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A THEOREM ON GRAPH EMBEDDING WITH A RELATION TO HYPERBOLIC VOLUME

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We prove that a planar cubic cyclically 4-connected graph of odd $\chi<0$ is the dual of a 1-vertex triangulation of a closed orientable surface. We explain how this result is related to (and applied to prove at a separate place) a theorem about hyperbolic volume of links: the maximal volume of alternating links of given $\chi<0$ does not depend on the number of their components.

1. Introduction

In this paper we will deal with embeddings of graphs on closed orientable surfaces [15,16,17]. More precisely, we will examine cellular embeddings of trivalent graphs. They become the dual of the 1-skeleton of triangulations of the surface. These triangulations have been mostly studied in the case of the sphere [7,25], but of special interest to us will be the opposite case: triangulations with one vertex.

One-vertex triangulations have received some treatment in the literature. In particular, in [26] a method was presented to encode the way a cubic graph becomes the dual 1-skeleton of such a triangulation in terms of words in a formal alphabet subject to certain conditions, called *Wicks forms*. (Thus for the same abstract cubic graph, several triangulations may, and in general do, exist.) This work was later applied in [3] to enumerate such triangulations.

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Here we are mainly concerned with the problem what cubic graphs G are realizable as dual 1-skeletons of one-vertex triangulations. In [26] a recursive procedure was given which algorithmically can decide the question for any given cubic graph G. But explicit criteria in terms of self-contained properties of G seem lacking. We will provide a sufficient condition, proving the following main result:

Theorem 1.1. *Let G be a planar cubic cyclically 4-connected (see Definition 2.1) graph of odd $\chi<0$. Then G is the dual of a 1-vertex triangulation of a closed orientable surface.*

Note that when G is the dual of a n-vertex triangulation of a closed orientable surface of Euler characteristic χ, then $1\le n\le 2-\chi$ and $n+\chi$ is even. For given χ we will say that such n are χ-*admissible.* There is an easy way to extend Theorem 1.1 to even χ and n-vertex triangulations for any χ-admissible n (Theorem 4.3).

We will see below that for planar graphs 3-connectedness is not sufficient to ensure the realizability of G by a 1-vertex triangulation (and, in fact, by an n-vertex triangulation for any given n; see Section 4.3). As for planarity, the proof of Theorem 1.1 given here makes essential use of this property, and it is not clear under what meaningful conditions Theorem 1.1 holds when planarity is dropped.

One main reason why planar (and 3-connected) graphs are of interest is the work of [24], where these graphs occurred in an enumeration problem of alternating knots. The present paper arose in, and is prepared as an application to, a continuation of that study, this time with emphasis on hyperbolic volume.

Let $\mathrm{vol}(L)$ be the hyperbolic volume of a link L in S^3 (and set $\mathrm{vol}(L)=0$ if the link is not hyperbolic). An observation, essentially due to Brittenham [9], is that there is a finite maximal volume

$$v_\chi := \sup\{\mathrm{vol}(L)\colon \chi(L)=\chi,\ L \text{ alternating}\} \tag{1}$$

for alternating links of given Euler characteristic χ of an orientable spanning (Seifert) surface. For similar homological reasons, such links have n components for n which are χ-admissible. Then we can define

$$v_{n,\chi} := \sup\{\mathrm{vol}(L)\colon \chi(L)=\chi,\ L \text{ alternating of } n \text{ components}\}. \tag{2}$$

The main motivation for Theorem 1.1 is unrelated to graph embedding, and occurred in an attempt to study an intimate relationship between the numbers v_χ and $v_{n,\chi}$ and cubic planar graphs, which was already partially elucidated in [24]. Theorem 1.1 will be essentially needed to prove the following concrete property.

Theorem 1.2. *We have $v_\chi = v_{n,\chi}$ for any χ-admissible n. In other words, the maximal volume of alternating links of given χ does not depend on the number of their components.*

In order not to make the transition to hyperbolic volume too abrupt, in Section 4.1 we will outline how Theorem 1.1 relates to Theorem 1.2. The full details will be given in a separate paper, which the inclusion of the proof of Theorem 1.1, we felt, rendered excessively long and unfocused.

2. General definitions and preliminaries

We begin with introducing many notations and recalling previous results that will be used throughout the paper. Most of these are well-known, but some are more specific, and build on our own previous work. They are given in the next section.

2.1. Miscellanea and polynomials

The expression $|S|$ denotes the cardinality of a (finite) set S. For any $S \subset \mathbb{R}$, we denote by $\sup S$ the supremum of S (with the natural convention that $\sup \emptyset = -\infty$).

Let $[Y]_{t^a} = [Y]_a$ be the *coefficient* of t^a in a polynomial $Y \in \mathbb{Z}[t]$. Let for $Y \in \mathbb{Z}[t] \setminus \{0\}$

$$\begin{aligned} \min\deg Y &= \min\{a \in \mathbb{N} \colon [Y]_a \neq 0\}, \\ \max\deg Y &= \max\{a \in \mathbb{N} \colon [Y]_a \neq 0\}, \\ \operatorname{spn} Y &= \max\deg Y - \min\deg Y \end{aligned}$$

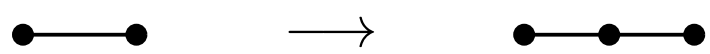
be the *minimal* and *maximal degree* and *span* (or breadth) of Y, respectively.

2.2. Graphs

A graph G will have for us possibly multiple edges but usually no loop edges (edges connecting one and the same vertex). If G has no multiple edges, we call G *simple*. $V(G)$ will be the set of vertices of G, and $E(G)$ the set of edges of G (each multiple edge counting as a set of single edges); $v(G)$ and $e(G)$ will be the number of vertices and edges of G (thus counted), respectively.

We call G to be *k-valent* (resp. $\geq k$-valent) if all vertices have valence k (resp. at least k). A 3-valent graph is also called *cubic*. A graph is k-l valent, if all vertices have valence k or l.

For a graph, let the operation



(adding a vertex of valence 2) be called *bisecting* and its inverse (removing such a vertex) *unbisecting* (of an edge). We call a graph G' a *bisection* of a graph G, if G' is obtained from G by a sequence of edge bisections. We call a bisection G' *reduced*, if it has no adjacent vertices of valence 2 (that is, each edge of G is bisected at most once if G is ≥ 3-valent). Contrarily, if G' is a graph, its *unbisected graph* G is the graph with no valence-2-vertices, of which G' is a bisection.

A *cut vertex* is a vertex which disconnects a graph, when removed together with all its incident edges.

A graph is *n-connected*, if n is the minimal number of edges needed to remove from it to disconnect it. (Thus connected means not 0-connected.) Such a collection of edges is called an *n-cut*. For every n-cut of a planar graph we can draw a *cut curve* γ in the plane, which intersects G only in interior points of the edges in the cut. This curve γ is determined up to isotopies of the plane which avoid intersection with vertices of G. We will often for convenience identify a cut with its cut curve.

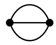
A *(cyclic) orientation* O of a graph G can be described as a map

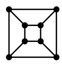
$$O\colon V(G) \to \bigcup_{n=0}^{e(G)} E(G)^n/\mathbb{Z}_n,$$

with $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ acting by cyclic permutation on $E(G)^n$. If $O(v) = (e_1, \ldots, e_n)$, then we demand that $n = \mathrm{val}_G(v)$ is the valence of v in G, and that e_i are the edges incident to v (with $e_i \neq e_j$ for $i \neq j$). The *opposite* orientation $-O$ is defined by $-O(v) = (e_n, \ldots, e_1)$. Any embedding p of G on an oriented surface S (in particular, any planar embedding of G) defines a *canonical orientation* O_p of G (corresponding to this embedding), given by listing the edges incident to v in counterclockwise order.

