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THE UNIVERSALITY OF HOM COMPLEXES OF GRAPHS

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It is shown that given a connected graph T with at least one edge and an arbitrary finite simplicial complex X, there is a graph G such that the complex $\mathrm{Hom}(T,G)$ is homotopy equivalent to X. The proof is constructive, and uses a nerve lemma. Along the way several results regarding Hom complexes, exponentials of graphs, and subdivisions are established that may be of independent interest.

1. Introduction

The Hom complex is a functorial way to assign a poset (and hence topological space) $\mathrm{Hom}(T,G)$ to a pair of graphs T and G. Versions of these spaces were introduced by Lovász in his proof of Kneser's conjecture ([9]), and later further investigated by Babson and Kozlov in [2] and [3]. The automorphism group of T naturally acts on the space $\mathrm{Hom}(T,G)$, and in the case that $T=K_2$ is an edge and G is graph without loops, the complex $\mathrm{Hom}(T,G)$ is a space with a *free* $\mathbb{Z}_2$-action. In [5] Csorba shows that *any* free $\mathbb{Z}_2$-space can be realized (up to $\mathbb{Z}_2$-homotopy type) as $\mathrm{Hom}(K_2,G)$ for some suitably chosen graph G. His proof involves a simple and elegant construction in which one obtains a graph G whose vertices are precisely those of the given $\mathbb{Z}_2$-simplicial complex.

A natural question to ask is what homotopy types can be realized as $\mathrm{Hom}(T,?)$ for other test graphs T. As Csorba points out, arbitrary homotopy types cannot be realized by Hom complexes of *loopless* graphs even with

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$T = K_2$ as the test graph; all such Hom complexes will be free $\mathbb{Z}_2$-spaces and hence will present topological obstructions (e.g. parity of the Euler characteristic). However, if we allow loops on our graphs, and do not concern ourselves with group actions, we are able to prove the following 'universality' of Hom complexes.

Theorem 1.1. *Let T be a connected graph with at least one edge, and suppose X is a finite simplicial complex. Then there exists a graph $G_{k,X}$ (depending on X and the diameter of T) and a homotopy equivalence*

$$\mathrm{Hom}(T, G_{k,X}) \simeq X.$$

The graph $G_{k,X}$ will be *reflexive* (that is, has loops on all the vertices), and hence the space $\mathrm{Hom}(T, G_{k,X})$ will no longer carry a free $\mathrm{Aut}(T)$ action. The idea behind our proof of this theorem will be to consider $X^k = \mathrm{bd}^k(X)$, a high enough (depending on the diameter of T) barycentric subdivision of the given simplicial complex X, and to define $G_{k,X}$ as the 1-skeleton of X^k with loops placed on each vertex. To show that $\mathrm{Hom}(T, G_{k,X})$ has the desired homotopy type, we will first replace it with a homotopy equivalent space X' (which will be the clique complex of some graph). We then determine the homotopy type of X' by covering it with a collection of contractible subcomplexes (with contractible intersections) and then employing a nerve lemma.

The structure of the paper is as follows. In section 2 we provide some necessary background on graphs, Hom complexes, and their properties. Section 3 is devoted to the proof of the main result and some related lemmas. We conclude in section 4 with some open questions.

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2. Main objects of study

In this section we record some basic facts about graphs and Hom complexes. For us, a *graph* $G = (V(G), E(G))$ consists of a vertex set $V(G)$ and an edge set $E(G) \subseteq V(G) \times V(G)$ such that if $(v, w) \in E(G)$ then $(w, v) \in E(G)$. Hence our graphs are undirected and do not have multiple edges, but may have loops (if $(v, v) \in E(G)$). If $(v, w) \in E(G)$ we will often say that v and w are *adjacent* and denote this as $v \sim w$. Given a pair of graphs G and H, a *graph homomorphism* (or *graph map*) $f \colon G \to H$ is a mapping of the vertex set

$f\colon V(G)\to V(H)$ that preserves adjacency: if $v\sim w$ in G, then $f(v)\sim f(w)$ in H. With these as our objects and morphisms we obtain a category of graphs which we will denote $\mathcal{G}$.

If v and w are vertices of a graph G, the *distance* $d(v,w)$ is the length of the shortest path in G from v to w. The *diameter* of a finite connected graph G, denoted $\mathrm{diam}(G)$ is the maximum distance between two vertices of G. The *neighborhood* of a vertex v, denoted $N_G(v)$ (or $N(v)$ if the context is clear), is the set of vertices that are precisely distance 1 from v (so that $v\in N(v)$ if and only if v has a loop). If v and w are vertices of a graph such that $N(v)\subseteq N(w)$ then we call the map $f\colon G\to G\setminus v$ that sends v to w a *folding* of the vertex v; we will also say that G *folds onto* the graph $G\setminus v$.

There are several simplicial complexes one can associate with a given graph G. One such construction is the *clique complex* $\Delta(G)$, a simplicial complex with vertices given by all *looped* vertices of G, and with faces given by all cliques (complete subgraphs) on the looped vertices of G.

