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SHORTER TOURS BY NICER EARS: 7/5-APPROXIMATION FOR THE GRAPH-TSP, 3/2 FOR THE PATH VERSION, AND 4/3 FOR TWO-EDGE-CONNECTED SUBGRAPHS

ANDRÁS SEBŐ*, JENS VYGEN†

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We prove new results for approximating the graph-TSP and some related problems. We obtain polynomial-time algorithms with improved approximation guarantees.

For the graph-TSP itself, we improve the approximation ratio to 7/5. For a generalization, the minimum T-tour problem, we obtain the first nontrivial approximation algorithm, with ratio 3/2. This contains the s-t-path graph-TSP as a special case. Our approximation guarantee for finding a smallest 2-edge-connected spanning subgraph is 4/3.

The key new ingredient of all our algorithms is a special kind of ear-decomposition optimized using forest representations of hypergraphs. The same methods also provide the lower bounds (arising from LP relaxations) that we use to deduce the approximation ratios.

1. Introduction

The traveling salesman problem is one of the most famous and notoriously hard combinatorial optimization problems [8]. For 35 years, the best known approximation algorithm for the metric TSP, due to Christofides [7], could not be improved. This algorithm computes a solution of length at most $\frac{3}{2}$ times the linear programming lower bound [33]. It is conjectured that a tour of length at most $\frac{4}{3}$ times the value of the subtour relaxation always exists: this is the ratio of the worst known examples. In these examples

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the length function on pairs of vertices is the minimum number of edges of a path between the vertices in an underlying graph. This natural, purely graph-theoretical special case received much attention recently, and is also the subject of the present work.

Notation and Terminology. All graphs in this paper are undirected. They can have parallel edges but no loops. For a graph G we denote by $V(G)$ and $E(G)$ its sets of vertices and edges, respectively. For $X \subseteq V(G)$ we write $\delta(X)$ for the set of edges with exactly one endpoint in X. We denote by $G[X]$ the subgraph induced by X. By the *components* of G we mean the vertex sets of the maximal connected subgraphs (so the components form a partition of $V(G)$). By $2G$ we denote the graph arising from G by doubling all its edges, and a *multi-subgraph* of G is a subgraph of $2G$.

If G is a graph and $T \subseteq V(G)$ with $|T|$ even, then a *T-join* in G is a set $F \subseteq E(G)$ such that $T = \{v \in V(G) : |\delta(v) \cap F| \text{ is odd}\}$. Edmonds [10] showed how to reduce the minimum (in fact, minimum weight) T-join problem to weighted matching, and thus it can be solved in $O(|V(G)|^3)$ time [15].

Definition 1.1. A *T-tour* in G is a T-join F in $2G$ such that $(V(G), F)$ is connected.[1] If $T = \emptyset$, F will be called a *tour*. The minimum cardinality of a T-tour in G is denoted by $\textsc{opt}(G,T)$, and the minimum cardinality of a tour by $\textsc{opt}(G) = \textsc{opt}(G, \emptyset)$.

The *metric closure* of a connected graph G is the pair $(\bar{G}, \bar{c})$, where $\bar{G}$ is the complete graph with $V(\bar{G}) = V(G)$, and $\bar{c}(\{v,w\})$ is the minimum number of edges in a v-w-path in G.

Problems. The *graph-TSP* can be described in any of the following ways. Given a connected graph G, find

- a shortest Hamiltonian circuit in the metric closure of G; or
- a minimum length closed walk in $2G$ that visits every vertex at least once; or
- a minimum cardinality tour in G.

It is easy to see and well-known that these formulations are equivalent; this is the unweighted special case of the "graphical TSP" (see [9]). We also consider two related problems. In the *minimum T-tour problem*, the input is a connected graph G and a set $T \subseteq V(G)$ of even cardinality, and we are looking for a minimum cardinality T-tour in G. The case $T = \emptyset$ is the graph-TSP. The case $|T| = 2$, say $T = \{s, t\}$, has also been studied; we call it the *s-t-path graph-TSP*. (By "Euler's theorem" a subset of $E(2G)$ is an

[1] In a preliminary version of this paper, we used the term "connected-T-join".

$\{s,t\}$-tour if and only if its edges can be ordered to form a walk from s to t that visits every vertex at least once.)

Note that more than two copies of an edge are never useful. However, the variants of the above problems that do not allow doubling edges have no approximation algorithms unless $P = NP$. To see this, note that in a 3-regular graph any tour without doubled edges is a Hamiltonian circuit, and the problem of deciding whether a given 3-regular graph is Hamiltonian is *NP*-complete [17].

A relaxation of the graph-TSP is the *2-edge-connected subgraph problem.* Given a connected graph G, we look for a 2-edge-connected spanning multi-subgraph with minimum number of edges. We denote this minimum by $\text{OPT}_{2\text{EC}}(G)$. A solution F will of course contain two copies of each bridge, and may at first contain parallel copies of other edges too. However, the latter can always be avoided: if an edge e is not a bridge but has two copies, either the second copy can be deleted from F, or the two copies form a cut in F and, since e is not a bridge in G, there is another edge f between the two sides of this cut; the second copy of e can then be replaced by f. Hence an equivalent formulation asks for a 2-edge-connected spanning subgraph, called *2ECSS*, with minimum number of edges, of a given 2-edge-connected graph G. Note that any tour in a 2-edge-connected graph G gives rise to a 2ECSS of G with at most the same number of edges.

Previous Results. All the above problems are *NP*-hard because the 2-edge-connected subgraphs of G with $|V(G)|$ edges are precisely the Hamiltonian circuits. A *ρ-approximation algorithm* is a polynomial-time algorithm that always computes a solution of value at most ρ times the optimum. For all our problems, a 2-approximation algorithm is trivial by taking a spanning tree and doubling all its edges (for TSP or 2ECSS) or some of its edges (for T-tours).

For the TSP with arbitrary metric weights (of which the graph-TSP is a proper special case), Christofides [7] described a $\frac{3}{2}$-approximation algorithm. No improvement on this has been found for 35 years, but recently there has been some progress for the graph-TSP.

A first breakthrough improving on the $\frac{3}{2}$ (by a very small amount) for a difficult subproblem appeared in Gamarnik, Lewenstein and Sviridenko [16]; they considered the graph-TSP for 3-connected cubic graphs. This result has been improved to $\frac{4}{3}$ and generalized to all cubic graphs by Boyd, Sitters, van der Ster and Stougie [4], who also survey other previous work on special cases. However, for general graphs there has not been any progress until 2011.

Then Oveis Gharan, Saberi and Singh [27] gave a $(\frac{3}{2}-\epsilon)$-approximation for a tiny $\epsilon > 0$, using a sophisticated probabilistic analysis. Mömke and Svensson [24] obtained a 1.461-approximation by a simple and clever polyhedral idea, which easily yields the ratio $\frac{4}{3}$ for cubic (actually subcubic) graphs, and will also be an important tool in the sequel. Mucha [26] refined their analysis and obtained an approximation ratio of $\frac{13}{9} \approx 1.444$.

The graph-TSP was shown to be *MAXSNP*-hard by Papadimitriou and Yannakakis [28].

Several of the above articles apply their method to the s-t-path graph-TSP as well, but we found no mention of the minimum T-tour problem. However, we note that the natural adaptation of Christofides' [7] idea provides a $\frac{5}{3}$-approximation algorithm for minimum weight T-tours for any non-negative weight function c on $E(G)$. This was shown for the special case $|T| = 2$ by Hoogeveen [18], but works in general as follows. Let F be the edge set of a minimum weight spanning tree, and T' such that F is a $(T \triangle T')$-join. Let J' be a minimum weight T'-join. Then the disjoint union $F \dot\cup J'$ (taking edges appearing in both sets twice) is a T-tour, and its cost is at most $\frac{5}{3}$ times the optimum. To see this, note that $c(F)$ is at the most the optimum. We now show that $c(J') \leq \frac{2}{3} c(J)$, where J is a minimum weight T-tour. Indeed, $F \dot\cup J$ is a T'-join, and can be partitioned into three T'-joins: $(V(G), F)$ is connected and thus contains a T'-join J_1, $(V(G), J)$ is connected and thus contains a T'-join J_2, and $J_3 := (F \setminus J_1) \dot\cup (J \setminus J_2)$ is a T'-join. We conclude that $3c(J') \leq c(J_1) + c(J_2) + c(J_3) = c(F) + c(J) \leq 2c(J)$.

An, Kleinberg and Shmoys [2] improved on Christofides' algorithm for the s-t-path version, obtaining an approximation ratio of 1.619 (for general weights)[2]. They also obtained a 1.578-approximation algorithm for the s-t-path graph-TSP.

For the 2ECSS problem, Khuller and Vishkin [19] gave a $\frac{3}{2}$-approximation algorithm, and Cheriyan, Sebő and Szigeti [6] improved the approximation ratio to $\frac{17}{12}$. Better approximation ratios have been claimed, but to the best of our knowledge, no complete proof has been published. For the weighted generalization (which we do not consider in this paper), i.e. the problem of finding a minimum weight 2ECSS of a given graph with nonnegative edge weights, the 2-approximation algorithm by Khuller and Vishkin [19] is still the best known.

[2] This was recently improved to 1.6 by Sebő [30]. See also Vygen [31] for a more detailed survey of the recent TSP approximation algorithms and more references.

Our Results and Methods. We describe polynomial-time algorithms with approximation ratio $\frac{7}{5}$ for the graph-TSP, $\frac{3}{2}$ for the minimum T-tour problem (including the s-t-path graph-TSP), and $\frac{4}{3}$ for the 2ECSS problem.

The classical work of Christofides [7] is still present: the roles of the edges in our work can most of the time be separated to working for "connectivity" or "parity". We begin by constructing an appropriate ear-decomposition, using a result of Frank [13] in a similar way as Cheriyan, Sebő and Szigeti [6]. For the graph-TSP, ear-decompositions can be combined in a natural way with an ingenious lemma of Mömke and Svensson [24], which corrects the parity not only by adding but also by deleting some edges, without destroying connectivity. This fits together with ear-decompositions surprisingly well. However, this is not always good enough. It turns out that short and "pendant" ears need special care. We can make all short ears pendant (Section 2) and optimize them in order to need a minimum number of additional edges for connectivity (Section 3). This subtask, which we call earmuff maximization, is related to matroid intersection and actually to the particular case of forest representations of hypergraphs. We use our earmuff theorem and the corresponding lower bound (Section 4) for all three problems that we study. We present our algorithms in Section 5.

Let us overview the four main assertions that are animating all the rest of the paper: a key result that will be used as a first construction for our three approximation results is that *a T-tour of cardinality at most $\frac{3}{2}\mathrm{OPT}(G,T)+\frac{1}{2}\varphi-\pi$ (and at most $\frac{3}{2}\mathrm{OPT}_{2\mathrm{EC}}(G)-\pi\le\frac{3}{2}\mathrm{OPT}(G)-\pi$ if $T=\emptyset$) can be constructed in polynomial time* (Theorem 7), where φ and π are "the number of even and the number of pendant ears in a suitable ear-decomposition". We postpone the precise details until Subsection 2.3, where the main optimization problem we have to solve is also explained. Section 3 is technically solving this optimization problem. The solution is used in Theorem 7 and in the lower bounds proving its quality. In the particular case $T=\emptyset$ this construction provides a tour, which can also be used for a 2ECSS.

