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TRANSLATIONAL TILINGS BY A POLYTOPE, WITH MULTIPLICITY

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We study the problem of covering $\mathbb{R}^d$ by overlapping translates of a convex polytope, such that almost every point of $\mathbb{R}^d$ is covered exactly k times. Such a covering of Euclidean space by a discrete set of translations is called a k-tiling. The investigation of simple tilings by translations (which we call 1-tilings in this context) began with the work of Fedorov [5] and Minkowski [15], and was later extended by Venkov and McMullen to give a complete characterization of all convex objects that 1-tile $\mathbb{R}^d$.

By contrast, for $k \geq 2$, the collection of polytopes that k-tile is much wider than the collection of polytopes that 1-tile, and there is currently no known analogous characterization for the polytopes that k-tile. Here we first give the necessary conditions for polytopes P that k-tile, by proving that if P k-tiles $\mathbb{R}^d$ by translations, then it is centrally symmetric, and its facets are also centrally symmetric. These are the analogues of Minkowski's conditions for 1-tiling polytopes, but it turns out that very new methods are necessary for the development of the theory. In the case that P has rational vertices, we also prove that the converse is true; that is, if P is a rational polytope, is centrally symmetric, and has centrally symmetric facets, then P must k-tile $\mathbb{R}^d$ for some positive integer k.

1. Introduction

Suppose we are given a convex polytope P, and a multiset of discrete translation vectors Λ. We wish to cover all of $\mathbb{R}^d$ by translating P using the translation vectors in Λ, such that each point $x \in \mathbb{R}^d$ is covered exactly k times. Along the boundary points of P there may be some technical lower-dimensional problems, but if we require that each point which does not lie

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on the boundary of any translate of P to be covered exactly k times, then we call such a covering of $\mathbb{R}^d$ a *k-tiling*. The traditional field of tilings of Euclidean space by translates of a single convex polytopes P has a long and rich history. The usual notion of a *simple tiling* by translations is thus equivalent to the notion of a 1-tiling. The reader is invited to consult the books by Alexandrov [1] and Gruber [7] for a nice overview of the problem of tiling space with translates of one convex polytope.

Tilings of $\mathbb{R}^d$ by translations of a single object have been extensively studied from as early as 1881 [5], by the mathematical crystallographer Fedorov, and are an active research area today. For example, even translational tilings of sets on the real line are highly non-trivial and have been studied in the 90's by Lagarias and Wang [12]. There is also a beautiful recent survey article on k-tilings and other related tilings, in various different mathematical contexts, by Kolountzakis and Matolcsi [11].

Minkowski [15] has shown that if a polytope P 1-tiles $\mathbb{R}^d$ with any discrete set of translation vectors, then it follows that P is a centrally symmetric polytope, with centrally symmetric facets. Venkov [18] and McMullen [13] proved that if P 1-tiles $\mathbb{R}^d$ by translation, then for each of its codimension two faces F there are either four or six faces which are translates of F; furthermore, they proved that the latter condition, taken together with Minkowski's centrally symmetric conditions, gives a complete characterization of 1-tiling polytopes.

There is a rich collection of convex polytopes that fail to 1-tile, but nevertheless do k-tile for some positive integer $k \geq 2$. A simple 2-dimensional example is the octagon in figure 1. Thus, the subject of convex polytopes that k-tile by translations is much more diverse. Various different aspects of k-tilings have been studied, for example the fascinating group-theoretic connections with the famous Minkowski facet-to-facet 1-tiling conjecture, was extended to k-tilings by cubes in 1936, by Furtwängler [6], as well as by Hajós [8], Robinson [16], and Gordon[3]. It was also known that in $\mathbb{R}^2$, every k-tiling convex polytope has to be a centrally symmetric polygon, and there exists a nice characterization by Bolle [19] of all *lattice* k-tilings of convex bodies in $\mathbb{R}^2$. In addition, Kolountzakis [10] proved that every k-tiling of $\mathbb{R}^2$ by a convex polygon P, which is not a parallelogram, is a k-tiling with a finite union of two-dimensional lattices.

Despite the work on k-tilings over the last 75 years, there has never been an analogue for the Minkowski–Venkov–McMullen characterization of 1-tiling polytopes. Even in $\mathbb{R}^2$ the situation is rather tricky and remains open in general. One reason for this difficulty is that Minkowski's methods of proof

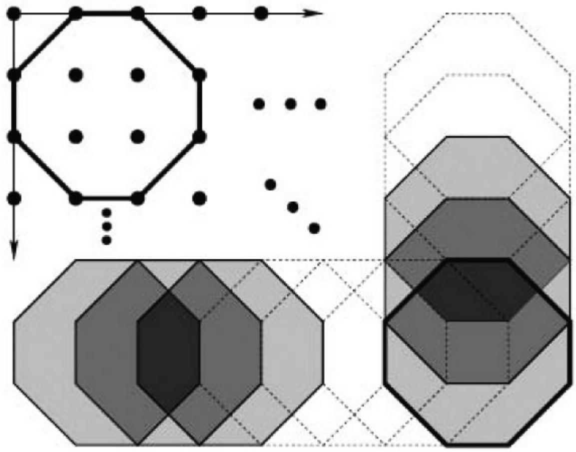

Figure 1. An octagon that 7-tiles, but does not 1-tile. Here, each point in the interior of the octagon is covered exactly 7 times, once we translate the octagon by all of the integer translation vectors.

rely heavily on the facet-to-facet condition, and Minkowski's methods work for 1-tilings but fail for higher k-tilings with $k \geq 2$.

Here we find such an analogue, in the case of general k-tilings, for any positive integer k. The main result of this paper can be stated as follows.

Theorem 1.1. *If a convex polytope k-tiles $\mathbb{R}^d$ by translations, then it is centrally symmetric and its facets are centrally symmetric.*

Our proof of the main theorem above introduce some new ideas, involving certain boundary operators that keep track of the number of points of Λ that are contained in various faces of P. These ideas are quite different from Minkowski's proof for 1-tilings. In the case that the polytope has rational coordinates for its vertices, we also prove that the converse of the main Theorem 1.1 is true.

Theorem 1.2. *Every rational polytope P that is centrally symmetric and has centrally symmetric facets must necessarily k-tile $\mathbb{R}^d$ with a lattice, for some positive integer k.*

Moreover, the polytope P must k-tile $\mathbb{R}^d$ with the rational lattice $\frac{1}{N}\mathbb{Z}^d$, where N is the lcm of the denominators of all the vertex coordinates of P.