We next recall the definition of our main object of study, the Hom complex. (Versions of) this construction were originally used by Lovász, Babson and Kozlov, and others to provide so-called *topological* lower bounds on the chromatic numbers of graphs (see [8] for a nice survey). We will use the following definition.

Definition 2.1. For graphs G and H, we define $\mathrm{Hom}(G,H)$ to be the poset whose elements are given by all functions $\eta\colon V(G)\to 2^{V(H)}\setminus\{\emptyset\}$, such that if $(x,y)\in E(G)$, then for any $\tilde{x}\in\eta(x)$ and $\tilde{y}\in\eta(y)$ we have $(\tilde{x},\tilde{y})\in E(H)$. The partial order is given by containment, so that $\eta\le\eta'$ if $\eta(x)\subseteq\eta'(x)$ for all $x\in V$.

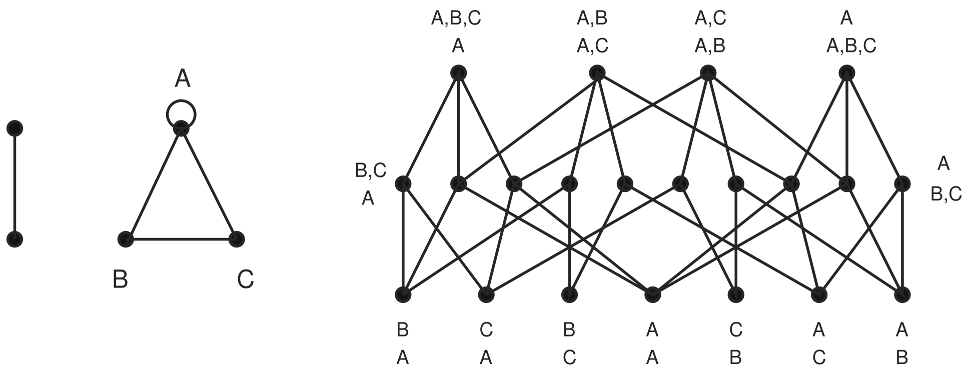

Figure 1. The graphs G and H, and the poset $\mathrm{Hom}(G,H)$.

One can check that for a fixed graph G, $\mathrm{Hom}(G,?)$ (resp. $\mathrm{Hom}(?,H)$) is a covariant (resp. contravariant) functor from the category of graphs to the category of posets. We will often speak of topological properties of the Hom *complex*. In this context we will mean the space obtained as the geometric realization of the *order complex* of the poset $\mathrm{Hom}(G,H)$. When the context is clear, we will refer to this topological space (realization of a simplicial complex) with the same $\mathrm{Hom}(G,H)$ notation.

Example 2.2. In Figure 1 we provide an example of graphs G and H with the associated poset $\mathrm{Hom}(G,H)$, where elements of the latter are labeled with the images of the top and bottom vertices of G. In Figure 2 we depict the complex whose face poset is $\mathrm{Hom}(G,H)$; hence the realization of $\mathrm{Hom}(G,H)$ will be the simplicial complex given by the barycentric subdivision of this space. We point out that the polyhedral complex depicted here is the original construction of $\mathrm{Hom}(G,H)$ described in [2]. For us it will be more convenient to work with the poset.

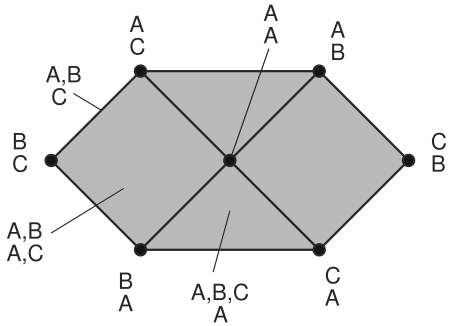

Figure 2. The realization of the poset $\mathrm{Hom}(G,H)$ (up to barycentric subdivision).

The category $\mathcal{G}$ has a product with a right adjoint given by the exponential graph construction. We recall these constructions below.

Definition 2.3. If G and H are graphs, then the *categorical product* $G\times H$ is a graph with vertex set $V(G)\times V(H)$ and adjacency given by $(g,h)\sim(g',h')$ in $G\times H$ if both $g\sim g'$ in G and $h\sim h'$ in H.

Definition 2.4. For graphs G and H, the *categorical exponential graph* H^G is a graph with vertex set $\{f\colon V(G)\to V(H)\}$, the collection of all vertex set maps, with adjacency given by $f\sim f'$ if whenever $v\sim v'$ in G we have $f(v)\sim f'(v')$ in H.

The exponential graph construction provides a right adjoint to the categorical product. This gives the category of graphs the structure of an *internal hom* associated with the (monoidal) categorical product (see [6] for the meaning of these statements). It turns out that the Hom complex interacts well with this adjunction, as described in the following proposition (see [8] or [6] for a proof).

Proposition 2.5. *For A, B, C any graphs,* $\mathrm{Hom}(A \times B, C)$ *can be included in* $\mathrm{Hom}(A, C^B)$ *so that* $\mathrm{Hom}(A \times B, C)$ *is a strong deformation retract of* $\mathrm{Hom}(A, C^B)$*. In particular, we have* $\mathrm{Hom}(A \times B, C) \simeq \mathrm{Hom}(A, C^B)$.