Then for our three different approximation algorithms we have three different second constructions for the case when π is "small". A simple inductive construction with respect to the ear-decomposition (Propositions 2.1 and 2.3) provides a T-tour of cardinality at most $\frac{3}{2}\mathrm{OPT}(G,T)-\frac{1}{2}\varphi+\pi$. We see that the smaller of the two T-tours has cardinality at most $\frac{3}{2}\mathrm{OPT}(G,T)$ (Theorem 8).

If $T=\emptyset$, our second construction applies the lemma of Mömke and Svensson [24] to our ear-decomposition, obtaining the bound $\frac{4}{3}\mathrm{OPT}(G)+\frac{2}{3}\pi$ (Lemma 5.3). Therefore the worst ratio is given by $\pi=\frac{1}{10}\mathrm{OPT}(G)$, when both constructions guarantee $\frac{7}{5}\mathrm{OPT}(G)$ (Theorem 10). We could use this bound for 2ECSS as well, but here a simple induction with respect to the number of ears obeys the stronger bound $\frac{5}{4}\textsc{opt}_{2\mathrm{EC}}(G)+\frac{1}{2}\pi$, and so $\pi=\frac{1}{6}\textsc{opt}_{2\mathrm{EC}}(G)$ provides the worst ratio of $\frac{4}{3}\textsc{opt}_{2\mathrm{EC}}(G)$ (Theorem 11).

Preliminaries. The natural LP relaxation of the 2ECSS problem is the following:

$$\textsc{lp}(G) := \min\left\{x(E(G))\colon x \in \mathbb{R}_{\geq 0}^{E(G)}, x(\delta(W)) \geq 2 \text{ for all } \emptyset \neq W \subset V(G)\right\},$$

where we abbreviate $x(S):=\sum_{e\in S} x_e$ as usual. Obviously we have:

Proposition 1.2. *For every connected graph G:*

$$\textsc{opt}(G) \geq \textsc{opt}_{2\mathrm{EC}}(G) \geq \textsc{lp}(G) \geq |V(G)|. \qquad \blacksquare$$

For the minimum T-tour problem, $\textsc{lp}(G)$ is not a valid lower bound; we need a more general setting. For a partition $\mathcal{W}$ of $V(G)$ we introduce the notation

$$\delta(\mathcal{W}) := \bigcup_{W\in\mathcal{W}} \delta(W),$$

that is, $\delta(\mathcal{W})$ is the set of edges that have their two endpoints in different classes of $\mathcal{W}$.

Let G be a connected graph, and $T\subseteq V(G)$ with $|T|$ even. The following takes an analogous role to $\textsc{lp}(G)$ for T-tours:

$$\begin{aligned}\textsc{lp}(G,T) := \min\Big\{ & x(E(G))\colon x \in \mathbb{R}_{\geq 0}^{E(G)},\\ & x(\delta(W)) \geq 2 \text{ for all } \emptyset \neq W \subset V(G) \text{ with } |W\cap T| \text{ even},\\ & x(\delta(\mathcal{W})) \geq |\mathcal{W}| - 1 \text{ for all partitions } \mathcal{W} \text{ of } V(G)\Big\}.\end{aligned}$$

Note that $\textsc{lp}(G,\emptyset)=\textsc{lp}(G)$. We obviously have as well:

Proposition 1.3. *For every connected graph G and $T \subseteq V(G)$ with $|T|$ even:*

$$\textsc{opt}(G,T) \geq \textsc{lp}(G,T) \geq |V(G)| - 1. \qquad \blacksquare$$

The bound can be tight as every spanning tree is a T-tour, where T is the set of its odd degree vertices. Surprisingly, in our lower bounds we will be satisfied by the relaxation of $\textsc{lp}(G,T)$ in which "$|W\cap T|$ even" is replaced by "$W\cap T=\emptyset$".

As a last preliminary remark we note that in all our problems, we can restrict our attention to 2-vertex-connected graphs because we can consider the blocks (i.e., the maximal 2-vertex-connected subgraphs) separately:

Proposition 1.4. *Let G_1 and G_2 be two connected graphs with $V(G_1) \cap V(G_2) = \{v\}$. Let $G := (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$, and let $T \subseteq V(G)$, $|T|$ even. Let T_i be the even set among $(T \cap V(G_i)) \setminus \{v\}$ and $(T \cap V(G_i)) \cup \{v\}$ ($i = 1,2$). Then* $\text{OPT}(G,T) = \text{OPT}(G_1,T_1) + \text{OPT}(G_2,T_2)$, $\text{OPT}_{2\text{EC}}(G) = \text{OPT}_{2\text{EC}}(G_1) + \text{OPT}_{2\text{EC}}(G_2)$, *and* $\text{LP}(G,T) = \text{LP}(G,T_1) + \text{LP}(G,T_2)$. *In particular, any approximation guarantee or integrality ratio valid for (G_1,T_1) and (G_2,T_2) is valid for (G,T).*

Proof. The T-tours in G are precisely the unions of a T_1-tour of G_1 and a T_2-tour of G_2. The same holds for 2ECSS.

We finally observe that the set of facet-defining constraints of $\text{LP}(G,T)$ is the union of the sets of facet-defining constraints of $\text{LP}(G_1,T_1)$ and $\text{LP}(G_2,T_2)$. This implies $\text{LP}(G,T) = \text{LP}(G_1,T_1) + \text{LP}(G_2,T_2)$. ∎

2. Ear-Decompositions

An *ear-decomposition* is a sequence $P_0, P_1, \dots, P_k$, where P_0 is a graph consisting of only one vertex (and no edge), and for each $i \in \{1,\dots,k\}$ we have:

(a) P_i is a circuit sharing exactly one vertex with $V(P_0) \cup \dots \cup V(P_{i-1})$, or
(b) P_i is a path sharing exactly its two different endpoints with $V(P_0) \cup \dots \cup V(P_{i-1})$.

$P_1, \dots, P_k$ are called *ears*. P_i is a *closed ear* if it is a circuit and an *open ear* if it is a path. A vertex in $V(P_i) \cap (V(P_0) \cup \dots \cup V(P_{i-1}))$ is called an *endpoint* of P_i, even if P_i is closed. An ear has one or two endpoints; its other vertices will be called *internal* vertices. The set of internal vertices of an ear Q will be denoted by $\text{in}(Q)$. We always have $|\text{in}(Q)| = |E(Q)| - 1$, while $|V(Q)|$ is $|E(Q)| + 1$ or $|E(Q)|$ depending on whether Q is an open or closed ear. If P and Q are ears and $q \in \text{in}(Q)$ is an endpoint of P, then we say that P is *attached* to Q (at q).

$P_0, P_1, \dots, P_k$ is called an ear-decomposition *of* the graph $P_0 + P_1 + \dots + P_k := (V(P_0) \cup \dots \cup V(P_k), E(P_1) \cup \dots \cup E(P_k))$. It is called *open* if all ears except P_1 are open.

A graph has an ear-decomposition if and only if it is 2-edge-connected. A graph has an open ear-decomposition if and only if it is 2-vertex-connected. The number of ears in any ear-decomposition of G is $|E(G)| - |V(G)| + 1$. These definitions and statements are due to Whitney [32].

We call $|E(P)|$ the *length* of a path or of an ear P. An *l-path* is a path of length l, and an *l-ear* is an ear of length l; an l-ear for $l > 1$ is said to be *nontrivial*. Minimizing the number of nontrivial ears is equivalent to the 2ECSS problem because deleting 1-ears maintains 2-edge-connectivity.

Given an ear-decomposition, we call an ear *pendant* if it is nontrivial and there is no nontrivial ear attached to it.

2.1. Even, Short, and Clean Ears

For an ear P let $\varphi(P)=1$ if $|E(P)|$ is even, and $\varphi(P)=0$ if it is odd. For a 2-edge-connected graph G, $\varphi(G)$ denotes the minimum number of even ears in an ear-decomposition of G, that is, the minimum of $\sum_{i=1}^{k}\varphi(P_i)$ over all ear-decompositions of G. This parameter was introduced by Frank [13], who proved that this minimum can be computed in polynomial time.

Another kind of ears that plays a particular role is 2-ears and 3-ears. We will call these *short* ears. Unlike the number of even ears, we do not know how to minimize the number of short ears efficiently. However, they can be useful in other ways (cf. Section 3). All short ears occurring in this paper will be open, except possibly for the first ear. Given also $T\subseteq V(G)$ with $|T|$ even, we call an ear P *clean* and write $\gamma(P)=1$ if P is short and $\mathrm{in}(P)\cap T=\emptyset$; otherwise we write $\gamma(P)=0$.

We will construct T-tours, and in our algorithms we will also add T-joins for parity correction similar to Christofides. One way to construct these, to be analyzed now, is to consider the ears of an ear-decomposition in reverse order and pick edges in a greedy way, satisfying parity conditions locally. This will in general not yield optimal T-tours or T-joins, but it allows us to prove simple upper bounds.

Let G be a 2-edge-connected graph with an ear-decomposition, $T\subseteq V(G)$, $|T|$ even, and P a pendant ear. Then P is subdivided into subpaths by the vertices of $\mathrm{in}(P)\cap T$. Let us color these subpaths blue and red alternatingly. To obtain a T-join in G, we could take the edges of the red subpaths and add them to an S-join (where we define S appropriately) in the subgraph induced by $V(G)\setminus\mathrm{in}(P)$. For a T-tour in G, we can take $E(P)$, double the edges of the red subpaths, and proceed as before. In this case we can in addition delete one pair of parallel edges if there is one.

This yields the following bounds.

Lemma 2.1. *Let G be a 2-edge-connected graph with an ear-decomposition, and $T\subseteq V(G)$, $|T|$ even. Let P be a pendant ear. Then there exist $F\subseteq E(P)$, $F'\subseteq E(2P)$ and $S,S'\subseteq V(G)\setminus\mathrm{in}(P)$ such that $|S|$ and $|S'|$ are even and:*

(a) $|F|\le\frac{1}{2}|\mathrm{in}(P)|+\frac{1}{2}\varphi(P)$, *and $F\cup J$ is a T-join in G for every S-join J in $G-\mathrm{in}(P)$.*
(b) $|F'|\le\frac{3}{2}|\mathrm{in}(P)|+\frac{1}{2}\varphi(P)+\gamma(P)-1$, *and $F'\cup J'$ is a T-tour in G for every S'-tour J' in $G-\mathrm{in}(P)$.*

Such sets F and F' can be computed in $O(|\mathrm{in}(P)|)$ time.

Proof. The vertices of $\mathrm{in}(P)\cap T$ subdivide P into subpaths, alternatingly colored red and blue. Let E_R and E_B denote the set of edges of red and blue subpaths, respectively; w.l.o.g., $|E_R| \le |E_B|$. Let T_R and T_B be the set of vertices having odd degree in $(V(P),E_R)$ and $(V(P),E_B)$, respectively. Note that $\{E_R,E_B\}$ is a partition of $E(P)$, and $T_R\cap\mathrm{in}(P)=T_B\cap\mathrm{in}(P)=T\cap\mathrm{in}(P)$.

Let $S := T\varDelta T_R$ and $F := E_R$. Then F and S satisfy the claims in (a) because $|F|\le\lfloor\frac{1}{2}|E(P)|\rfloor=\frac{1}{2}(|\mathrm{in}(P)|+\varphi(P))$.