We will define a few more properties of embedded graphs below. For now let us finish definitions of abstract graphs by fixing symbols for two instances which will continuously occur in the sequel.

Let

(3) $\theta =$  the theta-curve

(4) $\kappa =$  the cube net

The letters θ and κ will retain this meaning throughout the paper.

2.3. Graph embedding and triangulations

It will be enough in the following to consider surfaces S which are oriented and compact.

An embedding p of G on a surface S is *cellular* if all components of $S\backslash p(G)$ are discs. For such embeddings there is a notion of duality: the discs of $S\backslash p(G)$ are vertices of a dual graph $p^*(G^*)$ (understood as an embedding p^* on S of an abstract graph G^*) and vice versa.

If G is 3-connected, the dual $T=p^*(G^*)$ of $p(G)$ is (the 1-skeleton of a) cell decomposition of S. If G is additionally trivalent, then T is a triangulation of S.

From now on, unless otherwise noted, G will be a 3*-connected* 3*-valent planar graph.* (Planarity is not essential for the definitions here, but will play a central role in our later arguments.)

Clearly the number n of discs of $S\setminus p(G)$ is $v(T)=v(G^*)$, and $e(G)=e(G^*)=e(T)$. Note the relationship

$$\chi(S) = 2 - 2g(S) = v(G) - e(G) + n,$$

where χ is the Euler characteristic and g the genus of S. We will mostly focus below on the case $n=1$.

A 1-vertex triangulation of an oriented compact surface S is an embedded graph $T\subset S$ with exactly one vertex such that all connected components of $S\setminus T$ are adjacent to exactly 3 edges of T, that is, are triangles. Two such triangulations $T\subset S$ and $T'\subset S$ are *isomorphic* (or *equivalent*) if there exists an orientation-preserving homeomorphism $\phi\colon S \to S$ such that $\phi(T) = T'$. For the enumeration (up to equivalence, and reference to some other uses) of such triangulations see [3]. For other n and general cell decompositions T see, e.g., [14].

Now we consider *cyclically 4-connected* graphs G. This property turns out to be of considerable importance in the following, and so we give a formal (though not entirely standard) definition.

Definition 2.1. For 3-valent graphs G the common definition of *cyclic 4-connectedness* can be paraphrased to require that if some ≤ 3 edges disconnect G, then they are the 3 edges incident to some vertex of G. To save space, we will often write "c4c" for "cyclically 4-connected".

2.4. Seifert surfaces and Seifert graphs

A classical algorithm given by Seifert (discussed, e.g., in [1,18,11] or [12, Section 5]) associates to an oriented link L in S^3 an orientable surface S with $\partial S = L$. The algorithm constructs (one among many) S out of an arbitrary diagram D of L as follows.

First *smooth out* all crossings in D

(5) 

obtaining a set of (oriented) circles in the plane, the *Seifert circles*. Glue into each Seifert circle a disk, and reinstall the crossings of D by gluing the Seifert circles by half-twisted bands. The resulting surface is called the *canonical Seifert surface* of L associated to the diagram D.

The canonical Seifert surface can be represented by its *Seifert graph* $G(D)$, in which a vertex is chosen for each Seifert circle, and an edge connects Seifert circles along a crossing. The Seifert graph is a planar bipartite graph.

The *canonical Euler characteristic* of a link diagram D is called the Euler characteristic of D's canonical Seifert surface, and of its Seifert graph $G(D)$, for which we have

$$\chi(D) = -c(D) + s(D),$$

where $c(D)$ is the number of crossings of D, and $s(D)$ the number of its Seifert circles. The *canonical genus* $g(D)$ is given by

$$g(D) = \frac{1}{2}\big(2 - n(D) - \chi(D)\big),$$

where $n(D)$ is the number of components of (the link represented by) D, and D is connected. The canonical Euler characteristic and canonical genus of a link L are the maximal canonical Euler characteristic resp. minimal canonical genus of any diagram D of L.

The (classical, or Seifert) *Euler characteristic* $\chi(L)$ resp. *genus* $g(L)$ of a link L is called the maximal Euler characteristic resp. minimal genus of all its (not necessarily canonical for some diagram) Seifert surfaces. From their definition, we have the inequalities $\chi(L)\geq\chi_c(L)$ and $g(L)\leq g_c(L)$.

2.5. Markings

To express ourselves nicely, we need some more terminology. Most of it was already introduced in [24].

Definition 2.2. Take a 3-connected 3-valent planar graph G in a particular planar embedding p_0, which we keep in mind, but do not write. Let $D_{G,O}$ be the alternating diagram corresponding to G with choice of vertex orientation O. The diagram $D_{G,O}$ can be defined by having as Seifert graph a reduced bisection of G, and the orientation of the Seifert circles corresponding to the vertices of G being given by O. We denote the orientation $O(v)$ of $v\in V(G)$ by $+$ or $-$.

We call O also a *marking* of G. We will often not distinguish between a marking O and its diagram $D_{G,O}$ to simplify our language. Let $L_{G,O}$ be the link represented by $D_{G,O}$. We call O an *n-component marking* (or *knot marking* for $n=1$), if $n(L_{G,O})=n$. The marking O is said to be *even* or *odd* depending on the parity of the crossing number $c(L_{G,O})$. Let $T_{G,O}$ be the *thickening* of (G,O), i.e. the canonical Seifert surface of $D_{G,O}$ with $\partial T_{G,O}=L_{G,O}$.

Whenever a marking O is given, it induces an *edge coloring* of G into *even* and *odd* edges, depending on whether the two vertices connected have the same or opposite marking. (Note that then one can alternatively define a marking to be even or odd according to the parity of the number of its odd edges.)

It was explained in [5] (for the sphere, but higher genus is completely analogous), that an n-component marking O of G gives rise to a cellular embedding p of G on an oriented surface S, s.t. $p(G)$ is the 1-skeleton of the dual (of a) triangulation of S with n vertices. To obtain p, glue (abstractly) disks into the boundary components of $T_{G,O}$. On the opposite side, given p, one can recover $T_{G,O}$ by recording the cyclic orientation of any $v \in V(G)$ induced by p. Given a planar embedding p_0 of G, define O by putting a $+$ or $-$ in $v \in V(G)$ depending on whether $p_0(v) = p(v)$ or $p_0(v) = -p(v)$. Then $T_{G,O}$ is homeomorphic to a neighborhood of $p(G)$ on S.

In the following, it will be more convenient for us to express ourselves in terms of markings rather than cellular embeddings.

2.6. Wicks forms

A *maximal Wicks form* w is a cyclic word in the free group over an alphabet with the following 3 conditions:

1) Each letter a appears exactly once in w, and so does its inverse a^{-1}.
2) w has no subwords of the form $a^{\pm 1}a^{\mp 1}$.
3) If $a^{\pm 1}b^{\pm 1}$ and $b^{\pm 1}c^{\pm 1}$ are subwords of w (for some independently to choose signs), then $c^{\pm 1}a^{\pm 1}$ is also a subword of w (for proper to be chosen signs).

Two forms are *equivalent* if a cyclic permutation and a permutation of the letters (and between letters and their inverses) transforms the one form into the other.

Such words were first considered in [28]. Later they were studied in several contexts, e.g. [10,13], but most relevant here will be their description as duals of 1-vertex triangulations of oriented surfaces [3].

