COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA **35** (6) (2015) 655–668
DOI: 10.1007/s00493-015-2922-4

ON ORDER AND RANK OF GRAPHS

EBRAHIM GHORBANI, ALI MOHAMMADIAN,
BEHRUZ TAYFEH-REZAIE

Received January 12, 2012
Revised February 26, 2014
Online Frist May 21, 2015

The rank of a graph is defined to be the rank of its adjacency matrix. A graph is called reduced if it has no isolated vertices and no two vertices with the same set of neighbors. Akbari, Cameron, and Khosrovshahi conjectured that the number of vertices of every reduced graph of rank r is at most $m(r)=2^{(r+2)/2}-2$ if r is even and $m(r)=5\cdot 2^{(r-3)/2}-2$ if r is odd. In this article, we prove that if the conjecture is not true, then there would be a counterexample of rank at most 46. We also show that every reduced graph of rank r has at most $8m(r)+14$ vertices.

1. Introduction

For a graph G, we denote by $V(G)$ the vertex set of G. The *order* of G is the number of vertices of G and denoted by $|G|$. The *adjacency matrix* of G, denoted by $A(G)$, has its rows and columns indexed by $V(G)$ and its (u,v)-entry is 1 if the vertices u and v are adjacent and 0 otherwise. The *rank* of G, denoted by $\text{rank}(G)$, is the rank of $A(G)$.

The problem of bounding the order of a graph in terms of the rank was first studied by Kotlov and Lovász [9]. Their motivation was to determine the gap between the chromatic number and the rank of graphs originated from the rank-coloring conjecture of van Nuffelen [13]. The conjecture stated that the chromatic number of every graph with at least one edge does not exceed the rank. The first counterexample to the conjecture was obtained by Alon and Seymour [2]. A superlinear gap was found by Razborov [15] and a larger gap was provided by Nisan and Wigderson [12]. This problem, indeed,

Mathematics Subject Classification (2010): 05C50, 15A03 has a close connection with the log-rank conjecture by Lovász and Saks [10] from communication complexity which is equivalent to the statement that the logarithm of the chromatic number of any graph is bounded above by a polylogarithmic function of the rank, see [11].

The order of a graph with rank r is trivially bounded above by 2^r-1 as soon as we make the assumption that the graph is reduced; that is, it has no isolated vertices and no two vertices with the same set of neighbors. In fact, over the two element field this bound is achievable by a unique graph [6]. We only consider the rank of graphs over the field of real numbers. Kotlov and Lovász [9] proved that there exists a constant c such that the order of every reduced graph of rank r is at most $c\cdot 2^{r/2}$. Later on, Akbari, Cameron, and Khosrovshahi [1] made the following conjecture.

Conjecture 1. *For every integer $r \geqslant 2$, the order of any reduced graph of rank r does not exceed $m(r)$, where*

$$m(r)=\begin{cases}2^{(r+2)/2}-2 & \text{if } r \text{ is even,}\\ 5\cdot 2^{(r-3)/2}-2 & \text{if } r \text{ is odd.}\end{cases}$$

They also constructed some reduced graphs of rank r and order $m(r)$, for every integer $r \geqslant 2$. In this article, we show that if Conjecture 1 is not true, then there would be a counterexample of rank at most 46. From our arguments, it also follows that the order of every reduced graph of rank r is at most $8m(r)+14$.

Recently, some relevant results were obtained by a number of authors. Haemers and Peeters [7] proved Conjecture 1 for graphs containing an induced matching of size $r/2$ or an induced subgraph consisting a matching of size $(r-3)/2$ and a cycle of length 3. Royle [16] proved that the rank of every reduced graph containing no path of length 3 as an induced subgraph is equal to the order. In [4,5], we proved that the order of every reduced tree, bipartite graph, and non-bipartite triangle-free graph of rank r is at most $3r/2-1$, $2^{r/2}+r/2-1$, and $3\cdot 2^{\lfloor r/2\rfloor -2}+\lfloor r/2\rfloor$, respectively, and we characterized all the corresponding graphs achieving these bounds.

2. Notation and Preliminaries

For a vertex v of a graph G, let $N_G(v)$ denote the set of all vertices of G adjacent to v. By $\Delta_G(u,v)$ we mean the symmetric difference of $N_G(u)$ and $N_G(v)$. We will drop the subscript G when it is clear from the context. Two vertices u and v of G are called *duplicated vertices* if $N(u)=N(v)$. We say that G is *reduced* if it has no isolated vertex and no duplicated vertices. A subset S of $V(G)$ with $|S|>1$ is called a *duplication class* of G if $N(u)=N(v)$ for any $u,v\in S$. For a subset X of $V(G)$, $\langle X\rangle$ and $G-X$ represent the induced subgraphs of G on X and on $V(G)\setminus X$, respectively. We use the same notation if X is a subgraph of G. For a vertex $v\in V(G)$, we write $G-v$ for $G-\{v\}$. For a matrix M, we denote by $\mathrm{row}(M)$ the vector space generated by the row vectors of M over the field of real numbers. We use the notation $\boldsymbol{j}_k$ and $J_{r\times s}$ for the all one vector of length k and the $r\times s$ all one matrix, respectively. The complete graph of order n is denoted by K_n. For a graph G with at least one edge, let $\rho(G)$ denote the minimum number of vertices whose removal results in a graph with a smaller rank. If G is not a complete graph, then we denote by $\tau(G)$ the minimum number of vertices whose removal results in a graph with duplicated vertices.

