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PROOF OF A TOURNAMENT PARTITION CONJECTURE AND AN APPLICATION TO 1-FACTORS WITH PRESCRIBED CYCLE LENGTHS

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In 1982 Thomassen asked whether there exists an integer $f(k,t)$ such that every strongly $f(k,t)$-connected tournament T admits a partition of its vertex set into t vertex classes $V_1,\dots,V_t$ such that for all i the subtournament $T[V_i]$ induced on T by V_i is strongly k-connected. Our main result implies an affirmative answer to this question. In particular we show that $f(k,t)=O(k^7t^4)$ suffices. As another application of our main result we give an affirmative answer to a question of Song as to whether, for any integer t, there exists an integer $h(t)$ such that every strongly $h(t)$-connected tournament has a 1-factor consisting of t vertex-disjoint cycles of prescribed lengths. We show that $h(t)=O(t^5)$ suffices.

1. Introduction

1.1. Partitioning tournaments into highly connected subtournaments

There is a rich literature of results and questions relating to partitions of (di)graphs into subgraphs which inherit some properties of the original (di)graph. For instance, Hajnal [4] and Thomassen [10] proved that for every k there exists an integer $f(k)$ such that every $f(k)$-connected graph has a vertex partition into sets S and T so that both S and T induce k-connected graphs. Here we investigate a corresponding question for tournaments.

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A tournament is an orientation of a complete graph. A tournament is *strongly connected* if for every pair of vertices u,v there exists a directed path from u to v and a directed path from v to u. For any integer k we call a tournament T *strongly k-connected* if $|V(T)|>k$ and the removal of any set of fewer than k vertices results in a strongly connected tournament. We denote the subtournament induced on a tournament T by a set $U\subseteq V(T)$ by $T[U]$.

The following problem was posed by Thomassen (see [8]).

Problem 1.1. *Let $k_1,\dots,k_t$ be positive integers. Does there exist an integer $f(k_1,\dots,k_t)$ such that every strongly $f(k_1,\dots,k_t)$-connected tournament T admits a partition of its vertex set into vertex classes $V_1,\dots,V_t$ such that for all $i\in\{1,\dots,t\}$ the subtournament $T[V_i]$ is strongly k_i-connected?*

If $k_i = 1$ for all $i \in \{2,\dots,t\}$ then $f(k_1,\dots,k_t)$ exists and is at most k_1+3t-3. This follows by an easy induction on t, taking V_t to be a set inducing a directed 3-cycle. Chen, Gould and Li [3] showed that every strongly t-connected tournament with at least $8t$ vertices admits a partition into t strongly connected subtournaments. This gives the best possible connectivity bound in the case $k_1=\dots=k_t=1$ and $|V(T)|\geq 8t$. Until now even the existence of $f(2,2)$ was open. Our main result answers all cases of the above problem of Thomassen in the affirmative.

Theorem 1.2. *Let T be a tournament on n vertices and let $k,t\in\mathbb{N}$ with $t\geq 2$. If T is strongly $10^7k^6t^3\log(kt^2)$-connected then there exists a partition of $V(T)$ into t vertex classes $V_1,\dots,V_t$ such that for all $i\in\{1,\dots,t\}$ the subtournament $T[V_i]$ is strongly k-connected.*

The above bound is unlikely to be best possible. It would be interesting to establish the correct order of magnitude of $f(k_1,\dots,k_t)$ for all fixed k_i and t. In fact, we believe a linear bound may suffice.

Conjecture 1.3. *There exists a constant c such that the following holds. Let T be a tournament on n vertices and let $k,t\in\mathbb{N}$. If T is strongly ckt-connected then there exists a partition of $V(T)$ into t vertex classes $V_1,\dots,V_t$ such that for all $i\in\{1,\dots,t\}$ the subtournament $T[V_i]$ is strongly k-connected.*

It would also be interesting to know whether Theorem 1.2 can be generalised to digraphs.

Question 1.4. *Does there exist, for all $k,t\in\mathbb{N}$, a function $\hat{f}(k,t)$ such that for every strongly $\hat{f}(k,t)$-connected digraph D there exists a partition* *of $V(D)$ into t vertex classes $V_1,\dots,V_t$ such that for all $i\in\{1,\dots,t\}$ the subdigraph $D[V_i]$ is strongly k-connected?*

Instead of proving Theorem 1.2 directly, we first prove the following somewhat stronger result. It establishes the existence of small but powerful 'linkage structures' in tournaments, and Theorem 1.2 follows from it as an immediate corollary. These linkage structures are partly based on ideas of Kühn, Lapinskas, Osthus and Patel [6], who proved a conjecture of Thomassen by showing that for every k there exists an integer $\tilde{f}(k)$ such that every strongly $\tilde{f}(k)$-connected tournament contains k edge-disjoint Hamilton cycles.

Theorem 1.5. *Let T be a tournament on n vertices, let $k,m,t\in\mathbb{N}$ with $m\geq t\geq 2$. If T is strongly $10^7k^6t^2m\log(ktm)$-connected then $V(T)$ contains t disjoint vertex sets $V_1,\dots,V_t$ such that for every $j\in\{1,\dots,t\}$ the following hold:*

(i) $|V_j|\leq n/m$,
(ii) *for any set $R\subseteq V(T)\setminus\bigcup_{i=1}^{t}V_i$ such that $|V_j\cup R|>k$ the subtournament $T[V_j\cup R]$ is strongly k-connected.*

1.2. Partitioning tournaments into vertex-disjoint cycles

Theorem 1.5 also has an application to another problem on tournaments, this time concerning partitioning the vertices of a tournament into vertex-disjoint cycles of prescribed lengths.

Reid [7] proved that any strongly 2-connected tournament on $n\geq 6$ vertices admits a partition of its vertices into two vertex-disjoint cycles (unless the tournament is isomorphic to the tournament on 7 vertices which contains no transitive tournament on 4 vertices). Chen, Gould and Li [3] showed that every strongly t-connected tournament with at least $8t$ vertices admits a partition into t vertex-disjoint cycles. This answered a question of Bollobás (see [7]), namely what is the least integer $g(t)$ such that all but a finite number of strongly $g(t)$-connected tournaments admit a partition into t vertex-disjoint cycles? Song proved the following strengthening of Reid's result.