Note that, as a result of the proposition, we have $\mathrm{Hom}(G, H) \simeq \mathrm{Hom}(\mathbf{1}, H^G)$, where $\mathbf{1}$ is the graph with a single looped vertex. The latter space is homeomorphic to the (realization) of the clique complex $\Delta(H^G)$. Hence, up to homotopy type, the space $\mathrm{Hom}(G, H)$ is just the clique complex on the looped vertices of the graph H^G. We will use this identification in the proof of the main theorem.

3. Proof of the main theorem

In this section we provide the proof of Theorem 1.1. Note that if $T = \mathbf{1}$ is a single looped vertex, we have $\mathrm{Hom}(\mathbf{1}, G) \simeq \Delta(G)$, the clique complex on the looped vertices of G. Hence to obtain the result in this case, we define $G_{k,X}$ to be the graph obtained by taking the 1-skeleton of $\mathrm{bd}(X) = X^1$, the first barycentric subdivision of the given complex X. Since the barycentric subdivision of a simplicial complex is a flag complex, we get that the 1-skeleton provides an inverse to the Δ functor in this case, and hence $X \simeq X^1 = \Delta(G_{k,X})$. We note that the same graph works more generally if T is *dismantlable* (see [6]) since in this case we have $\mathrm{Hom}(T, G) \simeq \mathrm{Hom}(\mathbf{1}, G)$ by results of [2].

In the general case we will similarly obtain $G_{k,X}$ as the looped 1-skeleton of some iterated subdivision of X, but this time we have to take into account the diameter of the test graph T. Recall the setup: we are given a connected graph T with at least one edge, and a finite simplicial complex X. If $d = \mathrm{diam}(T)$ is the diameter of T, we fix an integer $k \geq 2$ such that

$$2^{k-1} - 1 \geq d.$$

Next, we let $X^k = \mathrm{bd}^k(X)$ denote the k^{th} barycentric subdivision of the simplicial complex X. We define $G_{k,X}$ to be the graph given by the 1-skeleton of X^k, with loops placed at every vertex.

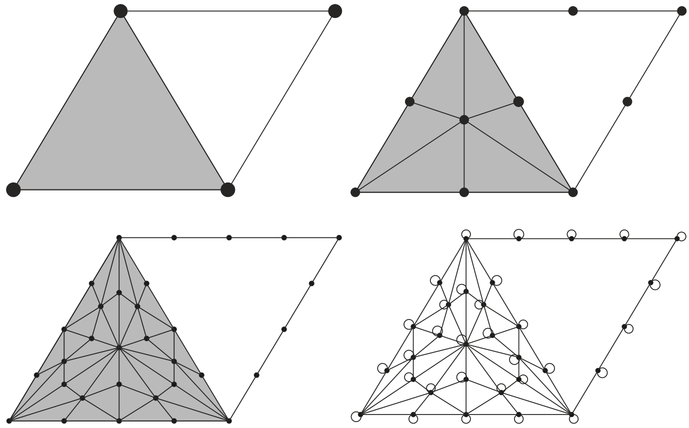

Figure 3. The complexes X, X^1, X^2, and the reflexive graph $G_{2,X}$.

We claim that $\mathrm{Hom}(T, G_{k,X}) \simeq X$. From Proposition 2.5, we have $\mathrm{Hom}(T, G_{k,X}) \simeq \mathrm{Hom}(\mathbf{1}, (G_{k,X})^T)$ (where $\mathbf{1}$ is the graph with one looped vertex). The latter space is homeomorphic to $\Delta((G_{k,X})^T)$, the clique complex on the (looped vertices of the) graph $\Delta((G_{k,X})^T)$. Hence to prove the main result (Theorem 1.1) it is enough to prove the following restatement.

Theorem 3.1. *Let T be an arbitrary connected graph with at least one edge, and let X be a finite simplicial complex. Then for all integers $k \geq \log_2(d+1)+1$ there is a homotopy equivalence*

$$X \simeq \Delta\big((G_{k,X})^T\big).$$

Proof. We consider subcomplexes of $\Delta((G_{k,X})^T)$ of the form $\Delta((G^x_{k,X})^T)$ (see Definition 3.2 below for the definition of the graph $G^x_{k,X}$). By Lemma 3.4 the collection of these subcomplexes form a cover of $\Delta((G_{k,X})^T)$, and by Lemma 3.5 the nerve of this cover is isomorphic to the simplicial complex X. By Lemma 3.7 and Lemma 3.8, these subcomplexes and all nonempty intersections are contractible. The result follows from the nerve lemma of [1]. ∎

We next turn to the definition of our subcomplexes and the proofs of the lemmas mentioned above. Recall that the simplicial complex $\Delta((G_{k,X})^T)$ is determined by its 1-skeleton $(G_{k,X})^T$, whose vertices are given by all graph maps $f \colon T \to G_{k,X}$, and with edges $\{f, f'\}$ whenever $f(t) \sim f'(t')$ for all $t \sim t'$ in T. We note that the vertices of the original complex X are naturally vertices of the graph $G_{k,X}$. We will work with certain graph theoretic 'open neighborhoods' of these vertices, as described in the following definition.

