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HOMOTOPY TYPES OF BOX COMPLEXES

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In [14] Matoušek and Ziegler compared various topological lower bounds for the chromatic number. They proved that Lovász's original bound [9] can be restated as $\chi(G) \geq \mathrm{ind}(\mathrm{B}(G)) + 2$. Sarkaria's bound [15] can be formulated as $\chi(G) \geq \mathrm{ind}(\mathrm{B}_0(G)) + 1$. It is known that these lower bounds are close to each other, namely the difference between them is at most 1. In this paper we study these lower bounds, and the homotopy types of box complexes. The most interesting result is that up to $\mathbb{Z}_2$-homotopy the box complex $\mathrm{B}(G)$ can be any $\mathbb{Z}_2$-space. This together with topological constructions allows us to construct graphs showing that the mentioned two bounds are different. Some of the results were announced in [14].

1. Introduction

In [14] Matoušek and Ziegler compared various topological lower bounds for the chromatic number. They reformulated Lovász's original bound [9] and Sarkaria's bound [15] in terms of the index of various box complexes:

Theorem 1.1 (The Lovász bound [14]). *For any graph G*

$$\chi(G) \geq \mathrm{ind}(\mathrm{B}(G)) + 2.$$

Theorem 1.2 (The Sarkaria bound [14]). *For any graph G*

$$\chi(G) \geq \mathrm{ind}(\mathrm{B}_0(G)) + 1.$$

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We will study these lower bounds in this paper, which is organized as follows.

Section 2 contains the definition of the box complexes of graphs and we fix some notation.

In Section 3 we prove that the box complex $\mathrm{B}_0(G)$ is $\mathbb{Z}_2$-homotopy equivalent to the suspension of $\mathrm{B}(G)$. This makes the connection between these two bounds explicit. Since $\mathrm{ind}(X) \leq \mathrm{ind}(\mathrm{susp}(X)) \leq \mathrm{ind}(X)+1$ the difference between the right side of the Lovász and the Sarkaria bound is at most one.

From topological point of view it is possible that these two bounds are not the same. We construct a $\mathbb{Z}_2$-space X_{2h} such that $\mathrm{ind}(\mathrm{susp}(X_{2h})) = \mathrm{ind}(X_{2h})$ in Section 6.

However we need a graph such that its box complex $\mathrm{B}(G)$ has this property. In Section 4 we show that the homotopy type of the box complex $\mathrm{B}(G)$ (which is homotopy equivalent to the neighborhood complex) can be 'arbitrary'; in Section 5 we extend this result to $\mathbb{Z}_2$-homotopy equivalence. This allows us to construct a graph G such that the gap between these two bounds is 1. This means that the Lovász bound can be better than the Sarkaria bound, which answers a question of Matoušek and Ziegler [14].

Finally in Section 7 we show that both of these topological lower bounds can be arbitrarily bad. Our examples are purely topological.

2. Preliminaries

In this section we recall some basic facts of graphs and simplicial complexes and topology to fix notation. The interested reader is referred to [13] or [2] and [6] for details.

Graphs. Any graph G considered will be assumed to be finite, simple, connected, and undirected, i.e., G is given by a finite set $V(G)$ of *vertices* and a set of *edges* $E(G) \subseteq \binom{V(G)}{2}$. A *graph coloring with n colors* is a homomorphism $c : G \to K_n$, where K_n is the complete graph on n vertices and the *chromatic number* $\chi(G)$ of G is the smallest n such that there exists a graph coloring of G with n colors. The *common neighborhood* of $A \subseteq V(G)$ is $\mathrm{CN}(A) = \{v \in V(G) \colon \{a, v\} \in E(G) \text{ for all } a \in A\}$; we define $\mathrm{CN}(\emptyset) := V(G)$. For two disjoint sets of vertices $A, B \subseteq V(G)$ we define $G[A, B]$ as the (not necessarily induced) subgraph of G with $V(G[A, B]) = A \cup B$ and $E(G[A, B]) = \{\{a, b\} \in E(G) \colon a \in A, b \in B\}$.

Simplicial Complexes. A *simplicial complex* $\mathcal{K}$ is a finite hereditary set system. We denote its vertex set by $V(\mathcal{K})$ and its barycentric subdivision by $\mathrm{sd}(\mathcal{K})$. The *star* of $\sigma \in \mathcal{K}$ is $\mathrm{star}_{\mathcal{K}}(\sigma) = \{\tau \in \mathcal{K} \colon \tau \cup \sigma \in \mathcal{K}\}$.

For sets A, B define $A \uplus B := \{(a, 0) \colon a \in A\} \cup \{(b, 1) \colon b \in B\}$.

Neighborhood Complex. The *neighborhood complex* is $\mathrm{N}(G) = \{S \subseteq V(G) \colon \mathrm{CN}(S) \neq \emptyset\}$.

Box Complex. The *box complex* $\mathrm{B}(G)$ of a graph G (the one introduced by Matoušek and Ziegler [14]) is defined by

$$\mathrm{B}(G) := \big\{A \uplus B \colon A, B \subseteq V(G),\ A \cap B = \emptyset,\\ G[A,B] \text{ is complete bipartite}, \mathrm{CN}(A) \neq \emptyset \neq \mathrm{CN}(B)\big\}.$$

The vertices of the box complex are $V_1 := \{v \uplus \emptyset \colon v \in V(G)\}$ and $V_2 := \{\emptyset \uplus v \colon v \in V(G)\}$. The subcomplexes of $\mathrm{B}(G)$ induced by V_1 and V_2 are disjoint subcomplexes of $\mathrm{B}(G)$ that are both isomorphic to the neighborhood complex $\mathrm{N}(G)$. We refer to these two copies as *shores* of the box complex. The box complex is endowed with a $\mathbb{Z}_2$-action which interchanges the shores.