Theorem 1.6. [9] *Let T be a tournament on $n\geq 6$ vertices and let $3\leq L\leq n-3$. If T is strongly 2-connected then T contains two vertex-disjoint cycles of lengths L and $n-L$ (unless T is isomorphic to the tournament on 7 vertices which contains no transitive tournament on 4 vertices).*

Song [9] also posed a question that generalises the question of Bollobás. Namely, for any integer t, what is the least integer $h(t)$ such that all but a finite number of strongly $h(t)$-connected tournaments admit a partition into t vertex-disjoint cycles of prescribed lengths? Until now, for $t \geq 3$, even the existence of $h(t)$ remained open. The following consequence of Theorem 1.5 settles this question in the affirmative.

Theorem 1.7. *Let T be a tournament on n vertices, let $t \in \mathbb{N}$ with $t \geq 2$ and let $L_1, \dots, L_t \in \mathbb{N}$ with $L_1, \dots, L_t \geq 3$ and $\sum_{j=1}^{t} L_j = n$. If T is strongly $10^{10} t^4 \log t$-connected then T contains t vertex-disjoint cycles of lengths $L_1, \dots, L_t$.*

Camion's theorem (see [2]) states that every strongly connected tournament contains a Hamilton cycle. So certainly $g(1) = h(1) = 1$. Note that Song [9] showed that $g(2) = h(2) = 2$. Clearly $g(k) \leq h(k)$ for all k. Song [9] conjectured that $g(k) = h(k)$ for all k. Showing that $h(k)$ is linear would already be a very interesting step towards this.

Theorem 1.7 has a similar flavour to the El-Zahar conjecture. This determines the minimum degree which guarantees a partition of a graph into vertex-disjoint cycles of prescribed lengths and was proved for all large n by Abbasi [1]. A related result to Theorem 1.7 for oriented graphs (where the assumption of connectivity is replaced by that of high minimum semidegree) was proved by Keevash and Sudakov [5].

The rest of the paper is organised as follows. In Section 2 we lay out some notation, set out some useful tools, and prove some preliminary results. Section 3 is the heart of the paper in which we prove Theorem 1.5. In Section 4 we deduce Theorem 1.7.

2. Notation, tools and preliminary results

We write $|T|$ for the number of vertices in a tournament T. We denote the in-degree of a vertex v in a tournament T by $d_T^-(v)$, and we denote the out-degree of v in T by $d_T^+(v)$. We say that a set $A \subseteq V(T)$ *in-dominates* a set $B \subseteq V(T)$ if for every vertex $b \in B$ there exists a vertex $a \in A$ such that there is an edge in T directed from b to a. Similarly, we say that a set $A \subseteq V(T)$ *out-dominates* a set $B \subseteq V(T)$ if for every vertex $b \in B$ there exists a vertex $a \in A$ such that there is an edge in T directed from a to b. We denote the *minimum semidegree* of T (that is, the minimum of the minimum in-degree of T and the minimum out-degree of T) by $\delta^0(T)$. We say that a tournament T is *transitive* if we may enumerate its vertices $v_1, \dots, v_m$ such that there is an edge in T directed from v_i to v_j if and only if $i<j$. In this case we call v_1 the *source* of T and v_m the *sink* of T. The *length* of a path is the number of edges in the path. If $P=x_1\dots x_\ell$ is a path directed from x_1 to x_ℓ then we denote the set $\{x_1,\dots,x_\ell\}\setminus\{x_1,x_\ell\}$ of interior vertices of P by $\mathrm{Int}(P)$, and if $1\le i<j\le \ell$ we say that x_i is an *ancestor* of x_j in P and that x_j is an *descendant* of x_i in P. We say that an ordered pair of vertices (x,y) is *k-connected* in a tournament T if the removal of any set $S\subseteq V(T)\setminus\{x,y\}$ of fewer than k vertices from T results in a tournament containing a directed path from x to y. A tournament T is called *k-linked* if $|T|\ge 2k$ and whenever $x_1,\dots,x_k,y_1,\dots,y_k$ are $2k$ distinct vertices in $V(G)$ there exist vertex-disjoint paths $P_1,\dots,P_k$ such that P_i is a directed path from x_i to y_i for each $i\in\{1,\dots,k\}$. For clarity we may sometimes refer to a strongly connected tournament as a strongly 1-connected tournament. Throughout the paper we write $\log x$ to mean $\log_2 x$.

We now collect some preliminary results that will prove useful to us. The following proposition follows straightforwardly from the definition of linkedness.

Proposition 2.1. *Let $k\in\mathbb{N}$. Then a tournament T is k-linked if and only if $|T|\ge 2k$ and whenever $(x_1,y_1),\dots,(x_k,y_k)$ are ordered pairs of (not necessarily distinct) vertices of T, there exist distinct internally vertex-disjoint paths $P_1,\dots,P_k$ such that for all $i\in\{1,\dots,k\}$ we have that P_i is a directed path from x_i to y_i and that $\{x_1,\dots,x_k,y_1,\dots,y_k\}\cap V(P_i)=\{x_i,y_i\}$.*

Proposition 2.2. *Let $k,s\in\mathbb{N}$ and let T be a ks-linked tournament. Let $(x_1,y_1),\dots,(x_k,y_k)$ be ordered pairs of (not necessarily distinct) vertices of T. Then there exist distinct internally vertex-disjoint paths $P_1,\dots,P_k$ such that for all $i\in\{1,\dots,k\}$ we have that P_i is a directed path from x_i to y_i with $\{x_1,\dots,x_k,y_1,\dots,y_k\}\cap V(P_i)=\{x_i,y_i\}$ and such that $|\mathrm{Int}(P_1)\cup\cdots\cup\mathrm{Int}(P_k)|\le|T|/s$.*

Proof. By Proposition 2.1 T contains ks distinct internally vertex-disjoint paths $P_1^1,\dots,P_k^s$ such that for all $i\in\{1,\dots,k\}$ and $j\in\{1,\dots,s\}$ we have that P_i^j is a directed path from x_i to y_i and that $\{x_1,\dots,x_k,y_1,\dots,y_k\}\cap V(P_i^j)=\{x_i,y_i\}$. The disjointness of the paths implies that there is a $j\in\{1,\dots,s\}$ with $|\mathrm{Int}(P_1^j)\cup\cdots\cup\mathrm{Int}(P_k^j)|\le|T|/s$. So the result follows by setting $P_i:=P_i^j$ for all $i\in\{1,\dots,k\}$. ∎

We will also use the following theorem from [6] in proving Theorem 1.5.