The number of letters of a maximal Wicks form w is always $6g-3$ for some $g > 0$. Such a form w gives rise to a triangulation of an oriented surface S. First label the edges of a $6g-3$-gon X by the letters of w and reverse the orientation induced from the one of X on edges corresponding to inverses of letters. Then identify the edges labelled by each letter and its inverse according to their orientation. The surface S thus obtained from X is orientable and of genus g. We call g also the *genus* of the Wicks form. The boundary of X gives a certain 3-valent graph G embedded on S, which is the 1-skeleton of a 1-face cell decomposition (or dual of a 1-vertex triangulation). The edges of G correspond to letters $\{a, a^{-1}\}$ of w, while the vertices to triples of such pairs occurring as in property 3) of the above description of Wicks forms. Thus G comes from a Wicks form if and only if it admits a knot marking.

In [26] three elementary operations to construct Wicks forms of genus $g+1$ out of Wicks forms of genus g were introduced. They were called α,

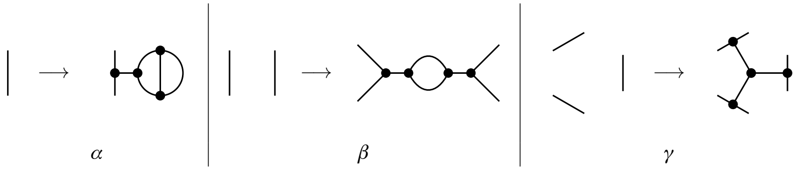


Figure 1. The three Vdovina constructions. (The segments on the left of the moves β and γ do not necessarily belong to different edges.)

β and γ construction (or transformation). The effect of these operations on the graphs of the Wicks form are given in Figure 1 (see also Figure 1 of [3]). We will call these graph moves also graphic α, β and γ construction (or transformation). Their importance is that they are exhaustive in the following sense. (This property is a consequence of the work of [3].)

Theorem 2.3. *([3]) If $\chi(G)$ is odd, then G admits a knot marking iff it can be obtained recursively from θ by a sequence of (graphic) α, β and γ transformations.*

In [24] we defined *maximal planar Wicks forms* to be those, whose graph G is planar and 3-connected. We explained how a maximal planar Wicks form bijectively corresponds to a knot marking of its 3-valent graph G. (The reverse map is easily described: take the Gauß code of the marking and remove all cyclic occurrences of b in $\cdots a^{\pm 1} b^{\pm 1} \cdots b^{\pm 1} a^{\pm 1} \cdots$.) We also introduced, in [23], the Gauß diagram (see [20]) version of the form and its knot marking.

3. Proof of main result

3.1. Initial arguments

Theorem 1.1 was suggested by the following initial computational verification.

Proposition 3.1. *Theorem 1.1 is true for $\chi \geq -21$.*

Proof. The work is mainly a question of efficient programming. The program of Brinkmann and McKay [8] can generate the graphs rather rapidly (despite their large number).

Checking for the existence of a knot marking can be done quickly. We generated in lexicographic order a vector $(a_1, \ldots, a_n) \in \{-1, 1\}^n$, where

$n = -2\chi(G)$ is the number of vertices of G, and associated to it the marking where the sign of vertex i is $(-1)^i a_i$. (The alternation was introduced to avoid almost uncolored markings, which contain many cycles.) For this marking, we start at an arbitrary vertex and trace the component C of $D_{G,O}$ through it. If it closes prematurely, we take the lexicographically next vector (a_i), in which at least the entry for the minimum vertex in the detected loop C is changed (so that the loop is broken). ∎

We explained already that there is an exact recursive description of the cubic (not necessarily planar) graphs G for which the assertion of Theorem 1.1 holds. (See Theorem 2.3 and Figure 1.) But we noticed that this recursion is rather intractable with respect to the c4c property. For the proof we will use a different set of transformations (though not disjoint; γ is included). It cannot be ascertained to generate all graphs, but it is sufficient for the ones we are interested in, and it makes a recursive work with the c4c property possible (even if not easy).

Proof of Theorem 1.1. As outlined, it goes by induction on χ. The induction start is not a problem. We refer to the verification of enough simple graphs in Proposition 3.1.

The work consists in the induction step. We fix a (c4c planar odd χ) graph G. Our approach will be that we can obtain a knot marking of G from knot markings of some simpler graphs G'. There are several transformations between G and G', which will be concretely specified. A universally applied one is the γ-construction in Figure 1. We recall that in this case G' is obtained from G by removing all three edges incident to a vertex, i.e., its *star*.

Whenever we can perform a simplification between G and G' so that G' is c4c, we are done by induction assumption. Thus we will assume that for various simplifications, G' fails to be c4c. We will argue, by encountering numerous contradictions, that this can happen only in a few isolated cases, which can be (and have been) explicitly checked. This is mainly achieved by looking at 3-cuts in G' and studying their position in G.

From now we stipulate, following Section 2.2, that a cut is identified with its cut curve. We take the freedom to regard cut curves up to plane isotopy, which keeps intersections with edges transversal and does not meet any vertices. (We will use somewhat more general isotopies to modify cuts at some places.)

The following terminology will be used throughout the proof.

Definition 3.2. We call a 3-cut *inessential* or *trivial* if it consists of the three edges incident from (i.e., the star of) a vertex. Otherwise is called *essential* or *non-trivial.* A 4-cut is inessential (or trivial) if one of the two components of its complement is a single edge, and otherwise essential (or non-trivial).

Thus G' is not c4c iff an essential 3-cut γ exists in G'. We will study how several such γ look while undoing the simplification $G \mapsto G'$.

It is worth pointing out again that for a given G various, even if not variously designated, G' will be used. Each G' is determined by the cut γ it gives rise to. Thus for example, when we say that γ is inessential in G', we will of course mean the G' which originally, by contradiction, let us find this same γ as an essential 3-cut in it.

The procedure of moving a cut through a vertex (i.e., through an isotopy which crosses one vertex in G exactly once) to decrease its size by 1 will be called below *reducing* the cut.

We will often implicitly use the following argument. If G is c4c, and a 4-cut γ reduces to a 3-cut in G, then γ is a trivial 4-cut in G. Consequently γ cannot become an essential 3-cut in G'.

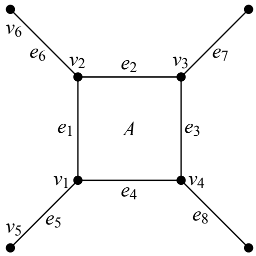
The nomenclature will be as follows: cut(curve)s will be denoted by (variously sub- or superscripted) Greek letters γ, $\mathit{\Gamma}$ or δ. Small latin letters will be used to designate edges (if toward the beginning of the alphabet) and vertices (if toward the end). Capital latin letters will stand for regions (connected components of the planar complement of graphs) or fragments (parts of graphs specified only by the position of their outgoing edges).

In the diagrams below, cut curves will be usually drawn dashed, or in a few cases by thickened lines. Fragments will be indicated by grey areas. Vertices of graphs will often be drawn as beads, but sometimes we will omit the beads to save space. We will occasionally change the infinite region of the planar embedding. (For 3-connected cubic graphs, by Whitney's theorem, a spherical embedding is unique, so the change of infinite region is the only freedom to switch between planar embeddings.)

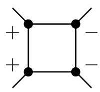
There are two major cases of the induction step, depending on whether a 4-gonal face exists in G, or not.

3.2. The 4-gonal face

Case 1. G has a 4-gonal face A.

(6) 

By choosing markings



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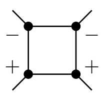
we see that we can obtain a knot marking of G when the there is a knot marking of the graph G_1 obtained from G by replacing (6) by) (. Similarly



will inherit a knot marking from the graph G_2 in which (6) is replaced by .