We note that in $\mathbb{R}^3$ the two conditions of the main Theorem 1.1 are enough for a convex polytope to necessarily be a *zonotope*, which is the projection of some l-dimensional cube. Equivalently, it turns out that a zonotope is a polytope with the property that all of its k-dimensional faces are centrally symmetric, for all $1 \le k \le d$. For example, the zonotopes in $\mathbb{R}^2$ are the centrally symmetric polygons.

However, in dimension 4 the conclusion that a k-tiling polytope must be a zonotope is no longer true. A counterexample is furnished by the 24-*cell*, which is a polytope in $\mathbb{R}^4$ which 1-tiles $\mathbb{R}^4$, is centrally symmetric, has centrally symmetric facets, but is not a zonotope because it has 2-dimensional faces that are triangles. The 24-cell is by definition the Voronoi region for the root lattice D_4, and the reader may consult Coxeter [4] for more details.

Very little is known about the precise classification of polytopes which k-tile $\mathbb{R}^d$ by translations. We outline some specific open questions in the last section that pertain to the current state of affairs along these lines. (see [7], pages 463-479 for more details about 1-tiling polytopes and some open problems).

A *parallelotope* is, by definition, a convex polytope that 1-tiles $\mathbb{R}^d$ facet-to-facet, with a lattice. That is, its multiset of discrete translation vectors Λ is in fact given by a lattice in this case. It was proved by McMullen [13] that if a polytope 1-tiles $\mathbb{R}^d$ with a discrete multiset of translations Λ, then it must also admit a facet-to-facet tiling with a *lattice*. In other words, McMullen showed that every 1-tiler must be a parallelotope.

It is clear that not all zonotopes are parallelotopes, an easy example being furnished by the octagon (see fig.1) in two dimensions, which clearly does not 1-tile by a lattice of translation vectors; conversely, not all parallelotopes are zonotopes, as evidenced by the example of the 24-cell given above. In fact, McMullen has given a beautiful characterization of those parallelotopes which are zonotopes, in terms of unimodular systems (see [14] for more details).

The paper is organized as follows. Section 3 is devoted to the proof of the main result, namely Theorem 1.1, and comprises the main body of the paper. Section 4 is short, and is devoted to the proof of Theorem 1.2. In section 5 we provide a more analytic approach of the main result, using Fourier techniques. Although it is not crucial to supply another proof of the main result, this approach provides a Fourier lens through which we can view our results. Kolountzakis has also studied this problem using the Fourier approach, and indeed our Fourier approach borrows some techniques from his work.

In section 6 we give another necessary and sufficient condition for a polytope to k-tile $\mathbb{R}^d$, this time in terms of the solid angles of the vertices of P.

Finally, in section 7 we mention some of the important open problems concerning polytopes that k-tile $\mathbb{R}^d$.

2. Definitions and preliminaries

We adopt the usual conventions and notation from combinatorial geometry. First, we recall that the *Minkowski sum* of two multisets $A \subset \mathbb{R}^d$ and $B \subset \mathbb{R}^d$ is the set $A+B = \{a+b \colon a \in A, b \in B\}$, and that the Minkowski difference is defined similarly by $A-B = \{a-b \colon a \in A, b \in B\}$.

For any set $A \subset \mathbb{R}^d$, its *opposite set* is defined as $-1 \cdot A = \{-a \colon a \in A\}$. We are particularly interested in the case that both A and B are polytopes. We are also keenly interested in the case that A is a polytope and B is a discrete set of vectors, so that here $A+B$ is a set of translated copies of the polytope A.

Given a convex polytope $P \subseteq \mathbb{R}^d$, ∂P denotes the boundary of P. The standard convention for ∂P includes the fact that it has (d-dimensional) Lebesgue measure 0, with respect to the Lebesgue measure of $\mathbb{R}^d$. We let the interior of a body P be denoted by $\mathrm{Int}(P)$. Throughout the paper, Λ denotes an infinite discrete multiset of vectors in $\mathbb{R}^d$, which is not necessarily a lattice.

We say that a polytope P *k-tiles* $\mathbb{R}^d$ with the discrete multiset Λ, if after translating P by each vector $\lambda \in \Lambda$, almost every point of $\mathbb{R}^d$ (except for the boundary points of translated copies of P) is covered by exactly k of these translated copies of P. This condition can be written more concisely as follows:

$$\sum_{\lambda \in \Lambda} 1_{P+\lambda}(v) = k,$$

for all $v \notin \partial P + \Lambda$.

We also recall that a *facet* of a d-dimensional polytope is any one of its $(d-1)$-dimensional faces. We let $V^k(F)$ denote the k-dimensional volume of a k-dimensional object F, even if F resides in a higher dimensional ambient space, and sometimes we simply write $V(F)$ for the d-dimensional volume of a d-dimensional object $F \subset \mathbb{R}^d$. Finally, $\#(A)$ denotes the cardinality of any finite multiset A.

3. Proof of Theorem 1.1

To simplify the ensuing notation, we will assume that $-1 \cdot P$ k-tiles $\mathbb{R}^d$. We do not lose any generality, because $-1 \cdot P$ k-tiles $\mathbb{R}^d$ if and only if P also k-tiles $\mathbb{R}^d$.

We say that $v \in \mathbb{R}^d$ is in *general position* if there are no points of Λ on the boundary of $P+v$. In other words, $v \notin \Lambda - \partial P$. We first prove the following elementary but useful lemma, giving an equivalent condition for k-tiling in terms of the number of Λ-points that lie in a 'typical' translate of P.

Lemma 3.1. *A convex polytope $-1 \cdot P$ k-tiles $\mathbb{R}^d$ by translations with a multiset Λ if and only if $\#(\Lambda \cap \{P+v\}) = k$ for every v in general position.*

Proof. Suppose that $-1 \cdot P$ k-tiles $\mathbb{R}^d$. Then for every $v \notin \partial(-1 \cdot P) + \Lambda$ we can write

$$k = \sum_{\lambda \in \Lambda} 1_{\{-1 \cdot P + \lambda\}}(v) = \sum_{\lambda \in \Lambda} 1_{P+v}(\lambda) = \#(\Lambda \cap \{P+v\}).$$

It remains to mention that $\partial(-1 \cdot P) + \Lambda = \Lambda - \partial P$. The proof in the other direction is identical. ∎

We need to introduce some useful and natural notation for the theorems that follow. Let $\mathfrak{P}$ be the vector space of the real linear combinations of indicator functions of all convex polytopes in $\mathbb{R}^d$. Thus, for example, if P is any convex k-dimensional polytope and Q is any convex m-dimensional polytope, then $\frac{1}{3} \cdot 1_P - 2 \cdot 1_Q \in \mathfrak{P}$.

