functions in FIN_k supported below n, there exists a block sequence in $\mathrm{FIN}_k(n)$ of length m that generates a monochromatic combinatorial subspace.*

Theorem 2. *For all natural numbers m,k,r there exists a natural number n such that for every r-coloring c of $\mathrm{FIN}_k^{\pm}(n)$, the functions in $\mathrm{FIN}_k^{\pm}$ supported below n, there exist $i<r$ and a block sequence $(f_i)_{i<m}$ in $\mathrm{FIN}_k^{\pm}(n)$ such that $\langle f_i\rangle_{i<m}^{\pm}\subseteq(c^{-1}(i))_1$.*

Let $g_k(m,r)$ and $g_k^{\pm}(m,r)$ be the minimal numbers satisfying the conditions of Theorems 1 and 2, respectively. While these results follow easily from the corresponding infinite versions by a compactness argument, such an argument gives no information about the bounds of the function $g_k(m,r)$ or $g_k^{\pm}(m,r)$. These compactness arguments could be formalized within second order arithmetic and would yield second order proofs of the finite FIN_k Theorem and of the finite $\mathrm{FIN}_k^{\pm}$ Theorem, provided there were proofs of the infinite versions within second order arithmetic. However, the existing proofs of the infinite versions use ultrafilter dynamics and have not been formalized in second order arithmetic. Our proofs of the finite versions use only induction and can be written in PA.

In the notation above, the bounds we find for $k>1$ are

$$g_k(m,2) \leq f_{4+2(k-1)} \circ f_4(6m-2),$$
$$g_k^{\pm}(m,2) \leq f_{4+2(k-1)} \circ f_4(12m-2),$$

where for $i\in\mathbb{N}$, f_i denotes the i-th function in the Ackermann Hierarchy. Notice that these bounds have an Ackermann type dependence on k. Recently K. Tyros [11] obtained primitive recursive upper bounds.

The paper is organized as follows: In Section 2 we introduce notation related to the structures FIN_k and $\mathrm{FIN}_k^{\pm}$. We also state Ramsey's Theorem and van der Waerden's Theorem and introduce some notation that we shall use in our proof and in the calculation of the upper bounds. In Section 3 we prove Theorem 1 and in Section 4 we prove Theorem 2. Section 5 contains some concluding remarks.

2. Notation

We start by fixing some notation. We denote by $\mathbb{N}$ the set of natural numbers starting at zero and use the von Neumann identification of a natural number n with the set of its predecessors, $n=\{0,1,\dots,n-1\}$. Let $k\in\mathbb{N}$ be given. For $N,d\in\mathbb{N}$ we define the finite version of FIN_k and its d-dimensional version by:

$$\mathrm{FIN}_k(N) = \{f \in \mathrm{FIN}_k\colon \max(\mathrm{supp}(f)) < N\}$$
$$\mathrm{FIN}_k(N)^{[d]} = \{(f_i)_{i<d} \mid f_i \in \mathrm{FIN}_k(N) \text{ and } f_i < f_j \text{ for } i < j < d\}.$$

Where for $f,g \in \mathrm{FIN}_k(N)$, we write $f < g$ when $\max(\mathrm{supp}(f)) < \min(\mathrm{supp}(g))$. The elements of $\mathrm{FIN}_k(N)^{[d]}$ are called block sequences. The combinatorial space $\langle f_i\rangle_{i<d}$ generated by a sequence $(f_i)_{i<d}\in\mathrm{FIN}_k(N)^{[d]}$ is the set of elements of $\mathrm{FIN}_k(N)$ of the form $T^{l_0}(f_{i_0})+\ldots+T^{l_{n-1}}(f_{i_{n-1}})$ where $n\in\mathbb{N}$, $i_0<\ldots<i_{n-1}<d$ and $\min\{l_0,\dots,l_{n-1}\}=0$. A block subsequence of $(f_i)_{i<d}$ is a block sequence contained in $\langle f_i\rangle_{i<d}$. Just as we defined the d-dimensional version of $\mathrm{FIN}_k(N)$, if $(f_i)_{i<l}$ is a block sequence, we define $(\langle f_i\rangle_{i<l})^{[d]}$ to be the collection of block subsequences of $(f_i)_{i<l}$ of length d.