Thus we are done (by induction assumption) if one of G_1 or G_2 is c4c. We assume thus now that neither G_1 nor G_2 is c4c.

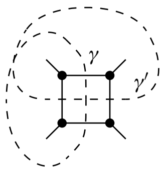
If G_1 is not c4c, then there exists a non-trivial 3-cut C of G_1. In G, either (case 1.1.) one of e_1, e_3 is in a 4-cut γ of G, or (case 1.2.) both e_1 and e_3 are both in a ≤ 5-cut γ of G.

Case 1.1. Let e_1 be in a 4-cut γ. Then one of e_2, e_3, e_4 is also in the 4-cut. Let us exclude e_3, since otherwise we have case 1.2. (below).

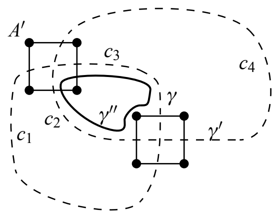
Then replacing e_1, e_2 in the 4-cut γ by e_6, or e_1, e_4 by e_5 gives a 3-cut γ' of G. Since G is c4c, this 3-cut γ' must be trivial, i.e. consist of the three edges around v_5 or v_6. Then γ in G_1 also becomes trivial, in contradiction to our assumption.

Case 1.2. Now e_1 and e_3 are in a ≤ 5-cut γ of G, and similarly we may assume e_2 and e_4 are in a ≤ 5-cut γ' of G.

(7)



Since γ and γ' intersect in A, they intersect also in another face A' of G. Obviously $A' \neq A$, and even A' is not a neighbor face to (i.e., does not share an edge with) A. Otherwise, G is not 3-connected.

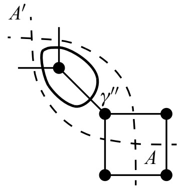


At least one of c_1 and c_3 intersects at most one edge outside ∂A, and at least one of c_2 and c_4 does (since c_1 and c_3 intersect in total 3 edges outside ∂A, and similarly do c_2 and c_4). In the above picture we assume w.l.o.g. $x_1, x_2 \leq 1$, with

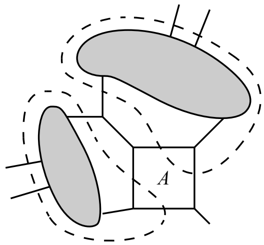
$$x_1 = |c_2 \cap (G \setminus \partial A)|, \quad \text{and} \quad x_2 = |c_3 \cap (G \setminus \partial A)|\,. \tag{8}$$

Moreover, these two numbers are in fact equal to 1, since we argued that A' and A are not neighbored.

Thus one can modify the cuts γ and γ' to a ≤ 3-cut γ'' of G. Since G is c4c, γ'' must be trivial.

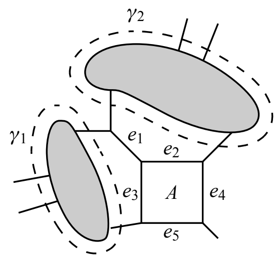


This means then that γ and γ' can be modified to ≤ 5-cuts that intersect neighbored edges in ∂A,



and then reduce to ≤ 4-cuts γ_1 and γ_2. (Above we have omitted putting beads for the vertices of G.)

(9)

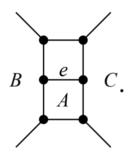


Next, we can exclude the situation that one of γ_i is a 3-cut, and hence with case 1.1. that some edge in ∂A is in a 4-cut. This follows from c4c and the below lemma, which will be applied several times later.

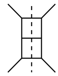
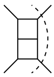
Lemma 3.3. *We can exclude the cases that the 4-gonal face A has a 4-gonal neighbor face, i.e. one sharing a common edge e in its boundary.*

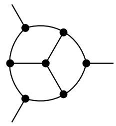
Proof. Assume A has a 4-gonal neighbor face:

(10)

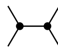
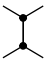


Lemma 3.4. *Let $G' = G \setminus e$ (we remove also the endpoints of e, so that again G' is trivalent). If G is c4c, then G' is c4c, or G is the cube net, shown in* (3)*, which we will call κ.*

Proof. If G' is not c4c, then e must be in a 4-cut . This cut simplifies to a 3-cut , and since G is c4c, it must contain the fragment

(11) .

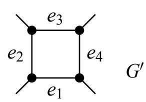
The only such c4c graph is κ. ∎

Observe that one can gain a knot marking of G from knot markings of G_1 and G_2, where G_i are obtained from G by replacing (10) by  and .

Thus Lemma 3.3 is proved with the following lemma.

Lemma 3.5. *When G is c4c and $\chi(G) < -5$, then at least one of G_1 or G_2 is c4c.*

Proof. We have $G \neq \kappa$. Then G' is c4c by Lemma 3.4.



If neither of G_1 and G_2 are c4c, then there is a 4-cut of G' going through e_1 and e_3, and one such cut going through e_2 and e_4. The argument that leads to (9) then modifies to show that γ_1 and γ_2 are 3-cuts, which are trivial, and then we see that one of B or C in (10) is also a 4-gon. Then we have the fragment (11), and so G' is the cube net κ, whence $\chi(G)+1=\chi(G')=-4$, which we excluded. ∎

Lemma 3.5 concludes the proof of Lemma 3.3. ∎

Now we return to the picture (9) and assume also that the graph G' obtained from G by removing e_1, e_2 and e_3 has an essential 3-cut Γ.

Case 1.2.1. If Γ is a 5-cut in G intersecting e_2 and e_3, then it simplifies to a 4-cut intersecting e_1, which will be dealt with in Case 3.2 below.

Case 1.2.2. If Γ is a 4-cut intersecting e_2 and e_4, or e_3 and e_5, then it reduces to a 3-cut Γ' in G, which must be trivial, and then we see that Γ is not essential in G', a contradiction.

Case 1.2.3. If Γ is a 4-cut that intersects e_2 and e_5, or e_3 and e_4, then the argument that led to (9) modifies to show that one of γ_1 or γ_2 is a 3-cut. We dealt with this case in Lemma 3.3 (see the remark before the lemma).

Case 1.2.4. We consider now the situation that Γ intersects e_1, but none of e_2 and e_3 (and Γ is a 4-cut).

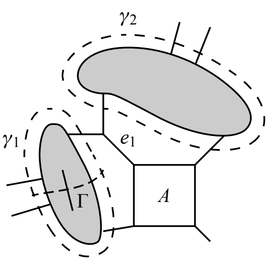
Assume Γ intersects none of the grey regions in (9), which we call $\operatorname{int}\gamma_1$ and $\operatorname{int}\gamma_2$. Then either (1) it can be simplified to a $\leq$3-cut, which must be trivial in G, and so Γ is not essential in G'; or (2) Γ is the cut around A, which is also not essential in G'.

We assume now thus that in (9), w.l.o.g. Γ intersects $\operatorname{int}\gamma_1$. We have that Γ is a 4-cut intersecting e_1. Thus it intersects $\operatorname{int}\gamma_1$ in at most 3 edges.

Case 1.2.4.1. The option that Γ intersects $\operatorname{int}\gamma_1$ in 3 edges is easily ruled out. In this case, the only possibility for Γ would be to go through $\operatorname{int}\gamma_2$ but not cross any edge there. In this situation, by 3-connectedness of G, we see that $\operatorname{int}\gamma_2$ contains just the inner parts of two edges of G (and no vertices). This means that the cut γ' in (7) contains only one vertex of G' in its interior, and cannot be essential in G', in opposition to our assumption. (We will less explicitly apply such an argument several times below.)