A *different* box complex $\mathrm{B}_0(G)$ [14] is defined by:

$$\mathrm{B}_0(G) \ = \big\{A \uplus B \colon A, B \subseteq V(G),\ A \cap B = \emptyset,\\ G[A,B] \text{ is complete bipartite}\big\}.$$

The *cones over the shores* complex $\mathrm{B}_{\mathcal{C}}(G)$ is (only for technical reason):

$$\mathrm{B}_{\mathcal{C}}(G) = \mathrm{B}(G) \cup \big\{(x, A \uplus \emptyset) \colon A \subseteq V(G),\ \mathrm{CN}(A) \neq \emptyset\big\}\\ \cup \big\{(\emptyset \uplus B, y) \colon B \subseteq V(G),\ \mathrm{CN}(B) \neq \emptyset\big\},$$

where we assume that $x, y \notin V(G)$. ($\mathrm{B}(G)$, $\mathrm{B}_0(G)$, $\mathrm{B}_{\mathcal{C}}(G)$ are $\mathbb{Z}_2$-spaces.)

Examples. For the complete graph K_n its neighborhood complex $\mathrm{N}(K_n)$ is the boundary complex of the $n-1$ dimensional simplex. Its box complex $\mathrm{B}_0(K_n)$ is the boundary complex of the n-dimensional cross polytope; while $\mathrm{B}(K_n)$ is the boundary complex of the n-dimensional cross polytope, with two opposite facets removed. $\mathrm{B}_{\mathcal{C}}(K_n)$ can be obtained from $\mathrm{B}(K_n)$ by attaching cones over its boundary components.

$\mathbb{Z}_2$-space. A $\mathbb{Z}_2$-*space* is a pair (X, ν) where X is a topological space and $\nu \colon X \to X$, called the $\mathbb{Z}_2$-action, is a homeomorphism such that $\nu^2 = \nu \circ \nu = \mathrm{id}_X$. If (X_1, ν_1) and (X_2, ν_2) are $\mathbb{Z}_2$-spaces, a $\mathbb{Z}_2$-*map* between them is a continuous mapping $f \colon X_1 \to X_2$ such that $f \circ \nu_1 = \nu_2 \circ f$. The sphere S^n is understood as a $\mathbb{Z}_2$-space with the antipodal homeomorphism $x \to -x$. We will consider only finite dimensional *free* $\mathbb{Z}_2$-complexes (free means that the $\mathbb{Z}_2$-action ν has no fixed point).

$\mathbb{Z}_2$-index. We define the $\mathbb{Z}_2$-*index* of a $\mathbb{Z}_2$-space (X,ν) by

$$\mathrm{ind}(X) = \min\{n \geq 0\colon \text{ there is a } \mathbb{Z}_2\text{-map } (X,\nu) \to (S^n,-)\}$$

(the $\mathbb{Z}_2$-action ν will be omitted from the notation if it is clear from the context). The Borsuk–Ulam Theorem can be re-stated as $\mathrm{ind}(S^n)=n$.

Another index-like quantity of a $\mathbb{Z}_2$-space, the *dual index* can be defined by

$$\mathrm{coind}(X) = \max\{n \geq 0\colon \text{ there is a } \mathbb{Z}_2\text{-map } S^n \xrightarrow{\mathbb{Z}_2} X\}.$$

The consequence of the Borsuk–Ulam Theorem is that $\mathrm{coind}(X) \leq \mathrm{ind}(X)$. We call a free $\mathbb{Z}_2$-space *tidy* if $\mathrm{coind}(X)=\mathrm{ind}(X)$.

A $\mathbb{Z}_2$-map $f\colon X \to Y$ is a $\mathbb{Z}_2$-equivalence if there exist a $\mathbb{Z}_2$-map $g\colon Y \to X$ such that $g \circ f$ and $f \circ g$ are $\mathbb{Z}_2$-homotopic to id_X and id_Y respectively. A general reference for $\mathbb{Z}_2$-spaces is the textbook of Bredon [4].

3. The connection between $\mathrm{B}_{\mathcal{C}}(G)$, $\mathrm{B}_0(G)$ and $\mathrm{B}(G)$

In this section we will prove that $\mathrm{B}_0(G)$ and $\mathrm{susp}(\mathrm{B}(G))$ are $\mathbb{Z}_2$-homotopy equivalent. The reason is that the box complex is 'nearly' $\mathrm{N}(G) \times [0,1]$.

Theorem 3.1. $\mathrm{B}_{\mathcal{C}}(G)$ *is* $\mathbb{Z}_2$*-homotopy equivalent to* $\mathrm{susp}(\mathrm{B}(G))$.

Theorem 3.1 considerably strengthen the statement of Matoušek and Ziegler [14]. It will be proven in Lemmas 3.3 and 3.4.

Remark 3.2. It follows from Lovász's original proof [9] of Kneser's conjecture [8] that the box complexes of Kneser graphs are tidy spaces. The box complexes of Schrijver graphs are tidy as well (spheres up to homotopy [3]). This means that one can prove Kneser's conjecture using Sarkaria's bound (or any higher suspension of the box complex).

Lemma 3.3. $\mathrm{B}_{\mathcal{C}}(G)$ *is* $\mathbb{Z}_2$*-homotopy equivalent to* $\mathrm{B}_0(G)$.

Proof. $\mathrm{B}_{\mathcal{C}}(G)$ was obtained from $\mathrm{B}(G)$ by attaching two cones C_1, C_2 over the shores, while $\mathrm{B}_0(G)$ is $\mathrm{B}(G)$ plus two simplices Δ_1, Δ_2 covering the shores.

We consider the following two quotient CW-complexes. $(\mathrm{B}_{\mathcal{C}}(G)/C_1)/C_2$ and $(\mathrm{B}_0(G)/\Delta_1)/\Delta_2$ (the order of the factorization does not matter since we collapse disjoint subcomplexes). It is obvious that they are the same CW-complexes and since C_i, Δ_i are contractible spaces $\mathrm{B}_{\mathcal{C}}(G)$ and $\mathrm{B}_0(G)$ are $\mathbb{Z}_2$-homotopy equivalent. ∎

Lemma 3.4. $\mathrm{B}_{\mathcal{C}}(G)$ *is a* $\mathbb{Z}_2$*-deformation retract of* $\mathrm{susp}(\mathrm{B}(G))$.