One of the most important operators for us is the following boundary operator, with respect to a vector $n \in \mathbb{R}^d$. It is the function $\partial_n \colon \mathfrak{P} \to \mathfrak{P}$, defined as follows:

$$\partial_n 1_P = 1_{F^+} - 1_{F^-},$$

where F^+ and F^- are the (possibly degenerate) facets of P with outward pointing normals n and $-n$, respectively. It is a standard vector space verification that this operation is also well-defined on $\mathfrak{P}$.

We also define this boundary operator on all of $\mathfrak{P}$, by letting it act as a linear operator on the linear combinations of indicator functions of polytopes. For example, another iteration of this operator on $P \subset \mathbb{R}^3$ yields $\partial_{n_2}(\partial_{n_1} P) = \partial_{n_2}(1_{F^+} - 1_{F^-}) = (1_{E_1} - 1_{E_2}) - (1_{E_3} - 1_{E_4})$, where E_1, E_2 are the edges (which are by definition the 1-dim'l faces) of F^+, and E_3, E_4 are the edges of F^-. In this case, each of the four edges is orthogonal to both of the vectors n_1 and n_2, as is seen in Figure 2 below.

For the sake of convenience, we also define the action of the boundary operator ∂_n on convex polytopes P as follows:

$$\partial_n P = \operatorname{supp}(\partial_n 1_P) = \{v \in \mathbb{R}^d \mid \partial_n 1_P(v) \neq 0\},$$

so that the same symbol now acts on the subset P. However, we note that the more salient operator for our discussions is still $\partial_n 1_P$. It is useful to

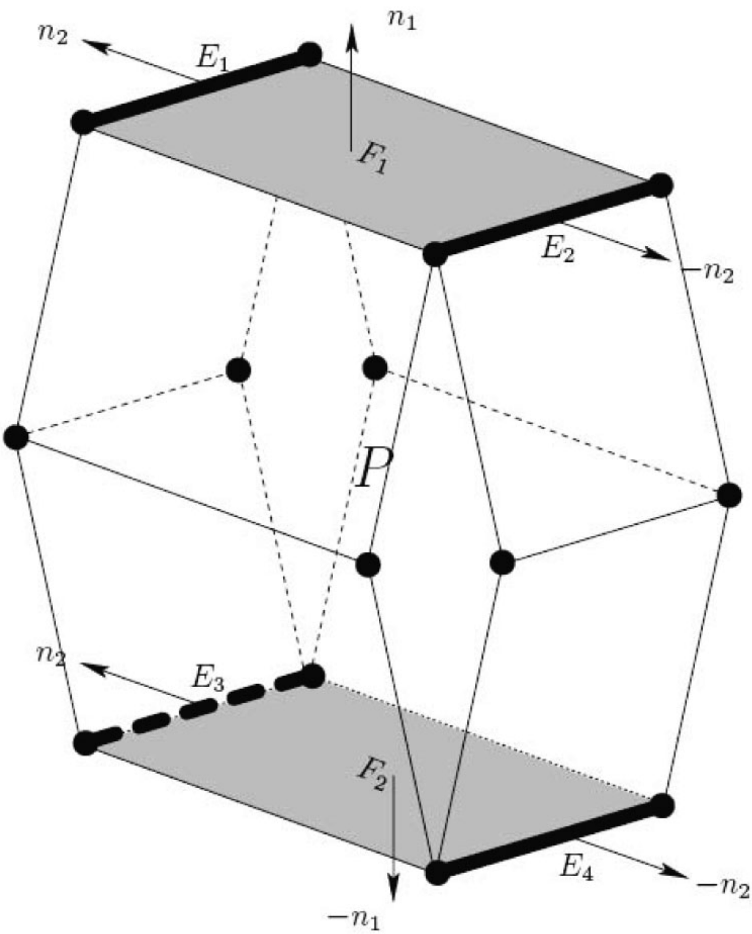

Figure 2. The boundary operator with respect to n_1 picks out the two facets F^+ and F^-, illustrating the definition of $\partial_n 1_P = 1_{F^+} - 1_{F^-}$. A second iteration of the boundary operator, this time with respect to n_2, picks out the four edge vectors E_1, E_2, E_3, and E_4, thus visually illustrating the identity $\partial_{n_2}(\partial_{n_1} P) = \partial_{n_2}(1_{F^+} - 1_{F^-}) = (1_{E_1} - 1_{E_2}) - (1_{E_3} - 1_{E_4})$.

utilize both of these actions, the first being an action on indicator functions, and the second being an action on subsets of points $P \subset \mathbb{R}^d$.

We call a sequence $\mathfrak{n} = (n_1, \dots, n_m)$ of vectors in $\mathbb{R}^d$ an orthogonal frame if they are pairwise orthogonal to each other. We denote it by $\mathfrak{n}^\perp$ the subspace of $\mathbb{R}^d$ consisting of those vectors which are orthogonal to every vector in the orthogonal frame $\mathfrak{n}$.

We define $\partial_{\mathfrak{n}} := \partial_{n_m} \dots \partial_{n_1}$, a composition of boundary operators that is read from right to left. In case $m = 0$, when an orthogonal frame $\mathfrak{n}$ is empty, we define $\partial_{\mathfrak{n}}$ to be an identity operator. Similarly to $\partial_n P$ we define a boundary operator relative to a whole frame $\mathfrak{n} = (n_1, \dots, n_m)$:

$$\partial_{\mathfrak{n}} P = \operatorname{supp}(\partial_{\mathfrak{n}} 1_P) = \{v \in \mathbb{R}^d \mid \partial_{\mathfrak{n}} 1_P(v) \neq 0\}.$$

Note that all the faces whose indicator functions appear in $\partial_{\mathfrak{n}} 1_P$ must have codimension at least m (degeneracies might occur), they must be parallel to each other, and they must have outward pointing normals n_m or $-n_m$ in $\partial_{n_{m-1}} \dots \partial_{n_1} P$.

We can now separate $\partial_{\mathfrak{n}}$ into two parts: $\partial_{\mathfrak{n}}^+$ and $\partial_{\mathfrak{n}}^-$, corresponding to faces with outward normals n_m or $-n_m$, so that $\partial_{\mathfrak{n}} 1_P = \partial_{\mathfrak{n}}^+ 1_P - \partial_{\mathfrak{n}}^- 1_P$. For example, when the frame $\mathfrak{n}$ consists of a single vector n, then the identity $\partial_n 1_P = 1_{F^+} - 1_{F^-}$ means by definition that $\partial_n^+ 1_P = 1_{F^+}$, and $\partial_n^- 1_P = 1_{F^-}$.