Case 1.2.4.2. If Γ intersects no edge in $\operatorname{int}\gamma_1$, then the argument of Case 3.2 applies with $\operatorname{int}\gamma_1$ and $\operatorname{int}\gamma_2$ interchanged and γ' in (7) replaced by γ. This leads to a similar contradiction to the property that γ is an essential 3-cut in G'.

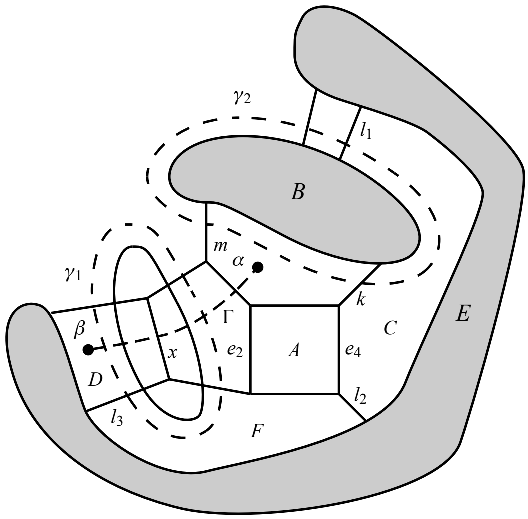
Case 1.2.4.3. Assume Γ intersects $\operatorname{int}\gamma_1$ in one edge (and Γ is not movable out of $\operatorname{int}\gamma_1$ through isotopy):



(above only the part of Γ in $\operatorname{int}\gamma_1$ was drawn).

Then int γ_1 splits into two 3-cuts, which must be trivial.

(12)



Still Γ is drawn only partially, between α and β (which are not vertices of G).

Next we analyze how to close Γ between α and β. If Γ passes through k in (12), we can modify Γ to a 4-cut through e_2 and e_4, which we excluded (see remark above Lemma 3.3). If Γ passes through m, it reduces to a 3-cut (and is not essential in G').

Thus Γ passes through $B = \text{int}\,\gamma_2$. It can obviously intersect at most two edges there.

If Γ intersects no edge in B, then B becomes the parts of two edges of G, and by going back to γ and γ' in (7), we see that some of them is not essential in G'.

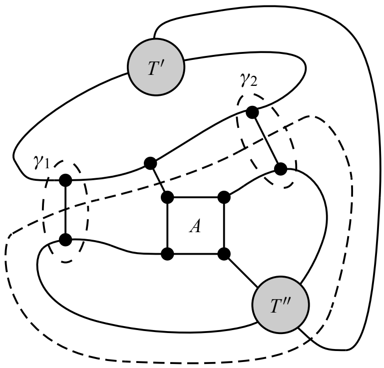
If Γ intersects two edges in B, it must pass through the fragment E without intersecting any edge there. In that case, by c4c, the edges l_1, l_2 and l_3 become the star of a vertex of G, and F turns into a 4-gonal neighbor face of A, a situation we already finished off.

Thus Γ intersects exactly one edge in B. Note that the regions C and D are not the same, for otherwise $\{x, e_2, e_4\}$ would be an essential 3-cut in G. Thus Γ cannot intersect l_1. The other possible intersections of Γ with edges outside E are even easier to exclude. Then Γ must also pass through E and intersect exactly one edge there.

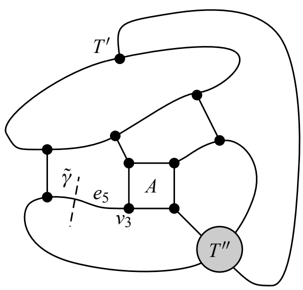
With this argument, we see that there is only one interesting way to close $\varGamma$ in (12) between α and β with 2 edges disjoint from ∂A. (Interesting means that all other options lead to cases we have excluded or dealt with.)

This interesting case is shown below.

(13)



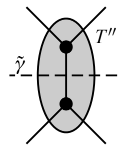
By cyclic 4-connectivity of G, the fragment T' is trivial.



Now remove the three edges incident from v_3, and call the resulting graph G'. This shows that e_5 must be in a 4-cut $\tilde{\gamma}$, which reduces to an essential 3-cut in G'. (We considered already the cases that some edge in ∂A is in a 4-cut; see comment above Lemma 3.3.)

A direct check shows that $\tilde{\gamma}$ must go through T'' and intersect at most one edge there (otherwise it is not essential in G'). Then, using that G is c4c, we see that the only option is up to rotation

(14)

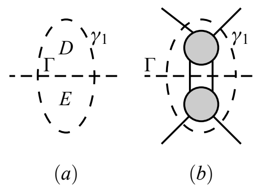


.

Thus T'' contains only two vertices, and G becomes some concrete simple graph of $\chi=-6$, which we chose not to consider.

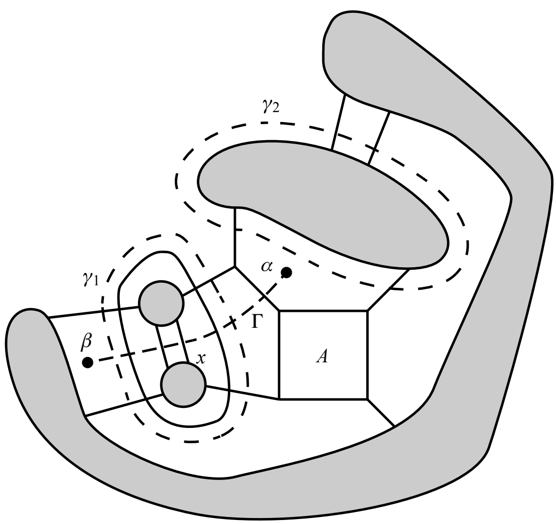
Case 1.2.4.4. This is the situation that Γ intersects 2 edges in $\operatorname{int}\gamma_1$. Again we use the cyclic 4-connectedness of G. In the below picture (15) part (a), both D and E contain each two ends of $\operatorname{int}\gamma_1$ (otherwise Γ reduces to a 3-cut in G, and is trivial in G').

(15)



Then $\operatorname{int}\gamma_1$ looks like part (b). Consequently, (12) modifies to

(16)



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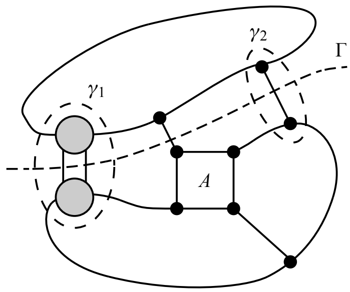
and then the same problem with connecting α and β to close Γ occurs (now there is only one edge extra to intersect). By an argument similar to the one leading to (13), we are left with the only possible situation



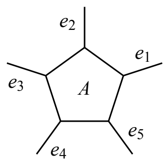
,

which is dealt with by the observation that A has a 4-gonal neighbor face.

3.3. The 5-gonal face

Case 2. G has no 4-gonal face. Then it must have a pentagonal face A.

(17)



We use two types of simplifications. We can obtain a knot marking of G, if there is a knot marking of a graph G' of one of the following two sorts.

1) G' is obtained from G by removing two non-neighbored edges in ∂A. We see this by choosing the sign (in a marking) of two neighbored vertices in ∂A opposite to the three others.
2) G' is obtained from G by removing the three edges incident to a vertex in ∂A (using γ-construction).

So we are done if some of these G' is cyclically 4-connected. We assume for all G' in 1) and 2) that G' is not cyclically 4-connected.

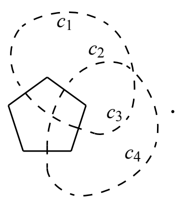
For each option we fix an essential 3-cut in G' and analyze its position in G. Let us also stipulate that we fix the (essential) 3-cut in G' so that it is a cut of a size as small as possible in G. It is always a ≤ 5-cut (in G) for 1) and a ≤ 4-cut for 2). It cannot be a 3-cut in either case, since it would be trivial in G, and hence in G'. Thus in particular it is a 4-cut for 2). We make a case distinction with regard to the size for 1).