Proof. $\mathrm{B}_{\mathcal{C}}(G)$ is a subcomplex of $\mathrm{susp}(\mathrm{B}(G))$. The idea of the proof is to start with $\mathrm{susp}(\mathrm{B}(G))$, and get rid of the extra simplexes one by one (using deformation retraction) such that finally we get $\mathrm{B}_{\mathcal{C}}(G)$. We will work with one cone (half) of the suspension. Since we want a $\mathbb{Z}_2$-retraction, on the other cone we have to do the $\mathbb{Z}_2$-pair of each step.

Let x be the apex of the cone over the first shore in $\mathrm{susp}(\mathrm{B}(G))$ (y is the other apex). We will define (by induction) sequences of simplicial complexes such that

$$\mathrm{susp}(\mathrm{B}(G)) =: X_0 \supset X_1 \supset \cdots \supset X_N = \mathrm{B}_{\mathcal{C}}(G)$$

and X_{i+1} is a $\mathbb{Z}_2$-deformation retraction of X_i.

Let assume that we already defined X_n. We choose a simplex $\sigma \in X_n$ such that

1. $x \in \sigma$, and the rest of the vertices of σ are from the second shore,
2. no other simplex in X_n containing x has *more* vertices from the second shore, and it has at least one vertex from the second shore.

The vertex set of σ will be $\{x, \emptyset \uplus b_{j_1}, \ldots, \emptyset \uplus b_{j_{l-1}}\}$ for some $B = \{b_{j_1}, \ldots, b_{j_{l-1}}\} \subseteq V(G)$. Let $A := \mathrm{CN}(B) = \{a_{i_1}, \ldots, a_{i_k}\}$ and $\tilde{\sigma}$ be the $\mathbb{Z}_2$-pair of σ with vertex set $\{y, b_{j_1} \uplus \emptyset, \ldots, b_{j_{l-1}} \uplus \emptyset\}$. We are ready to define X_{n+1}:

$$X_{n+1} := X_n \setminus \big\{\tau \in X_n : \sigma \in \tau \text{ or } \tilde{\sigma} \in \tau\big\}.$$

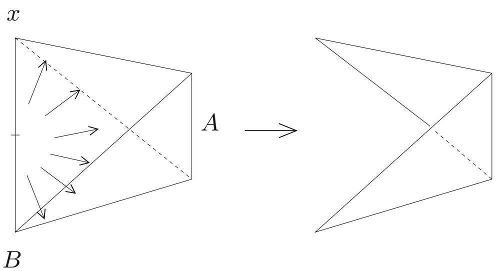

Figure 1. A deformation retraction.

We have to only show that X_{n+1} is the deformation retract of X_n. We know the local structure of our complex X_n around σ. Let assume that it is a face of a bigger simplex Δ with vertex set $\{x, \emptyset \uplus b_{j_1}, \ldots, \emptyset \uplus b_{j_{l-1}}, c\}$. c can not be the other apex. If c were from the second shore, then we would choose Δ instead of σ to define X_{n+1}. So c can be only from the first shore and then $c \in A$. This means that σ is on the boundary of X_n; it is on the boundary of the simplex s with vertex set $\{x, \emptyset \uplus b_{j_1}, \ldots, \emptyset \uplus b_{j_{l-1}}, a_{i_1} \uplus \emptyset, \ldots, a_{i_k} \uplus \emptyset\}$. Moreover every simplex which has σ as face is on the boundary of X_n. So what we delete to get X_{n+1} is on the boundary (except s). This deformation retraction of the simplex $\{v_1 := a_{i_1} \uplus \emptyset, \ldots, v_k := a_{i_k} \uplus \emptyset, w_1 := \emptyset \uplus b_{j_1}, \ldots, w_{l-1} := \emptyset \uplus b_{j_{l-1}}, w_l := x\}$ is indicated on Figure 1 and can be explicitly given by:

$$h_t\left(\sum t_i v_i + \sum s_j w_j\right) = \sum \left(\frac{l \cdot t}{k} + t_i\right) v_i + \sum (s_j - t)\, w_j,$$

where $\sum t_i + \sum s_j = 1$. It starts with $h_0 = id$, and ends (for a particular point), just when the first coefficient of w_j become zero. This retraction 'kills' those simplices, which has as a face the simplex $\{w_1, \ldots, w_l\}$, and retracts the 'interior' points to the remaining simplices. ∎

4. Neighborhood complex

We consider the following natural question about the neighborhood complex. Given a simplicial complex $\mathcal{K}$, is there a graph G such that its neighborhood complex is the given complex, $\mathrm{N}(G) = \mathcal{K}$?

For example, if $\mathcal{K}_1$ is the complex on Figure 2 then the answer is *no*! The reason is that there is a topological obstruction. The neighborhood complex is homotopy equivalent to the box complex which is a free $\mathbb{Z}_2$-simplicial complex so it has clearly even Euler characteristic. But $\chi(\mathcal{K}_1) = -1$ is odd.

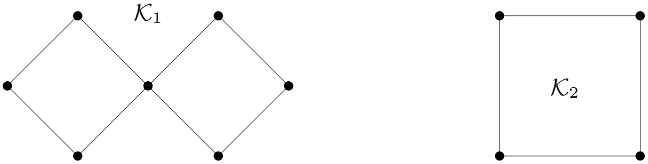

Figure 2. Two simplicial complexes $\mathcal{K}_1$ and $\mathcal{K}_2$.

Another example if $\mathcal{K}_2$ is the complex of Figure 2. Now the answer is no again, but there is no topological reason. With the usual antipodal map $\mathcal{K}_2$ becomes a free $\mathbb{Z}_2$-simplicial complex. On the other hand the graph G with $\mathrm{N}(G) = \mathcal{K}_2$ should have 4 vertices, and by brute force one can check that $\mathcal{K}_2$ is not a neighborhood complex.

Unfortunately we can not answer this question, but we will show that up to homotopy everything is possible.