The finite version of $\mathrm{FIN}_k^{\pm}$ and its d-dimensional version are defined similarly. The combinatorial space $\langle f_i\rangle_{i<d}^{\pm}$ generated by a sequence $(f_i)_{i<d}\in\mathrm{FIN}_k^{\pm}(N)^{[d]}$ is the set of elements of $\mathrm{FIN}_k^{\pm}(N)$ of the form $\delta_0T^{l_0}(f_{i_0})+\ldots+\delta_{n-1}T^{l_{n-1}}(f_{i_{n-1}})$ where $n\in\mathbb{N}$, $i_0<\ldots<i_{n-1}<d$, $\min\{l_0,\dots,l_{n-1}\}=0$ and $\delta_i=\pm1$ for $i<n$.

We first prove Theorem 1 and then prove Theorem 2 using an intermediate lemma whose proof closely resembles the proof of Theorem 1. The idea to obtain Theorem 2 from a variation of Theorem 1 comes from [6].

The following definition is important when coding an element of FIN_k in a sequence of elements of FIN_{k-1}. Given $\mathbf{f}=(f_i)_{i<m}\in\mathrm{FIN}_k^{[m]}$, for

$$g=\sum_{i<m}T^{l_i}f_i,$$

we define $\mathrm{supp}_k^{\mathbf{f}}(g)$ to be the set of all $i<m$ such that $l_i=0$. The cardinality of this set determines the length of the sequence we need in order to code g, as we shall describe in detail later on. The proof is by induction on k. The starting point is Folkman's Theorem. In the inductive step, the idea is to code an element of FIN_k in a finite sequence of elements of FIN_{k-1} and apply the result for FIN_{k-1} and its higher dimensional versions.

After each step of the proof we shall sketch some calculations that will allow us to obtain at the end bounds for the functions $g_k(m,2)$. Since we will focus on 2-colorings for the analysis of the upper bounds, we shall omit the parameter corresponding to the number of colors and write $g_k(m)$ for $g_k(m,2)$. We adopt the same convention for any other numbers defined in the course of the proof that have the number of colors as a parameter.

We will use Ramsey's Theorem and van der Waerden's Theorem in our arguments, therefore we will need upper bounds for the numbers corresponding to these two theorems. For Ramsey's Theorem, $R_d(m)$ is the minimal n such that if $[n]^d$, the collection of subsets of n of cardinality d, is 2-colored then there exists a monochromatic set of cardinality m. For van der Waerden's Theorem, $W(m)$ is the minimal n such that if n is 2-colored, then there exists a monochromatic arithmetic progression of length m.

For a discussion of Ramsey numbers and van der Waerden numbers, see [4, Chapter 4]. It turns out that these numbers grow very rapidly and, in order to deal with such rapidly growing functions, we use the Ackermann Hierarchy. The Ackermann hierarchy is the sequence of functions $f_i\colon \mathbb{N}\to\mathbb{N}$ defined as follows:

$$f_1(x) = 2x$$
$$f_{i+1}(x) = f_i^{(x)}(1).$$

The Ackermann function is obtained by diagonalization and grows even faster than any f_i, $i\in\mathbb{N}$:

$$f_\omega(x) = f_x(x).$$

There is a slight variation of the function TOWER in the Ackermann hierarchy, it is useful to express upper bounds for the Ramsey numbers $R_d(m)$ and the van der Waerden numbers $W(m)$. The tower functions $t_i(x)$ are defined inductively by

$$t_1(x) = x$$
$$t_{i+1}(x) = 2^{t_i(x)}.$$

We shall use the following well known upper bounds for $R_d(m)$ and $W(m)$:

$$R_d(m) \le t_d(c_d m), \tag{1}$$

$$W(m) \le 2^{2^{2^{2^{2^{m+9}}}}} = t_6(m+9). \tag{2}$$

Where c_d is a constant that depends on d. See [4, Section 4.7] for a deduction of the bound for $R_d(m)$. The bound for $W(m)$ was found by Gowers in [3].

3. The positive case

We are now ready to start the proof of Theorem 1. We identify canonically FIN_1 with FIN, the collection of finite nonempty subsets of $\mathbb{N}$. The case $k=1$ of the finite FIN_k Theorem, phrased in terms of finite sets and finite unions, is a variation of Folkman's Theorem. We require a sequence of sets which are in block position, not just pairwise disjoint. We include the proof for the sake of completeness and more importantly because we are interested in analysing the corresponding upper bounds. The proof presented here is based on the proof presented in [4, Section 3.4]. For the best known upper bounds for Folkman's Theorem see [9].