Theorem 2.3. [6] *For all $k\in\mathbb{N}$ with $k\ge 2$ every strongly $10^4k\log k$-connected tournament is k-linked.*

The following lemma, which we will also use in proving Theorem 1.5, is very similar to Lemma 8.3 in [6]. The proof proceeds by greedily choosing vertices $v_1 = v, v_2, \dots, v_i$ such that the size of their common in-neighbourhood is minimised at each step. We omit the proof since it is almost identical to the one in [6].

Lemma 2.4. *Let T be a tournament, let $v \in V(T)$ and suppose $c \in \mathbb{N}$. Then there exist disjoint sets $A, E \subseteq V(T)$ such that the following properties hold:*

(i) $1 \leq |A| \leq c$ *and $T[A]$ is a transitive tournament with sink v,*
(ii) *either $E = \emptyset$ or E is the common in-neighbourhood of all vertices in A,*
(iii) *A out-dominates $V(T) \setminus (A \cup E)$,*
(iv) $|E| \leq (1/2)^{c-1} d^-_T(v)$.

The next lemma follows immediately from Lemma 2.4 by reversing the orientations of all edges.

Lemma 2.5. *Let T be a tournament, let $v \in V(T)$ and suppose $c \in \mathbb{N}$. Then there exist disjoint sets $B, E \subseteq V(T)$ such that the following properties hold:*

(i) $1 \leq |B| \leq c$ *and $T[B]$ is a transitive tournament with source v,*
(ii) *either $E = \emptyset$ or E is the common out-neighbourhood of all vertices in B,*
(iii) *B in-dominates $V(T) \setminus (B \cup E)$,*
(iv) $|E| \leq (1/2)^{c-1} d^+_T(v)$.

The following well-known observation will be useful in proving the subsequent technical lemma, which is essential to the proof of Theorem 1.5.

Proposition 2.6. *Let $k \in \mathbb{N}$ and let T be a tournament. Then T contains less than $2k$ vertices of out-degree less than k, and T contains less than $2k$ vertices of in-degree less than k.*

We call a non-empty tournament Q a *backwards-transitive path* if we may enumerate the vertices of Q as $q_1, \dots, q_{|Q|}$ such that there is an edge in Q from q_i to q_j if and only if either $j = i+1$ or $i \geq j+2$. The following lemma shows that if a tournament T can be split into vertex-disjoint backwards transitive paths then there exist small (not necessarily disjoint) sets U and W which are 'quickly reachable in a robust way'.

Lemma 2.7. *Let $k, \ell \in \mathbb{N}$ and let T be a tournament on vertex set $V = Q_1 \dot\cup \dots \dot\cup Q_\ell$, with $|Q_j| \geq k+1$ for all $j \in \{1, \dots, \ell\}$. Suppose that, for each $j \in \{1, \dots, \ell\}$, $T[Q_j]$ is a backwards-transitive path. Then there exist sets U, W, U', W' satisfying the following properties:*

- *$U \subseteq U' \subseteq V(T)$ and $W \subseteq W' \subseteq V(T)$,*
- *$|U|,|W| \leq 2k(k+1)$ and $|U'|,|W'| = \ell(k+1)$,*
- *for any set $S \subseteq V(T)$ of size at most $k-1$, and for every vertex v in $V(T)\setminus S$, there exists a directed path (possibly of length 0) in $T[(U' \cup \{v\}) \setminus S]$ from v to a vertex in U and a directed path in $T[(W' \cup \{v\}) \setminus S]$ from a vertex in W to v.*

Proof. We prove only the existence of U, U'; the existence of W, W' follows by a symmetric argument. Let the backwards-transitive paths $T[Q_j]$ have vertices enumerated $q_j^1, \ldots, q_j^{|Q_j|}$ such that there is an edge in $T[Q_j]$ from q_j^a to q_j^b if and only if either $b = a+1$ or $a \geq b+2$. For $i \in \{1, \ldots, k+1\}$ let $T_i := T[\{q_1^i, \ldots, q_\ell^i\}]$. Thus $|T_i| = \ell$. Let $U_i \subseteq V(T_i)$ be a set of $\min\{2k, \ell\}$ vertices of lowest out-degree in T_i, let $U' := V(T_1) \cup \cdots \cup V(T_{k+1})$, and let $U := U_1 \cup \cdots \cup U_{k+1}$. Then clearly $|U| \leq 2k(k+1)$ and $|U'| = \ell(k+1)$. Now suppose $S \subseteq V(T)$ is of size at most $k-1$ and $v \in V(T) \setminus S$. We need to show that there exists a directed path (possibly of length 0) in $T[(U' \cup \{v\}) \setminus S]$ from v to a vertex in U. We consider four cases:

(i) If $v \in U$ then we are clearly done.

(ii) If $v \in V(T_i) \setminus U$ for some $i \in \{1, \ldots, k+1\}$ and $V(T_i) \cap S = \emptyset$, then let $u \in U \cap V(T_i) = U_i$. Since the vertices of each U_i were picked to have minimal out-degree in T_i, we have that $d^+_{T_i}(u) \leq d^+_{T_i}(v)$, so there is an edge in T from either v or one of its out-neighbours in T_i to u. So there is a directed path in T_i of length at most two from v to u and we are done.

(iii) If $v \in V(T_i) \setminus U$ for some $i \in \{1, \ldots, k+1\}$ and $V(T_i) \cap S \neq \emptyset$, then first note that since $v \in V(T_i) \setminus U$, it must be that $\ell = |T_i| > 2k$. Note then that by Proposition 2.6 and our choice of U we have that $d^+_{T_i}(v) \geq k$. Hence, since $|S| \leq k-1$, there is at least one $j \in \{1, \ldots, \ell\}$ such that q_j^i is an out-neighbour of v and such that $Q_j \cap S = \emptyset$. Also since $|S| \leq k-1$, there is some $i' \in \{1, \ldots, k+1\}$ such that $V(T_{i'}) \cap S = \emptyset$. Since $T[Q_j]$ is a backwards-transitive path, there is a directed path in $T[Q_j \cap U']$ from q_j^i to $q_j^{i'}$, and by (i), (ii) there is a directed path (possibly of length 0) in $T_{i'}$ from $q_j^{i'}$ to a vertex in U. So piecing these paths together gives us a directed path P in $T[U' \setminus S]$ from v to U as required. (Indeed, note that P avoids S since both Q_j and $T_{i'}$ avoid S.)

(iv) If $v \in V(T) \setminus U'$ then note that $v = q_j^i$ for some $j \in \{1, \ldots, \ell\}$ and some $i > k+1$. Now since $T[Q_j]$ is a backwards-transitive path, there are edges in T directed from v to each of the vertices $q_j^1, \ldots, q_j^k$. Since $|S| \leq k-1$, there is some $i \in \{1, \ldots, k\}$ such that $q_j^i \notin S$. By (i)–(iii) there is a directed path in $T[U' \setminus S]$ from q^i_j to a vertex in U. So this path together with the edge directed from v to q^i_j is the directed path required.