For (b) let $S' := T\varDelta T_B$. We distinguish two cases. If $E_R=\emptyset$, then let $F':=E_B=E(P)$. Then $|F'|=|E(P)|=|\mathrm{in}(P)|+1\le\frac{3}{2}|\mathrm{in}(P)|+\frac{1}{2}\varphi(P)+\gamma(P)-1$.

If $E_R\neq\emptyset$, then let F' result from $E(P)$ by doubling the edges of E_R and then removing one arbitrary pair of parallel edges. Using (a) we have

$$|F'| = |E(P)| + |E_R|-2 = |\mathrm{in}(P)| + 1 + |F|-2 \le \frac{3}{2}|\mathrm{in}(P)| + \frac{1}{2}\varphi(P)-1. \blacksquare$$

Proposition 2.2 (Frank [13]). *Let G be a 2-edge-connected graph, and $T\subseteq V(G)$, $|T|$ even. Then there exists a T-join of cardinality at most*

$$\frac{1}{2}(|V(G)| + \varphi(G) - 1).$$

Proof. Let $P_0,\dots,P_k$ be an ear-decomposition with $\varphi(G)$ even ears. Apply Lemma 2.1(a) to the ears $P_k,\dots,P_1$ (in reverse order). Summing up the obtained inequalities, we get the claim. ∎

The number $|V(G)|+\varphi(G)-1$ is even, since an even ear adds an odd number of vertices. The bound of the Proposition is tight for every 2-edge-connected graph G in the following sense:

Theorem 1 (Frank [13]). *Let G be a 2-edge-connected graph. Then there exists $T\subseteq V(G)$, $|T|$ even, such that the minimum cardinality of a T-join is $\frac{1}{2}(|V(G)|+\varphi(G)-1)$. Such a T and an ear-decomposition with $\varphi(G)$ even ears can be found in $O(|V(G)||E(G)|)$ time.*

Now we prove a similar statement to Proposition 2.2 for T-tours:

Proposition 2.3. *Let G be a 2-edge-connected graph and an ear-decomposition of G with $\varphi(G)$ even ears, among which there are π_2 2-ears. Then for every $T\subseteq V(G)$, $|T|$ even, a T-tour with at most*

$$\frac{3}{2}(|V(G)| - 1) + \pi_2 - \frac{1}{2}\varphi(G)$$

edges can be found in $O(|E(G)|)$ time.

Proof. Apply Lemma 2.1(b) to the nontrivial ears in reverse order. Summing up the obtained inequalities, we get a T-tour with at most $\frac{3}{2}(|V(G)|-1)+\frac{1}{2}\varphi(G)-l$ edges, where l is the number of nontrivial ears that are not short. Note that l is at least the number of even ears that are not short, that is, at least $\varphi(G)-\pi_2$. The claim follows. ∎

2.2. Nice Ear-Decompositions

We need ear-decompositions with particular properties:

Definition 2.4. Let G be a graph. An ear-decomposition of G is called *nice* if

(i) the number of even ears is $\varphi(G)$;
(ii) all short ears are pendant;
(iii) internal vertices of different short ears are non-adjacent in G.

The following is essentially Proposition 4.1 of Cheriyan, Sebő and Szigeti [6]:

Lemma 2.5. *For any 2-vertex-connected graph G there exists a nice ear-decomposition, and such an ear-decomposition can be computed in $O(|V(G)||E(G)|)$ time.*

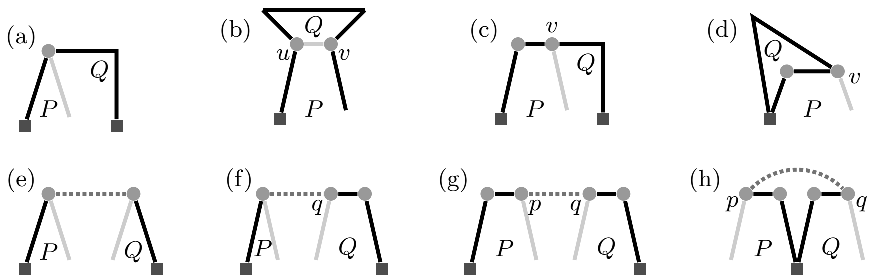

Figure 1. Proof of Lemma 2.5. Squares and circles represent distinct vertices; moreover, vertices represented by circles are internal vertices of short, pendant ears. Grey edges become 1-ears.

Proof. Take any open ear-decomposition with $\varphi(G)$ even ears. This can be done by Proposition 3.2 of Cheriyan, Sebő and Szigeti [6]. (Its proof, briefly: start with Theorem 1, then subdivide an arbitrary edge on each even ear, apply Theorem 5.5.2 of Lovász and Plummer [23] to construct an open odd ear-decomposition of this 2-connected factor-critical graph; finally undo the subdivisions.)

We will now satisfy the conditions (ii) and (iii) by successively modifying the ear-decomposition. Each of the operations that we will use decreases the number of nontrivial ears, and does not increase the number of even ears. Moreover pendant ears vanish or remain pendant in each operation.

First we make all 2-ears pendant. If a 2-ear P is not pendant, let Q be the first nontrivial ear attached to it (Figure 1(a)). Then we can replace P and Q by the ear $Q+e$ and the 1-ear e', where $\{e,e'\}=E(P)$, and e is chosen so that $Q+e$ is open. The new nontrivial ear $Q+e$ can be put at the place of Q in the ear-decomposition.

Next we make all 3-ears pendant. As long as this is not the case, we do the following. Let P be the first non-pendant 3-ear, and let Q be the first nontrivial (open) ear attached to P. Let $\text{in}(P) = \{u,v\}$, and let v be an endpoint of Q. If the other endpoint of Q is u, then we can form an ear R with $E(R)=E(Q)\cup E(P)\setminus\{\{u,v\}\}$ (Figure 1(b)). Otherwise we form R by Q plus the 2-subpath of P ending in v (Figure 1(c),(d)). We replace P and Q by R and a new 1-ear. The new nontrivial ear R has length at least 4; it can be open or closed. It can be put at the place of Q in the ear-decomposition. Since P was the first non-pendant 3-ear, we maintain the property that no closed ear is attached to any 3-ear.

Now all short ears are pendant. This also implies that there are no edges connecting internal vertices of 2-ears: otherwise one could replace the two (pendant) 2-ears and the 1-ear connecting them by an open pendant 3-ear and two 1-ears (Figure 1(e)), reducing the number of even ears by two.

We still have to obtain property (iii). If there is an edge e that connects the internal vertex of a 2-ear P with an internal vertex q of a 3-ear Q, let Q' be the 2-subpath of Q with endpoint q. Form a new open 4-ear R by Q', e, and one edge of P (Figure 1(f)). We replace P, Q, and the 1-ear consisting of e by R and two new 1-ears. The new nontrivial ear R is pendant, so it can be put at the end of the ear-decomposition, followed only by 1-ears.

Finally, if there is an edge $e=\{p,q\}$ that connects internal vertices of two different 3-ears P and Q, we form a new 5-ear R by the edge e and the 2-subpaths of P and Q ending in p and q respectively (Figure 1(g),(h)). We replace P, Q, and the 1-ear consisting of e by R and two new 1-ears. Note that R can be open or closed, but it is always pendant, so it can be put at the end of the ear-decomposition, followed only by 1-ears.

Since the number of nontrivial ears decreases by each of these operations, the algorithm will terminate after less than $|V(G)|$ iterations. At the end, the ear-decomposition is nice. ∎

2.3. How to Switch to Nicer Ears?

Our approximation algorithms will begin by computing a nice ear-decomposition. Lemma 2.1(b) indicates that clean ears are more expensive than others. We will make up for this by "optimizing" them, in order to serve best for connectivity.

An *eardrum* in G is the set M of components of an induced subgraph in which every vertex has degree at most 1. Let $V_M := \bigcup M$ be the vertex set of this subgraph. That is, M contains only one-element and two-element sets, and the two-element sets are the only edges in $G[V_M]$.

Given a nice ear-decomposition and $T \subseteq V(G)$ with $|T|$ even, we say that M is the eardrum *associated* with the ear-decomposition and T if M is the set of components of the subgraph induced by the set of internal vertices of the clean ears. (Note that clean ears are short by definition. Hence, due to conditions (ii) and (iii) of Definition 2.4, M is indeed an eardrum.)

Consider a graph G with a nice ear-decomposition, and let M be the eardrum associated with it and the given set $T \subseteq V(G)$. So M contains a 1-element set $\{v\}$ for each clean 2-ear, where v is the internal vertex of the 2-ear, and a 2-element set $\{v, w\}$ for each clean 3-ear where $\{v, w\}$ is the set of internal vertices of the 3-ear. Let again $V_M = \bigcup M$. Note that $V_M \cap T = \emptyset$. There may be 1-ears connecting V_M and $V(G) \setminus V_M$, and these can be used to replace some of the clean ears by "more useful" clean ears of the same length.

Proposition 2.6. *Let G be a 2-edge-connected graph, and $T \subseteq V(G)$ with $|T|$ even. Let a nice ear-decomposition be given, and let M be the eardrum associated with it and T. For $f \in M$ let P_f be the ear with f as set of internal vertices, and let Q_f be any path in G in which f is the set of internal vertices. Then replacing the ears $(P_f)_{f\in M}$ by the ears $(Q_f)_{f\in M}$ and changing the set of 1-ears accordingly, we get a nice ear-decomposition again with the same associated eardrum.*

Proof. Since all 2-ears and 3-ears were already pendant, no new pendant short ears, except of course the ears Q_f that replace P_f ($f \in M$), can arise by this change. Moreover, no vertex of V_M can be an endpoint of any path Q_f ($f \in M$). Hence the new ear-decomposition is also nice, and the eardrum associated with the ear-decomposition and T remains the same. ∎

We will choose the paths Q_f ($f \in M$) such that $(V(G), \bigcup_{f\in M} E(Q_f))$ has as few components as possible. We will show how in the next section. Let us denote this minimum by $c(G, M)$. Then adding $c(G, M) - 1$ edges to the $|M| + |V_M|$ edges of $\bigcup_{f\in M} E(Q_f)$ yields a connected spanning subgraph in which all vertices in V_M have even degree. It is not difficult to see (and we will show it in Corollary 4.2 below) that there is no such subgraph with fewer edges.

In the following section we will solve this optimization problem for an arbitrary eardrum in G, although we will apply it only to the eardrum associated with the initially computed nice ear-decomposition and T.

3. Earmuffs

Let G be a graph and M an eardrum in G. For each $f \in M$, let $\mathcal{P}_f$ be the set of $(|f|+1)$-paths in G in which f is the set of internal vertices. In other words, for $|f| = 2$ (or $|f| = 1$), $\mathcal{P}_f$ is the set of possible 3-ears (or 2-ears) containing f as middle edge (or the unique element of f as middle vertex, respectively). As explained in Subsection 2.3, we want to pick an element $P_f \in \mathcal{P}_f$ for each $f \in M$ such that we need to add as few further edges as possible to the graph $(V(G), \bigcup_{f\in M} E(P_f))$ in order to make it connected. Ideally, if this graph is a forest, then $|V(G)| - 1 - |M| - |V_M|$ further edges suffice. This motivates the following definitions:

Definition 3.1. Let G be a graph and M an eardrum in G. For $f \in M$ let $\mathcal{P}_f$ denote the set of paths P in G with $\mathrm{in}(P) = f$. An *earmuff* (for M in G) is a set of paths $\{P_f : f \in F\}$, where $F \subseteq M$ and $P_f \in \mathcal{P}_f$ for $f \in F$, such that $(V(G), \bigcup_{f\in F} E(P_f))$ is a forest.