Lemma 2. [8,9] *For any reduced graph G, the following hold.*

(i) *For every $v\in V(G)$, $\mathrm{rank}(G-N(v))\leqslant\mathrm{rank}(G)-2$.*
(ii) *For every adjacent vertices $u,v\in V(G)$, $\mathrm{rank}(G-\Delta(u,v))\leqslant\mathrm{rank}(G)-1$.*
(iii) *For every non-adjacent vertices $u,v\in V(G)$, $\mathrm{rank}(G-\Delta(u,v))\leqslant\mathrm{rank}(G)-2$.*
(iv) *If H is an induced subgraph of G with $|H|=|G|-\rho(G)$ and $\mathrm{rank}(H)<\mathrm{rank}(G)$, then $\mathrm{rank}(H)\geqslant\mathrm{rank}(G)-2$ and the equality occurs whenever H is not reduced.*

Corollary 3. *For any reduced graph G,*

$$\begin{aligned}\rho(G)&\leqslant\tau(G)\\&=\min\big\{|\Delta(u,v)|\,\big|\,u \text{ and } v \text{ are distinct non-adjacent vertices of } G\big\}.\end{aligned}$$

The following lemma which has a key role in our proofs is inspired from [9].

Lemma 4. *Let G be a reduced graph and let H be an induced subgraph of G with the maximum possible order subject to that H has duplicated vertices. Let $\mathrm{rank}(H)\geqslant\mathrm{rank}(G)-3$. Then the following properties hold.*

(i) *If c is an isolated vertex of H, then $N(c)=V(G-H)$.*
(ii) *Every duplication class of H has two elements and H has at most one isolated vertex.*
(iii) *One may label the duplication classes of H as $\{v_1,v_1'\},\dots,\{v_s,v_s'\}$ so that there exist two disjoint sets T_1 and T_2 such that $V(G-H)=T_1\cup T_2$, $T_1\subseteq N(v_i)\setminus N(v_i')$ and $T_2\subseteq N(v_i')\setminus N(v_i)$, for all $i\in\{1,\dots,s\}$.*
(iv) *If both T_1 and T_2 are non-empty, then H has no isolated vertex.*

Proof. For (i), suppose that $X = V(G)\setminus(V(H)\cup N(c))$ is non-empty. Let $K = G - N(c)$. If u and v are duplicated vertices of H, then by the definition of H, we find that $V(G-H) = \varDelta_G(u,v)$ and so $X = \varDelta_K(u,v)$. Therefore, Lemma 2 implies that $\operatorname{rank}(H) \leqslant \operatorname{rank}(K) - 2 \leqslant \operatorname{rank}(G) - 4$, a contradiction. For (ii), if H has a duplication class containing three distinct vertices x, y, z, then for every vertex $t \in V(G-H)$, at least one of $\varDelta(x,y)$, $\varDelta(x,z)$, $\varDelta(y,z)$ does not contain t. This contradicts the maximality of H. The second part of (ii) follows from (i). For (iii), note that, by the definition of H, every vertex in $V(G-H)$ is adjacent to exactly one vertex in each duplication class. If (iii) does not hold, then $A(G)$ contains

$$\left[\begin{array}{c|c|c|c|c|cc} & & & & & 1 & 1\\ & & & & & 0 & 0\\ \boldsymbol{x} & \boldsymbol{x} & \boldsymbol{y} & \boldsymbol{y} & \star & 1 & 0\\ & & & & & 0 & 1\\ & & & & & \star & \star\\ \hline 1&0&1&0&\star&0&\star\\ 1&0&0&1&\star&\star&0\end{array}\right]$$

as a principle submatrix, where the left-upper corner is $A(H)$. This directly yields that $\operatorname{rank}(H) \leqslant \operatorname{rank}(G) - 4$, a contradiction. For (iv), assume that both T_1 and T_2 are non-empty and H has an isolated vertex. Then, by (i), $A(G)$ contains

$$\left[\begin{array}{c|c|c|c|cc} & & & & 1 & 0\\ \boldsymbol{x} & \boldsymbol{x} & \star & \mathbf{0} & 0 & 1\\ & & & & \star & \star\\ 0 & 0 & \mathbf{0} & 0 & 1 & 1\\ \hline 1&0&\star&1&0&\star\\ 0&1&\star&1&\star&0\end{array}\right]$$

as a principle submatrix, where again the left-upper corner is $A(H)$. This directly implies that $\operatorname{rank}(H) \leqslant \operatorname{rank}(G) - 4$, a contradiction. ∎

Notice that for every integer $r \geqslant 4$, we have $m(r) = 2m(r-2) + 2$. Using this equality, we can prove the following lemma which will be frequently used in the sequel.

Lemma 5. *Let r and k be two positive integers.*

(i) *If $r \geqslant 6$ and $3 \leqslant k \leqslant r-3$, then $m(k) + m(r-k) \leqslant m(r-2) + 1$.*
(ii) *If $r \geqslant 10$ and $4 \leqslant k \leqslant r-3$, then $m(k) + m(r-k+1) \leqslant m(r-2)$.*

Proof. For (i), we prove the statement by induction on r. For $r \in \{6,7,8,9\}$, (i) can be easily verified. If $k \in \{3,4,r-4,r-3\}$, then the inequality in (i) is clearly true. For $5 \leqslant k \leqslant r-5$, by the induction hypothesis, we have

$$\begin{aligned} m(k)+m(r-k) &= 2m(k-2)+2m\big(r-4-(k-2)\big)+4 \\ &\leqslant 2m(r-6)+6 \\ &= m(r-4)+4 \\ &< m(r-2)+1. \end{aligned}$$