This covers all cases and we are done. ∎

3. Proof of Theorem 1.5

The purpose of this section is to prove Theorem 1.5. Very briefly, the proof strategy is as follows: suppose for simplicity that $k=t=m=2$. We aim to construct small disjoint out-dominating sets $A_1,\dots,A_4$ (i.e. for every vertex $v \in V(T)$ there is an edge from each A_i to v) so that each A_i induces a transitive subtournament of T. Similarly, we aim to construct small disjoint in-dominating sets B_i. Then for each i we find a short path P_i joining the sink of B_i to the source of A_i, using the assumption of high connectivity. Let $V_1 := D_1 \cup D_2$ and $V_2 := D_3 \cup D_4$, where $D_i := A_i \cup V(P_i) \cup B_i$ for $i=1,\dots,4$.

Now it is easy to check that Theorem 1.5(ii) holds: consider R as in (ii) and delete an arbitrary vertex s from $V_1 \cup R$ to obtain a set W. To prove (ii) we have to show that for any $x,y \in W$ there is a path from x to y in $T[W]$. To see this note that, without loss of generality, W still contains all of D_1 (otherwise we consider D_2 instead). Since B_1 is in-dominating, there is an edge from x to some $b \in B_1$. Similarly, there is an edge from some $a \in A_1$ to y. Since A_1 and B_1 induce transitive tournaments, we can now find a path from b to a in $T[D_1]$ by utilizing P_1 (see Claim 1).

The main problem with this approach is that one cannot quite achieve the above domination property: for every A_i there is a small exceptional set which is not out-dominated by A_i (and similarly for B_i). We overcome this obstacle by using the notion of 'safe' vertices introduced before Claim 2. With this notion, we can still find a short path from an exceptional vertex x to B_i (rather than a single edge).

Proof of Theorem 1.5. Let $x_1,\dots,x_{kt}$ be kt vertices of lowest in-degree in T. Let $y_1,\dots,y_{kt}$ be kt vertices in $V(T) \setminus \{x_1,\dots,x_{kt}\}$ whose out-degree in T is as small as possible. Define

$$\hat{\delta}^-(T) := \min_{v \in V(T) \setminus \{x_1,\dots,x_{kt}\}} d^-_T(v) \quad \text{and} \quad \hat{\delta}^+(T) := \min_{v \in V(T) \setminus \{y_1,\dots,y_{kt}\}} d^+_T(v).$$

Let $c := \lceil \log(32k^2tm) \rceil$. We may repeatedly apply Lemmas 2.4 and 2.5 with parameter c (removing the dominating sets each time) to obtain disjoint sets of vertices $A_1,\dots,A_{kt},B_1,\dots,B_{kt}$ and sets of vertices

$E_{A_1},\dots,E_{A_{kt}},E_{B_1},\dots,E_{B_{kt}}$ satisfying the following properties for all $i \in \{1,\dots,kt\}$, where we write $D := \bigcup_{i=1}^{kt}(A_i \cup B_i)$.

(i) $1 \le |A_i| \le c$ and $T[A_i]$ is a transitive tournament with sink x_i,
(ii) $1 \le |B_i| \le c$ and $T[B_i]$ is a transitive tournament with source y_i,
(iii) either $E_{A_i} = \emptyset$ or E_{A_i} is the common in-neighbourhood of all vertices in A_i,
(iv) either $E_{B_i} = \emptyset$ or E_{B_i} is the common out-neighbourhood of all vertices in B_i,
(v) $T[A_i]$ out-dominates $V(T)\setminus(D \cup E_{A_i})$,
(vi) $T[B_i]$ in-dominates $V(T)\setminus(D \cup E_{B_i})$,
(vii) $|E_{A_i}| \le (1/2)^{c-1}\hat{\delta}^-(T)$,
(viii) $|E_{B_i}| \le (1/2)^{c-1}\hat{\delta}^+(T)$.

For $j \in \{1,\dots,t\}$ define $j^* := \{(j-1)k+1,\dots,(j-1)k+k\}$, define $A_j^* := \bigcup_{i\in j^*} A_i$, and similarly define $B_j^* := \bigcup_{i \in j^*} B_i$. Define $E_A := E_{A_1} \cup \cdots \cup E_{A_{kt}}$ and $E_B := E_{B_1} \cup \cdots \cup E_{B_{kt}}$. Finally define $E := E_A \cup E_B$. Note that

$$|E_A| \le kt\left(\frac{1}{2}\right)^{c-1}\hat{\delta}^-(T) \le \frac{1}{16km}\hat{\delta}^-(T), \tag{3.1}$$

by our choice of c. Similarly, $|E_B| \le \hat{\delta}^+(T)/(16km)$.

For the remainder of the proof we will assume that $|E_A| \le |E_B|$. The case $|E_A| > |E_B|$ follows by a symmetric argument. Note then that

$$|E| \le |E_A| + |E_B| \le 2|E_B| \le \hat{\delta}^+(T)/(8km). \tag{3.2}$$

Our aim is to use the dominating sets A_i, B_i to construct the sets V_i required. Roughly speaking, for each $i \in \{1,\dots,kt\}$ our aim is to use the high connectivity of T in order to find vertex-disjoint paths P_i in $T-D$ directed from the sink of B_i to the source of A_i. We will then form disjoint vertex sets $V_1,\dots,V_t$ with

$$A_j^* \cup B_j^* \cup \bigcup_{i\in j^*} V(P_i) \subseteq V_j. \tag{3.3}$$

Claim 1. *Suppose that $j \in \{1,\dots,t\}$ and that $V_j \subset V(T)$ satisfies (3.3). Then for any pair of vertices $x \in V(T)\setminus(D\cup E_B)$ and $y \in V(T)\setminus(D \cup E_A)$, the ordered pair (x,y) is k-connected in $T[V_j \cup \{x,y\}]$.*

Indeed, if we delete an arbitrary set $S \subset V_j \setminus \{x,y\}$ of at most $k-1$ vertices then there is some $i \in j^*$ such that $S \cap (A_i \cup B_i \cup V(P_i)) = \emptyset$. So there is an edge from x to some vertex $b \in B_i$ (since B_i is in-dominating and $x \notin D \cup E_{B_i}$) and an edge from b to the sink of B_i (if b is not the sink of B_i); and similarly there is an edge from some vertex $a \in A_i$ to y and an edge from the source of A_i to a (if a is not the source of A_i). Then these at most four edges together with P_i form a directed walk from x to y in $T[(V_j \setminus S) \cup \{x,y\}]$, which we can shorten if necessary to find a directed path from x to y in $T[(V_j \setminus S) \cup \{x,y\}]$, as required.