Folkman's Theorem. *For every $m,r\in\mathbb{N}$ there exists $N\in\mathbb{N}$ such that for every $c\colon \mathrm{FIN}(N)\to r$ there exists $(x_i)_{i<m}\in\mathrm{FIN}(N)^{[m]}$ such that $c\restriction\langle x_i\rangle_{i<m}$ is constant.*

It easily follows from the Pigeon-Hole principle that the theorem reduces to the following:

Lemma 3. *For every $m,r\in\mathbb{N}$ there exists $N\in\mathbb{N}$ such that for all $c\colon \mathrm{FIN}(N)\to r$ there exists $(x_i)_{i<m}\in\mathrm{FIN}(N)^{[m]}$ such that $c\restriction\langle x_i\rangle_{i<m}$ is* min*-determined. That is, if $x=\bigcup_{i\in s}x_i, y=\bigcup_{i\in t}x_i$ with $s,t\subseteq m$ such that $\min s=\min t$ then $c(x)=c(y)$.*

We denote by $N(m,r)$ the minimal N satisfying the conditions of Lemma 3. We shall use van der Waerden's Theorem. For $n,r\in\mathbb{N}$, let $W(n,r)$ be the minimal m such that for any r-coloring of m there is a monochromatic arithmetic progression of length n.

Proof. We fix the number of colors $r\in\mathbb{N}$ and proceed by induction on m, the length of the desired sequence. The base case $m=1$ is clear, so we suppose the statement holds for m and prove it for $m+1$. By a repeated application of Ramsey's theorem, we fix $N\in\mathbb{N}$ such that given any r-coloring of $\mathrm{FIN}(N)$, there exists $A\subseteq N$ of cardinality $W(N(m,r),r)$ such that for all $i<W(N(m,r),r)$, the coloring c is constant on $[A]^i$, the color possibly depending on i.

Now let $c\colon \mathrm{FIN}(N)\to r$ be given and let $A\subset N$ be as above with $c\restriction[A]^i$ constant with value $c_i<r$. Define $d\colon W(N(m,r),r)\to r$ by

$$d(i)=c_i.$$

We use van der Waerden's Theorem to find $\alpha,\lambda<W(N(m,r),r)$ and $i_0<r$ such that $d\restriction\{\alpha+\lambda j\colon j<N(m,r)\}$ is constant with value c_{i_0}. Let x_0 be the set consisting of the first α elements of A and let $y_1<\ldots<y_{N(m,r)}$ be

a block sequence of subsets of $A \setminus x_0$ each one of which has cardinality λ. Note that the combinatorial space generated by $(y_i)_{i<N(m,r)}$ is canonically isomorphic to $\mathrm{FIN}(N(m,r))$, therefore by induction hypothesis there exists a block subsequence $x_1 < \ldots < x_m$ of $(y_i)_{0<i\leq N(m,r)}$ such that $c \restriction \langle x_i \rangle_{i=1}^m$ is min-determined.

We shall see that $(x_i)_{i<m+1}$ is the sequence we are looking for. Fix $x, y \in \langle x_i \rangle_{0 \leq i \leq m}$ with the same minimum. Suppose first that $x_0 \subseteq x$ then also $x_0 \subseteq y$ and $\#x = m + \lambda i, \#y = m + \lambda j$ for some $i, j < N(m,r)$. Hence $c(x) = c(y) = c_{i_0}$. Now suppose $x_0 \nsubseteq x$ then the same holds for y and consequently $x, y \in \langle x_i \rangle_{i=1}^m$. By the choice of $(x_i)_{i=1}^m$ it follows that $c(x) = c(y)$. ∎

In the proof of Lemma 3 we had to apply Ramsey's Theorem in dimensions $1, 2, \ldots, W(N(m))$ in order to obtain a suitable set of cardinality $W(N(m))$. To easily iterate the upper bound (1), note that for any $i \in \mathbb{N}$ and any given constant c, if x is big enough, then we have that $t_i(cx) \leq t_{i+1}(x)$. Using these estimates we get the recursive inequality

$$N(m+1) \leq t^3(N(m)).$$

From the recursive inequality for $N(m)$, we get that

$$\begin{aligned} N(m) &\leq (t^3)^m(1) \\ &\leq f_4(3m). \end{aligned}$$

In order to obtain Folkman's Theorem from Lemma 3, we applied the Pigeon-Hole principle, and so we have that

$$g_1(m) \leq N(2(m-1)+1) \tag{3}$$
$$\leq f_4(6m-3). \tag{4}$$

We now prove Theorem 1 in its multidimensional form.

