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THE LITTLEWOOD–OFFORD PROBLEM IN HIGH DIMENSIONS AND A CONJECTURE OF FRANKL AND FÜREDI

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We give a new bound on the probability that the random sum $\xi_1 v_1+\cdots+\xi_n v_n$ belongs to a ball of fixed radius, where the ξ_i are i.i.d. Bernoulli random variables and the v_i are vectors in $\mathbf{R}^d$. As an application, we prove a conjecture of Frankl and Füredi (raised in 1988), which can be seen as the high dimensional version of the classical Littlewood-Offord-Erdős theorem.

1. Introduction

Let $V=\{v_1,\dots,v_n\}$ be a (multi-)set of n vectors in $\mathbf{R}^d$. Consider the random sum

$$X_V := \xi_1 v_1 + \cdots + \xi_n v_n$$

where ξ_i are i.i.d. Bernoulli random variables (each ξ_i takes values 1 and -1 with probability 1/2 each).

The famous *Littlewood–Offord problem* (posed in 1943 [10]) is to estimate the *small ball probability*

$$p_d(n,\Delta) = \sup_{V,B} \mathbf{P}(X_V \in B)$$

where the supremum is taken over all multi-sets $V=\{v_1,\dots,v_n\}$ of n vectors of length at least one and all closed balls B of radius Δ (this problem is also

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sometimes referred to as the *small ball problem* in the literature). Here and later, d and Δ are fixed. The asymptotic notation $X = O(Y)$ or (equivalently) $X \ll Y$ will be used with the assumption that n tends to infinity; thus the implied constant in the $O()$ notation can depend on d and Δ but not on n.

The more combinatorial (but absolutely equivalent) way to pose the problem is to ask for the maximum number of subsums of V falling into a ball of radius $\Delta/2$. We prefer the probabilistic setting as it is more convenient and easier to generalize.

Shortly after the paper of Littlewood-Offord, Erdős [1] determined $p_1(n,\Delta)$, solving the problem completely in one dimension. Define $s := \lfloor \Delta \rfloor + 1$.

Theorem 1.1 (Erdős' Littlewood–Offord inequality). *Let $S(n,m)$ denote the sum of the largest m binomial coefficients $\binom{n}{i}, 0 \le i \le n$. Then*

$$p_1(n,\Delta) = 2^{-n} S(n,s).$$

The situation for higher dimension is more complicated, and there has been a series of papers devoted to its study (see [6,7,8,9,4,5,3,11,12] and the references therein). In particular, Frankl and Füredi [3], sharpening several earlier results, proved

Theorem 1.2 (Frankl–Füredi's Littlewood–Offord inequality). *For any fixed d and Δ*

$$p_d(n,\Delta) = (1+o(1)) 2^{-n} S(n,s). \tag{1}$$

This result is asymptotic. In view of Theorem 1.1, it is natural to ask if one can have the exact estimate

$$p_d(n,\Delta) = 2^{-n} S(n,s), \tag{2}$$

which can be seen as the high dimensional generalization of Erdős' result. However, it has turned out that in general this is not true. It was observed in [8,3] that (2) fails if $s \ge 2$ and

$$\Delta > \sqrt{(s-1)^2+1}. \tag{3}$$

Take $v_1 = \cdots = v_{n-1} = e_1$ and $v_n = e_2$, where e_1, e_2 are two orthogonal unit vectors. For this system, there is a ball B of radius Δ such that $\mathbf{P}(X_V \in B) > S(n,s)$.

Frankl and Füredi conjectured ([3, Conjecture 5.2])

Conjecture 1.3. Let Δ, d be fixed. If $s-1 \le \Delta < \sqrt{(s-1)^2+1}$ and n is sufficiently large, then

$$p_d(n,\Delta) = 2^{-n} S(n,s).$$

The conjecture has been confirmed for $s=1$ by an important result of Kleitman [7] and for $s=2,3$ by Frankl and Füredi [3] (see the discussion prior to [3, Conjecture 5.2]). For all other cases, the conjecture has been open. On the other hand, Frankl and Füredi showed that (2) holds under a stronger assumption that $s-1 \le \Delta \le (s-1) + \frac{1}{10s^2}$.