For (ii), note that if $k \in \{4, r-3\}$, then the inequality is clearly valid. If $5 \leqslant k \leqslant r-4$, then using (i), we have

$$\begin{aligned} m(k)+m(r-k+1) &= 2m(k-2)+2m\big(r-3-(k-2)\big)+4 \\ &\leqslant 2m(r-5)+6 \\ &= m(r-3)+4 \\ &\leqslant m(r-2). \end{aligned}$$

∎

3. Spherical codes

In this section, we recall some results on spherical codes. Let n be a positive integer and $\varphi \in (0,\pi]$. An (n,M,φ)-*spherical code* $\mathscr{C}$ is a set of M unit vectors in $\mathbb{R}^n$ for which $\cos^{-1}(\langle \boldsymbol{x},\boldsymbol{y}\rangle) \geqslant \varphi$ for every pair $\boldsymbol{x},\boldsymbol{y} \in \mathscr{C}$, where $\langle\ ,\ \rangle$ indicates the inner product of two vectors. Let $M(n,\varphi)$ denote the maximum possible value M for given n and φ such that an (n,M,φ)-spherical code exists. We proceed to verify the following lemma which is essential in the proof of our main theorem.

Lemma 6. *For every integer* $n \geqslant 47$, $M\big(n,\cos^{-1}(\sqrt{2}-1)\big) < 5\cdot 2^{(n-4)/2}-2$.

The following theorem is due to Rankin [14].

Theorem 7. *Let n be a positive integer and $\varphi \in (0,\pi]$. Then $M(n,\frac{\pi}{2}) = 2n$ and*

$$M(n,\varphi) \leqslant \begin{cases} n+1 & \text{if } \varphi > \frac{\pi}{2}, \\ \dfrac{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)\sin\alpha\tan\alpha}{2\Gamma\left(\frac{n}{2}\right)\int_0^{\alpha}(\sin\theta)^{n-2}(\cos\theta-\cos\alpha)\,d\theta} & \text{if } \varphi < \frac{\pi}{2}, \end{cases}$$

where $\alpha = \sin^{-1}\big(\sqrt{2}\sin\frac{\varphi}{2}\big)$ and Γ denotes the Gamma function.

From [17, p. 97], we have

$$\int_0^\alpha (\sin\theta)^{n-2}(\cos\theta - \cos\alpha)d\theta = \frac{(\sin\alpha)^{n+1}}{(n^2-1)\cos^2\alpha}\left(1 - \frac{3\xi\tan^2\alpha}{n+3}\right), \qquad (1)$$

for some $\xi \in [0,1]$. If $n > \max\{6\tan^2\alpha - 3, 5\}$, then $1 - \frac{3\xi\tan^2\alpha}{n+3} > \frac{1}{2}$ and since Γ is increasing on $[2,+\infty)$, Theorem 7 and (1) yield that

$$M(n,\varphi) < \frac{n^2-1}{(\sin\alpha)^n} = (n^2-1)\left(\sqrt{2}\sin\frac{\varphi}{2}\right)^{-n}. \qquad (2)$$

Let $\varphi_0 = \cos^{-1}(\sqrt{2}-1)$. Then, by (2), we obtain that

$$M(n,\varphi_0) < (n^2-1)\left(1+\frac{1}{\sqrt{2}}\right)^{\frac{n}{2}},$$

for every integer $n > 5$. So, it is now easily checked that $M(n,\varphi_0) < 5 \cdot 2^{(n-4)/2} - 2$, for every integer $n \geqslant 118$.

For smaller values of n, we have to employ another upper bound for $M(n,\varphi)$ given by Levenšteĭn. To present the Levenšteĭn bound, we first recall that the *Gegenbauer polynomials* $Q_0(t), Q_1(t), \ldots$ which are defined by the recurrence relation

$$\begin{cases} Q_0(t) = 1; \\ Q_1(t) = t; \\ Q_{k+1}(t) = \dfrac{(2k+n-2)tQ_k(t) - kQ_{k-1}(t)}{k+n-2}, \text{ for all } k \geqslant 1. \end{cases}$$

Now, let

$$Q_k^{1,0}(t) = \frac{(n-1)\big(Q_k(t) - Q_{k+1}(t)\big)}{(2k+n-1)(1-t)}$$

and

$$Q_k^{1,1}(t) = \frac{(n-1)\big(Q_k(t) - Q_{k+2}(t)\big)}{(2k+n)(1-t^2)}.$$

For every integer $k \geqslant 1$, denote by $t_k^{1,0}$ and $t_k^{1,1}$ the largest zeros of $Q_k^{1,0}(t)$ and $Q_k^{1,1}(t)$, respectively, and let $t_0^{1,1} = -1$. We know from [3, p. 51] that $t_{k-1}^{1,1} < t_k^{1,0} < t_k^{1,1}$, for every integer $k \geqslant 1$, and $\{[t_{k-1}^{1,1}, t_k^{1,1}) \mid k \geqslant 1\}$ is a partition of $[-1,1)$. The following theorem is called the Levenšteĭn bound [3, p. 57].

Theorem 8. *Let $n \geqslant 3$ and $\varphi \in (0, \pi]$. Then*

$$M(n,\varphi) \leqslant \begin{cases} \binom{k+n-3}{k-1}\left(\frac{2k+n-3}{n-1} - \frac{Q_{k-1}(s)-Q_k(s)}{(1-s)Q_k(s)}\right) & \text{if } s \in \left[t_{k-1}^{1,1}, t_k^{1,0}\right), \\ \binom{k+n-2}{k}\left(\frac{2k+n-1}{n-1} - \frac{(1+s)(Q_k(s)-Q_{k+1}(s))}{(1-s)(Q_k(s)+Q_{k+1}(s))}\right) & \text{if } s \in \left[t_k^{1,0}, t_k^{1,1}\right), \end{cases}$$

where $s = \cos\varphi$.