We say that $v \in \mathbb{R}^d$ is in general position w.r.t. the orthogonal frame $\mathfrak{n}$, if all the non-degenerate components of $\partial_{\mathfrak{n}}(P+v)$, which have codimension m, contain no points of Λ on their boundary, and any degenerate components (having larger codimension) contain no points of Λ at all.

A more formal description which we will have occasion to use below is that $v \notin \Lambda - \partial \partial_{\mathfrak{n}} P$. Even though we only need to consider orthogonal frames of size at most two in order to prove the theorem 1.1, we will prove the following two lemmas in the general case, for an orthogonal frame of any size.

Lemma 3.2. *Suppose $\#(\Lambda \cap \{P+v\}) = k$ for every v in general position. Let $\mathfrak{n} = (n_1, \dots, n_m)$ be an orthogonal frame in $\mathbb{R}^d$. Then for any v in general position w.r.t the orthogonal frame $\mathfrak{n}$ the following formula holds:*

$$\sum_{\lambda \in \Lambda} \partial_{\mathfrak{n}} 1_{P+v}(\lambda) = 0. \tag{1}$$

Proof. We proceed by induction on m. We remark that for $m = 0$ the hypothesis tells us that $\sum\limits_{\lambda \in \Lambda} \partial_{\mathfrak{n}} 1_{P+v}(\lambda) = k$, and for $m = 0$ this operator is by definition the identity operator. However, for each $m \geq 1$, we will show that $\sum\limits_{\lambda \in \Lambda} \partial_{\mathfrak{n}} 1_{P+v}(\lambda) = 0$.

Suppose that for an $(m-1)$-dimensional orthogonal frame $\mathfrak{n}' = (n_1, \dots, n_{m-1})$ and for every v in general position w.r.t. $\mathfrak{n}'$ the formula holds:

$$\sum_{\lambda \in \Lambda} \partial_{\mathfrak{n}'} 1_{P+v}(\lambda) = const$$

Now consider any m-dimensional orthogonal frame $\mathfrak{n}=(n_1,\dots,n_m)$, and v in general position w.r.t $\mathfrak{n}$. We know that all Λ-points of $v+\partial_{\mathfrak{n}}P$ lie in $v+Int(\partial_{\mathfrak{n}}P)$. Therefore, one can pick sufficiently small ϵ', such that no ϵ'-perturbation of v by a vector in $\mathfrak{n}^{\perp}$ removes or adds any Λ-points to $v+\partial_{\mathfrak{n}}P$. Clearly, by doing so we do not change $\sum_{\lambda\in\Lambda}\partial_{\mathfrak{n}}1_{P+v}(\lambda)$. On the other hand, we may choose an ϵ'-perturbation, $v_{\epsilon'}$, such that all Λ-points in $v_{\epsilon'}+\partial\partial_{\mathfrak{n}'}P$ get either inside or outside of $v_{\epsilon'}+\partial_{\mathfrak{n}'}P$ (see fig. 3).

Then consider two small perturbations of $v_{\epsilon'}$ in the directions n_m and $-n_m$: $v_{\epsilon'}^{+}=v_{\epsilon'}+\epsilon n_m$ and $v_{\epsilon'}^{-}=v_{\epsilon'}-\epsilon n_m$, such that $v_{\epsilon'}^{+}$ and $v_{\epsilon'}^{-}$ are in general position w.r.t $\mathfrak{n}'$, and ϵ small enough so that there are no points of Λ that lie in $P+v_{\epsilon'}^{\pm}$ and do not lie in $P+v_{\epsilon'}$ (such an ϵ can be found, because Λ is discrete).

By induction, $\sum_{\lambda\in\Lambda}\partial_{\mathfrak{n}'}1_{P+v_{\epsilon'}^{+}}(\lambda)=const=\sum_{\lambda\in\Lambda}\partial_{\mathfrak{n}'}1_{P+v_{\epsilon'}^{-}}(\lambda)$.

On the other hand, recalling that by definition $\partial_{\mathfrak{n}}P=\partial_{n_m}\partial_{\mathfrak{n}'}P$,

$$\sum_{\lambda\in\Lambda}\partial_{\mathfrak{n}'}1_{P+v_{\epsilon'}^{+}}(\lambda)-\sum_{\lambda\in\Lambda}\partial_{n_m}^{+}\partial_{\mathfrak{n}'}1_{P+v_{\epsilon'}}(\lambda)=\sum_{\lambda\in\Lambda}\partial_{\mathfrak{n}'}1_{P+v_{\epsilon'}}(\lambda)\cdot 1_{Int(\partial_{\mathfrak{n}'}P)}(\lambda) \tag{2}$$

$$=\sum_{\lambda\in\Lambda}\partial_{\mathfrak{n}'}1_{P+v_{\epsilon'}^{-}}(\lambda)-\sum_{\lambda\in\Lambda}\partial_{n_m}^{-}\partial_{\mathfrak{n}'}1_{P+v_{\epsilon'}}(\lambda). \tag{3}$$

It follows that

$$\sum_{\lambda\in\Lambda}\partial_{n_m}^{+}\partial_{\mathfrak{n}'}1_{P+v}(\lambda)=\sum_{\lambda\in\Lambda}\partial_{n_m}^{+}\partial_{\mathfrak{n}'}1_{P+v_{\epsilon'}}(\lambda) \tag{4}$$

$$=\sum_{\lambda\in\Lambda}\partial_{n_m}^{-}\partial_{\mathfrak{n}'}1_{P+v_{\epsilon'}}(\lambda) \tag{5}$$

$$=\sum_{\lambda\in\Lambda}\partial_{n_m}^{-}\partial_{\mathfrak{n}'}1_{P+v}(\lambda), \tag{6}$$

which gives us (1), since $\partial_{\mathfrak{n}}=\partial_{n_m}^{+}\partial_{\mathfrak{n}'}-\partial_{n_m}^{-}\partial_{\mathfrak{n}'}$. ∎

For any polytope in $\mathbb{R}^d$ lying in an affine subspace parallel to $\mathfrak{n}^{\perp}$, we may consider a naturally defined lower dimensional volume $\mathrm{Vol}_{\mathfrak{n}}$. For example, if $d=3$ and $\mathfrak{n}=(n_1,n_2)$, we get $\mathrm{Vol}_{\mathfrak{n}}$ to be just a length of a line segment in $\mathbb{R}^3$. As we know $\partial_{\mathfrak{n}}1_P$ is a finite sum of indicator functions of polytopes lying in affine subspaces parallel to $\mathfrak{n}^{\perp}$ taken with $+$ or $-$ signs. For each such indicator function 1_F let us denote by $\mathrm{Vol}_{\mathfrak{n}}(1_F)$ the volume of polytope F. Note that we can take any measurable object in the affine subspace parallel to $\mathfrak{n}^{\perp}$ instead of F.