In this short paper, we first prove the following general estimate:

Theorem 1.4. *Let $V = \{v_1, \dots, v_n\}$ be a multi-set of vectors in $\mathbf{R}^d$ with the property that for any hyperplane H, one has $\operatorname{dist}(v_i, H) \ge 1$ for at least k values of $i = 1, \dots, n$. Then for any unit ball B, one has*

$$\mathbf{P}(X_V \in B) = O(k^{-d/2}).$$

The hidden constant in the $O()$ notation here depends on d, but not on k and n.

As an application, we prove Conjecture 1.3 in full generality and also give a new proof for Theorem 1.2. This will be done in the next section. The remaining two sections are devoted to the proof of Theorem 1.4.

2. Proof of Theorem 1.2 and Conjecture 1.3

We now assume Theorem 1.4 is true, and use it to first prove Theorem 1.2. We will induct on the dimension d. The case $d=1$ follows from Theorem 1.1, so we assume that $d \ge 2$ and that the claim has already been proven for smaller values of d. The lower bound

$$p_d(n,\Delta) \ge p_1(n,\Delta) = 2^{-n} S(n,s)$$

is clear, so it suffices to prove the upper bound

$$p_d(n,\Delta) \le (1+o(1)) 2^{-n} S(n,s).$$

Fix Δ, and let $\varepsilon > 0$ be a small parameter to be chosen later. Suppose the claim failed, then there exists $\Delta > 0$ such that for arbitrarily large n, there exists a family $V = \{v_1, \dots, v_n\}$ of vectors in $\mathbf{R}^d$ of length at least 1 and a ball B of radius Δ such that

$$\mathbf{P}(X_V \in B) \ge (1+\varepsilon) 2^{-n} S(n,s). \tag{4}$$

In particular, from Stirling's approximation one has

$$\mathbf{P}(X_V \in B) \gg n^{-1/2}.$$

Assume n is sufficiently large depending on d, ε, and that V, B is of the above form. Applying the pigeonhole principle, we can find a ball B' of radius $\frac{1}{\log n}$ such that

$$\mathbf{P}(X_V \in B') \gg n^{-1/2} \log^{-d} n.$$

Set $k := n^{2/3}$. Since $d \geq 2$ and n is large, we have

$$\mathbf{P}(X_V \in B') \geq Ck^{-d/2}$$

for any fixed constant C. Applying Theorem 1.4 in the contrapositive (rescaling by $\log n$), we conclude that there exists a hyperplane H such that $\operatorname{dist}(v_i, H) \leq 1/\log n$ for at least $n-k$ values of $i = 1, \dots, n$.

Let V' denote the orthogonal projection to H of the vectors v_i with $\operatorname{dist}(v_i, H) \leq 1/\log n$. By conditioning on the signs of all the ξ_i with $\operatorname{dist}(v_i, H) > 1/\log n$, and then projecting the sum X_V onto H, we conclude from (4) the existence of a $d-1$-dimensional ball B' in H of radius Δ such that

$$\mathbf{P}(X_{V'} \in B') \geq (1+\varepsilon)2^{-n} S(n,s).$$

On the other hand, the vectors in V' have magnitude at least $1 - 1/\log n$. If n is sufficiently large depending on d, ε this contradicts the induction hypothesis (after rescaling the V' by $1/(1-1/\log n)$ and identifying H with $\mathbf{R}^{n-1}$ in some fashion). This concludes the proof of (1).

Now we turn to the proof of Conjecture 1.3. We can assume $s \geq 3$, as the remaining cases have already been treated. If the conjecture failed, then there exist arbitrarily large n for which there exist a family $V = \{v_1, \dots, v_n\}$ of vectors in $\mathbf{R}^d$ of length at least 1 and a ball B of radius Δ such that

$$\mathbf{P}(X_V \in B) > 2^{-n} S(n,s). \tag{5}$$

By iterating the argument used to prove (1), we may find a one-dimensional subspace L of $\mathbf{R}^d$ such that $\operatorname{dist}(v_i, L) \ll 1/\log n$ for at least $n - O(n^{2/3})$ values of $i = 1, \dots, n$. By reordering, we may assume that $\operatorname{dist}(v_i, L) \ll 1/\log n$ for all $1 \leq i \leq n-k$, where $k = O(n^{2/3})$.

Let $\pi \colon \mathbf{R}^d \to L$ be the orthogonal projection onto L. We divide into two cases. The first case is when $|\pi(v_i)| > \frac{\Delta}{s}$ for all $1 \leq i \leq n$. We then use the trivial bound

$$\mathbf{P}(X_V \in B) \leq \mathbf{P}(X_{\pi(V)} \in \pi(B)).$$

If we rescale Theorem 1.1 by a factor slightly less than s/Δ, we see that

$$\mathbf{P}(X_{\pi(V)} \in \pi(B)) \leq 2^{-n} S(n,s)$$

which contradicts (5).