By Theorem 8 and using Maple for computations, we find that $M(n,\varphi_0) < 5 \cdot 2^{(n-4)/2} - 2$, for every integer $47 \leqslant n \leqslant 118$. This discussion completes the proof of Lemma 6.

4. Main Results

In this section, we present our main results. We remark that Conjecture 1 was verified for all graphs of rank at most 8 by computation [1]. We have extended this result to all graphs of rank 9 by a computer search.

Lemma 9. *Let G be a reduced graph of order n and rank $r \geqslant 46$. If $n \geqslant 5 \cdot 2^{(r-3)/2} - 2$, then $\rho(G) < \left(1 - \frac{1}{\sqrt{2}}\right)n$.*

Proof. Suppose that $\rho(G) \geqslant \left(1 - \frac{1}{\sqrt{2}}\right)n$. Let M be the matrix resulting from replacing all 0 by -1 in $A(G)$. Clearly, $\text{rank}(M) \leqslant r+1$ and by Corollary 3,

$$\langle x, y\rangle \leqslant \frac{n - 2\rho(G)}{n} \leqslant \sqrt{2} - 1,$$

for every pair x, y of the row vectors of $\frac{1}{\sqrt{n}}M$. It turns out that there are n vectors in $\mathbb{R}^{r+1}$ where the angle between each pair of them is at least $\cos^{-1}\left(\sqrt{2} - 1\right)$. In view of Lemma 6, we have $n < 5 \cdot 2^{(r-3)/2} - 2$, a contradiction. ∎

Lemma 10. *Let G be a reduced graph of order n and rank $r \geqslant 6$. If $n > m(r)$, then $\rho(G) < n/2$.*

Proof. If $\rho(G) \geqslant n/2$, then $\frac{n-2\rho(G)}{n} \leqslant 0$. This, similar to the proof of Lemma 9, implies the existence of n vectors in $\mathbb{R}^{r+1}$ such that the angle between each pair of which is at least $\frac{\pi}{2}$. From Theorem 7, it follows that $m(r) < n \leqslant 2(r+1)$, which contradicts $r \geqslant 6$. ∎

In what follows, we assume that $\mathbb{G}$ is a counterexample to Conjecture 1 with the minimum possible order. Let $n = |\mathbb{G}|$, $r = \text{rank}(\mathbb{G})$, $\tau = \tau(\mathbb{G})$, and let H be an induced subgraph of $\mathbb{G}$ of order $n-\tau$ with duplicated vertices. If $\mathrm{rank}(H)\geqslant r-3$, then by Lemma 4 (iii), we may assume that $\{v_1,v_1'\},\ldots,\{v_s,v_s'\}$ are the duplication classes of H. For simplicity, let $S=\langle\{v_1,\ldots,v_s\}\rangle$ and $S'=\langle\{v_1',\ldots,v_s'\}\rangle$. Further, put $T=\mathbb{G}-H$ and let T_1 and T_2 be the sets given in Lemma 4 (iii) with sizes t_1 and t_2, respectively. We denote the number of isolated vertices of H by ϵ. Note that by Lemma 4 (ii), $\epsilon\in\{0,1\}$. Finally, we set $P=H-(V(S)\cup V(S')\cup\{c\})$ and $p=|P|$, where c is the possible isolated vertex of H.

Lemma 11. $n=m(r)+1$.

Proof. Let $v\in V(\mathbb{G})$. If $\mathbb{G}-v$ is reduced, then by the minimality of $\mathbb{G}$, we have $|\mathbb{G}-v|\leqslant m(r)$ and so $n=m(r)+1$. If $\mathbb{G}-v$ is not reduced, then either there is a vertex $x\in V(\mathbb{G})$ such that $N(x)=\{v\}$ or there are two non-adjacent vertices $y,y'\in V(\mathbb{G})$ such that $\Delta(y,y')=\{v\}$. Hence, by Lemma 2, $\mathrm{rank}(\mathbb{G}-v)=r-2$. Therefore, Lemma 4 (iii) yields that every duplication class of H has two vertices. Thus $\frac{n}{2}-1\leqslant m(r-2)$. This is a contradiction as $m(r)=2m(r-2)+2$. ∎

Lemma 12. *If* $\tau\leqslant m(r-2)+2$, *then* $\mathrm{rank}(H)\geqslant r-3$.

Proof. Suppose that $\tau\leqslant m(r-2)+2$ and $\mathrm{rank}(H)\leqslant r-4$. Add a vertex from $V(\mathbb{G}-H)$ to H and call the resulting graph K. Obviously, K has no duplicated vertices and $\mathrm{rank}(K)\leqslant r-2$. Thus $n-\tau+1-\epsilon\leqslant m(r-2)$. This implies that $n\leqslant m(r)$, a contradiction. ∎

Theorem 13. *Suppose that* $\mathrm{rank}(H)\geqslant r-3$ *with* $r\geqslant 10$. *Then* $\epsilon=0$ *and one of the following holds.*

(i) $S=K_1$ *and* $\tau\geqslant m(r-2)+2$.
(ii) $S=K_2$ *and* $\tau\geqslant m(r-2)+1$.
(iii) $S=K_3$ *and* $\tau=m(r-2)$.