Theorem 4. *For every $k, m, r, d \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for every coloring $c\colon \mathrm{FIN}_k(n)^{[d]} \to r$ there exists $(f_i)_{i<m} \in \mathrm{FIN}_k(n)^{[m]}$ such that $c \restriction (\langle f_i \rangle_{i<m})^{[d]}$ is constant.*

Let $g_{k,d}(m,r)$ be the minimal n satisfying the conditions of Theorem 4. We prove the theorem by induction on k. Note that if we have the theorem for some k and $d = 1$, we can deduce the theorem for k and dimensions $d > 0$ using a standard diagonalization argument that we describe below. We include the dimensions in the statement of the theorem because they play an important role in the proof and because we are interested in calculating upper bounds for $g_{k,d}(m,r)$.

Proof. The base case $k=1$ in dimension 1 is Folkman's Theorem. We now sketch the diagonalization argument to obtain the 2-dimensional result in the case $k=1$. The argument for higher dimensions and larger values of k is similar. Let $c\colon \mathrm{FIN}(N)^{[2]} \to 2$ be given and let us calculate how big should N be in order to ensure the existence of a block sequence of length m generating a monochromatic combinatorial subspace. We define block sequences $S_0, \dots, S_{p-1}$ and $a_0 < \dots < a_{p-1}$ where $p = g_{1,1}(m)$ with the following properties:

(i) $S_0 = \{\{0\}, \dots, \{N-1\}\}$,
(ii) a_j is the first element of S_j,
(iii) For $j>0$, S_j is a block subsequence of S_{j-1} such that for each $x \in \langle a_i \rangle_{i<j}$, the coloring $c_x\colon \mathrm{FIN}(N \setminus x) \to 2$ of the finite subsets of $N \setminus x$ defined by

$$y \mapsto c(x,y)$$

is constant with value i_x when restricted to $\langle S_j \rangle$,
(iv) the sequence S_{p-1} has length 2.

Each S_j, $0<j<p$ can be obtained by a repeated application of Theorem 4 for $k=1$ in dimension 1. Let $S=\{a_j : j<p\}$ and consider the coloring $d\colon \langle S \rangle \to 2$ defined by

$$x \mapsto i_x.$$

By the choice of p, we can find a block subsequence of S of length m that generates a d-monochromatic combinatorial subspace, and by construction this sequence will also generate a c-monochromatic combinatorial subspace. Since the total number of refinements to obtain the sequences $(S_j)_{j<p}$ is 2^p-1, it suffices to start with $N \ge g_{1,1}^{2^p-1}(2)$ and so

$$g_{1,2}(m) \le g_{1,1}^{2^p-1}(2). \tag{5}$$

One can prove by induction on l that $g_{1,1}^l(m) \le f_4^l(6m+l-4)$, so we have that

$$g_{1,2}(m) \le f_5^2(7m). \tag{6}$$

In general the recursive inequality resulting from the diagonalization argument is

$$g_{1,d+1}(m) \le g_{1,1}^{h_d(g_{1,d}(m))}(2), \tag{7}$$

where $h_d(l)$ for $l \in \mathbb{N}$, is the cardinality of $\mathrm{FIN}_1^{[d]}(l)$. Note that $h_d(l) \le 2^{ld}$. For $d \ge 2$ we get

$$g_{1,d}(m) \le f_5^d(7m+2(d-1)). \tag{8}$$

Now suppose the theorem holds for k and all $m,r,d\in\mathbb{N}$. We work to get the result for $k+1$. We need the following preliminary result:

Claim 5. *For every $N,r\in\mathbb{N}$ there exists $\bar{N}$ such that for every $c\colon \mathrm{FIN}_{k+1}(\bar{N})\to r$ there exists $\mathbf{h}=(h_i)_{i<N}\in\mathrm{FIN}_{k+1}(\bar{N})^{[N]}$ such that for*

$$f=\sum_{i<N}T^{s_i}(h_i)$$
$$g=\sum_{i<N}T^{t_i}(h_i),$$

$c(f)=c(g)$ whenever $\mathrm{supp}^{\mathbf{h}}_{k+1}(f)=\mathrm{supp}^{\mathbf{h}}_{k+1}(g)$, that is, whenever for all $i<N$, $s_i=0$ if and only if $t_i=0$.