In the second case, we assume $|\pi(v_n)| \leq \Delta/s$. We let V' be the vectors $v_1, \dots, v_{n-k}$, then by conditioning on the $\xi_{n-k+1}, \dots, \xi_{n-1}$ we conclude the existence of a unit ball B' such that

$$\mathbf{P}(X_{V'} + \xi_n v_n \in B') \geq \mathbf{P}(X_V \in B).$$

Let $x_{B'}$ be the center of B'. Observe that if $X_{V'} + \xi_n v_n \in B'$ (for any value of ξ_n) then $|X_{\pi(V')} - \pi(x_{B'})| \leq \Delta + \frac{\Delta}{s}$. Furthermore, if $|X_{\pi(V')} - \pi(x_{B'})| > \sqrt{\Delta^2 - 1}$, then the parallelogram law shows that $X_{V'} + v_n$ and $X_{V'} - v_n$ cannot both lie in B', and so conditioned on $|X_{\pi(V')} - \pi(x_{B'})| > \sqrt{\Delta^2 - 1}$, the probability that $X_{V'} + \xi_n v_n \in B'$ is at most $1/2$.

We conclude that

$$\begin{aligned}\mathbf{P}(X_{V'} + \xi_n v_n \in B') &\leq \mathbf{P}\Big(|X_{\pi(V')} - \pi(x_{B'})| \leq \sqrt{\Delta^2 - 1}\Big) \\ &\quad + \frac{1}{2}\mathbf{P}\left(\sqrt{\Delta^2 - 1} < |X_{\pi(V')} - \pi(x_{B'})| \leq \Delta + \frac{\Delta}{s}\right) \\ &= \frac{1}{2}\Bigg(\mathbf{P}\left(|X_{\pi(V')} - \pi(x_{B'})| \leq \sqrt{\Delta^2 - 1}\right) \\ &\quad + \mathbf{P}\left(|X_{\pi(V')} - \pi(x_{B'})| \leq \Delta + \frac{\Delta}{s}\right)\Bigg).\end{aligned}$$

However, note that all the elements of $\pi(V')$ have magnitude at least $1 - 1/\log n$. Assume, for a moment, that Δ satisfies

$$\sqrt{\Delta^2 - 1} < s - 1 \leq \Delta < \Delta + \frac{\Delta}{s} < s. \tag{6}$$

From Theorem 1.1 (rescaled by $(1 - 1/\log n)^{-1}$), we conclude that

$$\mathbf{P}\left(|X_{\pi(V')} - \pi(x_{B'})| \leq \sqrt{\Delta^2 - 1}\right) \leq 2^{-(n-k)} S(n-k, s-1)$$

and

$$\mathbf{P}\left(|\pi(X_{V'}) - \pi(x_{B'})| \leq \Delta + \frac{\Delta}{s}\right) \leq 2^{-(n-k)} S(n-k, s).$$

On the other hand, by Stirling's formula (if n is sufficiently large) we have

$$\frac{1}{2}\big(2^{-(n-k)} S(n-k, s-1)\big) + \frac{1}{2} 2^{-(n-k)} S(n-k, s) = \sqrt{\frac{2}{\pi}} \frac{s - 1/2 + o(1)}{n^{1/2}}$$

while

$$2^{-n}S(n,s)=\sqrt{\frac{2}{\pi}}\frac{s+o(1)}{n^{1/2}}$$

and so we contradict (5).

An inspection of the above argument shows that all we need on Δ are the conditions (6). To satisfy the first inequality in (6), we need $\Delta<\sqrt{(s-1)^2+1}$. Moreover, once $s-1\le\Delta<\sqrt{(s-1)^2+1}$, one can easily check that $\Delta+\frac{\Delta}{s}<s$ holds automatically for any $s\ge3$, concluding the proof.

3. Proof of Theorem 1.4

Let d,n,k,V be as in Theorem 1.4. We allow all implied constants to depend on d.