Proof. We denote the possible isolated vertex of H by c. Also, let $k=\mathrm{rank}(S)$, $K=\langle V(T)\cup V(S)\rangle$ and $K'=\langle V(T)\cup V(S')\rangle$. We first establish the following steps.

Step 1. $s+p\leqslant m(r-2)$, $\tau+s\geqslant m(r-2)+3-\epsilon$, and $\tau\geqslant p+3-\epsilon$.

Applying Lemma 2 (iii), $\mathrm{rank}(H)\leqslant r$–2 and so $\mathrm{rank}(\langle V(S)\cup V(P)\rangle)\leqslant r$–2. By the definitions of S and P, $\langle V(S)\cup V(P)\rangle$ is a reduced graph and thus $s+p\leqslant m(r-2)$. Moreover, $n=m(r)+1$ and $n=\tau+2s+p+\epsilon$ imply that $\tau+s\geqslant m(r-2)+3-\epsilon$. By subtracting these inequalities, we obtain the last inequality.

Step 2. The graph S has no duplication classes.

By contradiction, suppose that there are two vertices $a,b \in S$ with $N_S(a) = N_S(b)$. Hence $\Delta(a,b) \subseteq V(P)$ and by Corollary 3, we obtain that $\tau \leqslant p$, which is a contradiction to Step 1.

Step 3. If S has isolated vertices, then both T_1 and T_2 are non-empty.

By contradiction, assume that v_1 is an isolated vertex of S and T_1 is empty. Thus $N(v_1) \subseteq V(P)$. We show that $\mathbb{G} - (N(v_1) \cup \{v_1\})$ is reduced. If $\mathbb{G} - (N(v_1) \cup \{v_1\})$ has an isolated vertex, say x, then x is not adjacent to v_1 and $\Delta(x,v_1) \subseteq N(v_1)$, and if $\mathbb{G} - (N(v_1) \cup \{v_1\})$ has a duplication class, say $\{y,y'\}$, then $\Delta(y,y') \subseteq N(v_1)$. Since $|N(v_1) \cup \{v_1\}| < p+3-\epsilon \leqslant \tau$, both cases contradict the minimality of τ using Lemma 2. So $\mathbb{G} - (N(v_1) \cup \{v_1\})$ is a reduced graph of order at least $n-p-1$ and rank at most $r-2$. This implies that $p \geqslant m(r-2)+2$, which is a contradiction to Step 1.

Step 4. Every duplication class of T consists of one vertex from T_1 and one from T_2.

Otherwise, without loss of generality, suppose that there are two vertices $a,b \in T_1$ such that $N_T(a) = N_T(b)$. Therefore, $\Delta(a,b) \subseteq V(P)$ and so $\tau \leqslant p$, which is a contradiction to Step 1.

Step 5. $\operatorname{rank}(K) \geqslant r-1$ and $\operatorname{rank}(K') \geqslant r-1$.

We only prove that $\operatorname{rank}(K) \geqslant r-1$. By Step 1, $|K| = \tau+s \geqslant m(r-2)+3-\epsilon$. We show that K has a reduced induced subgraph of order at least $m(r-2)+1$ which in turn implies that $\operatorname{rank}(K) \geqslant r-1$ by the minimality of $\mathbb{G}$. If K has no duplication classes, then K has at most one isolated vertex. Thus, after removing the possible isolated vertex from K, we obtain the desired subgraph. So, assume that K has duplication classes. By applying Steps 2, 3, and 4, it is easily checked that T_1 is non-empty and K has exactly one duplication class which is of the form $\{v_1, x\}$, for some $x \in T_2$. Hence K has at most one isolated vertex. Furthermore, Lemma 4 (iv) implies that $\epsilon = 0$. Now, after removing the possible isolated vertex from $K - v_1$, we obtain the desired subgraph.

Step 6. The graph T has no isolated vertices.

By contradiction, without loss of generality, assume that $N_T(a)$ is empty, for some $a \in T_1$. Then $N(a) \subseteq V(S) \cup V(P) \cup \{c\}$. Since $K' = \mathbb{G} - (V(S) \cup V(P) \cup \{c\})$, we deduce that $\operatorname{rank}(K') \leqslant \operatorname{rank}(\mathbb{G} - N(a)) \leqslant r-2$, which is a contradiction to Step 5.

Step 7. Both T_1 and T_2 are non-empty.

If T_1 is empty, then $\operatorname{rank}(\langle V(T) \cup \{c\} \rangle) + \operatorname{rank}(S) \leqslant r$. By Steps 2, 3, 4, and 6, $\langle V(T) \cup \{c\} \rangle$ and S are reduced graphs. So, Step 1 implies that

$m(r-2)+3 \leqslant \tau+s+\epsilon \leqslant m(r-k)+m(k)$, which contradicts Lemma 5 (i), since $2 \leqslant k \leqslant r-2$. Similarly, we see that T_2 is non-empty.

Step 8. $\epsilon=0$.

It immediately follows from Step 7 and Lemma 4 (iv).

We now proceed with the following cases.