Case 2.1. None of the cuts 1) is (choosable to be) a 4-cut in G for any pentagonal face A of G. We put forward the following four helpful observations.

- Under this case's assumption we have that *no edge in ∂A is in an essential 4-cut of G* (cf. Definition 3.2). This is because an essential 4-cut of G cannot go through two neighbored edges in ∂A, and so it must be of type 1).
- This implies then (using type 2) γ-construction) that in (17) *all e_i are in a(n essential) 4-cut,*
- *which in turn means that* A has no pentagonal neighbor face (because otherwise some e_i lying in the boundary of that face would give a contradiction to the first observation).
- Also, *each 4-cut through e_i does not intersect $e_{i\pm 1}$ (where $e_0 = e_5$ and $e_6 = e_1$). Otherwise, we find a 4-cut of type 1).*

Consider the 5-cuts corresponding to pairs in ∂A below

(18)

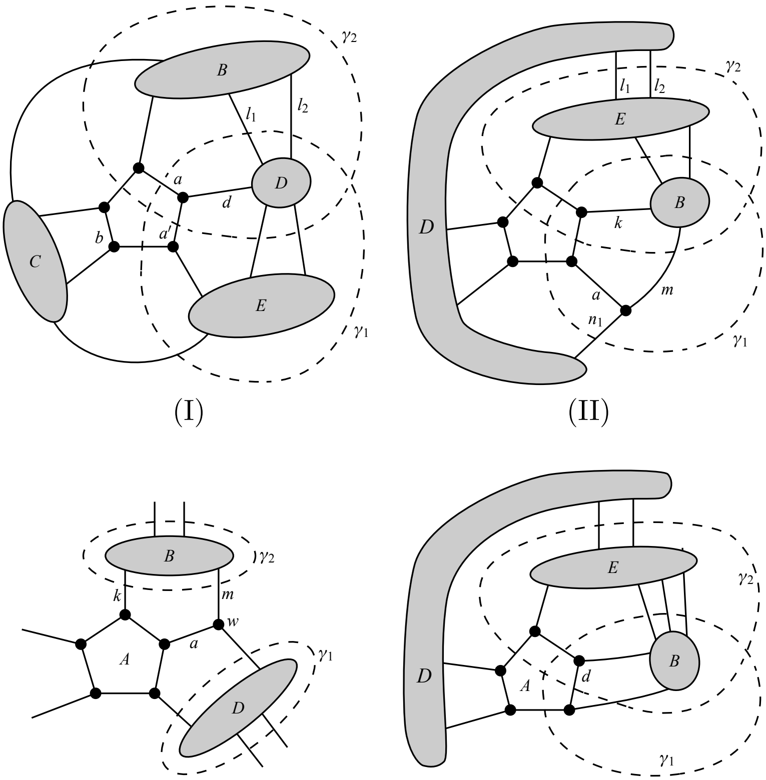


We consider again x_1, x_2 as in (8), now based on (18). Obviously, both numbers are between 0 and 3, and we may by symmetry assume that $x_1 \geq x_2$.

The options that (x_1, x_2) is one of $(0,0)$, $(3,0)$ and $(3,3)$ fail because of the 3-connectedness of G.

The case $(3,2)$ shows, by c4c property, that G has a triangular face. For $(2,0)$ and $(1,0)$, by c4c property of G, we see that γ_1 fails to be an essential 3-cut in G'.

There remain the options $(2,2)$, $(2,1)$, $(1,1)$ and $(3,1)$. These give rise to the diagrams (I) to (IV) in (19). (In (III), we have again rearranged and reduced the cuts, similarly to (9).)



(19) (I) (II)

(III) (IV)

Case 2.1.1. We consider first (II) and (III). Let γ be the 4-cut through a.

The option that γ passes through k was excluded in the fourth of our initial observations. If γ passes through m, then it reduces to a 3-cut (and is not essential in G'). Thus γ must go through B.

Case 2.1.1.1. Type (III). The symmetry between B and D allows us to argue similarly that γ must pass through D. Since γ passes in total at most 3 edges in B and D, by the same symmetry we may w.l.o.g. assume that γ intersects at most one edge in D.

If γ intersects D in no edge, D becomes parts of two edges of G. Then one can see that one of the two cuts in (18) (similar to γ and γ' in (7)) which simplified to γ_1 and γ_2 in (III) is not essential in (its) G'.

If γ intersects D in one edge, D becomes two vertices like T'' in (14), and A has a pentagonal neighbor face, which we observed how to deal with.

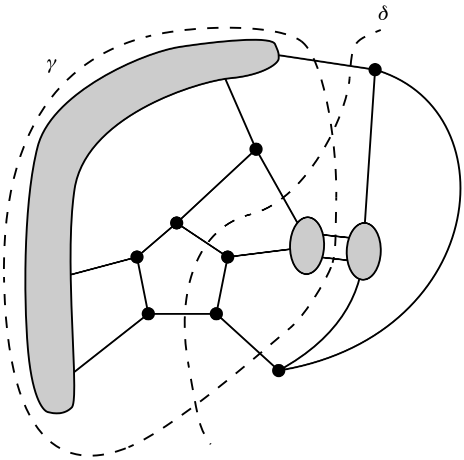
Thus we continue considering only type (II).

Case 2.1.1.2. If γ intersects three edges in B, then it must pass through D and intersect no edge there. Then, D simplifies to (the star of) a vertex and (part of) an extra edge of G. Then G has a triangular face (and is not c4c) or a 4-gonal face (which we already dealt with).

Thus γ intersects at most two edges in B. Then it must split the 4 outer B ends 2-2, as in part (b) of (15) (with Γ replaced by γ, and γ_1 by the boundary of B). Otherwise γ can be moved out of B (through an isotopy which crosses one vertex in B exactly once) to a 3-cut, which becomes trivial.

Case 2.1.1.3. Now assume γ intersects B in exactly two edges. Then, γ can intersect D and E in total in at most one edge. One easily sees that γ must go through both D and E. If γ intersects no edge in D, then we have a triangular or 4-gonal face of G.

Thus γ must intersect D in exactly one edge and E in no edge. In particular, E becomes again a vertex and an extra edge. One can check that the only option of having a c4c graph G and keeping the minimal length of γ_2 in (II) is (IIc).



(IIc)

Then, however, there is a 4-cut δ through two non-neighbored edges in ∂A, and we arrive at case 2.2 below.

Case 2.1.1.4. If γ in (II) does not intersect an edge in B at all, the interior of γ_1 will contain only 3 vertices of G, two of which are in ∂A. Thus this interior will remain with only one vertex of G' (where G' is the graph obtained in the argument which ascertained the existence of γ_1), and γ_1 will not be essential in G'. This is a contradiction.

Case 2.1.1.5. Since thus γ intersects B in exactly one edge, B becomes like T'' in (14), and A has a pentagonal neighbor face.

Case 2.1.2. In (I) consider the 5-cut γ that goes through a and b.

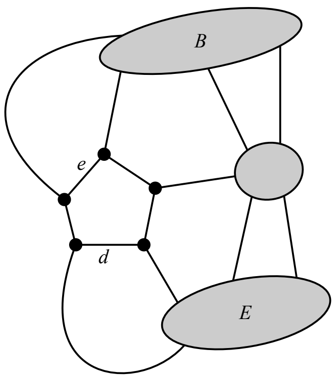
If γ intersects C in exactly one edge, then C becomes like T'' in (14), and A has a 4-gonal neighbor.