Definition 3.2. For a fixed vertex x of the original complex X, define $G^x_{k,X}$ to be the subgraph of $G_{k,X}$ induced by the vertices $\{w \in G_{k,X}\colon d(x,w) \leq 2^k - 1\}$.
Hence the vertices of the graph $G^x_{k,X}$ are the vertices of $G_{k,X}$ that are distance at most $2^k - 1$ from the vertex x.

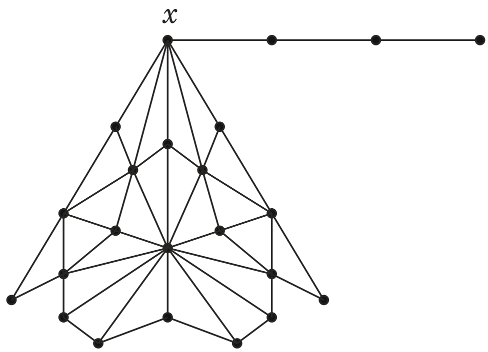

Figure 4. The graph $G^x_{2,X}$ (without the loops).

It is this collection of subcomplexes $\{\Delta((G^x_{k,X})^T)\}_{x\in V(X)}$ that we wish to show cover the complex $\Delta((G_{k,X})^T)$. For this we will need a general lemma regarding clique complexes of exponential graphs. For graphs T and G, and a simplex $\alpha = \{f_1,\dots,f_a\} \in \Delta(G^T)$, define G_α to be the subgraph of G induced by the vertices $\{f_i(t)\colon 1 \leq i \leq a, t \in V(T)\}$. We then make the following observation.

Lemma 3.3. *Let T be a finite connected graph with diameter $d = \operatorname{diam}(T)$, and suppose G is any graph. Then $\operatorname{diam}(G_\alpha) \leq \max\{2,d\}$ for all $\alpha \in \Delta(G^T)$.*

Proof. Suppose T and G are as above, and suppose $\alpha = \{f_1,\dots,f_a\}$ is a face of $\Delta(G^T)$. Let $v = f_i(t)$ and $v' = f_{i'}(t')$ be any two elements of G_α. We will find a path in G_α from v to v' of length $\leq d$. If $t \neq t'$, then by assumption we have a path in T from t to t' given by $(t = t_0, t_1, \dots, t_j = t')$, with $j \leq d$. If $t = t'$, we take our path to be $(t, t_1, t_2 = t)$, where t_1 is any neighbor of t. So we have $j \leq \max\{2,d\}$.

Now, since α is a clique (with looped vertices) in the graph G^T, we have that $f_i \sim f_j$ for all $1 \leq i,j \leq k$, and hence $f_i(t) \sim f_j(t')$ for all adjacent $t \sim t'$. Hence we can take our desired path to be $f_i(t) = f_i(t_0), f_i(t_1), \dots, f_i(t_{j-1}), f_{i'}(t_j) = f_{i'}(t')$. ∎

We can now show that our subcomplexes indeed form a cover.

Lemma 3.4. *The collection of complexes* $\{\Delta((G^x_{k,X})^T)\}_{x\in V(X)}$ *covers the complex* $\Delta((G_{k,X})^T)$.

Proof. To simplify indices, in our notation for graphs we will suppress reference to the integer k and the simplicial complex X, so that for this proof $G := G_{k,X}$ and $G^x := G^x_{k,X}$. If α is a face of $\Delta(G^T)$ then by Lemma 3.3 we have either $d = \text{diam}(T) = 1$ and $k = 2$, or else $\text{diam}(G_\alpha) \le 2^{k-1} - 1$ (where k is taken as in Theorem 3.1). We claim that $G_\alpha \subseteq G^x$ for some $x \in V(X)$, which would prove the lemma.

Let $m = \min\{d(w,x)\colon w \in G_\alpha, x \in V(X)\}$. Note that $m \le 2^{k-1}$ since every vertex of X^k is within distance 2^{k-1} of some vertex of the original complex X.

If $m = 0$ then we have $y \in G_\alpha$ for some vertex $y \in V(X)$. Hence $G_\alpha \subseteq G^y$ since G^y contains all vertices distance at most $2^k - 1$ from y (this number is at least 2 since $k \ge 2$).

If $m > 0$ let w be a vertex of G_α such that $d(w,x) = m$ for some vertex $x \in X$, and choose w such that it is contained in the interior of a face of X of minimum dimension. We need to show that $G_\alpha \subseteq G^x$. To see this, first consider the case that $k > 2$. By Lemma 3.3, all vertices w' in G_α are distance at most $d \le 2^{k-1} - 1$ from w. So all vertices of G_α are at most $m + d \le 2^{k-1} + 2^{k-1} - 1 = 2^k - 1$ away from x, which implies $G_\alpha \subseteq G^x$.