A *maximum earmuff* is one in which $|F|$, its *size*, is maximum, and this maximum is denoted by $\mu(G, M)$. See Figure 2 for an illustration.

We show now that a maximum earmuff can be computed in polynomial time. This can be reduced to matroid intersection (as explained in Section 3.1), and in fact to a more elementary special case (see Section 3.2): forest representative systems. Although general matroid intersection provides a quicker proof of polynomial-time solvability, it is worthwhile to exploit the properties of this special case: it leads to a faster algorithm and a more intuitive way of representing dual solutions (which will provide a lower bound).

3.1. Maximum Earmuffs by Matroid Intersection

Earmuffs are the common independent sets of two matroids. On the one hand, the sets $\mathcal{P}_f$ $(f \in F)$ partition the set $\bigcup_{f\in F} \mathcal{P}_f$. Therefore, the subsets of $\bigcup_{f\in F} \mathcal{P}_f$ that contain at most one element of each partition class form a *partition matroid* (see, e.g., [14]). On the other hand, the subsets of

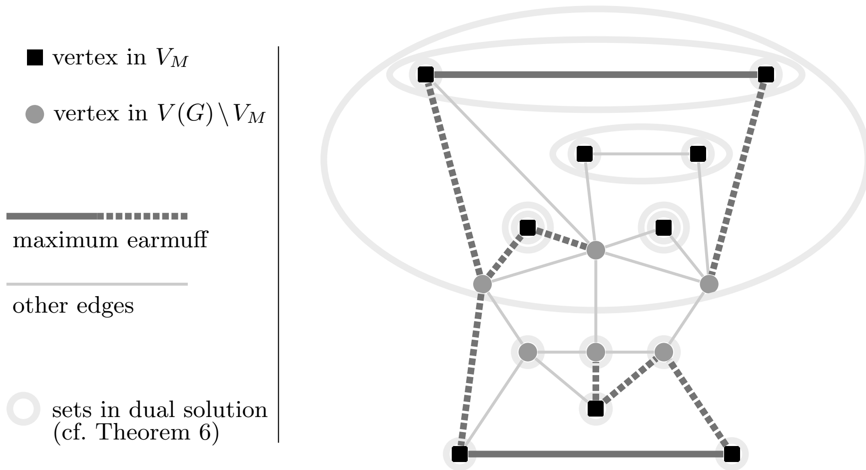

Figure 2. An eardrum, a maximum earmuff, and an optimum dual solution

$\bigcup_{f\in F}\mathcal{P}_f$ whose union is a forest form the *cycle matroid* of a graph (see below). Hence earmuff maximization reduces to matroid intersection, and so it can be solved in polynomial time [11].

The following special form of matroid intersection (where one of the matroids is a partition matroid) is easier to use for our purpose:

Theorem 2 (Rado [29]). *Let E be a finite set and r the rank function of a matroid on E. Let $E_1, E_2, \dots, E_k \subseteq E$. Then*

$$\max\{r(\{e_1,\dots,e_k\})\colon e_i\in E_i\ (i=1,\dots,k)\}$$
$$=\min\left\{r\left(\bigcup_{i\in I}E_i\right)+k-|I|\colon I\subseteq\{1,\dots,k\}\right\}.$$

It is an easy and well-known exercise to deduce this from the matroid intersection theorem [11]. Therefore one can find a set attaining the maximum in polynomial time using the matroid intersection algorithm.

In order to apply Rado's Theorem directly, we represent each path $P\in\mathcal{P}_f$ $(f\in M)$ by the set $e_P\in\binom{V(G)\setminus V_M}{2}$ of its two endpoints. Let r be the rank function of the cycle matroid of the complete graph on $V(G)\setminus V_M$. If we write $E_f := \{e_P\colon P\in\mathcal{P}_f\}$ for $f\in M$, then

$$\mu(G,M)=\max\{r(\{e_f\colon f\in M\})\colon e_f\in E_f\ (f\in M)\}.$$

Hence we can find a maximum earmuff in polynomial time.

3.2. Maximum Earmuffs and Forest Representatives

This section provides an alternative (more elementary and faster) solution to the earmuff maximization problem.

Let U and M be finite sets, and let $U_f \subseteq U$ for $f \in M$. Then $(e_f)_{f\in M}$ is called a *forest representative system* for $(U_f)_{f\in M}$ if $e_f \in \binom{U_f}{2}$ for all $f \in M$, $e_f \neq e_{f'}$ for $f \neq f'$, and the graph $(U, \{e_f : f \in M\})$ is a forest.

For our application, let M be an eardrum in G, and let $U := V(G) \setminus V_M \neq \emptyset$. We will denote by U_f the set of endpoints of paths in $\mathcal{P}_f$ ($f \in M$). We now show that it is sufficient to compute a forest representative system for $(U_f)_{f\in F}$ for a maximum possible subset F of M.

Lemma 3.2. *$\mu(G,M)$ is the maximum cardinality of a subset $F \subseteq M$ for which $(U_f)_{f\in F}$ has a forest representative system. Given a forest representative system, we can compute an earmuff of the same size in $O(|V(G)|^2)$ time.*

Proof. Given an earmuff $\{P_f : f \in F\}$ (with $F \subseteq M$ and $P_f \in \mathcal{P}_f$ for $f \in F$), then $\{e_{P_f} : f \in F\}$ is a forest representative system for $(U_f)_{f\in F}$.

Conversely, let $\{e_f : f \in F\}$ be a forest representative system for $(U_f)_{f\in F}$. We will successively replace each e_f ($f \in F$) by the edge set of a path $P_f \in \mathcal{P}_f$ and maintain a forest.

So let $f \in F$. Since $e_f \in \binom{U_f}{2}$, say $e_f = \{u, v\}$, there are paths $P, Q \in \mathcal{P}_f$ such that u is an endpoint of P and v is an endpoint of Q.

If $|f| = 1$, say $f = \{a\}$, then a is adjacent to u (in P, and thus in G) and to v (in Q, and thus in G). So let P_f be the 2-path with vertices u, a, v in this order.

If $|f| = 2$, suppose that the vertices of P are u, a, b, w in this order. Note that v is adjacent to a or b (in Q, and thus in G).

If v is adjacent to b, then let P_f be the 3-path with vertices u, a, b, v in this order. If v is adjacent to a, then consider the path R with vertices v, a, b, w in this order. Since the edge e_f (as every edge in a forest) is a bridge, at least one of the paths P and R can be chosen as P_f so that replacing e_f by $E(P_f)$ does not create a circuit. ∎

We will show how to find a maximum forest representative system efficiently. We begin with the following min-max theorem:

Corollary 3.3 (Lovász [21]). *Let U and M be finite sets, and let $\emptyset \neq U_f \subseteq U$ for $f \in M$. Then the maximum cardinality of a subset $F \subseteq M$ for which $(U_f)_{f\in F}$ has a forest representative system equals*

$$\min\left\{ |M| - \sum_{W\in\mathcal{W}} \big(|\{f \in M : U_f \subseteq W\}| - (|W|-1)\big) : \mathcal{W} \text{ is a partition of } U \right\}.$$

This is a variant of Corollary 1.4.6 of Lovász and Plummer [23], where bipartite matchings are used in the proof, convertible to an algorithm. It also follows directly from Rado's Theorem:

Proof. The inequality "$\le$" follows from the fact that for every partition $\mathcal{W}$ of U and each $W \in \mathcal{W}$ at most $|W|-1$ of the $f \in M$ with $U_f \subseteq W$ can be represented, and the sets $\{f \in M : U_f \subseteq W\}$ are pairwise disjoint for different sets $W \in \mathcal{W}$ because all U_f are nonempty.

For the other direction, apply Theorem 2 to the sets $\binom{U_f}{2}$ $(f \in M)$ and the cycle matroid of the complete graph on U. We get a forest representative system of size $r\big(\bigcup_{f\in F}\binom{U_f}{2}\big)+|M|-|F|$ for some $F \subseteq M$. Let $\mathcal{W}$ be the set of components of the graph $\big(U, \bigcup_{f\in F}\binom{U_f}{2}\big)$. We have $r\big(\bigcup_{f\in F}\binom{U_f}{2}\big) = \sum_{W\in\mathcal{W}}(|W|-1)$ and $|F| \le \sum_{W\in\mathcal{W}} |\{f \in M : U_f \subseteq W\}|$ because, by the definition of $\mathcal{W}$, for every $f \in F$ there is a $W \in \mathcal{W}$ with $U_f \subseteq W$. ∎

We give now an elementary and algorithmic proof of the nontrivial inequality of Corollary 3.3, giving rise to an efficient algorithm.

Let $F \subseteq M$ such that $(U_f)_{f\in F}$ has a forest representative system $(e_f)_{f\in F}$. A set $W \subseteq U$ will be called F-*closed* if $|\{f \in F : U_f \subseteq W\}| = |W|-1$. For any F-closed set W, the graph $(W, \{e_f : f \in F, U_f \subseteq W\})$ is a tree. Therefore the union of two F-closed sets with nonempty intersection is also F-closed. Moreover, every singleton is F-closed. We conclude that the set of maximal F-closed sets is a partition of U.

If F is a maximum subset of M such that $(U_f)_{f\in F}$ has a forest representative system, then this partition certifies maximality, as we shall prove now.

Lemma 3.4. *Let U and M be finite sets, and let $U_f \subseteq U$ for $f \in M$. Let $F \subseteq M$ and a forest representative system $(e_f)_{f\in F}$ for $(U_f)_{f\in F}$ be given, and let $g \in M \setminus F$. Then one can*

- *either find a forest representative system $(e'_f)_{f\in F\cup\{g\}}$ for $(U_f)_{f\in F\cup\{g\}}$*
- *or conclude that U_g is contained in an F-closed set*

in $O(\sum_{f\in M}|U_f|)$ time.

Proof. Let $E_F := \{e_f : f \in F\}$, and consider the forest (U, E_F). Let $\mathcal{C}$ be the set of components of (U, E_F). Let $T := \{f \in M : U_f \not\subseteq C \text{ for all } C \in \mathcal{C}\}$. Consider the digraph D on the vertex set M that contains an edge (f, f') if and only if $f \in M \setminus T$, $f' \in F$, and there exist $u, v \in U_f$ such that $e_{f'}$ lies on the unique u-v-path in (U, E_F). We call f *reachable from* g if there exists a directed path P from g to f in D.

Claim 1. If there is a $t \in T$ that is reachable from g, then $F \cup \{g\}$ has a forest representative system.

To prove this, let P be a shortest directed path from g to $t \in T$ in D. Let $g = f_0, f_1, \ldots, f_k = t$ be the vertices of P in this order. Set $e'_f := e_f$ for all $f \in F \setminus \{f_0, \ldots, f_k\}$.