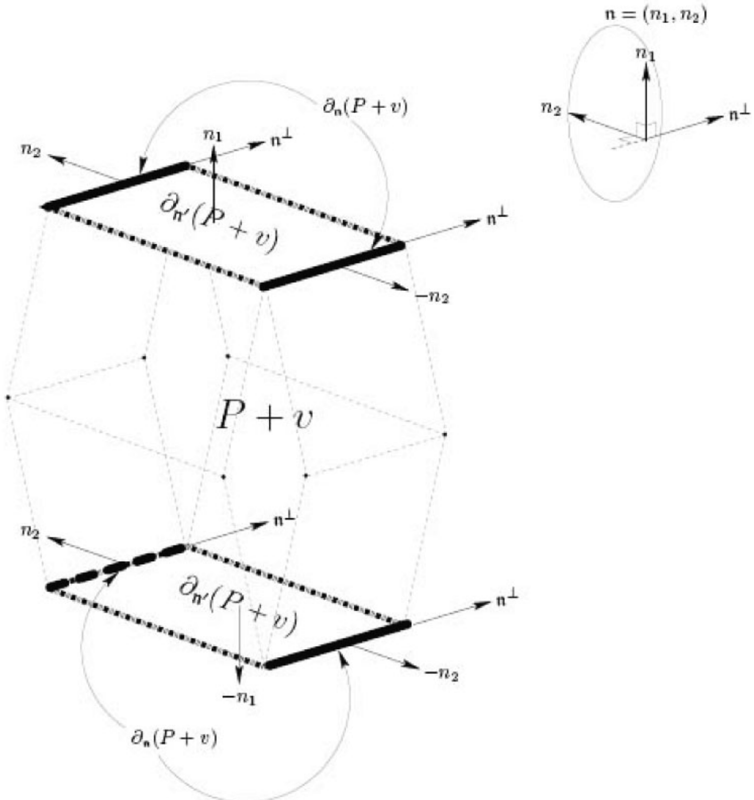

Figure 3. The ϵ' perturbation, along the $n^\perp$ direction, insures that all Λ points have been removed from the four dotted edges on the upper facet and lower facet of $P+v$, giving us the set $\partial_{\mathfrak{n}'}(P+v)$. Also, the ϵ perturbation, along the n_2 direction, insures that all Λ points on the right-hand bold edges, attached to the normal vector $-n_2$, will end up outside of the perturbed set, and that all Λ points on the left-hand bold edges, attached to the normal vector n_2, will end up inside the perturbed set.

We now extend the notion of $\mathrm{Vol}(S)$ to a more general notion of a signed linear combination of volumes. We let $V_{\mathfrak{n}}(\partial_{\mathfrak{n}} 1_P)$ denote the sum of the corresponding volumes taken with different signs, and in a similar way we can write $V_{\mathfrak{n}}$ for any sum of positive and negative indicator functions. The next Lemma extends equality (1) in Lemma 3.2 from a discrete measure of facets to a continuous measure of facets.

Lemma 3.3. *Under the same assumptions of lemma 3.2, the following formula holds:*

$$V_{\mathfrak{n}}(\partial_{\mathfrak{n}} 1_P) = 0.$$

Proof. Let us recall what we have so far. Lemma 3.2 tells us $\sum_{\lambda \in \Lambda} \partial_{\mathfrak{n}} 1_{P+v}(\lambda) = 0$ for any v with $v + \partial\partial_{\mathfrak{n}} P$ containing no Λ points.

For each $\lambda \in \Lambda$ let us consider a set S of vectors v enjoying the property that $\lambda \in v + \partial_{\mathfrak{n}} P$ and $\Lambda \cap \{v + \partial\partial_{\mathfrak{n}} P\} = \emptyset$. We call the set S $\mathfrak{n}$-interior w.r.t. λ. We can also realize the set S by excluding a finite number of lower dimensional polytopes (polytopes F with $V_{\mathfrak{n}}(F) = 0$) from $\lambda - \partial_{\mathfrak{n}} P$. We call a vector $\mathfrak{n}$-internal if it belongs to $\mathfrak{n}$-interior for some $\lambda \in \Lambda$.

Assume now that $V_{\mathfrak{n}}(\partial_{\mathfrak{n}} 1_P) = A_1 \neq 0$. Let us also write $V_{\mathfrak{n}}(|\partial_{\mathfrak{n}} 1_P|) = A_2 \geq |A_1| > 0$, where by $|\partial_{\mathfrak{n}} 1_P|$ we imply the sum of indicators of $\partial_{\mathfrak{n}} 1_P$ with all negative coefficients of indicators switched to their absolute value.

For any $R > 0$ we may consider a ball B_R in $\mathbb{R}^d$ with the center at origin and given radius R. Clearly, there is a constant $C = C(P)$, such that $B_R + (-1)\partial_{\mathfrak{n}} P \subset B_{R+C}$. For any positive real R we define $N(R) := \#\{B_R \cap \Lambda\}$. For each $\mathfrak{n}$-internal $v \in B_R$ we may rewrite the formula from lemma 3.2 and get

$$\sum_{\lambda \in B_{R+C} \cap \Lambda} \partial_{\mathfrak{n}} 1_{\lambda - P}(v) = 0.$$

This implies

$$V_{\mathfrak{n}} \left(1_{B_R} \cdot \sum_{\lambda \in B_{R+C} \cap \Lambda} \partial_{\mathfrak{n}} 1_{\lambda - P} \right) = 0.$$