Theorem 4.1. *Given a free $\mathbb{Z}_2$-simplicial complex $(\mathcal{K},\nu)$, there is a graph G such that its neighborhood complex is homotopy equivalent to the given complex,* $\mathrm{N}(G)\simeq\mathcal{K}$.

In order to prove it we will use the following construction of a graph from a $\mathbb{Z}_2$-simplicial complex. Actually we will show that $\mathrm{N}(G_{\mathrm{sd}(\mathcal{K})})\simeq\mathcal{K}$.

Construction 4.2 **($\mathcal{K}\to G_{\mathcal{K}}$).** Let $\mathcal{K}$ be a $\mathbb{Z}_2$-simplicial complex. The vertices of $G_{\mathcal{K}}$ are the vertices of $\mathcal{K}$, and each vertex is connected to its $\mathbb{Z}_2$-pair and the neighbors (neighbors in the 1-skeleton of $\mathcal{K}$) of the $\mathbb{Z}_2$-pair. Thus if $x,y\in V(G_{\mathcal{K}})=V(\mathcal{K})$ then there is an edge between them if and only if $\nu(x)=y$ or $\{x,\nu(y)\}\in\mathcal{K}$ (or $\{y,\nu(x)\}\in\mathcal{K}$).

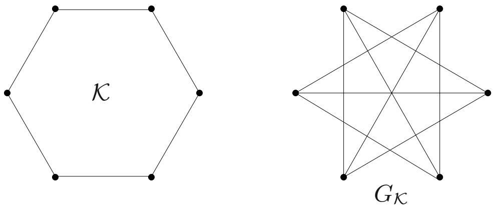

Figure 3. Example for Construction 4.2.

We need the *nerve theorem* as well.

Definition 4.3 **(Nerve).** Let $\mathcal{F}$ be a set-system. The *nerve* $\mathcal{N}(\mathcal{F})$ of $\mathcal{F}$ is defined as the simplicial complex whose vertices are the sets in $\mathcal{F}$, and $\{X_1,\dots,X_r\}\in\mathcal{N}(\mathcal{F})$ if and only if $X_1,\dots,X_r\in\mathcal{F}$ and $X_1\cap X_2\cap\cdots\cap X_r\neq\emptyset$.

Theorem 4.4 **(Nerve theorem [2]).** *Let $\mathcal{K}$ be a simplicial complex and $\mathcal{K}_i$ $(i\in I)$ a family of subcomplexes such that $\mathcal{K}=\bigcup_{i\in I}\mathcal{K}_i$. Assume that every nonempty finite intersection $\mathcal{K}_{i_1}\cap\cdots\cap\mathcal{K}_{i_r}$ is contractible. Then $\mathcal{K}$ and the nerve $\mathcal{N}(\bigcup\mathcal{K}_i)$ are homotopy equivalent.*

Proof of Theorem 4.1. For technical reason we need the first barycentric subdivision $\mathrm{sd}(\mathcal{K})$ of $\mathcal{K}$. The free simplicial $\mathbb{Z}_2$-action on $\mathrm{sd}(\mathcal{K})$ will be denoted by ν as well.

We use Construction 4.2 with $\mathrm{sd}(\mathcal{K})$ to obtain $G_{\mathrm{sd}(\mathcal{K})}$. Because of the barycentric subdivision the vertices of $G_{\mathrm{sd}(\mathcal{K})}$ denoted by subsets of $V(\mathcal{K})$. If $A,B\in V(G_{\mathrm{sd}(\mathcal{K})})$ then there is an edge between them if and only if $\nu(A)=B$ or $\nu(A)\subset B$ or $\nu(A)\supset B$.

We denote the vertices of $\mathcal{K}$ by $1,2,\dots,n$. Let $\mathrm{star}_{\mathrm{sd}(\mathcal{K})}(A)$ be the star of the vertex A in $\mathrm{sd}(\mathcal{K})$. The *nerve* of the set system $\big\{\mathrm{star}_{\mathrm{sd}(\mathcal{K})}(A)\colon A\in$

$V(G_{\mathrm{sd}(\mathcal{K})})\}$ is clearly the neighborhood complex of $G_{\mathrm{sd}(\mathcal{K})}$. (This is even true without any subdivision: $N(G_{\mathcal{K}}) = \mathcal{N}(\mathcal{S})$ where $\mathcal{S}$ is the set of the vertex stars in $\mathcal{K}$.)

We want to use the *nerve* theorem so we should prove that if $B \in \mathrm{star}_{\mathrm{sd}(\mathcal{K})}(A_1) \cap \cdots \cap \mathrm{star}_{\mathrm{sd}(\mathcal{K})}(A_r) \neq \emptyset$ then this intersection is contractible. We show that this is a cone. We have two cases:

1. If $A_i \subset B$ for all $i = 1, 2, \ldots, r$.
   In this case $\cup A_i$ is a vertex of the barycentric subdivision since it is a subset of B, and it is in the intersection as well. We show that the intersection can be contracted to this point. We construct this deformation retraction by letting each vertex to travel towards $\cup A_i$ with uniform speed. The only thing that we have to check is that whenever $B_1 \subset B_2 \subset \cdots \subset B_q$ is a simplex in the intersection, then with the special vertex $X := \cup A_i$ they form a simplex as well. First observe that there is an edge between X and B_l, $l \in \{1, \ldots, q\}$. If $B_l \subset A_i$ for some i then $B_l \subset X$ as well. Otherwise $X \subset B_l$. For the simplex $B_1 \subset B_2 \subset \cdots \subset B_q$ if $X \subset B_1$ or $X \supset B_q$ then they form a simplex with X. Otherwise there is an index k such that $B_k \subset X \subset B_{k+1}$. This means that $B_1, B_2, \ldots, B_q, X$ form a simplex.
2. If $B \subset A_{i_j}$ for some $j = 1, \ldots, k$ $(k \geq 1)$, and $A_i \subset B$ for the rest.
   In this case $B \subset \cap_{j=1}^{k} A_{i_j} \neq \emptyset$ is a vertex of the barycentric subdivision and the intersection as well ($B \supset \cup_{A_i \subset B} A_i$ would be good as before, but it does not have to exists). We show that the intersection can be contracted to this point. We construct this deformation retraction by letting each vertex to travel towards $\cap A_{i_j}$ with uniform speed. We have to show that whenever $B_1 \subset B_2 \subset \cdots \subset B_q$ is a simplex in the intersection, then with the special vertex $X := \cap A_{i_j}$ they form a simplex as well. First observe that there is an edge between X and B_l, $l \in \{1, \ldots, q\}$. If $B_l \supset A_{i_j}$ for some i_j then $B_l \supset X$ as well. Otherwise $X \supset B_l$. For the simplex $B_1 \subset B_2 \subset \cdots \subset B_q$ if $X \subset B_1$ or $X \supset B_q$ then it is true. Otherwise there is an index k such that $B_k \subset X \subset B_{k+1}$ which means that $B_1, B_2, \ldots, B_q, X$ form a simplex.