Let e'_t be a pair $\{u_t, v_t\}$ such that v_t is not in the same component of (U, E_F) as u_t. For each arc $a = (f_i, f_{i+1})$ of P we have $u_i, v_i \in U_{f_i}$ such that $e_{f_{i+1}}$ (but no e_{f_j} with $j > i+1$) lies on the unique u_i-v_i-path in (U, E_F), and we set $e'_{f_i} := \{u_i, v_i\}$. A straightforward induction shows that $(U, \{e_f : f \in F \setminus \{f_{j+1}, \ldots, f_k\}\} \cup \{e'_{f_j}, \ldots, e'_{f_k}\})$ is a forest for all $j = k, k-1, \ldots, 0$. For $j = 0$ this means that $(U, \{e'_f : f \in F \cup \{g\})$ is a forest, and Claim 1 is proved.

Claim 2. If no element of T is reachable from g, then U_g is contained in an F-closed set.

Indeed, if R is the set of vertices that are reachable from g in D, and $R \cap T = \emptyset$, then let $W := \bigcup \{U_f : f \in R\}$. Since $(W, \{e_f : f \in R \setminus \{g\}\})$ is a tree, $|R| = |W|$ holds, so W is F-closed.

The two Claims directly imply an algorithm: we perform a BFS search from g in D. To do this efficiently, we fix an element $r \in U_g$ (we may assume that U_g is nonempty), compute the components of (U, E_F), and orient the component C containing r as an arborescence rooted at r. We work with a queue Q that we initialize so that it contains only g, and do the following until we reach an element of T or cannot continue because Q is empty.

Remove the first element f from Q. For all $u \in U_f$, check whether $u \in C$ (if not, $f \in T$, and we are done) and traverse the u-r-path in (U, E_F) (always following the incoming arc in the arborescence) as long as we visit edges that we have not visited before. For each such edge $e_{f'}$ we insert f' at the end of the queue Q and store that f was the predecessor of f'.

Note that the set of visited edges always forms a tree containing r. If f' enters the queue with predecessor f, then (f, f') is an arc of D. The correctness and the claimed running time follow. ∎

Theorem 3. *Let U and M be finite sets, and let $U_f \subseteq U$ for $f \in M$. A maximum subset $F \subseteq M$ with a forest representative system for $(U_f)_{f \in F}$ can be computed in $O(|M| \sum_{f \in M} |U_f|)$ time.*

Proof. We may assume $U_f \neq \emptyset$ for all $f \in M$. Let $M = \{g_1, \ldots, g_n\}$. We run the greedy algorithm, beginning with $F_0 = \emptyset$. For $j = 1, \ldots, n$ we apply Lemma 3.4 to F_{j-1}, $M_j := \{g_1, \ldots, g_j\}$, and g_j. We either augment $F_j := F_{j-1} \cup \{g_j\}$, or we set $F_j := F_{j-1}$. In each case we have a forest representative system of

$(U_f)_{f\in F_j}$ and the property that U_f is contained in an F_j-closed set for all $f\in M_j\setminus F_j$. So each U_f $(f\in M_j\setminus F_j)$ is also contained in an element of $\mathcal{W}$, where $\mathcal{W}$ is the set of maximal F_j-closed sets, and we have

$$|M_j\setminus F_j| = \sum_{W\in\mathcal{W}} |\{f\in M_j\setminus F_j : U_f\subseteq W\}|.$$

Since all elements of $\mathcal{W}$ are F_j-closed, this implies

$$|M_j\setminus F_j| = \sum_{W\in\mathcal{W}} \big(|\{f\in M_j : U_f\subseteq W\}| - (|W|-1)\big).$$

By the trivial inequality of Corollary 3.3, this implies that F_j is a maximum subset of M_j with a forest representative system. ∎

This is an algorithmic reformulation of the following result of Lorea [20] (see Frank [14] for a direct proof): given a hypergraph, the sets of hyperedges that have a forest representative system form the independent sets of a matroid.

Now we have all that we need. Let M be an eardrum in G, let $U := V(G)\setminus V_M\neq\emptyset$, and denote by U_f the set of endpoints of paths in $\mathcal{P}_f$ $(f\in M)$. For $\emptyset\neq W\subseteq V(G)\setminus V_M$ we define the *surplus* of W as $\mathrm{sur}(W):=|\{f\in M: U_f\subseteq W\}|-(|W|-1)$. In particular, if $\mathcal{P}_f\neq\emptyset$ (and thus $|U_f|\geq 2$) for all $f\in M$ and $|W|=1$, then $\mathrm{sur}(W)=0$. We conclude:

Theorem 4. *Let G be a graph and M an eardrum in G with $\mathcal{P}_f\neq\emptyset$ for all $f\in M$. Then a maximum earmuff can be computed in $O(|V(G)||E(G)|)$ time, and its size is*

$$\mu(G,M) = \min\left\{|M| - \sum_{W\in\mathcal{W}} \mathrm{sur}(W) : \mathcal{W} \text{ is a partition of } V(G)\setminus V_M\right\}.$$

Proof. Follows directly from Lemma 3.2, Corollary 3.3, and Theorem 3. ∎

4. Lower Bounds

To prove the approximation guarantees of our algorithms, we need several lower bounds.

Theorem 5 (Cheriyan, Sebő and Szigeti [6]). *Let G be a 2-edge-connected graph. Then*

$$L_\varphi(G) := |V(G)| + \varphi(G) - 1 \leq \mathrm{LP}(G).$$

In particular, every 2-edge-connected spanning subgraph of G has at least $L_\varphi(G)$ edges.

Proof. By Theorem 1 there exists a $T \subseteq V(G)$ with $|T|$ even such that $\frac{1}{2}L_\varphi(G)$ is the minimum cardinality of a T-join in G. By a well-known result due to Edmonds and Johnson [12] and Lovász [22], this implies that there exists a multiset of $L_\varphi(G)$ T-cuts containing every edge at most twice. By summing the inequalities $x(\delta(W)) \geq 2$ for all these cuts, we obtain $\text{LP}(G) \geq L_\varphi(G)$. ∎

Consequently $L_\varphi(G) \leq \text{OPT}_{\text{2EC}}(G)$, and this can indeed be seen more easily: it holds since the number of even ears is at most the number of nontrivial ears in any ear-decomposition.

Recall that $\text{LP}(G)$ is not a valid lower bound for the minimum cardinality of a T-tour, nor are $L_\varphi(G)$ and $|V(G)|$. We use Proposition 1.3 and our "earmuff theorem" (Theorem 4) to establish another lower bound:

Theorem 6. *Let G be a connected graph, $T \subseteq V(G)$ with $|T|$ even, and M an eardrum in G with $V_M \cap T = \emptyset$ and $\mathcal{P}_f \neq \emptyset$ for all $f \in M$. Then*

$$L_\mu(G,M) := |V(G)| - 1 + |M| - \mu(G,M) \leq \text{LP}(G,T).$$

In particular, every T-tour in G has at least $L_\mu(G,M)$ edges.

Proof. We use Theorem 4. Let $\mathcal{W}$ be a partition of $V(G) \setminus V_M$ such that

$$\mu(G,M) = |M| - \sum_{W \in \mathcal{W}} \text{sur}(W).$$

Let I be the subset of M containing those sets f for which $U_f \subseteq W$ for some $W \in \mathcal{W}$. Consider the partition $\hat{\mathcal{W}}$ of $V(G)$ that contains

- the set $W \cup \bigcup_{f \in M:\, U_f \subseteq W} f$ for each $W \in \mathcal{W}$;
- the set $\{x\}$ for each $x \in f \in M \setminus I$.

Next, consider the following multiset $\mathcal{S}$ of nonempty proper subsets of $V(G)$:

- for each $x \in f \in I$, take the set $\{x\}$;
- for each $f \in I$, take the set f.

See Figure 2 for an illustration. Note that singletons in I appear and are counted twice in $\mathcal{S}$. Each of the sets of $\mathcal{S}$ induces a cut. None of these cuts contains an edge of $\delta(\hat{\mathcal{W}})$. Moreover, no edge belongs to more than two of these cuts.

Therefore every feasible solution x of $\text{LP}(G,T)$ satisfies

$$x(E(G)) = x(\delta(\hat{\mathcal{W}})) + x(E(G) \setminus \delta(\hat{\mathcal{W}}))$$

$$
\begin{aligned}
&\geq x(\delta(\hat{\mathcal{W}})) + \frac{1}{2}\sum_{S\in\mathcal{S}} x(\delta(S)) \\
&\geq |\hat{\mathcal{W}}| - 1 + |\mathcal{S}| \\
&= |\mathcal{W}| - 1 + |V_M| + |I| \\
&= |\mathcal{W}| - 1 + |V_M| + \sum_{W\in\mathcal{W}} (\mathrm{sur}(W) + |W| - 1) \\
&= |V(G)| - 1 + \sum_{W\in\mathcal{W}} \mathrm{sur}(W) \\
&= L_\mu(G, M).
\end{aligned}
$$

∎

For the special case $T = \emptyset$ we note:

Corollary 4.1. *Let G be a 2-edge-connected graph and M an eardrum in G with $\mathcal{P}_f \neq \emptyset$ for all $f \in M$. Then*

$$L_\mu(G, M) \leq \textsc{lp}(G).$$

In particular, every 2-edge-connected spanning subgraph of G has at least $L_\mu(G, M)$ edges.

Proof. This follows from Theorem 6 and $\textsc{lp}(G, \emptyset) = \textsc{lp}(G)$. ∎

The following statement will not be explicitly used but may be worth mentioning: it shows a problem close to ours but solvable in polynomial time.

Corollary 4.2. *Let G be a 2-edge-connected graph, and $T \subseteq V(G)$ with $|T|$ even. Let a nice ear-decomposition be given, and let M be the eardrum associated with it and T. Then $L_\mu(G, M)$ is the minimum number of edges of a connected spanning subgraph of $2G$ in which every vertex of V_M has even degree.*

Proof. Let $(P_f)_{f\in F}$ be a maximum earmuff for M in G, and for $f \in M \setminus F$ let P_f be the ear with internal vertices f. Taking all the $|M| + |V_M|$ edges in $\bigcup_{f\in M} E(P_f)$ results in a subgraph of G with $|V(G)| - |V_M| - |F|$ components, and every vertex of V_M has even degree. Adding $|V(G)| - |V_M| - |F| - 1$ edges of $G - V_M$ makes the graph connected. We have used $|M| + |V_M| + |V(G)| - |V_M| - |F| - 1 = L_\mu(G, M)$ edges in total.

For the converse, Proposition 1.3 and Theorem 6 establish $\textsc{opt}(G,T) \geq \textsc{lp}(G,T) \geq L_\mu(G, M)$ for all $T \subseteq V(G)$ with $T \cap V_M = \emptyset$. Thus also the minimum is at least $L_\mu(G, M)$. ∎

The construction in the first part of this proof will be repeated in the first part of the proof of Theorem 7.

5. Approximation Algorithms

All our approximation algorithms begin by computing a suitable ear-decomposition:

Lemma 5.1. *Let G be a 2-vertex-connected graph, and $T \subseteq V(G)$ with $|T|$ even. Then G has a nice ear-decomposition containing a maximum earmuff for the eardrum associated with it and T. Such an ear-decomposition can be computed in $O(|V(G)||E(G)|)$ time.*

Proof. Lemma 2.5 provides us with a nice ear-decomposition. Let M be the eardrum associated with this ear-decomposition and T. Compute a maximum earmuff $(Q_f)_{f\in F}$ $(F \subseteq M)$ for M in G (cf. Theorem 4). Let $(P_f)_{f\in F}$ be the original ears containing the elements of F. Change now the current ear-decomposition by replacing the ears $(P_f)_{f\in F}$ by $(Q_f)_{f\in F}$. By Proposition 2.6, the new ear-decomposition is nice, and the associated eardrum remains the same. Moreover, the new ear-decomposition contains a maximum earmuff for M. ∎

5.1. 3/2-Approximation for *T*-Tours

Before describing our three approximation algorithms, we first prove a theorem for T-tours that will be applied for all the three problems in the case when there are many pendant ears. "Many" is not the same quantity though for the three problems.