Also we know that

$$\left| V_{\mathfrak{n}} \sum_{\lambda \in B_{R+C} \cap \Lambda} \partial_{\mathfrak{n}} 1_{\lambda - P} \right| = N(R+C)|A_1|.$$

$$\begin{aligned}\left|V_{\mathfrak{n}}\left(\sum_{\lambda\in B_{R+C}\cap\Lambda}\partial_{\mathfrak{n}}1_{\lambda-P}\right)\right| &\le \left|V_{\mathfrak{n}}\left(1_{B_R}\cdot\sum_{\lambda\in B_{R+C}\cap\Lambda}\partial_{\mathfrak{n}}1_{\lambda-P}\right)\right| \\ &\quad+\left|V_{\mathfrak{n}}\left(\left(1_{B_{R+2C}}-1_{B_R}\right)\cdot\sum_{\lambda\in B_{R+C}\cap\Lambda}\partial_{\mathfrak{n}}1_{\lambda-P}\right)\right| \\ &=\left|V_{\mathfrak{n}}\left(\left(1_{B_{R+2C}}-1_{B_R}\right)\cdot\sum_{\lambda\in(B_{R+C}\setminus B_{R-C})\cap\Lambda}\partial_{\mathfrak{n}}1_{\lambda-P}\right)\right| \\ &\le V_{\mathfrak{n}}\sum_{\lambda\in(B_{R+C}\setminus B_{R-C})\cap\Lambda}\left|\partial_{\mathfrak{n}}1_{\lambda-P}\right| = A_2\cdot\left(N(R+C)-N(R-C)\right).\end{aligned}$$

Thus we get $\left(1-\frac{|A_1|}{A_2}\right)N(R+C)\geq N(R-C)$, which establishes an exponential growth of $N(R)$ in R. We can cover B_R by a disjoint union of $O(R^{2d})$ cubes whose side-length is $\frac{1}{R}$. Thus taking sufficiently large R we can find a cube K with side-length $\frac{1}{R}$, which contains more than k Λ-points. We can now translate P so that the cube K is contained in P, and therefore this translate of P now contains more than k Λ-points, a contradiction. ∎

In order to finish the proof of main theorem, we need the following theorem by Minkowski [15].

Theorem 3.4 (Minkowski). *A convex polytope in $\mathbb{R}^d$, with given facet normals and facet $(d-1)$-volumes, is unique up to translation.*

Proof of Theorem 1.1 We will first prove that P is centrally symmetric. Take any pair of facets of P, F^+ and F^-, with outward normals n and $-n$ respectively. Applying lemma 3.3 to $\mathfrak{n}=(n)$ we get $V_{\mathfrak{n}}(\partial_{\mathfrak{n}}1_P)=0$, which means that $V(F^+)=V(F^-)$. Since n can be chosen arbitrarily, polytopes P and $(-1)\cdot P$ have equal codimension 1 volumes of facets in every direction. By theorem 3.4 we get that $P=(-1)\cdot P+v$ for some translation vector v, so P is centrally symmetric.

Similarly we prove that every facet of P is centrally symmetric. Given a pair of opposite facets F_1 and F_2 of P with outward normals n_1 and $-n_1$ respectively, consider any direction $n_2\in(n_1)^\perp$ and two pairs of corresponding faces of codimension 2: F_1^+ and F_1^- are facets of F_1 with outward normals n_2 and $-n_2$ respectively, F_2^+ and F_2^- are facets of F_2 with outward normals n_2 and $-n_2$ respectively. Applying lemma 3.3 to $\mathfrak{n}=(n_1,n_2)$ we get $V_{\mathfrak{n}}(\partial_{\mathfrak{n}}1_P)=0$, which means that

$$(V(F_1^+)-V(F_1^-))-(V(F_2^+)-V(F_2^-))=0. \tag{7}$$

But since P is centrally symmetric, F_1^+ and F_2^- are symmetric to each other as well as F_1^- and F_2^+, so $V(F_1^+)=V(F_2^-)$ and $V(F_1^-)=V(F_2^+)$.

Combining the last three equations we get an equality for codimension 2 faces of P: $V(F_1^+)=V(F_1^-)$. It follows that as $(d-1)$-dimensional objects, F_1 and $(-1)\cdot F_1$, themselves have equal facets in every direction (in their affine span), and again by theorem 3.4 we get that F_1 is centrally symmetric. But since F_1 could be chosen arbitrarily among the facets of P, every facet of P is centrally symmetric, which concludes the proof of theorem 1.1. ∎

Remark. We note that Lemma 3.2 gives us interesting information about the relationship between the Λ points that lie in various faces, for any frame that has more than 2 vectors. In contrast, Lemma 3.3 does not give us any additional information about the codimension 3 volumes (or higher codimension volumes), because using a frame consisting of 3 vectors would simply replace the identity (7) by the identity $0=0$. Indeed, according to the Figure 2 the identity (7) can be rewritten as $V(E_1)-V(E_2)-V(E_3)+V(E_4)=0$. By the symmetry of P we had $V(E_1)=V(E_4)$ and $V(E_2)=V(E_3)$. Thus due to (7) we could conclude that $V(E_1)=V(E_2)$, $V(E_3)=V(E_4)$ and that F_1 and F_2 are symmetric. Now if we use a third direction n_3 (in the Figure 3 one can take $n_3\in n^{\perp}$) and denote the respective facets of $E_1,\dots,E_4$ by $E_1^+,E_1^-,\dots,E_4^+,E_4^-$, we get the identity

$$\begin{aligned}(8)\quad \big(V(E_1^+)-V(E_1^-)\big)-\big(V(E_2^+)-V(E_2^-)\big)&\\ -\big(V(E_3^+)-V(E_3^-)\big)+\big(V(E_4^+)-V(E_4^-)\big)&=0.\end{aligned}$$

Due to the symmetry of P we have the identities $V(E_1^+)=V(E_4^-)$, $V(E_1^-)=V(E_4^+)$, $V(E_2^+)=V(E_3^-)$, and $V(E_2^-)=V(E_3^+)$. Due to the symmetry of F_1 and F_2 we can further conclude that $V(E_1^+)=V(E_4^-)=V(E_2^-)=V(E_3^+)$ and $V(E_1^-)=V(E_4^+)=V(E_2^+)=V(E_3^-)$. Therefore, plugging these equalities into (8) we obtain equality $0=0$. It is for this reason that we cannot conclude that codimension 3 faces of a k-tiling polytope are centrally symmetric, and in fact they are not in general centrally symmetric, as the example of the 24-cell shows.

4. Proof of Theorem 1.2

Proof. We may assume, without loss of generality, that our rational polytope P is an integer polytope, by dilating it by the least common multiple of the denominators of all of the rational coordinates of its vertices. Now, given that P has integer vertices, we will show that the polytope P k-tiles $\mathbb{R}^d$ with $\Lambda=\mathbb{Z}^d$.