If $k = 2$, then we have $m = 1$ or $m = 2$. If $m = 1$ then all vertices of G_α are distance at most $1 + 2 = 3 = 2^k - 1$ away from x, as desired. If $m = 2$, then all vertices of G_α are distance at least 2 from *every* vertex of X. Now, w is contained in the interior of some face $F_w = \{x, x_1, \dots, x_j\}$ of the original complex X. If w' is any other vertex of G_α, then w' cannot be contained in any proper face of F_w since otherwise we would have taken $w = w'$. Hence w' is contained in the interior of F_w, so that $d(w', x) \le 2^k - 1$, as desired. This shows that $G_\alpha \subseteq G^x$. ∎

We next turn to the combinatorics of our cover. Recall that the *nerve* of a covering of a space by subspaces is the simplicial complex with vertices given by the subspaces and with faces corresponding to all non-empty intersections. We then have the following observation.

Lemma 3.5. *The nerve of the covering of* $\Delta((G_{k,X})^T)$ *given by the subcomplexes* $\Delta((G^x_{k,X})^T)$ *is isomorphic to the simplicial complex* X.

Proof. By construction, the vertices of the nerve determined by the $\Delta((G^x_{k,X})^T)$ are indexed by $V(X)$, the vertices of the simplicial complex X. A collection $I \subseteq V(X)$ of such subcomplexes has nonempty intersection if and only if there exists a vertex x within distance $2^k - 1$ from each $v \in I$

in X^k, the k^{th} barycentric subdivision of X. But this occurs if and only if the collection I of vertices form a face of X. ∎

Next we wish to show that each subcomplex $\Delta((G^x_{k,X})^T)$ is contractible. To do this we will show that each graph $G^x_{k,X}$ is in fact *dismantlable*. Recall that a finite graph G is called *dismantlable* if it can be folded down to the looped vertex $\mathbf{1}$ (see [4] and [6] for other characterizations). It follows from the results of [7] that if G is dismantlable, then $\mathrm{Hom}(S,G)$ is contractible for *any* graph S. Hence to show that the subcomplexes $\mathrm{Hom}(T,G^x_{k,X}) \simeq \Delta((G^x_{k,X})^T)$ are each contractible, it suffices to show that each graph $G^x_{k,X}$ is dismantlable.

For this we will describe a recursive folding procedure for the graph $G^x_{k,X}$. In our induction we will need the fact that barycentric subdivision preserves dismantlability, as described by the following lemma.

Lemma 3.6. *If G is a dismantlable graph and $\Delta(G)$ is its clique complex (on its looped vertices), then the looped one-skeleton of* $\mathrm{bd}(\Delta(G))$ *is again dismantlable.*

Proof. Suppose G is a dismantlable graph, and let G' denote the graph obtained by taking the looped one-skeleton of $\mathrm{bd}(\Delta(G))$. We can think of G' as the graph whose vertices are the elements of the poset $\mathrm{Hom}(\mathbf{1},G)$, with adjacency given by $x \sim y$ if x and y are comparable.

To show that G' is dismantlable, we proceed by induction on n, the number of looped vertices of G. If $n=1$ we have that $G=G'$ is a single looped vertex, and hence dismantlable.

Next suppose $n>1$, and let v and w be distinct looped vertices of G such that $N_G(v) \subseteq N_G(w)$. For future reference, we let $N_G(v) = \{v, w, v_1, \ldots, v_m\}$ denote the neighboring vertices of v in the graph G. We will use the following running example, in which the loops (present on all vertices) will be omitted for the sake of space.

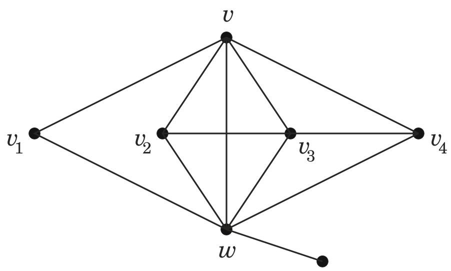

Figure 5. The containment $N_G(v) \subseteq N_G(w)$.

For the inductive step, we need to fold away all vertices in G' that are barycenters of simplices that have v as a vertex (including the vertex v itself). But this is precisely $N_{G'}(v)$, the collection of neighboring vertices of v in the graph G'.

We will first fold away the vertices in $N_{G'}(v)$ that are furthest from w. We let S denote the collection of vertices in $N_{G'}(v)$ that are barycenters of simplices that do *not* contain w. So S is the collection of vertices in $\mathrm{bd}(\Delta(G))$ that are barycenters of simplices with vertices among the set $\{v, v_1, \ldots, v_m\}$.

Each vertex $s \in S$ is the barycenter of a face of a certain dimension, and we will fold away the elements of S in descending order according to this dimension. If s is the barycenter of a face $\{v, v_{i_1}, \ldots, v_{i_r}\}$ of *maximal* dimension then we have $N_{G'}(s) \subseteq N_{G'}(y)$, where $y \in G'$ is the barycenter of the face $\{v, v_{i_1}, \ldots, v_{i_r}, w\}$; this collection forms a face of $\Delta(G)$ since $N_G(v) \subseteq N_G(w)$. Hence we can fold away s in this case.

In general, s is the barycenter of a face $F_s = \{v, v_{j_1}, \ldots, v_{j_\ell}\}$ and, as we have folded away the vertices of greater dimension in S (barycenters of faces that contain F_s), we have $N(s) \subseteq N(y)$ in the resulting graph, where again y is the barycenter of the face $\{v, v_{j_1}, \ldots, v_{j_\ell}, w\}$.