This completes the proof. ∎

5. Box complex

In this section we prove our main theorem. It is the $\mathbb{Z}_2$-extension of Theorem 4.1. After our work was completed, (it was already announced in [14], arXiv:math.CO/0208072v2) R. T. Živaljević [17] proved Theorem 5.1 and 3.1 by a different poset-theoretic route.

Theorem 5.1. *Given a free $\mathbb{Z}_2$-simplicial complex $(\mathcal{K},\nu)$, there is a graph G such that its box complex $\mathrm{B}(G)$ is $\mathbb{Z}_2$-homotopy equivalent to the given complex.*

We will show that $\mathrm{B}(G_{\mathrm{sd}(\mathcal{K})})\overset{\mathbb{Z}_2}{\simeq}\mathcal{K}$. First we need the $\mathbb{Z}_2$-carrier lemma.

Definition 5.2 (carrier). Let $(\mathcal{K},\nu)$ be a $\mathbb{Z}_2$-simplicial complex and (T,μ) a $\mathbb{Z}_2$-space. A function C taking faces σ of $\mathcal{K}$ to subspaces $C(\sigma)$ of T, satisfying $C(\nu(\sigma))=\mu(C(\sigma))$, is a $\mathbb{Z}_2$-*carrier* if $C(\sigma)\subseteq C(\tau)$ for all $\sigma\subseteq\tau$.

Lemma 5.3 ($\mathbb{Z}_2$-carrier lemma). *Assume that for a $\mathbb{Z}_2$-carrier C for any $\sigma\in\mathcal{K}$ $C(\sigma)$ is contractible. Then any two $\mathbb{Z}_2$-maps $f,g\colon\mathcal{K}\to T$ that are both carried by C are $\mathbb{Z}_2$-homotopic.*

Proof. The proof follows the proof of the usual carrier lemma by induction on the skeleton, as in [10, Theorem II.9.2]. ∎

Proof of Theorem 5.1. We will use the same notations as in the proof of Theorem 4.1. Similarly we obtain $G_{\mathrm{sd}(\mathcal{K})}$ by using Construction 4.2 with $\mathrm{sd}(\mathcal{K})$. We need to show that the box complex $\mathrm{B}(G_{\mathrm{sd}(\mathcal{K})})$ and $(\mathcal{K},\nu)$ are $\mathbb{Z}_2$-homotopy equivalent. In order to prove it we will define $\mathbb{Z}_2$-maps $f\colon\mathrm{sd}(\mathrm{B}(G_{\mathrm{sd}(\mathcal{K})}))\to\mathrm{sd}(\mathcal{K})$ and $g\colon\mathrm{sd}(\mathcal{K})\to\mathrm{B}(G_{\mathrm{sd}(\mathcal{K})})$, that are $\mathbb{Z}_2$-homotopy inverses.

The definition of g: This is an embedding. We map a vertex $A\in\mathrm{sd}(\mathcal{K})$ to $A\uplus\emptyset\in\mathrm{B}(G_{\mathrm{sd}(\mathcal{K})})$ and of course its $\mathbb{Z}_2$-pair $\nu(A)\in\mathrm{sd}(\mathcal{K})$ to $\emptyset\uplus A\in\mathrm{B}(G_{\mathrm{sd}(\mathcal{K})})$. Here we had to choose! If we pick $\nu(A)$ first than we mapped $\nu(A)$ to $\nu(A)\uplus\emptyset$ and A to $\emptyset\uplus\nu(A)$. So we have 2 choices for any $\mathbb{Z}_2$-pair $A,\nu(A)$. This defines a $\mathbb{Z}_2$-map g on the vertex level. We have to show that g is simplicial. Let $A_1\subset\cdots\subset A_l$ be a simplex σ in $\mathrm{sd}(\mathcal{K})$. Since $A_1\uplus\emptyset,\dots,A_l\uplus\emptyset$, $\emptyset\uplus\nu(A_1),\dots,\emptyset\uplus\nu(A_l)$ form a simplex in $\mathrm{B}(G_{\mathrm{sd}(\mathcal{K})})$ the image of σ is a simplex. (In $G_{\mathrm{sd}(\mathcal{K})}$ A_i is connected to $\nu(A_i)$ and since $A_i\subset A_j$ or $A_i\supset A_j$ it is connected to $\nu(A_j)$ as well. So $G_{\mathrm{sd}(\mathcal{K})}[\{A_1,\dots,A_l\};\{\nu(A_1),\dots,\nu(A_l)\}]$ is complete bipartite.)