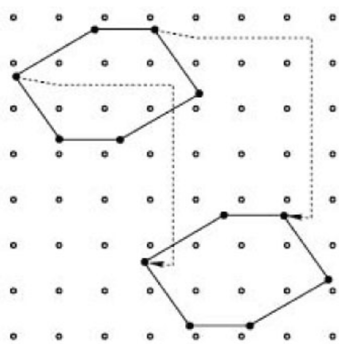

Figure 4. This polygon illustrates that fact that there is always a continuous path that a polygon P may take so that the vertices of P (and in general the codimension 2 faces of P) never pass through the discrete set of translations vectors Λ, shown here as a lattice.

We claim that in every general position P has an equal number of integer points on every pair of opposite facets. Indeed, since it is centrally symmetric and has centrally symmetric facets (and integer vertices), any two opposite facets are translations of one another by some integer vector. It follows that for every integer point on a facet there is a corresponding integer point on an opposite facet, so their numbers are equal.

Now, consider any two general positions of P, say $P+u$ and $P+v$. There exists some path from u to v such that when we translate P along this path, no integer point of $\mathbb{Z}^d$ collides with any co-dimension 2 face of the translates of P along this path (see fig. 4). But since in any general position the number of integer points on two opposite facets of P are equal, it follows that the number of points inside P along this path is constant. We conclude that any two general positions of P have the same number of interior integer points, say k. Thus, $-P$ k-tiles $\mathbb{R}^d$ with the lattice $\mathbb{Z}^d$. ∎

5. An analytic approach, using Fourier techniques

In this section we give a proof-outline of the main result, Theorem 1.1, using Fourier techniques, which exhibits some interesting connections between combinatorics and Fourier analysis. The reader may consult the classic reference [17] for background on Fourier techniques. We begin once again with the definition of a k-tiling. Thus, we suppose that a polytope P k-tiles $\mathbb{R}^d$

with some discrete multiset Λ. In other words, we assume that

$$\sum_{\lambda\in\Lambda} 1_{P+\lambda}(v) = k,$$

for all $v \notin \partial P + \Lambda$. We can rewrite this condition as a convolution of generalized functions, as follows:

$$1_P * \delta_\Lambda = k, \tag{9}$$

where $\delta_\Lambda := \sum_{\lambda\in\Lambda} \delta_\lambda$, where δ_λ is the unit point mass for the point $\lambda \in \Lambda$. We first differentiate both sides of (9), with respect to any $\xi \in \mathbb{R}^d$, obtaining

$$\frac{d}{d\xi}\left(1_P * \delta_\Lambda\right) = \left(\frac{d}{d\xi} 1_P\right) * \delta_\Lambda = 0. \tag{10}$$

Next, we take the Fourier transform of both sides of (10), obtaining

$$\left(\xi \hat{1}_P\right)\, \hat{\delta}_\Lambda = 0, \tag{11}$$

If we now have some more detailed knowledge about $\hat{1}_P$, then we can use (11) to proceed further. The next result is a useful combinatorial version of Stokes' formula, which holds for the Fourier transform of the indicator function of any polytope. This is a result about $\hat{1}_P$ that was used recently in [2], and does not appear to be well-known, so we include its statement here for completeness. For the transform of a function on $\mathbb{R}^d$, we use the standard definition:

$$\hat{1}_P(\xi) := \int_P \exp(2\pi i \langle \xi, x\rangle) dx,$$

valid for any $\xi \in \mathbb{R}^d$.

Theorem 5.1. *Let F be a k-dimensional polytope in $\mathbb{R}^d$, for any $k \le d$. Let $Proj_F(\xi)$ denote the orthogonal projection of ξ onto the k-dimensional subspace of $\mathbb{R}^d$ that is parallel to F. Moreover, for each $(k-1)$-dimensional face $G \in \partial F$, let n_G be its outward pointing normal vector. Then the Fourier transform of the indicator function of F can be written as follows:*

Case I. If $Proj_F(\xi) = 0$, then

$$\hat{1}_F(\xi) = V^k(F) \exp(2\pi i \Phi),$$

where Φ is the constant value of the function $\phi(x) := \langle \xi, x\rangle$ on F.

Case II. If $Proj_F(\xi)\neq 0$, *then*

$$\hat{1}_F(\xi) = -\frac{1}{2\pi i}\sum_{G\in\partial F}\frac{\langle Proj_F(\xi), n_F\rangle}{||Proj_F(\xi)||^2}\hat{1}_G(\xi).$$ ∎

Applying Theorem (5.1) above to the generalized function 1_P, we may continue from (11) to get the identity

$$\left(\sum_{F\in\partial P}\xi\frac{\langle\xi,n_F\rangle}{\langle\xi,\xi\rangle}\hat{1}_F\right)\hat{\delta}_\Lambda = 0, \tag{12}$$

valid for any nonzero $\xi\in\mathbb{R}^d$. We also note that the sum runs over all the (codimension 1) facets F of the boundary ∂P. It now follows, upon taking the inner product with ξ, that

$$\left(\sum_{F\in\partial P}\langle\xi,n_F\rangle\,\hat{1}_F\right)\hat{\delta}_\Lambda = 0. \tag{13}$$

Taking Fourier transforms again, we may rewrite the last equation as

$$\left(\sum_{F\in\partial P}\left(\frac{d}{d\,n_F}\right)(1_F)\right)*\delta_\Lambda = 0. \tag{14}$$

We now focus our attention on each *pair* of facets of P, as in the first section. Thus, we consider a facet F^+ with its outward pointing normal $n(F)$, and a parallel facet F^-, with its outward pointing normal $-n(F)$.

Lemma 5.2. *For each facet F of P, we have the identity*

$$(1_{F^+} - 1_{F^-})*\delta_\Lambda = 0.$$

Proof. We assume that $(1_{F^+}-1_{F^-})*\delta_\Lambda\neq 0$. Therefore there exists a small ball B_r, of radius r, such that for any nonnegative, nonzero test function f whose support is contained in B_r, we have $\langle(1_{F^+}-1_{F^-})*\delta_\Lambda, f\rangle\neq 0$. We may further assume that the support of f is disjoint from the support of $(1_{G^+}-1_{G^-})*\delta_\Lambda$, for any facet G of P where $G\neq F$. Indeed, the discreteness of Λ guarantees that we can find such a ball B_r on which f satisfies the above conditions.