Case 1. Assume that T has a duplication class. We prove that $S=K_2$, $\mathrm{rank}(T)=r-3$, $\tau=m(r-2)+1$, and $p=m(r-2)-2$. Since T has a duplication class, $(\boldsymbol{j}_{t_1},\mathbf{0}) \notin \mathrm{row}(A(T))$. By Step 4, the two row vectors of

$$X=\left[\begin{array}{c|c} A(T) & \begin{array}{c} J_{t_1\times s} \\ \hline O \end{array} \end{array}\right]$$

corresponding to a duplication class of T are linearly independent. Extend these vectors to a basis $\mathcal{B}$ of size $\mathrm{rank}(T)+1$ for $\mathrm{row}(X)$. It is straightforward to see that the row vectors of

$$Y=\left[\begin{array}{cc|c|c} \multicolumn{2}{c|}{\multirow{2}{*}{A(T)}} & J_{t_1\times s} & O \\ \cline{3-4} & & O & J_{t_2\times s} \\ \hline J_{s\times t_1} & O & A(S) & A(S) \\ \hline O & J_{s\times t_2} & A(S) & A(S) \end{array}\right]$$

corresponding to $\mathcal{B}$ along with the row vectors of Y corresponding to a basis for $\mathrm{row}(A(S))$ are linearly independent. This implies that $\mathrm{rank}(T)+\mathrm{rank}(S) \leqslant r-1$. Note that by Step 4, the maximum reduced subgraph of T has at least $\tau/2$ vertices. Moreover, since $\mathrm{rank}(K) \geqslant r-1$, it is not hard to show that $\boldsymbol{j}_s \in \mathrm{row}(A(S))$ and so by Step 2, S is reduced. Now, from Steps 1, 6, and 8, we have $m(r-2)+3 \leqslant \tau+s \leqslant 2m(r-k-1)+m(k) = m(r-k+1)+m(k)-2$. Applying Lemma 5 (ii), we find that $k=2$ and hence $S=K_2$. Since $\tau \geqslant m(r-2)+1$, we deduce that $\mathrm{rank}(T)=r-3$. If $\{a,b\}$ is a duplication class of T, then $\Delta(a,b) \subseteq V(H)$ and therefore by Corollary 3, $\tau \leqslant p+4$. On the other hand, by Step 1, we have $\tau \geqslant p+3$ and since $n=\tau+p+4$, it follows that $\tau=m(r-2)+1$ and $p=m(r-2)-2$, as required.

Case 2. Assume that T has no duplication classes.

Subcase 2.1. $(\boldsymbol{j}_{t_1},\mathbf{0}) \notin \mathrm{row}(A(T))$ and $\boldsymbol{j}_s \notin \mathrm{row}(A(S))$.

Since $\mathrm{rank}(X)=1+\mathrm{rank}(T)$ and $\boldsymbol{j}_s \notin \mathrm{row}(A(S))$, the row vectors of Y corresponding to a basis of $\mathrm{row}(X)$ along with the row vectors of Y corresponding to a basis of $\mathrm{row}(A(S))$ are linearly independent. This implies that $\mathrm{rank}(T)+\mathrm{rank}(S) \leqslant r-1$. So, by Steps 1, 2, 6, and 8, we have $m(r-2)+3 \leqslant \tau+s \leqslant m(r-k-1)+m(k)+1 \leqslant m(r-k)+m(k)$. Applying Lemma 5 (i), we find that $k=0$ and thus $S=K_1$. Hence $t \geqslant m(r-2)+2$ and thus $\mathrm{rank}(T) \geqslant r-1$. Since $(\boldsymbol{j}_{t_1},\mathbf{0}) \notin \mathrm{row}(A(T))$, we find that $\mathrm{rank}(K) \geqslant r+1$, a contradiction.

Subcase 2.2. $(\boldsymbol{j}_{t_1}, \mathbf{0}) \notin \mathrm{row}(A(T))$ and $\boldsymbol{j}_s \in \mathrm{row}(A(S))$.

Clearly, S has no isolated vertex. Since $\mathrm{rank}(T) + \mathrm{rank}(S) \leqslant r$, by Steps 1, 2, 6, and 8, we deduce that $m(r-2)+3 \leqslant \tau+s \leqslant m(r-k)+m(k)$. Applying Lemma 5 (i), we find that $k=0$, which contradicts $\boldsymbol{j}_s \in \mathrm{row}(A(S))$.

Subcase 2.3. $(\boldsymbol{j}_{t_1}, \mathbf{0}) \in \mathrm{row}(A(T))$ and $\boldsymbol{j}_s \notin \mathrm{row}(A(S))$.

Since $\mathrm{rank}(T) + \mathrm{rank}(S) \leqslant r$, by Steps 1, 2, 6, and 8, we deduce that $m(r-2)+3 \leqslant \tau + s \leqslant m(r-k)+m(k)+1$. Applying Lemma 5 (ii), we find that $k \in \{0, 2, r-2\}$. If $k = r-2$, then $T = K_2$, which contradicts $\tau \geqslant p+3$. If $k=2$, then $S = K_1 \cup K_2$. If a and b belong to the copies of K_1 and K_2 in S, respectively, then by Corollary 3, $\tau \leqslant |\Delta(a,b)| \leqslant p+2$, which is a contradiction to Step 1. Hence $k=0$, that is, $S = K_1$ and $\tau \geqslant m(r-2)+2$.

Subcase 2.4. $(\boldsymbol{j}_{t_1}, \mathbf{0}) \in \mathrm{row}(A(T))$ and $\boldsymbol{j}_s \in \mathrm{row}(A(S))$.