By Esséen's concentration inequality (see [5], [13], or [14, Lemma 7.17]), we have for any unit ball B that

$$\mathbf{P}(X_V\in B)\ll\int_{\zeta\in\mathbf{R}^d:\ |\zeta|\le1}|\mathbf{E}(e(\zeta\cdot X_V))|\ d\zeta.$$

and $e(x):=e^{2\pi\sqrt{-1}x}$. From the definition of X_V and independence we have

$$\mathbf{E}(e(\zeta\cdot X_V))=\prod_{j=1}^n\mathbf{E}(e(\zeta\cdot\xi_jv_j))=\prod_{j=1}^n\cos(\pi\zeta\cdot v_j).$$

Denoting by $\|\theta\|$ the distance from θ to the nearest integer and using the elementary bound $|\cos(\pi\theta)|\le\exp(-\frac{\|\theta\|^2}{100})$ (whose proof is left as an exercise), we reduce to showing the bound

$$Q\ll k^{-d/2}. \tag{7}$$

where

$$Q:=\int_{\zeta\in\mathbf{R}^d:\ |\zeta|\le1}\exp(-\frac{1}{100}\sum_{v\in V}\|\zeta\cdot v\|^2)\ d\zeta. \tag{8}$$

To show (8), our main technical tool is the following lemma, whose proof is deferred to the next section.

Lemma 3.1. *Let* $w_1, \dots, w_d \in \mathbf{R}^d$ *be such that*

$$\operatorname{dist}(w_j, \operatorname{Span}\{w_1, \dots, w_{j-1}\}) \geq 1$$

for each $1 \leq j \leq d$*, where* $\operatorname{Span}\{w_1, \dots, w_{j-1}\}$ *is the linear span of the* $w_1, \dots, w_{j-1}$*, and* dist *denotes Euclidean distance. Then for any* $\lambda > 0$*,*

$$\int_{\zeta \in \mathbf{R}^d:\ |\zeta| \leq 1} \exp\left(-\lambda \sum_{j=1}^{d} \|\zeta \cdot w_j\|^2\right) \, d\zeta = O((1+\lambda)^{-d/2}).$$

With this lemma in hand, we conclude the proof as follows. By shrinking k, we may assume that $k = dl$ for some integer l. Let $v_{0,1}, \dots, v_{0,l}$ be l elements of V, and let $V_1 := V \backslash \{v_{0,1}, \dots, v_{0,l}\}$. Then we can write

$$Q = \int_{\zeta \in \mathbf{R}^d:\ |\zeta| \leq 1} \exp\left(-\frac{1}{100} \sum_{v \in V_1} \|\zeta \cdot v\|^2\right) \prod_{j=1}^{l} \exp\left(-\frac{1}{100} \|\zeta \cdot v_{0,j}\|^2\right) \, d\zeta.$$

Applying Hölder's inequality, we conclude the existence of a $j = 1, \dots, l$ such that

$$Q \leq \int_{\zeta \in \mathbf{R}^d:\ |\zeta| \leq 1} \exp\left(-\frac{1}{100} \sum_{v \in V_1} \|\zeta \cdot v\|^2\right) \exp\left(-\frac{l}{100} \|\zeta \cdot v_{0,j}\|^2\right) \, d\zeta.$$

Write $w_1 := v_{0,j}$. If $d = 1$, we stop at this point. Otherwise, we choose l elements $v_{1,1}, \dots, v_{1,l}$ be l elements of V_1 which lie at a distance at least 1 from the span $\operatorname{Span}\{w_1\}$ of w_1; such elements can be found thanks to the hypotheses of Theorem 1.4. We write $V_2 := V_1 \backslash \{v_{1,1}, \dots, v_{1,l}\}$. By using Hölder's inequality as before, we can find $j = 1, \dots, l$ such that

$$Q \leq \int_{\zeta \in \mathbf{R}^d:\ |\zeta| \leq 1} \exp\left(-\frac{1}{100} \sum_{v \in V_2} \|\zeta \cdot v\|^2\right) \cdot \exp\left(-\frac{l}{100} \|\zeta \cdot w_1\|^2\right) \exp\left(-\frac{l}{100} \|\zeta \cdot v_{1,j}\|^2\right) d\zeta.$$

We then set $w_2 := v_{1,j}$. We repeat this procedure $d-1$ times, eventually obtaining

$$Q \leq \int_{\zeta \in \mathbf{R}^d:\ |\zeta| \leq 1} \exp\left(-\frac{1}{100} \sum_{v \in V_d} \|\zeta \cdot v\|^2\right) \exp\left(-\frac{l}{100} \sum_{i=1}^{d} \|\zeta \cdot w_i\|^2\right) d\zeta$$

for some $w_1, \ldots, w_d$ with the property that $\text{dist}(w_i, \text{Span}\{w_1, \ldots, w_{i-1}\}) \geq 1$ for all $1 \leq i \leq d$, and where V_d is a subset of V of cardinality at least $n-k$. If we then trivially bound $\exp(-\frac{1}{100}\sum_{v \in V_d} \|\zeta \cdot v\|^2)$ by one, the claim follows from Lemma 3.1.