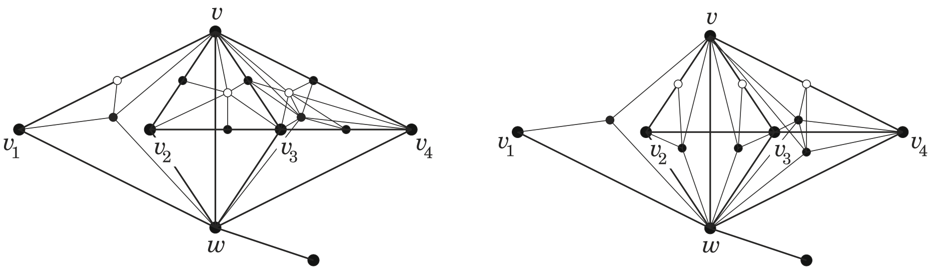

Figure 6. Folding away the vertices of S.

In the diagram above, the first step is to fold away the barycenters of $\{v, v_1\}$, $\{v, v_2, v_3\}$, and $\{v, v_3, v_4\}$ (the vertices in white). In the second step we fold away the barycenters of $\{v, v_2\}$, $\{v, v_3\}$, and $\{v, v_4\}$.

Next we fold away the vertex v. If $u \in N(v)$ is a neighbor of v in the graph at this stage of the folding, then u is the barycenter of some face that contains both v and w, and hence we have $N(v) \subseteq N(z)$, where z is the barycenter of $\{v, w\}$. We fold away v and now have that all neighbors of z are barycenters of faces that contain the vertex w. Hence we now have $N(z) \subseteq N(w)$, and we proceed to fold away the vertex z.

At this point, we are left with a subset $Y \subseteq N_{G'}(v)$ that consists of vertices that are barycenters of faces that contain v, w, and at least one

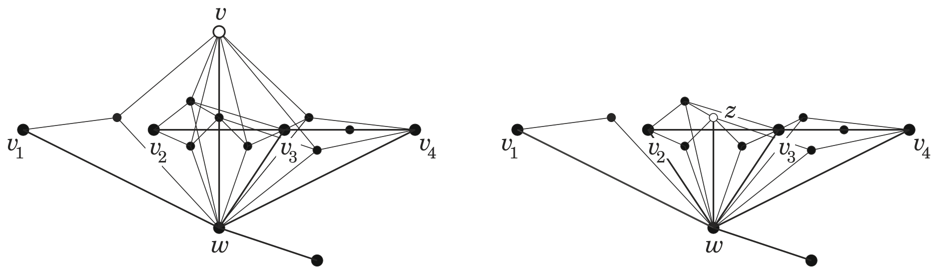

Figure 7. Folding away v and z.

vertex from $\{v_1, \ldots, v_m\}$. We fold away these vertices in *ascending* order according to their dimension. If $y \in Y$ is the barycenter of a face $\{v, w, v_i\}$ of *minimal* dimension, then $N(y) \subseteq N(z)$, where z is the barycenter of the face consisting of $\{w, v_i\}$ (since vertices that are barycenters of faces including v have been folded away). In the general case, y is the barycenter of a face $\{v, w, v_{i_1}, \ldots, v_{i_j}\}$ and, as we have folded away the vertices of smaller dimension in Y, we now have $N(y) \subseteq N(z)$, where again z is the barycenter of the face $\{w, v_{i_1}, \ldots, v_{i_j}\}$. See Figure 8. ∎

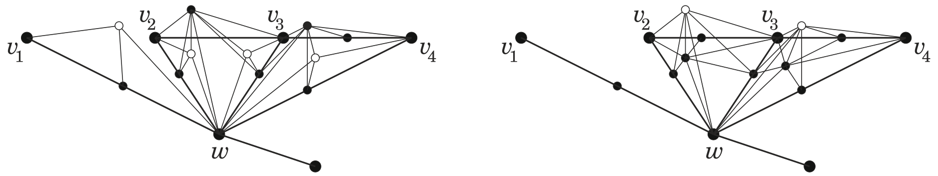

Figure 8. Folding away the remaining vertices of $N_{G'}(v)$.

We can now use this to prove the following result concerning the $G^x_{k,X}$ graphs.

Lemma 3.7. *For any vertex $x \in X$, the graph $G^x_{k,X}$ is dismantlable.*

Proof. Recall that $G^x_{k,X}$ is the subgraph of $G_{k,X}$ induced by the vertices that are distance at most $2^k - 1$ from x. We will prove the claim by induction on k. For $k = 1$ the graph $G^x_{k,X}$ consists of $N_{G_{k,X}}(x)$, the neighbors of the vertex x in $G_{k,X}$ (including x itself). Hence $G^x_{k,X}$ folds down to the single looped vertex x, as desired.

Next suppose $k > 1$. Our plan is to first fold away the vertices in $G^x_{k,X}$ that are distance *exactly* $2^k - 1$ from x. The resulting subgraph one obtains is the looped 1-skeleton of the barycentric subdivision of the clique complex $\Delta(G^x_{k-1,X})$ (this graph is called $(G^x_{k-1,X})'$ in the notation of the proof of Lemma 3.6). By induction, together with Lemma 3.6, this graph is dismantlable and hence our claim will be proved.