Claim 1 is a step towards constructing sets V_j as required in Theorem 1.5. However note that this construction so far ignores the problem of finding paths to or from the (relatively few) vertices in $D \cup E$ (in order to satisfy Theorem 1.5(ii)), and the problem of controlling the sizes of the vertex sets $V_1, \dots, V_t$ (in order to satisfy Theorem 1.5(i)). To address the former problem we will introduce the notion of 'safe' vertices and will construct the sets $V_1, \dots, V_t$ (which will eventually satisfy (3.3)) in several steps.

We will colour some vertices of $V(T)$ with colours in $\{1, \dots, t\}$, and at each step V_j will consist of all vertices of colour j. At each step we will call a vertex v in V_j *forwards-safe* if for any set $S \not\ni v$ of at most $k-1$ vertices, there is a directed path (possibly of length 0) in $T[V_j \setminus S]$ from v to $V_j \setminus (D \cup E_B \cup S)$. Similarly we will call a vertex v in V_j *backwards-safe* if for any set $S \not\ni v$ of at most $k-1$ vertices, there is a directed path (possibly of length 0) in $T[V_j \setminus S]$ to v from $V_j \setminus (D \cup E_A \cup S)$. We call a vertex *safe* if it is both forwards-safe and backwards-safe. We also call any vertex in $V(T) \setminus (V' \cup E)$ safe, where $V' := \bigcup_{j=1}^{t} V_j$. Note that the following properties are satisfied at every step:

- all vertices outside $D \cup E$ are safe,
- all vertices in $V' \setminus (D \cup E_B)$ are forwards-safe and all vertices in $V' \setminus (D \cup E_A)$ are backwards-safe,
- if $v \in V_j$ has at least k forwards-safe out-neighbours then v itself is forwards-safe; the analogue holds if v has at least k backwards-safe in-neighbours,
- if $v \in V_j$ is safe and in the next step we enlarge V_j by colouring some more (previously uncoloured) vertices with colour j then v is still safe.

Our aim is to first colour the vertices in D as well as some additional vertices in such a way as to make all coloured vertices safe (see Claim 3). We will then choose the paths P_i and colour the vertices on these paths, as well as some additional vertices, in such a way as to make all coloured vertices safe (see Claim 4). Finally we will colour all those vertices in E which are not coloured yet, as well as some additional vertices, in such a way as to make all coloured vertices safe (see Claim 5). The sets $V_1, \dots, V_t$ thus obtained will satisfy (3.3) and all vertices of T will be safe. So the next claim will then imply that the sets $V_1, \dots, V_t$ satisfy Theorem 1.5(ii). In order to ensure that

Theorem 1.5(i) holds as well, we will ensure that in each step we do not colour too many vertices.

Claim 2. *Suppose that $V_1,\dots,V_t$ satisfy (3.3) and that $j\in\{1,\dots,t\}$. Then for any pair of vertices $x,y\in V_j\cup(V(T)\setminus V')$ that are both safe, the ordered pair (x,y) is k-connected in $T[V_j\cup\{x,y\}]$.*

This is immediate from the definitions and Claim 1.

So our goal is to modify our construction so as to ensure that $V_1,\dots,V_t$ satisfy (3.3) and that every vertex in $V(T)$ is safe. We start with no vertices of T coloured, and we now begin to colour them. We first colour the vertices in $D=\bigcup_{j=1}^{t}(A_j^*\cup B_j^*)$ by giving every vertex in $A_j^*\cup B_j^*$ colour j. We now wish to ensure that every vertex in D is safe.

Claim 3. *We can colour some additional vertices of T in such a way that every coloured vertex is safe, and at most*

$$(k+1)^2(2ktc+4k^2t) \tag{3.4}$$

vertices are coloured in total.

To prove Claim 3 first note that, since T is by assumption strongly $10^7k^6t^2m\log(ktm)$-connected, it certainly holds that

$$\delta^0(T)\geq 10^7k^6t^2m\log(ktm). \tag{3.5}$$

Hence

$$\hat{\delta}^-(T)-|E_A| \overset{(3.1)}{\geq} \hat{\delta}^-(T)/2\geq \delta^0(T)/2 \overset{(3.5)}{\geq} 10^6k^6t^2m\log(ktm), \tag{3.6}$$

and similarly

$$\hat{\delta}^+(T)-|E| \overset{(3.2)}{\geq} \hat{\delta}^+(T)/2\geq \delta^0(T)/2 \overset{(3.5)}{\geq} 10^6k^6t^2m\log(ktm). \tag{3.7}$$

Since $|D|\leq 2ktc$, (3.5) implies that for each $v\in\{x_1,\dots,x_{kt},y_1,\dots,y_{kt}\}$ in turn we may greedily choose k uncoloured in-neighbours and k uncoloured out-neighbours, all distinct from each other, and colour them the same colour as v. Now the number of coloured vertices is at most $2ktc+4k^2t$. So we may greedily choose, for each coloured vertex v not in $\{x_1,\dots,x_{kt},y_1,\dots,y_{kt}\}$ in turn, k distinct uncoloured in-neighbours not in E_A, and colour them the same colour as v. Indeed, this is possible since by (3.6) the number of in-neighbours of v outside E_A is at least $(k+1)(2ktc+4k^2t)$. Now the number of coloured vertices is at most $(k+1)(2ktc+4k^2t)$, so by (3.7) we may greedily choose, for each coloured vertex v not in $\{x_1,\dots,x_{kt},y_1,\dots,y_{kt}\}$ in turn, k distinct uncoloured out-neighbours not in E, and colour them the same colour as v. Note that the number of coloured vertices is now at most $(k+1)^2(2ktc+4k^2t)$ and that every coloured vertex is safe, by construction.

We now wish to find the paths P_i discussed earlier and colour the vertices on these paths appropriately. For $i\in\{1,\dots,kt\}$ we define an *i-path* to be a directed path from the sink of B_i to the source of A_i.