Obviously, S has no isolated vertex. Choose $\mathrm{rank}(T)-1$ linearly independent row vectors of $A(T)$ in such a way that they do not generate $(\boldsymbol{j}_{t_1}, \mathbf{0})$. Now, the row vectors of $A(K)$ corresponding to these row vectors together with the row vectors of $A(K)$ corresponding to a basis for $\mathrm{row}(A(S))$ are linearly independent. This yields that $\mathrm{rank}(T)+\mathrm{rank}(S) \leqslant r+1$. So, by Steps 1, 2, 6, and 8, we have $m(r-2)+3 \leqslant \tau+s \leqslant m(r-k+1)+m(k)$. Applying Lemma 5 (ii), we find that $k \in \{2, 3, r-2\}$. If $k = r-2$, then $T = K_3$ and we may assume without loss of generality that $t_1 = 2$. Then by Lemma 2 (ii), $\mathrm{rank}(\mathbb{G} - T_1) \leqslant r-1$. However, this contradicts the minimality of $\mathbb{G}$ as $\mathbb{G} - T_1$ is a reduced graph of order $m(r)-1$. Therefore, $k \in \{2, 3\}$, which means that either $S = K_2$ or $S = K_3$. Using Step 1, if $S = K_2$, then $\tau \geqslant m(r-2)+1$, and if $S = K_3$, then $\tau = m(r-2)$ and $p = m(r-2)-3$, as desired. ∎

Now we are in the position to prove our main theorem.

Theorem 14. *Assume that Conjecture 1 is valid for all reduced graphs of rank at most* 46*. Then Conjecture 1 is true for every reduced graph.*

Proof. Assume that $r \geqslant 47$. Let $\rho = \rho(\mathbb{G})$ and L be an induced subgraph of $\mathbb{G}$ with $|L| = n - \rho$ and $\mathrm{rank}(L) < r$. By Lemma 2 (iv), $\mathrm{rank}(L) \geqslant r-2$. We consider the following two cases.

Case 1. $\mathrm{rank}(L) = r-2$.

If H has no duplicated vertices, then by Lemma 10 and the minimality of $\mathbb{G}$,

$$\frac{m(r)-1}{2} = \frac{n}{2} - 1 < |L| - 1 \leqslant m(r-2),$$

a contradiction. Hence L has duplicated vertices and so $L = H$. Furthermore, by Lemma 10 and Theorem 13, we obtain that $m(r-2) \leqslant \tau = \rho \leqslant m(r-2)+1$.

First suppose that $\tau=m(r-2)$. By Theorem 13, $S=K_3$ and so $p=m(r-2)-3$. For any pair $i,j\in\{1,2,3\}$, $\Delta(v_i,v_j)$ contains at least $p-1$ vertices of P. It follows that every vertex of S has at most three neighbors in P and so $p\leqslant 7$ implying that $m(r-2)\leqslant 10$, which is impossible for $r\geqslant 8$.

Next suppose that $\tau=m(r-2)+1$. By Theorem 13, $S=K_2$ and hence $p=m(r-2)-2$. Obviously, $|N_P(v_1)\cap N_P(v_2)|\leqslant 1$ and thus for either v_1 or v_2, say v_1, we have $|N_P(v_1)|\leqslant p/2$. We may assume that $t_1\leqslant t_2$ implying that $|N_T(v_1)|\leqslant(\tau-1)/2$. Hence $|N(v_1)|\leqslant m(r-2)+1$. By Lemma 2 (i), $\mathbb{G}-N(v_1)$ is of rank at most $r-2$ with an isolated vertex and no duplicated vertices. This means that $n\leqslant m(r)$, a contradiction.

Case 2. $\operatorname{rank}(L)=r-1$.

By Lemma 2 (iv), L is necessarily reduced. From Lemma 9, $\rho<\left(1-\frac{1}{\sqrt{2}}\right)n$ and therefore $|L|>\frac{n}{\sqrt{2}}>5\cdot 2^{(r-4)/2}-2$. Thus Lemma 9 implies that $\rho(L)<\left(1-\frac{1}{\sqrt{2}}\right)|L|$. Let L_0 be an induced subgraph of L with $|L_0|=|L|-\rho(L)$ and $\operatorname{rank}(L_0)<\operatorname{rank}(L)$. Put $T_0=\mathbb{G}-L_0$ and $t_0=|T_0|$. We have $|L_0|>\frac{1}{\sqrt{2}}|L|>\frac{n}{2}$ and $t_0<\frac{n}{2}$. If L_0 has no duplicated vertices, then $\frac{n}{2}-1<|L_0|-1\leqslant m(r-2)$, a contradiction. So, L_0 has duplicated vertices which in turn implies that $\operatorname{rank}(L_0)=r-3$ by Lemma 2 (iv). Hence $\tau\leqslant t_0\leqslant m(r-2)+1$. Using Lemma 12 and Theorem 13, it follows that $\tau\geqslant m(r-2)$. Therefore, either $t_0=\tau$ or $t_0=\tau+1$. Moreover, since $\tau(L)=\rho(L)$ and $\operatorname{rank}(L_0)=\operatorname{rank}(L)-2$, applying Lemma 4 (iii) for L, we deduce that each duplication class of L_0 consists of two vertices.

We claim that any two vertices from two distinct duplication classes of L_0 are adjacent. By contradiction, suppose that $U_1=\{u_1,u_1'\}$ and $U_2=\{u_2,u_2'\}$ are two distinct duplication classes of L_0 with no edges between them. Let $Q=V(T_0)\cap\Delta(u_1,u_1')\cap\Delta(u_2,u_2')$. In a similar manner to the one used in the proof of Lemma 4 (iii), we can show that there exist two disjoint sets Q_1 and Q_2 such that $Q=Q_1\cup Q_2$, $Q_1\subseteq N(u_i)\setminus N(u_i')$ and $Q_2\subseteq N(u_i')\setminus N(u_i)$, for $i=1,2$. From $t_0\leqslant\tau+1$, we deduce that for every duplication class $\{x,y\}$ of L_0, there is at most one vertex of T_0 which is not in $\Delta(x,y)$. This yields that $|T_0-Q|\leqslant 2$. Furthermore, by the maximality of L_0, it is easy to find two vertices $w_1\in U_1$ and $w_2\in U_2$ such that at most one vertex of T_0-Q is contained in $\Delta(w_1,w_2)$. Hence

$$\tau\leqslant|\Delta(w_1,w_2)|\leqslant\begin{cases}|L_0|-4 & \text{if } t_0=\tau,\\ |L_0|-3 & \text{if } t_0=\tau+1.\end{cases}$$

This implies that $\tau\leqslant n-\tau-4$, which contradicts $\tau\geqslant m(r-2)$. This establishes the claim.