The definition of f: Let $A_1\uplus\emptyset,\dots,A_l\uplus\emptyset$, $\emptyset\uplus B_1,\dots,\emptyset\uplus B_k$ be the vertices of a simplex σ in $\mathrm{B}(G_{\mathrm{sd}(\mathcal{K})})$. $G_{\mathrm{sd}(\mathcal{K})}[\mathcal{A};\mathcal{B}]$ is complete bipartite where $\mathcal{A}:=\{A_1,\dots,A_l\}$ and $\mathcal{B}:=\{B_1,\dots,B_k\}$. This means that $\mathcal{A}\subset\mathrm{star}_{\mathrm{sd}(\mathcal{K})}\nu(B_j)$ for any $j\in\{1,\dots,k\}$ so $\mathcal{A}\subset\bigcap_{j=1}^{k}\mathrm{star}_{\mathrm{sd}(\mathcal{K})}\nu(B_j)$. From the proof of Theorem 4.1 we know that $\bigcap_{j=1}^{k}\mathrm{star}_{\mathrm{sd}(\mathcal{K})}\nu(B_j)$ is a cone with apex X. Since $\mathcal{A},\nu(\mathcal{B})\subset\mathrm{star}_{\mathrm{sd}(\mathcal{K})}X$ we have that $Y:=\bigcap_{i=1}^{l}\mathrm{star}_{\mathrm{sd}(\mathcal{K})}A_i\bigcap\bigcap_{j=1}^{k}\mathrm{star}_{\mathrm{sd}(\mathcal{K})}\nu(B_j)\neq\emptyset$. From the proof of Theorem 4.1 we know that Y is a cone. We denote its apex by $X_{\mathcal{A}}^{\mathcal{B}}$ which can be chosen to be $\bigcap_{i=1}^{l}A_i\bigcap\bigcap_{j=1}^{k}\nu(B_j)$ if it is not the empty set. Now we are able to define f.

$$f(\mathcal{A}\uplus\mathcal{B}) := \begin{cases} \bigcap_{i=1}^{l} A_i \bigcap \bigcap_{j=1}^{k} \nu(B_j) & \text{if exist,} \\ X_{\mathcal{A}}^{\mathcal{B}} & \text{otherwise.} \end{cases}$$

By the construction it is $\mathbb{Z}_2$ on the vertex level. (We can choose $X_{\mathcal{B}}^{\mathcal{A}} := \nu(X_{\mathcal{A}}^{\mathcal{B}})$.) It is simplicial. An edge with two vertices $\mathcal{A}\uplus\mathcal{B}$ and $\tilde{\mathcal{A}}\uplus\tilde{\mathcal{B}}$ ($\tilde{\mathcal{A}}\subset\mathcal{A}$, $\tilde{\mathcal{B}}\subset\mathcal{B}$) is mapped to two vertices $S\subset R$ since $X_{\mathcal{A}}^{\mathcal{B}}$ is in the cone of $X_{\tilde{\mathcal{A}}}^{\tilde{\mathcal{B}}}$. Now a simplex is mapped to a chain (since every two vertex is comparable by inclusion).

Next we prove that $f\circ\mathrm{sd}(g)\colon \mathrm{sd}(\mathrm{sd}(\mathcal{K}))\to\mathrm{sd}(\mathcal{K})$ is $\mathbb{Z}_2$-homotopic to $\mathrm{id}_{\mathcal{K}}$. We will use the $\mathbb{Z}_2$-carrier lemma. We have to construct 'only' a contractible $\mathbb{Z}_2$-carrier for $f\circ\mathrm{sd}(g)$ and id. The image of the vertex $v=\{A_1,\dots,A_l\}$, $A_1\subset\cdots\subset A_l$ is $\mathrm{sd}(g)(v)=\{A_{i_1},\dots,A_{i_s}\}\uplus\{\nu(A_{j_1}),\dots,\nu(A_{j_r})\}$. And now $f(\mathrm{sd}(g)(v)) = A_1\cap\cdots\cap A_l = A_1$ in this case! The image of a simplex with vertex set $\{A_{i_1}\},\{A_{i_1},A_{i_2}\},\dots,\{A_{i_1},\dots,A_{i_l}\}$ is a face of the simplex $A_1\subset\cdots\subset A_l$. So for a simplex $\sigma\in\mathrm{sd}(\mathrm{sd}(\mathcal{K}))$ with its maximal vertex $\{A_1,\dots,A_l\}$ we define $C(\sigma):=\{A_1,\dots,A_l\}\in\mathrm{sd}(\mathcal{K})$. This C is a contractible $\mathbb{Z}_2$-carrier what we need. $f\circ\mathrm{sd}(g)$ and $\mathrm{id}_{\mathcal{K}}$ are $\mathbb{Z}_2$-homotopic.

Now we show that $g\circ f\colon \mathrm{sd}(\mathrm{B}(G_{\mathrm{sd}(\mathcal{K})}))\to\mathrm{B}(G_{\mathrm{sd}(\mathcal{K})})$ is $\mathbb{Z}_2$-homotopic to id. Again we construct a contractible $\mathbb{Z}_2$-carrier for $g\circ f$ and id. A vertex $\mathcal{A}\uplus\mathcal{B}$ is mapped to $X_{\mathcal{A}}^{\mathcal{B}}$ by f and to $X_{\mathcal{A}}^{\mathcal{B}}\uplus\emptyset$ or $\emptyset\uplus\nu(X_{\mathcal{A}}^{\mathcal{B}})$ by $g\circ f$. Let $\mathcal{A}_1\uplus\mathcal{B}_1,\dots,\mathcal{A}_n\uplus\mathcal{B}_n$ the vertex set of a simplex σ in $\mathrm{sd}(\mathrm{B}(G_{\mathrm{sd}(\mathcal{K})}))$. ($\mathcal{A}_1\subset\cdots\subset\mathcal{A}_n$, $\mathcal{B}_1\subset\cdots\subset\mathcal{B}_n$, $\mathcal{A}_n:=\{A_1,\dots,A_l\}$ and $\mathcal{B}_n:=\{B_1,\dots,B_k\}$.) We consider the subgraph H of $G_{\mathrm{sd}(\mathcal{K})}$ spanned by $A_1,\dots,A_l,B_1,\dots,B_k$, their $\mathbb{Z}_2$-image under ν and $X_{\mathcal{A}_i}^{\mathcal{B}_i},\nu(X_{\mathcal{A}_i}^{\mathcal{B}_i})$ for any $i\in\{1,\dots,n\}$. We will use H (actually $\mathrm{B}(H)$) to define the desired carrier. First of all $\mathrm{B}(H)$ contains the simplex with vertex set $A_1\uplus\emptyset,\dots,A_l\uplus\emptyset$, $\emptyset\uplus B_1,\dots,\emptyset\uplus B_k$ which contains σ. Moreover we defined H in such a way that $\mathrm{B}(H)$ contains $(g\circ f)(\sigma)$ as well. Observe that H is bipartite. The neighbors of the vertices $X_{\mathcal{A}_n}^{\mathcal{B}_n}$ and $\nu(X_{\mathcal{A}_n}^{\mathcal{B}_n})$ provides a partition of the vertex set of H. The neighborhood complex $\mathrm{N}(H)$ is the disjoint union of two simplices corresponding to this partition. So the box complex $\mathrm{B}(H)\subset\mathrm{B}(G_{\mathrm{sd}(\mathcal{K})})$ contains two disjoint contractible sets (since it is homotopy equivalent to $\mathrm{N}(H)$). One of these sets covers σ and $(g\circ f)(\sigma)$, so we define our contractible $\mathbb{Z}_2$-carrier $C(\sigma)$ to be this 'half' of $\mathrm{B}(H)$. ∎