Claim 4. *For every $j\in\{1,\dots,t\}$ and every $i\in j^*$ there exists an i-path P_i in T with previously uncoloured internal vertices, such that all such paths are vertex-disjoint from each other. Moreover, we can colour the internal vertices of P_i with colour j as well as colouring some additional (previously uncoloured) vertices of T in such a way that every coloured vertex is safe, and at most*

$$67k^4t^2\log m+n/(2m) \tag{3.8}$$

vertices are coloured in total.

We will prove Claim 4 in a series of subclaims. The paths P_i that we construct for Claim 4 will be either 'short' or 'long'; we deal with these two cases separately. Firstly, for every $j\in\{1,\dots,t\}$ and every $i\in j^*$ in turn we choose, if possible, an i-path of length at most $k+1$ with uncoloured internal vertices, vertex-disjoint from all previously chosen paths. For each $i\in\{1,\dots,kt\}$ for which we find such a path, let P_i be that path. Let $\mathcal{P}_{short}$ be the set of paths P_i of length at most $k+1$ found in this way, let $\mathcal{I}_{short}:=\{i\in\{1,\dots,kt\}:\mathcal{P}_{short}$ contains an i-path$\}$, and let $\mathcal{I}_{long}:=\{1,\dots,kt\}\setminus\mathcal{I}_{short}$. We colour the internal vertices of each i-path in $\mathcal{P}_{short}$ with colour j (where j is such that $i\in j^*$). Note that since some of these vertices may be in E, it is important that we ensure that they are safe.

Claim 4.1. *We may colour some (previously uncoloured) vertices of T in such a way that all coloured vertices are safe, and at most*

$$54k^4t^2\log m \tag{3.9}$$

vertices are coloured in total. In particular we can ensure that the internal vertices of all paths in $\mathcal{P}_{short}$ are safe.

We do this (similarly to before) as follows. By (3.4) the number of coloured vertices after colouring the short paths is at most $(k+1)^2(2ktc+4k^2t)+k^2t$, so by (3.6) we may greedily choose, for every path in $\mathcal{P}_{short}$

and every internal vertex v on that path in turn, k distinct uncoloured in-neighbours not in E_A, and colour them the same colour as v. (Note that $v \notin \{x_1,\dots,x_t,y_1,\dots,y_t\}$ since all the paths in $\mathcal{P}_{short}$ had uncoloured internal vertices when we chose them.) Now the number of coloured vertices is at most $(k+1)^2(2ktc+4k^2t)+(k+1)k^2t$, so by (3.7) we may greedily choose, for every path in $\mathcal{P}_{short}$ and every internal vertex v on that path, as well as the k in-neighbours of v just chosen, in turn, k distinct uncoloured out-neighbours not in E, and colour them the same colour as v. Note that the number of coloured vertices is now at most

$$(k+1)^2(2ktc+4k^2t)+(k+1)^2k^2t \le 54k^4t^2\log m$$

and that every coloured vertex is safe, by construction.

Now we must find i-paths P_i for all $i \in \mathcal{I}_{long}$; note that they will all be of length at least $k+2$. Initially, for every $j \in \{1,\dots,t\}$ and every $i \in j^* \cap \mathcal{I}_{long}$ we will in fact seek $13k^4t$ distinct internally vertex-disjoint i-paths with uncoloured internal vertices, such that for every $i' \in \mathcal{I}_{long} \setminus \{i\}$, all i-paths are vertex-disjoint from all i'-paths. We seek so many such paths because complications later in the proof may require us to colour some vertices in some of the i-paths with $i \in j^* \cap \mathcal{I}_{long}$ a colour other than j, so some spare paths are necessary. It is also important that we control the sizes of these paths so that we are able to control the sizes of the vertex sets $V_1,\dots,V_t$.

Claim 4.2. *For every $i \in \mathcal{I}_{long}$ we can find a set $\mathcal{P}_{i,long}$ of $13k^4t$ distinct internally vertex-disjoint i-paths with uncoloured internal vertices, such that for every $i' \in \mathcal{I}_{long} \setminus \{i\}$, all paths in $\mathcal{P}_{i,long}$ are vertex-disjoint from all paths in $\mathcal{P}_{i',long}$. Moreover, we may choose the sets $\mathcal{P}_{i,long}$ such that the total number of internal vertices on the paths in $\bigcup_{i\in\mathcal{I}_{long}} \mathcal{P}_{i,long}$ is at most $n/(2m)$.*

Indeed, consider the tournament T' induced on T by the uncoloured vertices as well as the sinks of B_i and the sources of A_i, for every $i \in \mathcal{I}_{long}$. By assumption T is strongly $10^7k^6t^2m\log(ktm)$-connected, so by (3.9) T' is certainly strongly $2.6\times10^5k^5t^2m\log(26k^5t^2m)$-connected. So by Theorem 2.3 T' is $26k^5t^2m$-linked. So since $|\mathcal{I}_{long}| \le kt$, Proposition 2.2 implies that we may find, for each $i \in \mathcal{I}_{long}$, the $13k^4t$ i-paths required, and we may do so in such a way that the total number of internal vertices on these paths is at most $|V(T')|/(2m) \le n/(2m)$, as required.

For each $i \in \mathcal{I}_{long}$, we obtain from each of the paths in $\mathcal{P}_{i,long}$ a possibly shorter path by deleting from the path any vertex v such that there is an edge in T directed from an ancestor of v in the path to a descendant of v in the path. We replace each of the paths in $\mathcal{P}_{i,long}$ by the corresponding shorter path obtained. Note that this ensures that each of the paths in $\mathcal{P}_{i,long}$ is now a backwards-transitive path of length at least $k+2$. As before, it is important that we now ensure that the internal vertices on these paths are coloured in such a way as to be safe, while also colouring them in accordance with the requirements of Claim 4; we do this as follows.

Claim 4.3. *For every $j \in \{1,\dots,t\}$ and every $i \in j^* \cap \mathcal{I}_{long}$ we may colour the internal vertices of all paths in $\mathcal{P}_{i,long}$ as well as some additional (previously uncoloured) vertices of T in such a way that every coloured vertex is safe and at least one path P_i in $\mathcal{P}_{i,long}$ has all vertices coloured with colour j. Moreover, we can do this so that at most*

$$67k^4t^2 \log m + n/(2m) \tag{3.10}$$

vertices are coloured in total.