If γ intersects C in 3 edges, then γ either passes through B and intersects no edge there, or it passes through D and intersects no edge there. In either situation, one can see that this is only possible so that l_1 and l_2 become parts of the same edge of G. This edge can be moved out of intersection with γ_1, and γ_1 simplifies to an essential 3-cut of G, which is impossible.

Now we argue in the same way with the 5-cut γ' through a' and b.

If γ and γ' intersect C in two edges (each), γ will intersect B in one edge e, and γ' will intersect E in one edge e'. Then $\{d, e, e'\}$ will become an essential 3-cut of G, which is impossible.

Then the only option is that γ (and γ') does not intersect any edge in C, as shown in (Ia).



(Ia)

In (Ia) argue with the 5-cut γ through e and d. It must go through B and E and intersect one of them in exactly one edge. (If γ intersects no edge inside B or E, then one of γ_1 or γ_2 in (19) is not essential in G'.) Then again A has a 4-gonal neighbor.

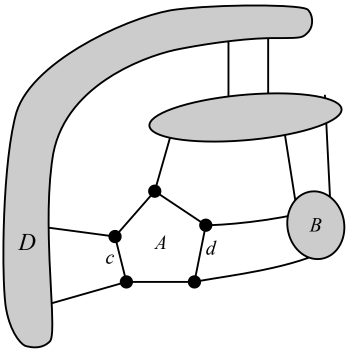
Case 2.1.3. Type (IV). The cut γ_2 reduces to a 4-cut of type 1), by being moved off d. This 4-cut is essential, unless D contains only parts of two edges of G, and then γ_2 is not an essential cut in G'.

Case 2.2. There is a 4-cut of type 1). The derivation of the four cases in (19) based on (18) goes as before. The difference is now that we may assume in each case (at least) one edge intersecting one of the cut curve segments c_1 or c_4 in (18) is missing.

Case 2.2.1. In (III) one of B or D collapses to a vertex, and we have a 4-gonal face.

Case 2.2.2. We consider the situation that in (II) some edge through one of c_1 or c_4 is missing.

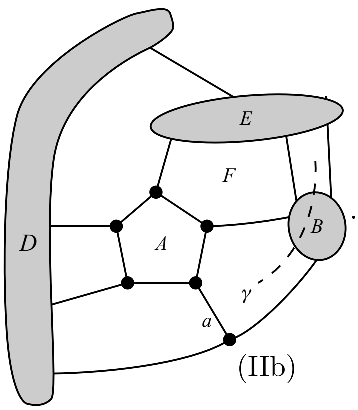
Case 2.2.2.1. If n_1 is missing, we have (IIa).



(IIa)

Then the 5-cut γ through c and d must cut B or D in exactly one edge, which would make it look as in (14), and so A has a 4-gonal neighbor. (If γ cuts B or D in no edge, one of γ_1 or γ_2 in (19) would not be essential in G'.)

Case 2.2.2.2. If one of l_1 or l_2 is missing in (II), we have:

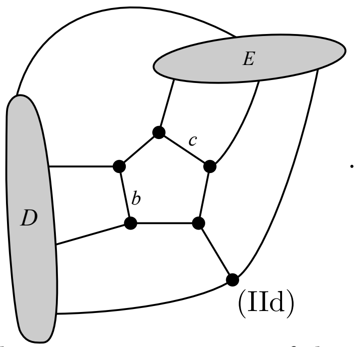


(IIb)

Consider the 4-cut γ through a. First we argue that γ must intersect D in at least two edges; otherwise, A has a triangular or 4-gonal neighbor face. Next, γ must pass through B, and it can intersect at most one edge in B.

If γ intersects an edge in B, it must pass through E and not intersect any edge there. Then the face F turns into a triangle.

If γ intersects no edge in B, then we have



The 5-cut through b and c must intersect one of the fragments D or E in one edge. Then the fragment collapses to (stars of) two vertices, and we have a 4-gonal face.

Case 2.2.3. Next we consider (IV) with some edge through c_1 in (18) missing (there are no edges through c_4). Then D collapses to a vertex, and G has a triangular face.

Case 2.2.4. Finally we consider the case that in (I) some edge through one of c_1 or c_4 is removed. Again, C collapses to a vertex, and we have a triangular face. ∎

4. Applications and generalizations

4.1. Limit links and graph volume

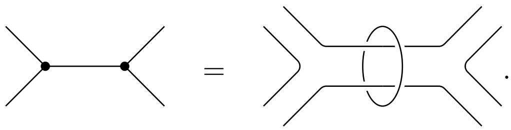
Recall the terminology on Seifert surfaces in Section 2.4.

A result of Crowell-Murasugi is that when L is an alternating link, then $g(L) = g_c(L)$ and $\chi_c(L) = \chi(L)$. We use this result to study $\chi(L)$ using a canonical Seifert surface. (What we say below is true for arbitrary links L, when χ is replaced by χ_c.)

For reasons elucidated in [21,22,24] (for knots, but the case of links is analogous), it is enough to study diagrams whose Seifert graphs are 2-3 valent. These correspond to a marking on a planar 3-valent graph G: attach signs to the vertices of G depending on what side of the canonical Seifert surface S is visible at the corresponding Seifert circle. Then bisect even edges.

Furthermore, from [24] it is suggestive why it is enough to consider only 3-connected (planar 3-valent) graphs G in this setting.

Define for a 3-connected 3-valent planar graph G the (*unoriented* limit) link L_G by

(20) 

These links can be used to calculate the maxima (1) and (2). Namely, using a result in [2], we found that (with both bracketed expressions omitted or both not)

$$v_{[n,]\chi} = \max\{\mathrm{vol}(L_G)\colon \chi(G) = \chi[, G \text{ has an } n\text{-component marking}]\}.$$

Now, as we will show in Section 4.1, not every G admits an n-component marking for every χ-admissible n. The way to Theorem 1.2 is to "rearrange" G via an operation we will call composition.

Definition 4.1. The *composition* '#' of two planar cubic graphs is defined by

In this way c4c means "prime" with respect to composition: G is c4c iff whenever $G = G_1 \# G_2$, one of G_1 or G_2 is θ.

Note that this operation is highly ambiguous. It depends not only on the choice of vertices in G_1 and G_2, but also on the (mutual) cyclic ordering of their incident edges.

It turns out that $\mathrm{vol}(L_G)$ is related to the graph volume $\mathrm{vol}(G)$ defined by v.d. Veen [27]. He explains how to make the complement of a graph embedded in S^3 into a cusped 3-manifold. The hyperbolicity of this manifold is not immediate in general, but for a planar graph $G \neq \theta$ it can be established rather easily. V.d. Veen argues that some sort of Mostov rigidity applies to such manifolds, so that there is a meaning to the *volume of a (planar cubic) graph* $\mathrm{vol}(G)$.

Graph volume is additive under composition, regardless of how composition is performed. This allows us to change G without changing $\mathrm{vol}(L_G)$.

The proof of Theorem 1.2 consists in decomposing a graph into c4c pieces, applying Theorem 1.1 to obtain a knot marking on the separate pieces, and then putting them together in a way so that the existence of a knot marking is preserved. (Higher n and even χ follow easily, using almost the same considerations as in Section 4.3 below.)

This implies then that for each G we can find a graph G' with $\chi(G')=\chi(G)$, $\mathrm{vol}(L_{G'})=\mathrm{vol}(L_G)$, and such that G' has an n-component marking for all χ-admissible n.