Let $\bar{N}_{k+1}(N,r)$ be the minimal $\bar{N}$ satisfying the conditions of Claim 5.

Proof of Claim 5. Let $N,r\in\mathbb{N}$ be given. By induction hypothesis, let $\bar{N}$ be such that for any sequence of r-colorings $(e_i)_{i<N}$ with $e_i\colon \mathrm{FIN}_k(\bar{N})^{[2i+3]}\to r$, there exists a block sequence $(f_j)_{j<3N}$ such that for each $i<N$, e_i is constant on $(\langle f_j\rangle_{j<3N})^{[2i+3]}$, its value possibly depending on i.

Let $c\colon \mathrm{FIN}_{k+1}(\bar{N})\to r$ be given. Define $U\colon \mathrm{FIN}_k\to\mathrm{FIN}_{k+1}$ by

$$(Uf)(i)=\begin{cases} f(i)+1 & \text{if } f(i)\neq 0\\ 0 & \text{otherwise.}\end{cases}$$

For each $i<N$ define the coloring $e_i\colon \mathrm{FIN}_k(\bar{N})^{[2i+3]}\to r$ by

$$e_i((h_j)_{j<2i+3})=c\left(\sum_{j<2i+3}U^{j \bmod 2}h_j\right).$$

By the choice of $\bar{N}$, we can find a block sequence $(f_j)_{j<3N}$ such that for each $i<N$, e_i is constant on $(\langle f_j\rangle_{j<3N})^{[2i+3]}$. We shall see that the sequence $(h_i)_{i<N}$ defined by $h_i=f_{3i}+Uf_{3i+1}+f_{3i+2}$ for $i<N$ is the sequence we are looking for. Let $g_1,g_2\in\langle h_i\rangle_{i<N}$ be such that $\mathrm{supp}^{\mathbf{h}}_{k+1}(g_1)=\mathrm{supp}^{\mathbf{h}}_{k+1}(g_2)$, let l be the cardinality of $\mathrm{supp}^{\mathbf{h}}_{k+1}(g_1)$. Then we can write g_1,g_2 as

$$g_1=\sum_{j<2(l-1)+3}U^{j \bmod 2}w_j$$
$$g_2=\sum_{j<2(l-1)+3}U^{j \bmod 2}w'_j$$

for some $(w_j)_{j<2(l-1)+3}, (w'_j)_{j<2(l-1)+3} \in (\langle f_j \rangle_{j<3N})^{[2(l-1)+3]}$. Since e_{l-1} is constant on $(\langle f_j \rangle_{j<3N})^{[2(l-1)+3]}$, it follows that $c(g_1) = c(g_2)$. ∎

We now verify that for the case $k+1$, $d=1$ in Theorem 4, we may take $n = \bar{N}_{k+1}(H, r)$ where $H = g_{1,1}(m, r)$. Let $c\colon \mathrm{FIN}_{k+1}(n) \to r$ be given. By the choice of n we can find $\mathbf{h} = (h_i)_{i<H}$ such that $c \restriction \langle h_i \rangle_{i<H}$ depends only on $\mathrm{supp}^{\mathbf{h}}_{k+1}$. Let

$$\begin{aligned} d\colon \mathrm{FIN}(H) &\to r \\ x &\mapsto c\left(\sum_{i\in x} h_i\right). \end{aligned}$$

By the choice of H we can find $x_0 < \ldots < x_{m-1}$ subsets of H such that $d \restriction \langle x_i \rangle_{i<m}$ is constant. For $i<m$ let $f_i = \sum_{j\in x_i} h_j$. Note that for $f \in \langle f_i \rangle_{i<m}$, $\mathrm{supp}^{\mathbf{h}}_{k+1}(f)$ is a finite union of $x_0, \ldots, x_{m-1}$. Therefore $c \restriction \langle f_i \rangle_{i<m}$ is constant.