Now we construct a test function g whose support is contained in B_r, with positive derivative $\left(\frac{d}{dn_F}\right)$ along the direction n_F, in a small ϵ vicinity of $B_r\cap\mathrm{Supp}((1_{F^+}-1_{F^-})*\delta_\Lambda):=D_\epsilon$. To construct such a g, we first restrict

f to D_ϵ, call it f_0. We now multiply f_0 by a one dimensional smooth bump function b whose derivative on $[-\epsilon,\epsilon]$ is positive, and whose support lives in $[-2\epsilon, 2\epsilon]$. Thus $g := f_0 \cdot b$ has positive derivative on D_ϵ. When we insert this g into (14), we arrive at a contradiction. Indeed $\langle (1_{G^+} - 1_{G^-}) * \delta_\Lambda, \frac{d}{dn_G} g\rangle = 0$ for $G \neq F$ because the choice of the support of g. On the other hand $\langle (1_{F^+} - 1_{F^-}) * \delta_\Lambda, \frac{d}{dn_F} g\rangle \neq 0$ by the construction, since $\frac{d}{dn_F} g$ is positive in the vicinity of the support of $(1_{F^+} - 1_{F^-}) * \delta_\Lambda$. ∎

We finish this section by remarking that iteration of Lemma 5.2 allows us to establish the same conclusion as Lemma 3.2. The next iteration would be applied to a normal vector to a facet of F^+ within the affine span of the facet F^+. The Fourier analogue of Lemma 3.3 involves the scalar product of $(1_{F^+} - 1_{F^-}) * \delta_\Lambda$ against an "approximate identity" function, compactly supported on a large ball. The main Theorem 1.1 now follows in a similar manner as in the previous section.

6. Another equivalent condition for k-tiling, using solid angles

Here we show that it is possible to reinterpret the condition that a polytope k-tiles $\mathbb{R}^d$ by considering all of the solid angles $\omega_P(\lambda)$ of the d-dimensional convex polytope P, at each point $\lambda \in \Lambda$. For any point $\lambda \in \mathbb{R}^d$, we define the solid angle at λ to be the proportion of a small sphere of radius R, centered at λ, which intersects P. More precisely, the solid angle is defined by

$$\omega_P(\lambda) = \lim_{R \to 0} \frac{V\big(\{\lambda + B_R)\} \cap P\big)}{V(B_R)},$$

where $V(S)$ is the d-dimensional volume of S. The following Theorem is of independent interest, showing another interesting equivalent condition for k-tiling Euclidean space.

Theorem 6.1. *A polytope P k-tiles $\mathbb{R}^d$ with the multiset Λ if and only if*

$$\sum_{\lambda \in \Lambda} \omega_{P+v}(\lambda) = k,$$

for every $v \in \mathbb{R}^d$.

Proof. Suppose that P k-tiles $\mathbb{R}^d$ with the multiset Λ. We know from Theorem 1.1 that P must be centrally symmetric, and therefore $-P$ k-tiles as well, with the multiset Λ. By Lemma 3.1 $\#(\Lambda \cap \{P + x\}) = k$ for almost every

$x \in \mathbb{R}^d$. We can therefore integrate this equality in the variable x, over a d-dimensional ball $B_R(v)$ with center in v and radius R, as follows:

$$\begin{aligned} k \cdot V(B_R(v)) = \int\limits_{B_R(v)} k\,dx &= \int\limits_{B_R(v)} \#(\Lambda \cap \{P+x\})dx \\ &= \int\limits_{B_R(v)} \sum_{\lambda \in \Lambda} 1_{\lambda - P}(x)dx \\ &= \sum_{\lambda \in \Lambda} \int\limits_{B_R(v)} 1_{\lambda - P}(x)dx \\ &= \sum_{\lambda \in \Lambda} V(B_R(v) \cap \{\lambda - P\}) \\ &= \sum_{\lambda \in \Lambda} V(\{\lambda - B_R\} \cap \{P+v\}) \end{aligned}$$

It follows that $k = \sum\limits_{\lambda \in \Lambda} \frac{V(\{\lambda - B_R\} \cap \{P+v\})}{V(B_R(v))}$, which approaches $\sum\limits_{\lambda \in \Lambda} \omega_{P+v}(\lambda)$ as R goes to 0.

In the other direction, the assumption that $\sum\limits_{\lambda \in \Lambda} \omega_{P+v}(\lambda) = k$ is, in general position, equivalent to the statement that $\#(\Lambda \cap \{P+x\}) = k$. By Lemma 3.1 we conclude that $-P$ k-tiles with the multiset Λ. Finally, by Theorem 1.1 we know that P is centrally symmetric, so that P k-tiles with the same multiset Λ. ∎

We note that a particularly interesting choice of v in this Theorem is the value $v = 0$, so that we can in fact have points in Λ coincide with some of the vertices of P. This equivalent condition allows us to consider such coincidences without having to translate P into general position.

7. Some open questions

We conclude our paper with some fascinating open questions which the main results of the present paper suggest as a natural research direction for k-tilings, a relatively new area.

1. Recall that the Venkov–McMullen condition for the existence of belts consisting of 4 or 6 parallel codimension 2 faces allowed an “if and only if” characterization for 1-tiling polytopes. Find the analogous additional condition that would give a complete characterization for k-tiling polytopes (without the assumption that P is a rational polytope).

2. Classify the combinatorial types of all polytopes which k-tile $\mathbb{R}^d$ by translations.

We note that for the classical question of 1-tiling $\mathbb{R}^d$ by parallelotopes (and parallelotopes are the only objects that can tile $\mathbb{R}^d$, by McMullen's theorem), there are exactly 5 combinatorially distinct parallelotopes in $\mathbb{R}^3$, and exactly 52 distinct parallelotopes in $\mathbb{R}^4$. It is still not known how many combinatorially distinct parallelotopes there are in dimensions 5 and higher. It is also not known how many facets a parallelotope may have in general (see [7] for references).

3. Prove or disprove that if any polytope k-tiles $\mathbb{R}^d$ by translations, then it also m-tiles $\mathbb{R}^d$ by a lattice, for a possibly different m.

This would give an analogue of the McMullen Theorem for 1-tiling parallelotopes in $\mathbb{R}^d$, but appears to be a very difficult problem.

4. Prove or disprove that if a 3-dimensional polytope, which is not a prism, k-tiles $\mathbb{R}^3$ by translations with a multiset Λ, then Λ is a union of a finite number of 3-dimensional lattices.

This would prove the 3-dimensional analogue of Kolountzakis' 2-dimensional result [9].

5. Find or estimate the smallest k for which any convex polytope P can k-tile $\mathbb{R}^d$, for each dimension $d \geq 2$.

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