4.2. A relation to the sl_N weight system

Markings are related to a quite different object, arising in the theory of Vassiliev invariants [4]. We will briefly explain this relation.

Recall that for any Lie algebra with *ad*-invariant non-degenerate scalar product, one can associate a *weight system*, an integer-valued invariant of 3-valent graphs subject to certain local relations (see [4]). The calculation of the weight system $W_N(G)$ of sl_N on a 3-valent graph G is described[1] in [4, Section 6.3.6]. It uses a construction very reminiscent to the even-odd coloring of edges in G, and can in our language be written as follows:

$$W_N(G)=W_{N,+}(G)-W_{N,-}(G),$$

with

$$W_{N,+}(G)=\sum_{O\text{ even}} N^{n(L_{G,O})},\quad\text{and}\quad W_{N,-}(G)=\sum_{O\text{ odd}} N^{n(L_{G,O})}.$$

Here the total number of summands of both sums is equal to the number $2^{-2\chi(G)}$ of choices of orientation O of the (Seifert circles of $D_{G,O}$ corresponding to the) $-2\chi(G)$ vertices of G. As in [4], it is useful to regard herein N as a variable rather than as some given number, so the W_N become polynomials in N.

We conclude the account on volume with the following question, which will be treated in detail in our subsequent work:

Question 4.2. *Does $W_N(G)$ determine* $\mathrm{vol}(G)$ *(or equivalently* $\mathrm{vol}(L_G)$*)?*

[1] Bar-Natan remarks that the *ad*-invariant non-degenerate scalar product on sl_N is unique up to scalars, so that the construction below is valid for a proper choice of constants.

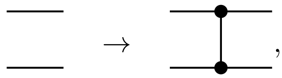
4.3. Even χ and higher n

Here we discuss the following generalization of Theorem 1.1.

Theorem 4.3. *Let G be a planar cubic cyclically 4-connected graph of Euler characteristic $\chi < 0$. Then G is the dual of an n-vertex triangulation of a closed orientable surface for any χ-admissible n.*

Proof. Again, let us think of the statement as a property of markings of G.

Let us first consider the case $n=2$ (and even χ). To obtain this case from the statement for $n=1$ in Theorem 1.1, observe that one can apply to a c4c graph G of even χ the move

(21) 

so that the resulting graph G' is c4c. Then using the odd-χ property on G', and removing in a knot marking of G' the band corresponding to the edge on the right of (21), gives a two component marking of G.

With this we obtained the cases $n=1,2$. The rest of the proof of Theorem 4.3 is accomplished by the following lemma. ∎

Lemma 4.4. *Let G have an n-component marking with $n < 2-\chi(G)$. Then G has an n'-component marking for all $2-\chi \geq n' \geq n$ with $n'-n$ even.*

Proof. Now it is to show that one can arbitrarily augment the number of components of L for fixed G by varying O.

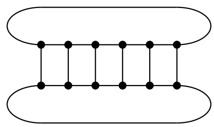
For this use that there is another even-odd edge coloring of G, namely all edges even, giving rise to a diagram of the maximal number $n = 2-\chi$ of components. To conclude the claim for the other n, note that the change of orientation of any Seifert circle changes the number of components by 0 or ± 2. ∎

4.4. Non-existence examples

The problem which graphs admit knot markings (or Wicks forms), although having a recursive solution [26], seems too difficult to solve explicitly. A generic graph would likely have a knot marking, but graphs without knot markings exist. In fact we have (using the notation of Section 4.2):

Proposition 4.5. *There exist 3-connected 3-valent planar graphs of both parities of* χ *with the minimal component number* $\min\deg_N W_{N,\pm}$ *of a marking arbitrarily large.*

Proof. We give a series of explicit examples. Consider the graph A_n, which is the net of the n-gonal prism (or wheel of n stairs/spokes [6]; depending on the embedding). The representative example for $n=6$ is



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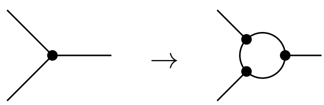
We build a graph B_n out of A_n by the local replacement

(22)



at each vertex. The graph B_n is planar, and 3-connected if $n \geq 3$, and we have $\chi(B_n) = -3n$, so both parities are covered.

We claim now that

$$\lim_{n\to\infty} \min\deg_N W_{N,\pm}(B_n) = \infty.$$

Assume the contrary, i.e. that there is a sequence of markings O_n of B_n such that $n(L_{B_n,O_n})$ remains bounded. If for some vertex of A_n the three vertices on the right of (22) have the same sign in O_n, then there is a loop within the cycle of length 3 they bound. Thus assume that the number of such vertices is bounded. Then arbitrarily large portions of B_n have 2-to-1 signing of the vertices on the right of (22). In this case one of the components of D_{B_n,O_n} forms a loop entering and exiting the length-3-cycle near the oppositely signed vertex, while the other two strands pass between the other two vertices.

Then, after changing orientation in O_n in a bounded (in n) number of vertices of B_n, we have that components of D_{B_n,O_n} correspond to some decomposition of A_n into edge-disjoint paths. Paths can either be closed paths (chains), in which all vertices have valence 2, or open paths, in which all vertices have valence 2 except exactly two, which have valence 1 or 3 (a path may connect onto itself). Each closed path will contribute two components of D_{B_n,O_n}, and each open path will contribute one.

Now, each (3-valent) vertex of A_n must be the beginning or end of some path. Thus A_n cannot be covered by a fixed number of (edge-disjoint) paths when n is large enough, and we have a contradiction. ∎

One can then obtain more graphs without low component markings by modifying locally B_n for n large enough. In particular we have:

Corollary 4.6. *There exist 3-connected 3-valent planar graphs G without knot markings when $\chi(G) \le -9$ odd.*

Proof. In fact, the argument for Proposition 4.5 shows the explicit inequality

$$\min\deg_N W_{N,\pm}(B_n) \ge \frac{1}{2}v(A_n) = \frac{1}{6}v(B_n). \tag{23}$$

Taking $n \ge 3$ odd, we handle genus $g \ge 5$ odd (where $\chi = 1-2g$). For $g \ge 8$ even, take the example for genus $g-1$ and apply twice the move (21). (We noticed that this move changes $\min\deg_N W_{N,\pm}$ by ± 1.) To deal with $g=6$, construct a graph from A_4 as B_4, but not applying (22) at one of the vertices of A_4. ∎

Remark 4.7. Since we need $n \ge 3$ in order B_n to be 3-connected, our examples with odd χ start with $\chi = -9$. We concluded in [22] from the enumeration of 3-connected 3-valent planar graphs (see [7,25], or [19, sequence A000109] for an extensive list of references), that 3-connected examples with odd $\chi > -9$ do not exist. As mentioned, despite such examples for smaller χ, most "generic" graphs still seem to have knot markings. For instance, among the 1249 planar 3-connected 3-valent graphs with $\chi = -9$ found by the program of Brinkmann and McKay [8], B_3 is indeed the only one without knot markings.

Inequality (23) holds more generally for B_n, whatever trivalent planar graphs A_n we build it from using (at every vertex) the move (21). Thus the argument for Proposition 4.5 works when A_n is an arbitrary sequence of graphs with increasing number of vertices.

One can also see that in fact for our particular examples (23) is sharp, and then ask whether they are the 'worst' possible.

Question 4.8. *Does for any simple G with $\chi(G) < -1$ the inequality hold $\min\deg_N W_{N,\pm}(G) \le \frac{1}{6}v(G)$?*

Note that such an inequality would give a wide range of n, for which the claim of Theorem 4.3 holds for simple G. For non-simple G we must replace '$\frac{1}{6}$' at least by '$\frac{1}{2}$'.

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