In general from the inductive step we get that for any $k \in \mathbb{N}$,

$$\bar{N}_{k+1}(N) \leq g_{k,2(N-1)+3} \circ \cdots \circ g_{k,5} \circ g_{k,3}(N), \text{ and} \tag{9}$$

$$g_{k+1,1}(m) \leq g_{k,2(g_{1,1}(m)-1)+3} \circ \cdots \circ g_{k,5} \circ g_{k,3}(g_{1,1}(m)). \tag{10}$$

The diagonalization argument used to increase the dimension from d to $d+1$ in the case $k=1$ is similar for larger values of k so we get that

$$g_{k,d+1}(m) \leq g_{k,1}^{h_{k,d}(g_{k,d}(m))}(2), \tag{11}$$

where $h_{k,d}(l)$ for $l \in \mathbb{N}$, is the cardinality of $\mathrm{FIN}^{[d]}_k(l)$. Note that $h_{k,d}(l) \leq dl^k$. Using (10) and (11), we obtain:

$$g_{k,1}(m) \leq f_{4+2(k-1)} \circ f_4(6m-2), \tag{12}$$

$$g_{k,d}(m) \leq f^d_{5+2(k-1)}(f_4(6m-2) + 2(d-1)), \tag{13}$$

where $d > 1$. ∎

4. The general case

We now prove the $\mathrm{FIN}^{\pm}_k$ Theorem using the following lemma:

Lemma 6. *For all natural numbers m, k, r there exists a natural number n such that for every r-coloring of $\mathrm{FIN}^{\pm}_k(n)$, the functions in $\mathrm{FIN}^{\pm}_k$ supported* *below n, there exists a block sequence $(f_i)_{i<m}$ of elements of* $\mathrm{FIN}_k^{\pm}(n)$ *such that the set*

$$\langle f_i\rangle_{i<m}^{(-T)} = \{(-T)^{l_0} f_{i_0} + \ldots + (-T)^{l_s} f_{i_s} : i_0 < \ldots < i_s \\ \min\{l_0, \ldots, l_s\} = 0, s \leq m\}$$

is monochromatic.

Proof. The lemma is a consequence of Theorem 1. Let $\mathrm{FIN}_k^-(n)$ be the set of functions $f\colon \mathbb{N} \to \{(-1)^j(k-j)\colon j = 0, 1, \ldots, k\}$ supported below n. Then the structures $(\mathrm{FIN}_k^-(n), +, -T)$ and $(\mathrm{FIN}_k(n), +, T)$ are isomorphic, as witnessed by the map $I\colon \mathrm{FIN}_k^-(n) \to \mathrm{FIN}_k(n)$ that sends each $f \in \mathrm{FIN}_k^-(n)$ to its pointwise absolute value. Note that the inverse image of a block sequence $(g_i)_{i<m}$ in $\mathrm{FIN}_k(n)$ is a block sequence in $\mathrm{FIN}_k^-(n)$, and $I^{-1}(\langle g_i\rangle_{i<m}) = \langle I^{-1}(g_i)\rangle_{i<m}^{(-T)}$. ∎

The following claim finishes the proof of Theorem 2:

Claim 7. *Fix $k \in \mathbb{N}$ and let $(f_i)_{i<2m}$ be a block sequence of elements of $\mathrm{FIN}_k^{\pm}$. If we set $h_i = f_{2i} - f_{2i+1}$ for each $i < m$, then $\langle h_i\rangle_{i<m}^{\pm} \subseteq (\langle f_i\rangle^{(-T)})_1$.*

The proof of this lemma is a slight variation of the proof of Lemma 9 in [6].

Proof. Let $(f_i)_{i<2m}$ and $(h_i)_{i<m}$ be as in the statement of the claim. To simplify the notation, set $f_i^0 = f_{2i}$ and $f_i^1 = f_{2i+1}$ for $i < m$. Let $f \in \langle h_i\rangle_{i<m}^{\pm}$ then f can be written as

$$\begin{aligned} f &= \delta_0 T^{l_0}(h_{n_0}) + \ldots + \delta_s T^{l_s}(h_{n_s}) \\ &= \delta_0 T^{l_0}(f_{n_0}^0) - \delta_0 T^{l_0}(f_{n_0}^1) + \ldots + \delta_s T^{l_s}(f_{n_l}^0) - \delta_s T^{l_s}(f_{n_s}^1). \end{aligned}$$

Let $g_i^j = (-1)^j \delta_i T^{l_i}(f_{n_i}^j)$ for $i \leq s$, $j < 2$. We consider two cases:

Case 1. If $\delta_i = 1$ and l_i is even, or $\delta_i = -1$ and l_i is odd, set $\bar{g}_i^0 = g_i^0 = (-T)^{l_i}(f_{n_i}^0)$ and $\bar{g}_i^1 = T(g_i^1) = (-T)^{l_i+1}(f_{n_i}^1)$.