We have the important inequality $L_\mu(G,M) \le \text{LP}(G,T) \le \text{OPT}(G,T)$, for all T. For $T = \emptyset$ this provides a lower bound for $\text{OPT}(G)$ and $\text{OPT}_{2\text{EC}}(G)$ as well. $L_\varphi(G)$ is also a lower bound for $\text{OPT}_{2\text{EC}}(G)$ and consequently for $\text{OPT}(G)$, but not for $\text{OPT}(G,T)$ in general. Nevertheless the following will prove useful also for computing T-tours.

Theorem 7. *Let G be a graph and $T \subseteq V(G)$ with $|T|$ even, given with a nice ear-decomposition of G containing a maximum earmuff for the eardrum M associated with it and T. Then a T-tour of cardinality at most $L_\mu(G,M)+\frac{1}{2}L_\varphi(G)-\pi$ can be constructed in $O(|V(G)|^3)$ time, where π is the number of pendant ears.*

Proof. Let $V_M = \bigcup M$ be the set of internal vertices of clean ears. Define V_D to be the set of internal vertices of pendant but not clean ears, and $V_I = V(G) \setminus (V_D \cup V_M)$. Note that $G[V_I]$ is 2-edge-connected. Let φ_M be the number of clean 2-ears, φ_D the number of even pendant ears that are not clean, and $\varphi_I = \varphi(G[V_I])$ the number of remaining even ears. Note that $\varphi(G) = \varphi_I + \varphi_D + \varphi_M$.

First, let E_1 denote the union of the edge sets of the clean ears. Since these contain a maximum earmuff, $(V_M \cup V_I, E_1)$ has $|V_I| - \mu(G,M)$ components. Note that $|E_1| = \frac{3}{2}|V_M| + \frac{1}{2}\varphi_M$.

Second, we add a set E_2 of $|V_I| - \mu(G,M) - 1$ edges of $G[V_I]$ such that $(V_M \cup V_I, E_1 \cup E_2)$ is connected.

Third, we apply Lemma 2.1(b) to all the remaining $\pi - |M|$ pendant ears. For each such ear P we add the corresponding edge set F'. Let E_3 denote the union of these sets. Now by Lemma 2.1, $(V(G), E_1 \cup E_2 \cup E_3)$ is connected, and for each such ear P we added at most $\frac{3}{2}|\mathrm{in}(P)| + \frac{1}{2}\varphi(P) - 1$ edges (since $\gamma(P) = 0$), so in total $|E_3| \le \frac{3}{2}|V_D| + \frac{1}{2}\varphi_D - (\pi - |M|)$.

Finally, we have to correct the parities of the vertices in V_I. Let T_0 be the set of vertices $v \in V_I$ for which $|(E_1 \cup E_2 \cup E_3) \cap \delta(v)|$ does not have the correct parity (odd if $v \in T$ and even if $v \notin T$). We add a minimum cardinality T_0-join E_4 in $G[V_I]$; recall that this graph is 2-edge-connected. By Proposition 2.2, $|E_4| \le \frac{1}{2}(|V_I| + \varphi_I - 1)$.

Now we have a T-tour with at most $|E_1| + |E_2| + |E_3| + |E_4|$ edges, which can be bounded as follows by substituting the bounds for each of these sets, and recalling $\varphi_I + \varphi_D + \varphi_M = \varphi(G)$:

$$\begin{aligned} &|E_1| + |E_2| + |E_3| + |E_4| \\ &\quad \le \frac{3}{2}|V_M| + \frac{1}{2}\varphi_M + |V_I| - \mu(G,M) - 1 + \frac{3}{2}|V_D| \\ &\qquad + \frac{1}{2}\varphi_D - (\pi - |M|) + \frac{1}{2}(|V_I| + \varphi_I - 1) \\ &\quad = \frac{3}{2}|V(G)| - 1 + |M| - \mu(G,M) + \frac{1}{2}(\varphi(G) - 1) - \pi \\ &\quad = L_\mu(G,M) + \frac{1}{2}L_\varphi(G) - \pi. \end{aligned}$$

∎

When the number of pendant ears is large, we will use this theorem for all the three problems. For the complementary case three different approaches will be needed for our three approximation algorithms. Our first approximation algorithm deals with the minimum cardinality of a T-tour:

Theorem 8. *There is a $\frac{3}{2}$-approximation algorithm for the minimum T-tour problem. For any connected graph G and $T \subseteq V(G)$ with $|T|$ even, it finds a T-tour of cardinality at most $\frac{3}{2}\mathrm{LP}(G,T)$ in $O(|V(G)|^3)$ time.*

Proof. We may assume that G is 2-vertex-connected (Proposition 1.4). We construct a nice ear-decomposition that contains a maximum earmuff for the eardrum M associated with it and T (using Lemma 5.1). Let π be the number of pendant ears.

If $\pi \geq \frac{1}{2}\varphi(G)$, we use Theorem 7 to find a T-tour of cardinality at most

$$L_\mu(G,M) + \frac{1}{2}L_\varphi(G) - \pi \leq L_\mu(G,M) + \frac{1}{2}(|V(G)| - 1),$$

which is at most $\frac{3}{2}\text{LP}(G,T)$ according to Theorem 6 and the second inequality of Proposition 1.3.

If $\pi \leq \frac{1}{2}\varphi(G)$, then we apply Proposition 2.3. Since $\pi_2 \leq \pi$, where π_2 is the number of 2-ears, we get a T-tour of cardinality at most

$$\frac{3}{2}(|V(G)| - 1) + \pi - \frac{1}{2}\varphi(G) \leq \frac{3}{2}(|V(G)| - 1).$$

By Proposition 1.3, this is at most $\frac{3}{2}\text{LP}(G,T)$, and $\text{LP}(G,T) \leq \text{OPT}(G,T)$. ∎

The result is tight as Figure 3 shows.

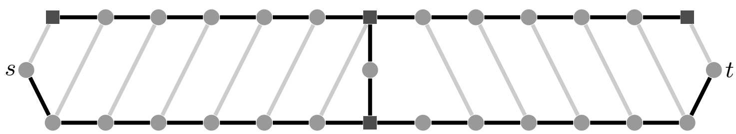

Figure 3. Example showing that the computed T-tour is not necessarily shorter than $\frac{3}{2}$ times the optimum. For each $k \in \mathbb{N}$, we have a graph G with $8k+5$ vertices and $12k+5$ edges. Two vertices are labeled s and t; they form the set $T = \{s,t\}$. The figure shows the case $k=3$. Note that there is a Hamiltonian s-t-path, and hence $\text{LP}(G,T) = \text{OPT}(G,T) = 8k+4$. Also note that $\varphi(G) = 2$ because G is not factor-critical. Suppose that we choose the ear-decomposition that begins with the circuit of length $8k+4$ and then has one pendant 2-ear (in the center). Then $\pi = 1 = \frac{1}{2}\varphi(G)$, so we have two choices in our algorithm. If we use Theorem 7, then our algorithm first takes the 2-ear and then adds edges to obtain a spanning tree, e.g., the one with thick edges. Then there are four vertices (shown as squares) whose degrees have the wrong parity, and we need another $4k+2$ edges to correct the parities. So we end up with a T-tour with $12k+6$ edges. If we use Proposition 2.3 instead, we could also end up with $12k+6$ edges.

5.2. 7/5-Approximation for the Graph-TSP

Our algorithm for the graph-TSP will first construct a nice ear-decomposition containing a maximum earmuff, then removes the 1-ears and computes a tour within each block of the resulting graph. Here we distinguish two cases. If there are many pendant ears, we get a short tour by Theorem 7. If there are few pendant ears, we use the following concept of Mömke and Svensson [24]:

Definition 5.2 (Definition 3.1 of Mömke and Svensson [24]). Given a connected graph G, a *removable pairing* of G is a pair $(R,\mathcal{P})$ of sets such that

- $R \subseteq E(G)$;
- for each $P \in \mathcal{P}$ there are three distinct edges $e, e', e'' \in E(G)$ and a vertex $v \in V(G)$ with $e, e', e'' \in \delta(v)$ and $P = \{e, e'\} \subseteq R$;
- for any two distinct pairs $P, P' \in \mathcal{P}$ we have $P \cap P' = \emptyset$;
- if $S \subseteq R$ and $|S \cap P| \leq 1$ for all $P \in \mathcal{P}$, then $(V(G), E(G) \setminus S)$ is connected.

We will call the elements of $\mathcal{P}$ simply *pairs*.

We need the following very nice lemma and include a variant of the proof:

Theorem 9 (Lemma 3.2 of Mömke and Svensson [24]). *Let G be a 2-vertex-connected graph and $(R,\mathcal{P})$ a removable pairing. Then G has a tour of cardinality at most $\frac{4}{3}|E(G)| - \frac{2}{3}|R|$. Moreover, such a tour can be found in $O((|V(G)| + |\mathcal{P}|)^3)$ time.*

Proof. An *odd join* in a graph G is a T-join in G where T is the set of odd degree vertices of G. For any odd join F in G that intersects each pair $P \in \mathcal{P}$ in at most one edge, we construct a tour from $E(G)$ by doubling the edges in $F \setminus R$ and deleting the edges in $F \cap R$. This tour has $|E(G)| + c(F)$ edges, where we define weights $c(e) = 1$ for $e \in E(G) \setminus R$ and $c(e) = -1$ for $e \in R$, and $c(F) = \sum_{e \in F} c(e)$.

To compute an odd join of weight at most $\frac{1}{3}|E(G)| - \frac{2}{3}|R|$, intersecting each pair at most once, we construct an auxiliary graph G' as follows. For each pair $P = \{\{v,w\},\{v,w'\}\} \in \mathcal{P}$ we add a vertex v_P and an edge $\{v, v_P\}$ of weight zero, and replace the two edges in P by $\{v_P, w\}$ and $\{v_P, w'\}$, keeping their weight.