Remark 5.4. For any free $\mathbb{Z}_2$-simplicial complex $(\mathcal{K},\nu)$ there is a graph G such that its Hom complex [1] $\mathrm{Hom}(K_2,G)$ is $\mathbb{Z}_2$-homotopy equivalent to the given complex since the box complex $\mathrm{B}(G)$ is $\mathbb{Z}_2$-homotopy equivalent to $\mathrm{Hom}(K_2,G)$. (The $\mathbb{Z}_2$-maps $f\colon \mathrm{sd}(\mathrm{B}(G)) \to \mathrm{sd}(\mathrm{Hom}(K_2,G))$ defined by

$$A \uplus B \to \begin{cases} (A, \mathrm{CN}(A)) & \text{if } B = \emptyset, \\ (\mathrm{CN}(B), B) & \text{if } A = \emptyset, \\ (A, B) & \text{otherwise,} \end{cases}$$

and $g\colon \mathrm{sd}(\mathrm{Hom}(K_2,G)) \to \mathrm{sd}(\mathrm{B}(G))$ given by $(A,B) \to A \uplus B$ are $\mathbb{Z}_2$-homotopy equivalences. $f \circ g = \mathrm{id}$ and $g \circ f$ is carried by id.)

6. The suspension and the index

In this section we will construct a $\mathbb{Z}_2$-space X such that $\mathrm{ind}(X) = \mathrm{ind}(\mathrm{susp}(X))$. This example is based on an earlier construction by Matoušek, Živaljević and the author [13, page 100]. Such examples appear in the homotopy theory literature, see e.g. [5], but we will give a simple and explicit example.

We proceed in the following way. Let $h\colon S^3 \to S^2$ be the Hopf map. It can be defined by considering S^3 as the unit sphere in $\mathbb{C}^2$ and $S^2 = \mathbb{CP}^1$. Now the Hopf map $h\colon S^3 \to S^2$ defined by $(z_1, z_2) \to [z_1, z_2] \in \mathbb{CP}^1$ [6, Example 4.45]. We note that h is a generator of $\pi_3(S^2) \cong \mathbb{Z}$.

We attach two 4-cells (the boundary of the 4-cell is S^3) to S^2 via $2h$ and $-2h$, where multiplies of maps are taken according to addition in $\pi_3(S^2) \cong \mathbb{Z}$. We denote this $\mathbb{Z}_2$-space by

$$X_{2h} := S^2 \underset{2h}{\cup} B^4 \underset{-2h}{\cup} B^4.$$

The $\mathbb{Z}_2$-action is the antipodality on $S^2 \subset X_{2h}$, and it interchanges the two 4-cells.

Now we compute the $\mathbb{Z}_2$-index of X_{2h} and $\mathrm{susp}(X_{2h})$. It is easy to see that $1 \le \mathrm{ind}(X_{2h}) \le 3$. A $\mathbb{Z}_2$-map $S^2 \subset X_{2h} \xrightarrow{\mathbb{Z}_2} S^1$ would contradict the Borsuk–Ulam Theorem. Let B^i be the unit ball in $\mathbb{R}^i$ centered at the origin. We assume that $2h\colon S^3 \to S^2$ maps the unit sphere, the boundary of the unit ball, into the unit sphere. We define a map $b\colon B^4 \to B^3$ such that it maps the origin of $\mathbb{R}^4$ into the origin of $\mathbb{R}^3$ and if $x \in B^4$, $\|x\| \neq 0$ then $b(x) := 2h\big(\frac{x}{\|x\|}\big) \cdot \|x\|$. Now we are ready to construct a $\mathbb{Z}_2$-map $f\colon X_{2h} \xrightarrow{\mathbb{Z}_2} S^3$. f maps $S^2 \subset X_{2h}$ into the equator of S^3. The remaining two 4-cells of X_{2h} are mapped to the upper and lower hemisphere of S^3 by b and $-b$.

It is slightly more difficult to prove that the index is 3. We will use the following:

Definition 6.1 **([6] Page 427, Section 4.B).** Let $f\colon S^{2n-1}\to S^n$, $(n\geq 2)$, and let $C_f=S^n\cup_f B^{2n}$ (we attach a $2n$-cell to S^n via f). The Hopf invariant of f (denoted by $\mathcal{H}(f)$) can be defined such that $\alpha\cup\alpha=\mathcal{H}(f)\cdot\beta$, where $\alpha\in H^n(C_f)=\mathbb{Z}$ and $\beta\in H^{2n}(C_f)=\mathbb{Z}$ are the generators of the corresponding cohomology groups and $\cup$ is the cup product.

We will use the following property of the Hopf invariant (see [6]).

- $\mathcal{H}\colon\pi_{2n-1}(S^n)\to\mathbb{Z}$ is a homomorphism. For n=2 it is an isomorphism.