Case 2. If $\delta_i = -1$ and l_i is even, or $\delta_i = 1$ and l_i is odd, set $\bar{g}_i^0 = T(g_i^0) = (-T)^{l_i+1}(f_{n_i}^0)$ and $\bar{g}_i^1 = g_i^1 = (-T)^{l_i}(f_{n_i}^1)$.

Note that $||g_i^j - \bar{g}_i^j||_\infty \leq 1$ for $i \leq l$ and $j < 2$. Also when $l_i = 0$, either $\bar{g}_i^0 = g_i^0$ or $\bar{g}_i^1 = g_i^1$. Therefore, if we set $\bar{f} = \sum_{i\leq l}(\bar{g}_i^0 + \bar{g}_i^1)$, we have that $\bar{f} \in \langle f_i\rangle_{i<2m}^{(-T)}$ and $||f - \bar{f}||_\infty \leq 1$. ∎

Note that this same method provides a reduction of the general case of Gowers' c_0 Theorem to the positive case. The bounds corresponding to Lemma 6 are the same as the bounds corresponding to Theorem 1 and from the last claim we get

$$g_k^{\pm}(m) \leq g_k(2m).$$

5. Concluding remarks

As far as we know there are no proofs of the infinite FIN_k Theorem or of the infinite $\mathrm{FIN}_k^{\pm}$ Theorem that avoid the use of idempotent ultrafilters. The proof we present of the finite version cannot be adapted to the infinite case. This is because when proving the result for $k+1$, we have to know how many dimensions of the inductive hypothesis we need, and this number depends on the desired length of the homogeneous (or almost homogeneous) sequence.

We also obtained upper bounds for the quantitative version of Milman's result about the stabilization of Lipschitz functions on finite dimensional Banach spaces, for the special case of the spaces ℓ_∞^n, $n \in \mathbb{N}$ and functions that do not depend on the sign of the canonical coordinates (the original statement of the Finite Stabilization Principle can be found in [7, p.6]). We obtain these bounds by providing a finitization of the proof of this particular case of Milman's stabilization principle presented in [8]. Namely we finitize the proof of:

Theorem 8. *For every $C,\epsilon>0$ and $m\in\mathbb{N}$ there is $n\in\mathbb{N}$ such that for every C-Lipschitz function $f\colon PS_{\ell_\infty^n}\to\mathbb{R}$ there is a positive block sequence $(\mathbf{y}_i)_{i<m}$ so that*

$$\operatorname{osc}(f \restriction PS_{[\mathbf{y}_i]_{i<m}}) < \epsilon.$$

Let $N(C,\epsilon,m)\in\mathbb{N}$ be the minimal n satisfying the conditions of Theorem 8. We obtained the following upper bound for $N(C,\epsilon,m)$:

$$N(C,\epsilon,m) \leq f_3\left(m^s \cdot \left\lceil \frac{C}{\epsilon} \right\rceil^{m^s}\right),$$

where $s=\log(\epsilon/12C)/\log(1-\epsilon/12C)+2$. For fixed ϵ and C, this upper bound is much slower growing than the bound we found for $g_k(m)$ for a fixed $k\geq 2$.

We still have to find lower bounds for the functions $g_k(m)$ and $g_k^{\pm}(m)$, $k\in\mathbb{N}$. This would amount to finding for any given $l\in\mathbb{N}$, a bad coloring of $\mathrm{FIN}_k(N)$ (or $\mathrm{FIN}_k^{\pm}(N)$) for some N, for which there is no sequence of length l generating a monochromatic (resp. almost monochromatic) combinatorial subspace. In this direction it would also be interesting to find a way for stepping up lower bounds for a given $k\in\mathbb{N}$ to larger values of k.

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