G' is 2-edge-connected. Hence the vector with all components $\frac{1}{3}$ is in the convex hull

$$\Big\{x \in [0,1]^{E(G')} \colon |F| - x(F) + x(\delta(W) \setminus F) \geq 1 \\ \text{for all } W \subseteq V(G') \text{ and } F \subseteq \delta(W) \text{ with } |\delta(W) \setminus F| \text{ odd}\Big\}$$

of incidence vectors of odd joins of G', and even in the face Q of this polytope defined by $x(\delta(v_P)) = 1$ for all $P \in \mathcal{P}$. So Q contains the incidence vector of an odd join J' in G' of weight at most $\frac{1}{3}c(E(G')) = \frac{1}{3}|E(G)| - \frac{2}{3}|R|$. By contracting the zero-weight edges, such a J' corresponds to an odd join J in G intersecting each pair at most once and having weight at most $\frac{1}{3}|E(G)| - \frac{2}{3}|R|$. To find such a J' and hence such a J, we add a large constant to all weights of edges incident to v_P for all $P \in \mathcal{P}$, and find a minimum weight odd join in G' with respect to these modified weights. ∎

We apply this in the following way:

Lemma 5.3. *Given a 2-vertex-connected graph G and an ear-decomposition in which all ears are nontrivial, a tour of cardinality at most $\frac{4}{3}(|V(G)|-1)+\frac{2}{3}\pi$ can be found in $O(|V(G)|^3)$ time, where π is the number of pendant ears.*

Proof. In order to apply Theorem 9, we define a removable pairing. For each non-pendant ear we define a pair of two edges of the ear that share a vertex that is an endpoint of another nontrivial ear. For each pendant ear we add any one of its edges to R. This defines a removable pairing with $|R|=2k-\pi$, where k is the number of ears. Note that $|E(G)| = |V(G)|+k-1$. From Theorem 9 we get then a tour of cardinality at most $\frac{4}{3}(|V(G)|+k-1)-\frac{2}{3}(2k-\pi)=\frac{4}{3}(|V(G)|-1)+\frac{2}{3}\pi$. ∎

Theorem 10. *There is a $\frac{7}{5}$-approximation algorithm for the graph-TSP. For any connected graph G it finds a tour of cardinality at most $\frac{7}{5}\textsc{lp}(G)$ in $O(|V(G)|^3)$ time.*

Proof. We may assume that G is 2-vertex-connected (Proposition 1.4). We construct a nice ear-decomposition containing a maximum earmuff for the eardrum M associated with it and $T=\emptyset$ (Lemma 5.1). Define $\Lambda(G,M) := \frac{2}{3}L_\mu(G,M)+\frac{1}{3}L_\varphi(G)$. By Corollary 4.1, Theorem 5 and Proposition 1.2 we have $\Lambda(G,M)\le \textsc{lp}(G)\le \textsc{opt}(G)$.

Let G' be the (2-edge-connected, spanning) subgraph resulting from G by deleting all 1-ears. Note that $\varphi(G')=\varphi(G)$, M is also the eardrum associated with the (nice) ear-decomposition without the 1-ears and $T=\emptyset$, and $\mu(G',M)=\mu(G,M)$. Therefore we also have $\Lambda(G',M)=\Lambda(G,M)$, and the following Claim implies the theorem.

Claim. Given a graph G' with a nice ear-decomposition without 1-ears, containing a maximum earmuff for the eardrum M associated with it and $T=\emptyset$, a tour of cardinality at most $\frac{7}{5}\Lambda(G',M)$ can be constructed in $O(|V(G')|^3)$ time.

We first prove the Claim in the case that G' is 2-vertex-connected. We use our two constructions for a tour.

If $\pi\le\frac{1}{10}\Lambda(G',M)$, then we use Lemma 5.3 and $|V(G')|-1\le\Lambda(G',M)$ to obtain a tour of cardinality at most $\frac{4}{3}\Lambda(G',M)+\frac{2}{3}\pi\le\frac{7}{5}\Lambda(G',M)$.

If $\pi \geq \frac{1}{10}\Lambda(G',M)$, then we apply Theorem 7 to G', $T=\emptyset$ and M: we obtain a tour of cardinality at most $\frac{3}{2}\Lambda(G',M)-\pi \leq \frac{7}{5}\Lambda(G',M)$.

The shorter one of the two tours has cardinality at most $\frac{7}{5}\Lambda(G',M)$.

To prove the Claim in the general case, we use induction on $|V(G')|$. Suppose $v \in V(G')$ is a cut-vertex, and G_1 and G_2 are graphs with $G' = (V(G_1)\cup V(G_2), E(G_1)\cup E(G_2))$ and $V(G_1)\cap V(G_2)=\{v\}$. Then the ears P with $\mathrm{in}(P) \subseteq V(G_i)$ form an ear-decomposition of G_i that contains a maximum earmuff for the eardrum M_i associated with it and $T=\emptyset$ (for each $i \in \{1,2\}$). Moreover, $|M_1|+|M_2|=|M|$, $\mu(G_1,M_1)+\mu(G_2,M_2)=\mu(G',M)$, and $|V(G_1)|+|V(G_2)|=|V(G')|+1$. Hence

$$\begin{aligned} L_\mu(G_1,M_1)+L_\mu(G_2,M_2) &= |V(G_1)|-1+|M_1|-\mu(G_1,M_1) \\ &\qquad +|V(G_2)|-1+|M_2|-\mu(G_2,M_2) \\ &= |V(G')|-1+|M|-\mu(G',M) = L_\mu(G',M). \end{aligned}$$

The ear-decompositions of G_1 and G_2 contain $\varphi(G_1)$ and $\varphi(G_2)$ even ears, respectively, and $\varphi(G_1)+\varphi(G_2)=\varphi(G')$. Therefore we have

$$\begin{aligned} L_\varphi(G_1)+L_\varphi(G_2) &= |V(G_1)|+\varphi(G_1)-1+|V(G_2)|+\varphi(G_2)-1 \\ &= |V(G')|+\varphi(G')-1 = L_\varphi(G'). \end{aligned}$$

Hence $\Lambda(G_1,M_1)+\Lambda(G_2,M_2)=\Lambda(G',M)$. By the induction hypothesis, a tour of cardinality at most $\frac{7}{5}\Lambda(G_i,M_i)$ can be constructed in G_i in polynomial time ($i=1,2$). The union of these two tours is a tour in G' of cardinality at most $\frac{7}{5}\Lambda(G_1,M_1)+\frac{7}{5}\Lambda(G_2,M_2)=\frac{7}{5}\Lambda(G',M)$. ∎

This result is tight as Figure 4 shows.

5.3. 4/3-Approximation for 2ECSS

Theorem 11. *There is a $\frac{4}{3}$-approximation algorithm for the minimum 2-edge-connected spanning subgraph problem. For any 2-edge-connected graph G it finds a 2-edge-connected spanning subgraph with at most $\frac{4}{3}\mathrm{LP}(G)$ edges in $O(|V(G)|^3)$ time.*

Proof. We may assume that our graph G is 2-vertex-connected (Proposition 1.4). We construct a nice ear-decomposition containing a maximum earmuff for the eardrum M associated with it and $T=\emptyset$ (Lemma 5.1). Let π denote again the number of pendant ears and π_3 the number of (pendant) 3-ears. We have $\pi_3 \leq \pi$.

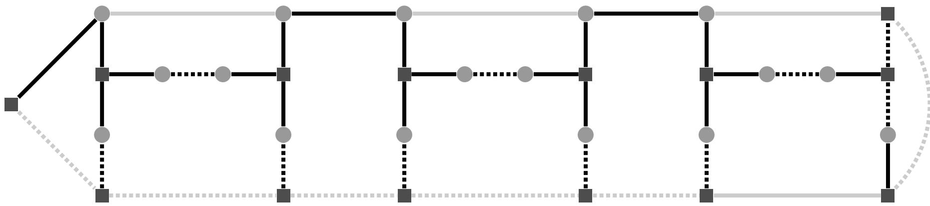

Figure 4. Example showing that the computed tour is not necessarily much shorter than $\frac{7}{5}$ times the optimum. For each $k\in\mathbb{N}$, we have a Hamiltonian graph with $10k+1$ vertices and $13k+1$ edges. The figure shows the case $k=3$. We have $\textsc{lp}(G)=\textsc{opt}(G)=10k+1$ and $\varphi(G)=0$. Construct a nice open ear-decomposition, starting with $2k$ 5-ears from left to right, each with three vertical edges, and then adding the k horizontal pendant 3-ears and the 1-ear (the rightmost edge). Let M be the eardrum associated with this ear-decomposition and $T=\emptyset$. The 3-ears form a maximum earmuff. After deleting the 1-ear, the graph remains 2-vertex-connected. We have $\Lambda(G,M)=10k$ and $\pi=k=\frac{1}{10}\Lambda(G,M)$, so we have two choices in our algorithm. If we use Theorem 7, then our algorithm takes first the 3-ears. Then we could choose the spanning tree consisting of the $10k$ black (solid and dashed) edges. The $4k+2$ odd degree vertices of this spanning tree are shown as squares. We then need another $4k$ edges to make all degrees even, obtaining a tour of cardinality $14k$. If we apply Theorem 9, we could define the removable set R as the dotted edges (without the 1-ear). We have $|R|=5k$, and Theorem 9 provides the bound $\frac{4}{3}13k-\frac{2}{3}5k=14k$. (In fact, if we define weights -1 on the dotted edges and 1 otherwise (cf. the proof of Theorem 9), then the minimum weight of an odd join in G that contains at most one dotted edge of each ear is k. Therefore, computing such an odd join does not help here.)

Claim. The number of edges in nontrivial ears is at most $\frac{5}{4}L_\varphi(G)+\frac{1}{2}\pi$.

Indeed, for any ear P with $|E(P)|\geq 5$ we have $|E(P)|\leq\frac{5}{4}|\mathrm{in}(P)|$, for any 2-ear and 4-ear we have $|E(P)|\leq\frac{5}{4}|\mathrm{in}(P)|+\frac{3}{4}$ (with equality for 2-ears), and for 3-ears we have $|E(P)|=\frac{5}{4}|\mathrm{in}(P)|+\frac{1}{2}$. Summing up for all ears (the sum of 2-ears and 4-ears being at most $\varphi(G)$), we get at most $\frac{5}{4}(|V(G)|-1)+\frac{3}{4}\varphi(G)+\frac{1}{2}\pi_3$ edges, implying the claim using $\pi_3\leq\pi$.

We have now two constructions for a 2ECSS, and the better of the two satisfies the claimed bound:

If $\pi\leq\frac{1}{6}\textsc{lp}(G)$, then we use the Claim and $L_\varphi(G)\leq\textsc{lp}(G)\leq\textsc{opt}_{2\mathrm{EC}}(G)$ (Theorem 5, Proposition 1.2) to obtain a 2ECSS with at most $\frac{5}{4}\textsc{lp}(G)+\frac{1}{2}\pi\leq\frac{4}{3}\textsc{lp}(G)\leq\frac{4}{3}\textsc{opt}_{2\mathrm{EC}}(G)$ edges.