Indeed, for each $j \in \{1,\dots,t\}$ consider the tournament induced on T by the set of all interior vertices of all paths in $\mathcal{P}_{i,long}$ for all $i \in j^* \cap \mathcal{I}_{long}$. Note that this tournament satisfies the assumptions of Lemma 2.7 (with $13k^4t \cdot |j^* \cap \mathcal{I}_{long}|$ playing the role of ℓ) since each of the paths in each of the sets $\mathcal{P}_{i,long}$ is a backwards-transitive path of length at least $k+2$. So consider the sets U, W each of size at most $2k(k+1)$ and the sets U', W' each of size at most $13k^5t(k+1)$ given by Lemma 2.7. Let us call them U_j, W_j, U'_j, W'_j respectively. By the properties of U_j, W_j, U'_j, W'_j and the definitions of forwards-safe and backwards-safe, it is clear that if every vertex in U'_j is coloured j and every vertex in U_j is forwards-safe, and every vertex in W'_j is coloured j and every vertex in W_j is backwards-safe, then for all $i \in j^* \cap \mathcal{I}_{long}$ every vertex on paths in $\mathcal{P}_{i,long}$ that is coloured j will be safe. So for each $j \in \{1,\dots,t\}$ we colour all vertices in $U'_j \cup W'_j$ with colour j, and we now aim to make every vertex in U_j forwards-safe and every vertex in W_j backwards-safe; we accomplish this (similarly to the way we have made vertices safe before) as follows. By (3.9) the number of coloured vertices is at most $54k^4t^2 \log m + 26k^5t^2(k+1)$, so by (3.6) we may greedily choose, for every $j \in \{1,\dots,t\}$ and for each vertex in W_j in turn, k distinct uncoloured in-neighbours not in E_A, and colour them j. Now, the number of coloured vertices is at most $54k^4t^2 \log m + 26k^5t^2(k+1) + 2k^2(k+1)t$, so by (3.7) we may greedily choose, for every $j \in \{1,\dots,t\}$ and for each vertex in U_j and each of the k in-neighbours of each of the vertices in W_j just chosen in turn, k distinct uncoloured out-neighbours not in E, and colour them j. Let Z be the set of all those vertices that we have just coloured to make all vertices in each U_j forwards-safe and all vertices in each W_j backwards-safe. Note that $|Z| \leq 2k^2(k+1)t + k(2k(k+1)t + 2k^2(k+1)t) < 13k^4t$.

Note also that some of the vertices in Z may be contained in some of the paths in $\mathcal{P}_{i,long}$ for some $i \in \mathcal{I}_{long}$; this is the reason for which we found spare paths. For each $i \in \mathcal{I}_{long}$, since $|\mathcal{P}_{i,long}| = 13k^4t$, there is at least one path in $\mathcal{P}_{i,long}$ that contains no vertices in Z; let P_i be one such path. Colour any uncoloured vertices remaining in paths in the sets $\mathcal{P}_{i,long}$ with colour j, where j is such that $i \in j^*$. In particular the vertices of P_i all have colour j. So we have now found our paths P_i for all $i \in \mathcal{I}_{long}$, and every coloured vertex is safe by construction. Also note that the number of coloured vertices is now at most

$$54k^4t^2 \log m + 13k^4t + n/(2m) \le 67k^4t^2 \log m + n/(2m),$$

as required for Claim 4.3.

This completes the proof of Claim 4. ∎

Now that we have built all of the structure required, it remains for us to colour the uncoloured vertices in E in such a way as to ensure that they are safe. This is essential as, recalling the definition, uncoloured vertices in E are not safe.

Claim 5. *We can colour the uncoloured vertices in E as well as some additional (previously uncoloured) vertices of T in such a way that every coloured vertex is safe, and at most n/m vertices are coloured in total.*

In order to prove Claim 5 we colour all the uncoloured vertices $v \in E$ by distinguishing three cases. We first colour all uncoloured vertices $v \in E$ which satisfy the assumptions of Case 1, then we colour all uncoloured vertices $v \in E$ which satisfy the assumptions of Case 2, and then we colour all uncoloured vertices $v \in E$ which satisfy the assumptions of Case 3.

Case 1. *There exist (not necessarily distinct) $j_1, j_2 \in \{1, \dots, t\}$ such that*

$$|\{i \in j_1^* : v \in E_{A_i}\}| \le |\{i \in j_1^* : v \in E_{B_i}\}|$$

and

$$|\{i \in j_2^* : v \in E_{A_i}\}| \ge |\{i \in j_2^* : v \in E_{B_i}\}|.$$

Note that by (3.2) it certainly holds that $|E| \le n/(8km)$. So by (3.8) the number of uncoloured vertices not in E is at least

$$n\left(1 - \frac{1}{2m} - \frac{1}{8km}\right) - 67k^4t^2 \log m \ge n - \frac{3n}{4m}. \tag{3.11}$$

Either there are k such vertices that are all out-neighbours of v, or there are not, in which case there must be k such vertices that are all in-neighbours of v.

Case 1.1. If v has k uncoloured out-neighbours not in E, we colour them and v with colour j_1. This ensures that v is forwards-safe. To see that v is backwards-safe too, note that if $v \notin E_{A_i}$ then there is an edge in T directed to v from a (safe) vertex in A_i, but similarly that if $v \in E_{B_i}$ then there is an edge in T directed to v from a (safe) vertex in B_i. Together with our assumption that $|\{i \in j_1^* \colon v \in E_{A_i}\}| \leq |\{i \in j_1^* \colon v \in E_{B_i}\}|$ this ensures that v has k safe in-neighbours of its colour. So v is backwards-safe.

Case 1.2. If v does not have k uncoloured out-neighbours outside E then v must have k uncoloured in-neighbours not in E; we colour them and v with colour j_2. This ensures that v is backwards-safe. To see that v is forwards-safe too, note that if $v \notin E_{B_i}$ then there is an edge in T directed from v to a (safe) vertex in B_i, but similarly that if $v \in E_{A_i}$ then there is an edge in T directed from v to a (safe) vertex in A_i. Together with our assumption that $|\{i \in j_2^* \colon v \in E_{A_i}\}| \geq |\{i \in j_2^* \colon v \in E_{B_i}\}|$ this ensures that v has k safe out-neighbours of its colour. So v is forwards-safe.

By (3.11) we can repeat this process greedily for all vertices $v \in E$ which satisfy the assumptions of Case 1. Note that after this step all coloured vertices are safe.

Case 2. *For all* $j \in \{1, \ldots, t\}$ *it holds that*

$$|\{i \in j^* \colon v \in E_{A_i}\}| < |\{i \in j^* \colon v \in E_{B_i}\}|.$$

We consider two sub-cases:

Case 2.1. If v has k uncoloured out-neighbours not in E then colour them and v with colour 1.