Let V_x denote the collection of vertices in $G^x_{k,X}$ that are distance exactly 2^k-1 from x; it is this collection of vertices that we wish to fold away. First we set up some notation. Note that every vertex v in the graph $G_{k,X}$ has a pair of parameters $\alpha(v)=(i,j)$ associated with it, where i is the dimension of the face in X that v lies in, and where j is the dimension of the face of X^{k-1} that v is the barycenter of (note that $j \le i$). We will say that v is *of type* (i,j) if $\alpha(v)=(i,j)$.

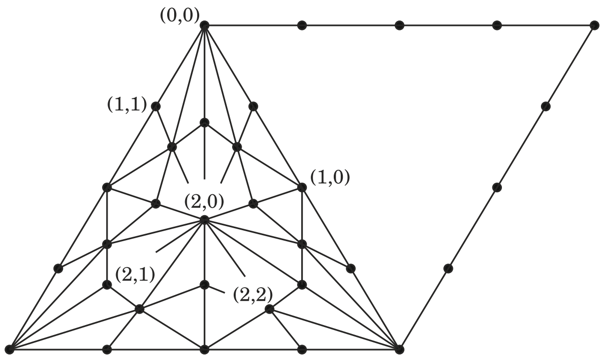

Figure 9. The types (i,j) of various vertices in the graph $G_{k,X}$.

We will fold away the vertices of $V_x \subseteq G^x_{k,X}$ in lexicographic order according to their type (i,j). First note that if $v \in V_x$ is of type (i,j) then $j \ge 1$, and hence our base case to consider is when $v \in V_x$ is of type $(1,1)$. In this case v is the barycenter of an edge $\{a,b\}$ in X^{k-1}, where b is a vertex of X, and a is distance $2^{k-1}-2$ from x. Any neighbor $w \in N_{G^x_{k,X}}(v)$ of v is a barycenter of a simplex that has a as a vertex; hence we have $w \sim a$. We conclude that v can be folded onto the neighboring vertex a.

Next we consider the case v is of type (i,j), where $i > 1$ is fixed. We proceed by induction on j. If $j=1$ then v is the barycenter of an edge $\{c,d\}$, where $c \notin V_x$ and d is of type $(i,0)$ and is distance 2^k-2 from x. Any other neighbor $w \in N_{G^x_{k,X}}(v)$ is the barycenter of a simplex that has d as a vertex; we conclude that $w \sim d$. Hence in this case v can be folded onto d.

For the same fixed $i>1$, we next consider the case that v is of type (i,j), where $j>1$. By induction, we have that all vertices in V_x of type (i',k) and of type (i,j') have been folded away, where $i'<i$ and $j'<j$. Pick a vertex $w \in N(v)$ in the neighborhood of v such that $w \in G^x_{k-1,X}$ and such that the

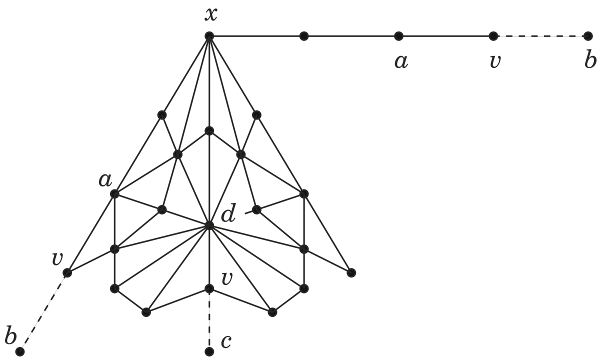

Figure 10. Folding away the vertex v when v is of type $(i,1)$.

type of w is largest in the lexicographic order – that is, of type (i,j) where j is maximum among maximum i.

We claim that $N_{G^x_{k,X}}(v) \subseteq N_{G^x_{k,X}}(w)$, so that the vertex v can be folded onto w. To see this, suppose $u \in N_{G^x_{k,X}}(v)$. If $u \in V_x$ (so that $d(u,x) = 2^k - 1$), then by induction we know that u is of type (i',j'), where either $i' > i$ or else $i' = i$ and $j' > j$. In either case we see that u is the barycenter of a simplex U that contains the vertex w, and hence $u \sim w$ as claimed. If $u \notin V_x$, so that $d(u,x) = 2^k - 2$, then either $u = w$ or else the type of u is lexicographically smaller than the type of w. In this latter case u is the barycenter of a simplex U' that contains the vertex w, and hence again $u \sim w$. We conclude that $u \in N_{G^x_{k,X}}(w)$ and hence $N_{G^x_{k,X}}(v) \subseteq N_{G^x_{k,X}}(w)$ as desired.

