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ON THE CHROMATIC NUMBER OF PENTAGON-FREE GRAPHS OF LARGE MINIMUM DEGREE

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We prove that, for each fixed real number $c > 0$, the pentagon-free graphs of minimum degree at least cn (where n is the number of vertices) have bounded chromatic number. This problem was raised by Erdős and Simonovits in 1973. A similar result holds for any other fixed odd cycle, except the triangle for which there is no such result for $c < 1/3$.

1. Introduction

Hajnal (see [2]) used the Kneser graphs to show that there are triangle-free graph of arbitrarily large chromatic number and of minimum degree close to $n/3$, where n is the number of vertices. Erdős and Simonovits [2] conjectured that this is best possible, and this was verified in [4] in the following sense: For each natural number t, let c_t be the smallest number such that every triangle-free graph with n vertices and minimum degree $> c_t n$ has chromatic number $< t$. Then $c_t \to 1/3$ as $t \to \infty$. For related results and problems, see Brandt [1].

In contrast to this we prove in the present note that for every odd natural number $k > 3$ and every real number c, $0 < c < 1$, there exists a natural number $\chi(c,k)$ such that the following holds: If G is a graph with n vertices and minimum degree $> cn$, then either G contains the cycle C_k of length k or else G has chromatic number at most $\chi(c,k)$. In particular, $\chi(c,5) \le 6/c$. This answers the last question in the paper of Erdős and Simonovits [2].

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2. Bounding the chromatic number

Theorem 2.1. *Let c be any fixed positive real number. Let k be any fixed odd natural number, $k \geq 5$. Then there exists a natural number $\chi(c,k)$ such that every graph G with n vertices and minimum degree at least cn either contains a cycle C_k of length k or has chromatic number at most $\chi(c,k)$ (or both). In particular, $\chi(c,5) \leq 6/c$.*

Proof. Let $A = \{a_1, a_2, \ldots, a_m\}$ be a maximal set of vertices of G such that no two distinct vertices of A are joined by a path of length 1 or 2. Let $N(a_i, G)$ denote the set of neighbors of a_i for $i = 1, 2, \ldots, m$. If z is a vertex in G but not in A or any of $N(a_i, G)$ for $i = 1, 2, \ldots, m$, then the maximality of A implies that z is joined to some $N(a_i, G)$ for $i = 1, 2, \ldots, m$. Let A_i be the set of those such vertices z that are joined to some vertex of $N(a_i, G)$ but not to any vertex of $N(a_j, G)$ when $j < i$. Then the sets $\{a_i\} \cup N(a_i, G) \cup A_i$, $i = 1, 2, \ldots, m$ form a partition of the vertex set $V(G)$ of G.

Let us consider the case $k = 5$. Let us assume that G contains no C_5. For each $i = 1, 2, \ldots, m$, the subgraph of G induced by $N(a_i, G)$ is 3-colorable since otherwise it contains a path of length 3. (This follows because any minimal graph of chromatic number 4 has minimum degree at least 3.) This path together with a_i gives a C_5, a contradiction. Also, the subgraph of G induced by A_i is 3-colorable since otherwise it contains a path $v_1 v_2 v_3 v_4$ of length 3. If the vertices of this path all have a common neighbor in $N(a_i, G)$, we get a C_5 as in the previous argument. So we may assume that two consecutive vertices of $v_1 v_2 v_3 v_4$ have distinct neighbors in $N(a_i, G)$. This results in a C_5 containing a_i, a contradiction. It follows that the subgraph of G induced by $\{a_i\} \cup N(a_i, G) \cup A_i$ is 6-colorable for $i = 1, 2, \ldots, m$. Since the sets $N(a_i, G)$, $i = 1, 2, \ldots, m$, are pairwise disjoint and of cardinality at least cn each, it follows that $cnm < n$, and hence G has chromatic number at most $6m < 6/c$. This proves the theorem for $k = 5$.

Assume now that $k > 5$ and that G contains no C_k. As in the previous case, we conclude that, for each $i = 1, 2, \ldots, m$, the subgraph of G induced by $N(a_i, G)$ is $(k-2)$-colorable since otherwise it contains a path of length $k-2$ which together with a_i gives a C_k, a contradiction. Corollary 4 in [3] says that for each natural number p, there exists a natural number $\theta(p)$ such that every 2-connected, nonbipartite graph of minimum degree at least $\theta(p)$ contains cycles of all lengths modulo k. We claim that the subgraph of G induced by A_i is $\theta(2k-8)$-colorable since otherwise it contains a cycle C $v_1 v_2 v_3 \ldots$ of length 1 modulo $2k-8$. Consider a subpath P of C of length $k-4$. Both ends have a neighbor in $N(a_i, G)$. These two neighbors must be identical since otherwise, G has a cycle C_k through a_i. Since this holds for every P, it follows that all vertices of C have a common neighbor in $N(a_i,G)$. Then clearly G has a C_k, a contradiction. This proves the claim that the subgraph of G induced by A_i is $\theta(2k-8)$-colorable. Hence G has chromatic number at most $m(k-2)\theta(2k-8) < (k-2)\theta(2k-8)/c$. ∎

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