Theorem 6.2 **([7] Theorem 9.5.9).** *Let $f\colon S^{2n-1}\to S^n$ and $g\colon S^n\to S^n$ be continuous maps. Then: $\mathcal{H}(g\circ f)=\deg(g)^2\cdot\mathcal{H}(f)$.*

Theorem 6.3 **([6] Proposition 2B.6).** *Every $\mathbb{Z}_2$-map $f\colon S^n\xrightarrow{\mathbb{Z}_2}S^n$ has odd degree.*

Lemma 6.4. $\operatorname{ind}(X_{2h})=3$.

Proof. By contradiction assume that $\operatorname{ind}(X_{2h})\leq 2$ which means that there is a $\mathbb{Z}_2$-map $f\colon X_{2h}\xrightarrow{\mathbb{Z}_2}S^2$. We restrict this map to $S^2\subset X_{2h}$ obtaining $g\colon S^2\to S^2$. We claim that $g\circ 2h\colon S^3\to S^2$ is null-homotopic. In X_{2h} we attached a 4-cell to S^2 via $2h$. This gives us a map $i\colon B^4\to X_{2h}$ and $f\circ i\colon B^4\to S^2$. The restriction of $f\circ i$ into $S^3=\partial B^4$ is $g\circ 2h$. So the map $g\circ 2h$ extends into B^4 which proves that $g\circ 2h$ is null-homotopic.

On the other hand Theorem 6.3 tells us that $\deg(g)$ is odd. (We need now only that it is non-zero.) Using Theorem 6.2 we have that $\mathcal{H}(g\circ 2h)=\deg(g)^2\cdot\mathcal{H}(2h)$. Since $\deg(g)\neq 0$ and $\mathcal{H}(2h)=2$ we have that $\mathcal{H}(g\circ 2h)\neq 0$. This means that $g\circ 2h$ is not null-homotopic, contradiction. ∎

Lemma 6.5. $\operatorname{ind}(\operatorname{susp}(X_{2h}))=3$.

Proof. $\operatorname{susp}(X_{2h})$ can be obtained similarly as X_{2h}: we attach two 5-cells (the boundary of the 5-cell is S^4) to S^3 via $\operatorname{susp}(2h)$ and $-\operatorname{susp}(2h)$. The Freudenthal Theorem ([6] Corollary 4.24) tells us that $\operatorname{susp}\colon\pi_3(S^2)\to\pi_4(S^3)$, which is actually $\mathbb{Z}\to\mathbb{Z}_2$, is surjective. So $\operatorname{susp}(2h)$ is null-homotopic which means that $\operatorname{susp}(X_{2h})$ is $\mathbb{Z}_2$-homotopy equivalent to S^3 so its index is 3. ∎

The generalization of this construction provides infinitely many examples of $\operatorname{ind}(X)=\operatorname{ind}(\operatorname{susp}(X))$.

Using a simplicial model for $2h\colon S^3_{12}\to S^2_4$ [12,11] one can obtain a simplicial complex model for X_{2h} as well.

7. The topological lower bound can be arbitrarily bad

It is well known (see [16]) that the topological lower bound for the chromatic number can be arbitrarily bad. But now we are able to construct graphs using spaces with unusual homotopy properties.

Definition 7.1. For a graph G let G^+ be the graph obtained from G by adding an extra vertex w and connecting it by edges to all the vertices of G, i.e., $V(G^+) = V(G) \cup \{w\}$ and $E(G^+) = E(G) \cup \{(v,w)\colon v \in V(G)\}$.

Lemma 7.2. $\mathrm{B}(G^+)$ *is* $\mathbb{Z}_2$*-homotopy equivalent to* $\mathrm{susp}(\mathrm{B}(G))$.

Proof. $\mathrm{susp}(\mathrm{B}(G))$ is a subcomplex of $\mathrm{B}(G^+)$. The difference is only two big simplices (and some of their faces) $V(G) \uplus w$ and $w \uplus V(G)$. We will get rid of the extra simplices one by one using deformation retraction. We will work with one shore, on the other shore we have to do the $\mathbb{Z}_2$-pair of each step.

We will define (by induction) sequences of simplicial complexes such that

$$\mathrm{B}(G^+) =: X_0 \supset X_1 \supset \cdots \supset X_N = \mathrm{susp}(\mathrm{B}(G))$$

and X_{i+1} is a $\mathbb{Z}_2$-deformation retraction of X_i.

Let assume that we already defined X_n. We choose $A \subseteq V(G)$ such that $A \uplus w \in X_n$, and there is no $A \subset B \subseteq V(G)$ such that $B \uplus w \in X_n$. We define X_{n+1}:

$$X_{n+1} := X_n \setminus \{A \uplus w, w \uplus A, A \uplus \emptyset, \emptyset \uplus A\}.$$

By the definition of X_{n+1} it is clearly a $\mathbb{Z}_2$-deformation retract of X_n since $A \uplus \emptyset$ is on the boundary of X_n. (Map the barycenter of $A \uplus \emptyset$ to $\emptyset \uplus w$.) ∎

Now we are ready to construct a graph such that $\chi(H) \geq \mathrm{ind}(\mathrm{B}(H)) + 2 + k$. Let X_k be a $\mathbb{Z}_2$-space (a simplicial complex) such that $\mathrm{ind}(\mathsf{X}_k) = \mathrm{ind}(\mathrm{susp}^k(\mathsf{X}_k))$.

Proposition 7.3. *For* $H := (G_{\mathrm{sd}(\mathsf{X}_k)})^{+k}$ *the difference between* $\chi(H)$ *and the topological lower bound (Theorem 1.1) is at least* k.

Proof. Let $G := G_{\mathrm{sd}(\mathsf{X}_k)}$. For G we have that $\chi(G) \geq \mathrm{ind}(\mathrm{B}(G)) + 2 = \mathrm{ind}(\mathsf{X}_k) + 2$. Clearly $\chi(G) + k = \chi(H)$ and $\mathrm{ind}(\mathrm{B}(H)) = \mathrm{ind}(\mathrm{susp}^k(\mathrm{B}(G))) = \mathrm{ind}(\mathrm{susp}^k(\mathsf{X}_k)) = \mathrm{ind}(\mathsf{X}_k)$. So $\chi(H) \geq \mathrm{ind}(\mathrm{B}(H)) + 2 + k$. ∎

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