From the previous paragraph, it follows that L_0 contains a copy of K_ℓ, where ℓ is the number of duplication classes of L_0. Since $\mathrm{rank}(L_0)=r-3$, we conclude that $\ell \leqslant r-3$. Thus

$$n-1-\big(m(r-2)+1\big)-(r-3)\leqslant n-1-t_0-\ell\leqslant m(r-3).$$

This in turn implies that $m(r-2)\leqslant m(r-3)+r-4$, which is impossible for $r\geqslant 10$.

Therefore, we obtain contradictions in both cases and the proof is complete. ∎

We finally mentation that, similar to the proofs of Lemmas 6 and 9, one can verify the following Lemmas.

Lemma 15. *For every integer* $n\geqslant 2$, $M\big(n,\cos^{-1}(\sqrt{2}-1)\big)<5\cdot 2^{(n+2)/2}-2.$

Lemma 16. *Let* G *be a reduced graph of order* n *and rank* r. *If* $n\geqslant 5\cdot 2^{(r+3)/2}-2$, *then* $\rho(G)<\left(1-\frac{1}{\sqrt{2}}\right)n.$

For every integer $r\geqslant 2$, define $m'(r)=8m(r)+14$. Notice that $m'(r)=2m'(r-2)+2$, whenever $r\geqslant 4$. Now, using this equality, Lemmas 15 and 16 as well as the approach given in this section, we are able to establish the following theorem.

Theorem 17. *For every integer* $r\geqslant 2$, *the order of any reduced graph of rank* r *is at most* $m'(r)$.

Acknowledgments. This research was in part supported by grants from IPM to the first author (No. 92050114) and the second author (No. 92050405). The authors are grateful to anonymous referees for valuable comments on an earlier draft of this article.

References

[1] S. Akbari, P. J. Cameron and G. B. Khosrovshahi: Ranks and signatures of adjacency matrices, unpublished manuscript.

[2] N. Alon and P. D. Seymour: A counterexample to the rank-coloring conjecture, *J. Graph Theory* **13** (1989), 523–525.

[3] T. Ericson and V. Zinoviev: *Codes on Euclidean Spheres*, North-Holland Mathematical Library, 63, North-Holland Publishing Co., Amsterdam, 2001.

[4] E. Ghorbani, A. Mohammadian and B. Tayfeh-Rezaie: Maximum order of trees and bipartite graphs with a given rank, *Discrete Math.* **312** (2012), 3498–3501.

[5] E. Ghorbani, A. Mohammadian and B. Tayfeh-Rezaie: Maximum order of triangle-free graphs with a given rank, *J. Graph Theory,* in press

[6] C.D. Godsil and G. F. Royle: Chromatic number and the 2-rank of a graph, *J. Combin. Theory Ser. B* **81** (2001), 142–149.

[7] W.H. Haemers and M. J. P. Peeters: The maximum order of adjacency matrices of graphs with a given rank, *Designs, Codes and Cryptography* **65** (2012), 223–232.

[8] A. Kotlov: Rank and chromatic number of a graph, *J. Graph Theory* **26** (1997), 1–8.

[9] A. Kotlov and L. Lovász: The rank and size of graphs, *J. Graph Theory* **23** (1996), 185–189.

[10] L. Lovász and M. Saks: Lattices, Möbius functions and communication complexity, in: *Proceedings of the 29th Annual IEEE Symposium on Foundations of Computer Science*, 1988, 81–90.

[11] L. Lovász and M. Saks: Communication complexity and combinatorial lattice theory, *J. Comput. System Sci.* **47** (1993), 322–349.

[12] N. Nisan and A. Wigderson: On rank vs. communication complexity, *Combinatorica* **15** (1995), 557–565.

[13] C. van Nuffelen: A bound for the chromatic number of a graph, *Amer. Math. Monthly* **83** (1976), 265–266.

[14] R. A. Rankin: The closest packing of spherical caps in n dimensions, *Proc. Glasgow Math. Assoc.* **2** (1955), 139–144.

[15] A. A. Razborov: The gap between the chromatic number of a graph and the rank of its adjacency matrix is superlinear, *Discrete Math.* **108** (1992), 393–396.

[16] G. F. Royle: The rank of a cograph, *Electron. J. Combin.* **10** (2003).

[17] C. Zong: *Sphere Packings*, Springer-Verlag, New York, 1999.

Ebrahim Ghorbani

Department of Mathematics
K.N. Toosi University of Technology
P.O. Box 16315-1618, Tehran, Iran

and

School of Mathematics
Institute for Research in
Fundamental Sciences (IPM)
P.O. Box 19395-5746, Tehran, Iran
e_ghorbani@ipm.ir

Ali Mohammadian, Behruz Tayfeh-Rezaie

School of Mathematics
Institute for Research in
Fundamental Sciences (IPM)
P.O. Box 19395-5746, Tehran, Iran
{ali_m,tayfeh-r}@ipm.ir