Case 2.2. Otherwise, since (3.7) implies that $\hat{\delta}^+(T) \geq kt + k + |E|$, an averaging argument shows that there is some $j \in \{1, \ldots, t\}$ such that v has k out-neighbours of colour j (recall that all currently coloured vertices are safe), in which case we colour v with colour j.

In either case it is clear that v is now forwards-safe. A similar argument as in Case 1.1 shows that v is backwards-safe too.

Case 3. *For all* $j \in \{1, \ldots, t\}$ *it holds that*

$$|\{i \in j^* \colon v \in E_{A_i}\}| > |\{i \in j^* \colon v \in E_{B_i}\}|.$$

We consider two sub-cases:

Case 3.1. If v has k uncoloured in-neighbours not in E_A then colour them and v with colour 1. (Note that none of these in-neighbours w can lie in E_B. Indeed, if $w \in E_B$ then w satisfies the assumptions of one of the first two cases (as $w \notin E_A$ implies $|\{i \in j^* : v \in E_{A_i}\}| = 0$) and so w would have already been coloured.)

Case 3.2. Otherwise, since (3.6) implies that $\hat{\delta}^-(T) \geq kt + k + |E_A|$, an averaging argument shows that there is some $j \in \{1,\dots,t\}$ such that v has k in-neighbours of colour j (recall that all currently coloured vertices are safe), in which case we colour v with colour j.

In either case it is clear that v is now backwards-safe. Again, a similar argument as in Case 1.2 shows that v is forwards-safe too.

This covers all cases, so we have now coloured all vertices in E in such a way that all coloured vertices are safe. Note that for each of the at most $|E| \leq n/(8mk)$ vertices in E that were uncoloured at the start of the proof of Claim 5 we have coloured at most k (previously uncoloured) vertices not in E in this step. So by (3.11) the total number of coloured vertices is at most $3n/(4m) + (k+1)|E| \leq n/m$, as required.

Now the only uncoloured vertices remaining are not in E and so they are safe. So all vertices in T are now safe. This completes the construction of the vertex sets required, where the colour classes of colours $1,\dots,t$ correspond to the vertex sets $V_1,\dots,V_t$ respectively. Since the number of coloured vertices is at most n/m, the size of each V_j is certainly at most n/m. And since we have ensured that every vertex in T is safe, Claim 2 implies that the V_j satisfy the requirements of Theorem 1.5. ∎

4. Partitioning tournaments into vertex-disjoint cycles

The purpose of this section is to derive Theorem 1.7 from Theorem 1.5.

Proof of Theorem 1.7. Note that by averaging there is at least one value $j \in \{1,\dots,t\}$ for which $L_j \geq n/t$. Without loss of generality let $L_1 \geq n/t$. Let $\tilde{J} := \{j \in \{1,\dots,t\} : L_j < n/(2t^2)\}$. For $j \in \tilde{J}$ let $L'_j := \lceil n/t^2 \rceil$. For $j \in \{2,\dots,t\} \setminus \tilde{J}$ let $L'_j := L_j$. Let $L'_1 := L_1 - \sum_{j=2}^t (L'_j - L_j)$. Note that $L'_1 \geq n/t^2$ and that $\sum_{j=1}^t L'_j = n$.

Since $10^{10} t^4 \log t \geq 10^7 2^6 t^2 (2t^2) \log(2t(2t^2))$, we have by Theorem 1.5 that $V(T)$ contains t disjoint sets of vertices, $V_1,\dots,V_t$, such that for every $j \in \{1,\dots,t\}$ the following hold:

(i) $|V_j| \leq n/(2t^2)$,

(ii) for any set $R \subseteq V(T) \setminus \bigcup_{i=1}^{t} V_i$ the subtournament $T[V_j \cup R]$ is strongly 2-connected.

Construct a partition $V'_1, \dots, V'_t$ of the vertices of T, such that for every $j \in \{1, \dots, t\}$ it holds that $V_j \subseteq V'_j$ and that $|V'_j| = L'_j$. This is possible, since for every $j \in \{1, \dots, t\}$ we have $L'_j \geq n/(2t^2) \geq |V_j|$. Note that, for every $j \in \{1, \dots, t\}$, $T[V'_j]$ is strongly 2-connected.

Now, since $n/t^2 > 7$, we have by Theorem 1.6 that for each $j \in \tilde{J}$, $T[V'_j]$ contains two vertex-disjoint cycles of lengths L_j and $L'_j - L_j$. The cycle of length L_j we call C_j and the cycle of length $L'_j - L_j$ we call C'_j. Since for every $j \in \tilde{J}$ we have that $|C'_j| = L'_j - L_j > n/2t^2 \geq |V_j|$, there is at least one vertex in $V(C'_j) \cap (V'_j \setminus V_j)$. Call one such vertex v_j. Let R be the set of all vertices v_j for $j \in \tilde{J}$.

Now let $V''_1 := V'_1 \cup \bigcup_{j \in \tilde{J}} V(C'_j)$. Note that $|V''_1| = L_1$. Note also that (ii) implies that $T[V'_1 \cup R]$ is strongly 2-connected; so certainly it is strongly 1-connected. We now claim that $T[V''_1]$ is strongly 1-connected. Indeed, suppose $x, y \in V''_1$, and we wish to find a path directed from x to y in $T[V''_1]$. First note that if $x \notin V'_1$ then $x \in V(C'_j)$ for some $j \in \tilde{J}$, so there is a path Q_j in $T[V(C'_j)]$, possibly of length 0, from x to $v_j \in R$. Similarly note that if $y \notin V'_1$ then $y \in V(C'_i)$ for some $i \in \tilde{J}$, so there is a path Q'_i in $T[V(C'_i)]$, possibly of length 0, to y from $v_i \in R$. Since $T[V'_1 \cup R]$ is strongly 1-connected there exists a path P in $T[V'_1 \cup R]$ directed from v_j to v_i. So $Q_j P Q'_i$ is a walk in $T[V''_1]$ directed from x to y. So indeed $T[V''_1]$ is strongly 1-connected.

Note also that for every $j \in \{2, \dots, t\} \setminus \tilde{J}$ we have that $T[V'_j]$ is strongly 2-connected, so certainly strongly 1-connected. So by Camion's theorem $T[V''_1]$ contains a Hamilton cycle, C_1 say, and for every $j \in \{2, \dots, t\} \setminus \tilde{J}$ we have that $T[V'_j]$ contains a Hamilton cycle, C_j say.

Now the cycles $C_1, \dots, C_t$ are vertex-disjoint and are of lengths $L_1, \dots, L_t$ respectively, so this completes the proof. ∎

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