If $\pi\geq\frac{1}{6}\textsc{lp}(G)$, then we apply Theorem 7 to G, $T=\emptyset$ and M: using Theorem 6, Theorem 5 and Proposition 1.2 as before, we obtain a tour, and hence a 2ECSS, of cardinality at most $\frac{3}{2}\textsc{lp}(G)-\pi\leq\frac{4}{3}\textsc{lp}(G)\leq\frac{4}{3}\textsc{opt}_{2\mathrm{EC}}(G)$. ∎

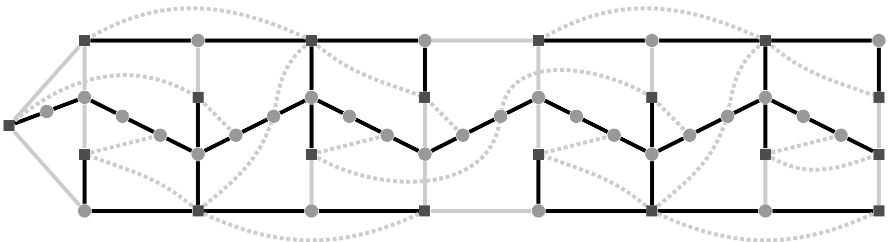

Figure 5. Example showing that the computed 2ECSS is not necessarily much shorter than $\frac{4}{3}$ times the optimum. For each $k\in\mathbb{N}$, we have a Hamiltonian graph with $24k$ vertices and $44k-2$ edges. The figure shows the case $k=2$. We have $\text{LP}(G)=\text{OPT}(G)=24k$ and $\varphi(G)=1$. Construct a nice ear-decomposition from left to right, starting with $4k$ 5-ears (with black and solid grey edges), and finally the $4k-1$ pendant 3-ears (with solid black edges), the pendant 2-ear (on the left), and the 1-ears (dashed grey edges). Then $\pi=4k=\frac{1}{6}\text{LP}(G)$, so we have two choices in our algorithm. If we use the Claim (first case of the proof of Theorem 11), we take all $32k-1$ edges of the $8k$ nontrivial ears. If we apply Theorem 7 (note that the pendant ears constitute a maximum earmuff), we first take the pendant ears (the 2-ear and all the 3-ears), and then add edges to obtain a spanning tree, say the one with the $24k-1$ black edges. The $8k+2$ odd degree vertices are shown as squares. We then need another $8k$ edges to make all degrees even, and a possible choice consists of the curved dashed edges. Then the result is a 2ECSS with $32k-1$ edges. In fact, in both cases the computed 2ECSS is minimal.

We remark that the first case of this proof follows directly from Cheriyan, Sebő and Szigeti [6]. The result is tight as Figure 5 shows.

6. Remarks on Integrality Ratios

For a family $\mathcal{P}$ of polyhedra (say $P\subseteq\mathbb{R}^{n_P}$ for $P\in\mathcal{P}$), the *integrality ratio* of $\mathcal{P}$ is the supremum of the ratios $\min\{\sum_{i=1}^{n_P} c_i x_i \colon x\in P\cap\mathbb{Z}^{n_P}\}/\min\{\sum_{i=1}^{n_P} c_i x_i\colon x\in P\}$ over all $P\in\mathcal{P}$ and all $c\in\mathbb{R}^{n_P}$. In this paper the objective functions are unit vectors. By the *unit integrality ratio* of $\mathcal{P}$ we mean the supremum of $\min\{\sum_{i=1}^{n_P} x_i\colon x\in P\cap\mathbb{Z}^{n_P}\}/\min\{\sum_{i=1}^{n_P} x_i\colon x\in P\}$ over all $P\in\mathcal{P}$.

Denote by $P(G)$ and $P(G,T)$ the polyhedra of feasible solutions of the linear programs defining $\text{LP}(G)$ and $\text{LP}(G,T)$, respectively (see the Introduction). Note that linear functions can be optimized over these polyhedra in polynomial time with the ellipsoid method: this follows using optimization on spanning trees in polynomial time (implying separation on the corresponding polyhedron in polynomial time), and in addition using the maxflow-min-cut theorem, and the algorithm of Barahona and Conforti [3] for finding a minimum weight T-even cut for non-negative weight functions in polynomial time.

Corollary 6.1. *For any connected graph G, the integer vectors in $P(G) \cap [0,2]^{E(G)}$ correspond exactly to the 2-edge-connected spanning subgraphs of $2G$. The minimal integer vectors in $P(G)$ correspond exactly to the minimal 2-edge-connected spanning subgraphs of $2G$. The unit integrality ratio of $\{P(G)\colon G \text{ connected graph}\}$ is at most $\frac{4}{3}$.*

Proof. The first two statements are obvious, and by Theorem 11 there always exists a 2ECSS with at most $\frac{4}{3}\text{LP}(G)$ edges. ∎

The integrality ratio of $\{P(G)\colon G \text{ connected graph}\}$ was conjectured by Carr and Ravi [5] to be $\frac{4}{3}$, and Corollary 6.1 gives some support to this. Alexander, Boyd and Elliott-Magwood [1] showed that it is at most $\frac{3}{2}$ and at least $\frac{6}{5}$ (see the example in Figure 1 of their paper). The same example with unit weights shows that the unit integrality ratio is at least $\frac{9}{8}$. We know no better lower bound.

For T-tours it does not seem useful to study the (unit) integrality ratio of $P(G,T)$ itself, because in general not all minimal integer vectors in $P(G,T)$ correspond to T-tours in G, not even in the case $T=\emptyset$ (indeed, $P(G,\emptyset)= P(G)$ and see Corollary 6.1). Therefore we intersect $P(2G,T)$ with the T-join polytope $Q(2G,T)$ of $2G$. The T-join polytope of a connected graph G is

$$\begin{aligned} Q(G,T) = \big\{&x \in \mathbb{R}^{E(G)}\colon 0 \le x_e \le 1 \text{ for all } e \in E(G), \\ &|F| - x(F) + x(\delta(W) \setminus F) \ge 1 \\ &\text{for all } W \subseteq V(G) \text{ and } F \subseteq \delta(W) \text{ with } |W \cap T| + |F| \text{ odd}\big\}. \end{aligned}$$

We get:

Corollary 6.2. *For any connected graph G and $T \subseteq V(G)$ with $|T|$ even, the integer vectors in $P(2G,T) \cap Q(2G,T)$ are exactly the incidence vectors of T-tours of G. The unit integrality ratio of $\{P(2G,T) \cap Q(2G,T)\colon G \text{ connected graph}, T \subseteq V(G), |T| \text{ even}\}$ is exactly $\frac{3}{2}$.*

Proof. The first statement is obvious, and by Theorem 8 there always exists a T-tour of cardinality at most $\frac{3}{2}\text{LP}(G,T)$. This yields the upper bound. For the lower bound, let $n \in \mathbb{N}$ and consider a circuit G of length $2n$ and two vertices s and t at distance n. The vector with all $4n$ components equal to $\frac{1}{2}$ is in $P(2G,\{s,t\}) \cap Q(2G,\{s,t\})$, but a minimum $\{s,t\}$-tour has $3n$ edges. ∎

Corollary 6.3. *For any connected graph G, the integer vectors in $P(2G) \cap Q(2G,\emptyset)$ are exactly the incidence vectors of tours. The unit integrality ratio of $\{P(2G) \cap Q(2G,\emptyset) \colon G$ connected graph$\}$ is at most $\frac{7}{5}$ and at least $\frac{4}{3}$.*

Proof. The upper bound follows from Theorem 10.

To prove the lower bound, we consider the standard example: let $k \in \mathbb{N}$ and define a graph G as the union of three internally vertex-disjoint paths of length k, all with the same endpoints. Then the vector $x \in \mathbb{R}^{E(2G)}$ with all components $\frac{1}{2}$ is in $P(2G) \cap Q(2G,\emptyset)$ and has $x(E(2G)) = |E(G)| = 3k$, but $\textsc{opt}(G) = 4k$. ∎

For a connected graph G, let $(\bar{G},\bar{c})$ again denote the metric closure of G, and let $S(\bar{G}) := \{x \in [0,1]^{E(\bar{G})} \cap P(\bar{G}) \colon x(\delta(v)) = 2$ for all $v \in V(\bar{G})\}$ be the subtour polytope of $\bar{G}$. Since $\textsc{lp}(G) = \min\{\sum_{e\in E(\bar{G})} \bar{c}(e)x_e \colon x \in P(\bar{G})\} \leq \min\{\sum_{e\in E(\bar{G})} \bar{c}(e)x_e \colon x \in S(\bar{G})\}$, Corollary 6.3 implies an upper bound of $\frac{7}{5}$ of the integrality ratio of the subtour polytope restricted to such "graph metrics" $\bar{c}$. No better bound than $\frac{3}{2}$ (which is due to Wolsey [33]) is known for general metric weight functions.

For general T-tours we have $\textsc{lp}(G,T) = \min\{\sum_{e\in E(\bar{G})} \bar{c}(e)x_e \colon x \in P(\bar{G},T)\}$ and the ratio $\frac{3}{2}$.[3]

We conclude with a remark concerning the relation between the 2ECSS problem and the graph-TSP:

Theorem 12. *Let $\rho \geq 1$. If there is a ρ-approximation algorithm for the 2ECSS problem, then there is a $\frac{2}{3}(\rho+1)$-approximation algorithm for the graph-TSP. If the unit integrality ratio of $\{P(G) \colon G$ connected graph$\}$ is ρ, then the unit integrality ratio of $\{P(2G) \cap Q(2G,\emptyset) \colon G$ connected graph$\}$ is at most $\frac{2}{3}(\rho+1)$.*

Proof. Let G be a connected graph, and let G' be a 2ECSS of $2G$.

Claim. G has a tour of cardinality at most $\frac{2}{3}(|E(G')| + |V(G)| - 1)$.

We prove the Claim by induction on the number of vertices. If G' is 2-vertex-connected, find any ear-decomposition of G', and define a removable

[3] **Added in proof:** Very recently, Z. Gao (An LP-based $\frac{3}{2}$-approximation algorithm for the s-t path graph Traveling Salesman Problem, Operations Research Letters 41 (2013), 615–617) gave a beautiful alternative $\frac{3}{2}$-approximation algorithm for the minimum T-tour problem for $|T| = 2$ (s-t-path graph-TSP), with a surprisingly simple proof (but longer running time, including the solution of an LP).
Note also that for $|T| \leq 2$ our lower bound $\textsc{lp}(G,T)$ is the same as the lower bound of [2] and of Gao: indeed, the partition constraints of $P(G,T)$ are then easily implied as the sum of cut constraints.

pairing $(R,\mathcal{P})$ by including one edge of each ear in R and setting $\mathcal{P}=\emptyset$. We have $|R|=|E(G')|-|V(G')|+1$. By Theorem 9 we get a tour of cardinality at most $\frac{4}{3}|E(G')|-\frac{2}{3}|R|=\frac{2}{3}(|E(G')|+|V(G')|-1)$ as required. If G' has a cut vertex v, we apply the induction hypothesis to two graphs that share only v and whose union is G' (as in the proof of Theorem 10). The Claim follows.

The proof is finished easily using the Claim and Proposition 1.2 as follows. If G' has at most $\rho\,\textsc{opt}_{2\text{EC}}(G)$ edges, then our tour has cardinality at most $\frac{2}{3}(\rho\,\textsc{opt}_{2\text{EC}}(G)+\textsc{opt}(G))\le\frac{2}{3}(\rho+1)\textsc{opt}(G)$. If G' has at most $\rho\,\textsc{lp}(G)$ edges, then our tour has cardinality at most $\frac{2}{3}(\rho\,\textsc{lp}(G)+\textsc{lp}(G))=\frac{2}{3}(\rho+1)\textsc{lp}(G)$. ∎

This partly strengthens a result of Monma, Munson and Pulleyblank [25] who gave the bound $\frac{4}{3}\rho$ instead of $\frac{2}{3}(\rho+1)$; but their result also holds for the weighted case. We conclude that any ρ-approximate 2ECSS with $\rho<\frac{11}{10}$ leads to a tour with less than $\frac{7}{5}\textsc{opt}(G)$ edges.

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András Sebő

Laboratoire G-SCOP
CNRS - Univ. Grenoble Alpes
46, Avenue Félix Viallet
F-38000 Grenoble, France
`Andras.Sebo@grenoble-inp.fr`

Jens Vygen

Research Institute for
Discrete Mathematics
University of Bonn
Lennéstr. 2
53113 Bonn, Germany
`vygen@or.uni-bonn.de`