Remark 3.2. An inspection of the argument reveals that Theorem 1.4 still holds if one replaces the Bernoulli random variables by more general ones. For example, it suffices to assume that $\xi_1, \ldots, \xi_n$ are independent random variables satisfying $|\mathbf{E}e(x_i t)| \leq (1-\mu) + \mu \cos \pi t$ for any real number t, where $0 < \mu \leq 1$ is a constant. Indeed, with this assumption we have

$$|\mathbf{E}e(x_i t)| \leq \exp(-c_\mu \|t\|^2)$$

for all t and some $c_\mu > 0$, and the rest of the argument can then be continued with c_μ playing the role of the constant $1/100$.

It is easy to see that if there are constants K, ϵ such that the support of every ξ_i belongs to $\{-K, \ldots, K\}$, and $\mathbf{P}(\xi = j) \leq 1 - \epsilon$ for all $-K \leq j \leq K$, then all ξ_i are μ-bounded for some $0 < \mu \leq 1$ depending on K and ϵ.

4. Proof of Lemma 3.1

The only remaining task is to show Lemma 3.1. We are going to prove this lemma in the following, slightly more general but more convenient form.

Lemma 4.1. *Let $w_1, \ldots, w_d \in \mathbf{R}^d$ be such that $\text{dist}(v_j, \text{Span}\{w_1, \ldots, w_{j-1}\}) \geq 1$, for each $1 \leq j \leq d$. Let $u_1, \ldots, u_d$ be arbitrary numbers. Then for any $\lambda > 0$,*

$$\int_{\zeta \in \mathbf{R}^d:\ |\zeta| \leq 1} \exp\left(-\lambda \sum_{j=1}^{d} \|\zeta \cdot w_j + u_j\|^2\right) d\zeta \ll (1+\lambda)^{-d/2}. \tag{9}$$

Again, we allow all implied constants to depend on d.

We first consider the case $d=1$. It this case the claim is equivalent to

$$\int_{\zeta \in \mathbf{R}; |\zeta - u_1| \leq w_1} \exp(-\lambda \|\zeta\|^2) d\zeta = O\left(\frac{|w_1|}{\sqrt{1+\lambda}}\right),$$

which follows from periodicity of the function $\|\zeta\|$ and the elementary estimate

$$\int_{-1}^{1} \exp(\| - \lambda\zeta\|^2) d\zeta = O\left(\frac{1}{\sqrt{1+\lambda}}\right),$$

whose proof is left as an exercise.

To handle the general case, we use Fubini's theorem and induction on d. By Gram-Schmidt orthogonalization, we can find an orthonormal basis $\{e_1,\dots,e_d\}$ of $\mathbf{R}^d$, such that $\mathrm{Span}\{w_1,\dots,w_j\}=\mathrm{Span}\{e_1,\dots,e_j\}$, for all $1\le j\le d$. Suppose that the desired claim holds for $d-1$. For a vector $\zeta\in\mathbf{R}^d$, write

$$\zeta := \zeta' + \zeta_d e_d$$

where $\zeta'\in\mathrm{Span}\{e_1,\dots,e_{d-1}\}$ and $\zeta_d\in\mathbf{R}$. The left hand side of (9) can be rewritten as

$$\int_{|\zeta'|\le 1}\Bigg[\exp\Big(-\lambda\sum_{j=1}^{d}\|\zeta\cdot w_j+u_j\|^2\Big)\cdot\int_{|\zeta_d|\le 1}\exp\left(-\lambda\|\zeta_d(e_d\cdot w_d)+(\zeta'\cdot w_d+u_d)\|^2\right)d\zeta_d\Bigg]d\zeta'.$$

By the case $d=1$, the inner integral is $O\left(\frac{1}{\sqrt{\lambda+1}}\right)$, uniformly in ζ'. The claim now follows from the induction hypothesis.

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