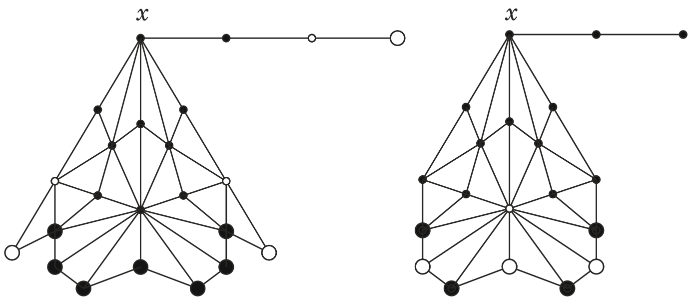

Figure 11. Folding away vertices of type $(1,1)$ and of type $(2,1)$ in $G^x_{2,X}$.

This completes the induction on j and hence we have now folded away all vertices of V_x that are of type $(i,?)$. This in turn completes the induction on i and we conclude that all vertices in V_x can be folded away. As we

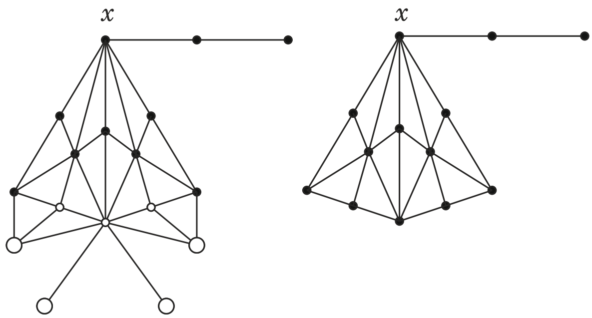

Figure 12. Folding away vertices of type $(2,2)$ and the resulting $(G^x_{1,X})'$.

noted above, the resulting graph is $(G^x_{k-1,X})'$, the barycentric subdivision of $G^x_{k-1,X}$, which we conclude is dismantlable by induction on k and by applying Lemma 3.6. The result follows. ∎

The final step in proving our theorem is to consider the intersections of the subcomplexes $G^x_{k,X}$.

Lemma 3.8. *All nonempty intersections of the subcomplexes*

$$\{(G^x_{k,X})^T\}_{x\in V(X)}$$

are contractible.

Proof. We prove this in much the same way as we handled the contractibility of the subcomplexes themselves. In particular it is enough to show that the subgraphs obtained as nonempty intersections of $\{G^x_{k,X}\}_{x\in V(X)}$ are dismantlable. A vertex of such a graph is, by definition, within a distance of 2^k-1 of every vertex $x\in V(X)$ in some index set $I\subseteq V(X)$. Note that this intersection is empty unless the vertices of I constitute a face of X.

Suppose $G^I_{k,X}$ is such a graph. Again, we will show that $G^I_{k,X}$ is dismantlable by induction on k. If $k=1$ then the vertices of the graph $G^I_{k,X}$ are all barycenters of the faces in the *star* of I, the collection of all faces which contain I as a subface. In the induced graph every such vertex is adjacent to the single looped vertex which is the barycenter of the face of X defined by I. Hence the graph is dismantlable in this case.

For the case $k>1$ we will, as above, fold away the vertices of $G^I_{k,X}$ that are distance 2^k-1 from some vertex $x\in I$. We will refer to these vertices as V_I, so that $V_I=\{v\in G^I_{k,X} : d(v,x)=2^k-1 \text{ for some } x\in I\}$.

Again, we fold away the vertices of V_I in lexicographic order according to their type (i,j). Since $V_I\subseteq V_x$ (for any $x\in I$), we can follow the same procedure as we described in the proof of Lemma 3.7. In particular, to fold away a vertex $v \in V_I$ of type (i,j), we choose a vertex $w \in N_{V_x}(v)$ in the neighborhood of v such that $w \in G^I_{k-1,X}$ and such that the type of w is largest in the lexicographic order.

We just need to check that w is within $2^k - 1$ of *every* vertex $x' \in I$, so that indeed $w \in V_I$. But this follows from the choice of w: since v is in the interior of the face of X determined by the vertices I, any neighbor w' of v that lies outside of V_I will be of type (i',j), where $i' < i$. But v has neighbors in $G^I_{k-1,X}$ that are of type (i,j'), so that the choice of w will indeed lie in V_I.

Hence the double induction follows through in this case, and we are left with a graph $(G^I_{k-1,X})'$, the barycentric subdivision of the graph $G^I_{k-1,X}$ (informally speaking). Once again we employ Lemma 3.6 and by induction we get that this graph is also dismantlable. ∎

4. Further questions

Having constructed our graph $G_{k,X}$ as the 1-skeleton of the k^{th} iterated subdivision of X, a natural question to ask is if this choice of k is best possible. We have a feeling that it is not, and in fact, for the case $\mathrm{diam}(T) = 1$ (so that T is a complete graph with possibly some loops) we conjecture that $k = 1$ will do the job.

Conjecture 4.1. If X is a finite simplicial complex, and T is a finite connected graph with $\mathrm{diam}(T) = 1$, then there is a homotopy equivalence

$$\mathrm{Hom}(T, G_{1,X}) \simeq X.$$

Another thing to consider would be simplicial complexes with a specified group action.

Question 4.2. Given a graph T with automorphism group $\Gamma = Aut(T)$, and a Γ-simplicial complex X, can one find a (loopless) graph G such that $\mathrm{Hom}(T, G)$ is Γ-homotopy equivalent to X?

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