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MAXIMAL TOTAL LENGTH OF k DISJOINT CYCLES IN BIPARTITE GRAPHS

HONG WANG

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Let k, s and n be three integers with $s \geq k \geq 2$, $n \geq 2k+1$. Let $G=(V_1,V_2;E)$ be a bipartite graph with $|V_1|=|V_2|=n$. If the minimum degree of G is at least $s+1$, then G contains k vertex-disjoint cycles covering at least $\min(2n,4s)$ vertices of G.

1. Introduction

We discuss only finite simple graphs and use standard terminology and notation from [1] except as indicated. Let k be an integer with $k \geq 2$. Let G be a graph of order $n \geq 3$. P. Erdős and T. Gallai [5] showed that if G is 2-connected and every vertex of G with at most one exception has degree at least k, then G contains a cycle of length at least $\min(2k,n)$. Corrádi and Hajnal [2] investigated the maximum number of vertex-disjoint cycles in a graph. They proved that if G is a graph of order at least $3k$ with minimum degree at least $2k$, then G contains k vertex-disjoint cycles. In particular, when the order of G is exactly $3k$, then G contains k vertex-disjoint triangles. Motivated by these results, we conjectured [12] that if t, k and n are three integers with $k \geq 2$, $t \geq 2k$ and $n \geq 3k$, and G is a graph of order n with minimum degree at least t, then G contains k vertex-disjoint cycles covering at least $\min(2t,n)$ vertices of G. This conjecture was verified for $k=2$ in [12]. Yoshimi Egawa, Kenichi Kawarabayashi and Hong Wang provided a complete proof of this conjecture in [13]. The result is as follows:

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Theorem A ([13]). *If t, k and n are three integers with $k \geq 3$, $t \geq 2k$ and $n \geq 3k$, and G is a graph of order n such that $d(x,G)+d(y,G) \geq 2t$ for each pair of non-adjacent vertices x and y, then G contains k vertex-disjoint cycles covering at least $\min(2t,n)$ vertices of G.*

In [10], we showed that if $G=(V_1,V_2;E)$ is a 2-connected bipartite graph with minimum degree larger than $(k+1)/2$, then G contains a cycle of length at least $\min(2a,2k)$ where $a=\min(|V_1|,|V_2|)$. We also showed [11] that if $|V_1|=|V_2|>2k$ and the minimum degree is at least $k+1$, then G contains k vertex-disjoint cycles. In this paper, we prove an analogous result (with Theorem A) for bipartite graphs. We will show:

Theorem B. *Let k, s and n be three integers with $s \geq k \geq 2$, $n \geq 2k+1$. Let $G=(V_1,V_2;E)$ be a bipartite graph with $|V_1|=|V_2|=n$. If the minimum degree of G is at least $s+1$, then G contains k vertex-disjoint cycles covering at least $\min(2n,4s)$ vertices of G.*

To demonstrate the sharpness of the minimum degree condition in Theorem B, we construct the bipartite graphs $H_{s,m}$ for positive integers s and m with $m \geq s+1$ as follows. Let $H_1=(A,B;E_1)$ and $H_2=(X,Y;E_2)$ be two vertex-disjoint complete bipartite graphs with $|A|=|Y|=s-1$ and $|B|=|X|=m$. Let b be a fixed vertex of B and x a fixed vertex of X. Then $H_{s,m}$ consists of H_1 and H_2 such that b is adjacent to every vertex of $X-\{x\}$ and x is adjacent to every vertex of $B-\{b\}$. Clearly, $\delta(H_{s,m})=s$. It is easy to see that any k vertex-disjoint cycles of $H_{s,m}$ contains no more than $4s$ vertices of $H_{s,m}$. When $s=k$, the examples $G_{k,m}$ in [11] shows that $\delta(G_{k,m})=k$ but it does not have k vertex-disjoint cycles. When $s>k$, we do not have appropriate bipartite graphs G such that $\delta(G)=s$ but any k vertex-disjoint cycles of G contains no more than $4s-1$ vertices.

We shall use the following terminology and notation. Let G be a graph. For a vertex $u \in V(G)$ and a subgraph H of G, $N(u,H)$ is the set of neighbors of u contained in H, i.e., $N(u,H)=N(u)\cap V(H)$. We let $d(u,H)=|N(u,H)|$. Thus $d(u,G)$ is the degree of u in G. Similarly, we define $N(u,X)$ and $d(u,X)$ for a subset X of $V(G)$. For a subset U of $V(G)$, $G[U]$ denotes the subgraph of G induced by U. Let C be a cycle of G. If u is a vertex of C and w is a vertex of $G-V(C)$ such that $C-u+w$ is hamiltonian, we say that u is replaceable by w (in C). We use $e(G)$ to denote the number of edges of G. If A and B are two disjoint subsets of $V(G)$, we use $e(A,B)$ to denote the number of edges of G between A and B.

2. Lemmas

Let $G=(V_1,V_2;E)$ be a bipartite graph in the following.

Lemma 2.1. *Let C be a cycle of order $2m$ in G. Let P be a path of even order in $G-V(C)$. Let x and y be the two endvertices of P. If $d(x,C)+d(y,C)\geq m$, then $G[V(C\cup P)]$ is hamiltonian unless either $d(x,C)=0$ or $d(y,C)=0$.*

Proof. Let $A=N(u,C)$ and $B=\{v|uv\in E(C)$ and $u\in A\}$. Clearly, if $N(y,C)\cap B\neq\emptyset$ then $G[V(C\cup P)]$ is hamiltonian. It is also clear that if $A\neq\emptyset$ and $|A|\neq m$ then $|B|>|A|$. If $G[V(C\cup P)]$ is not hamiltonian, then $d(x,C)+d(y,C)\leq |A|+m-|B|$. Furthermore, we see that either $A=\emptyset$ or $|A|=m$ since $d(x,C)+d(y,C)\geq m$, and therefore either $d(x,C)=0$ or $d(y,C)=0$. ∎

Lemma 2.2. *Let C be a cycle of order $2m$ in G and let x and y be two vertices in $G-V(C)$ such that $x\in V_1$ and $y\in V_2$. Suppose that $d(x,C)+d(y,C)\geq m+1$ and $G[V(C)\cup\{x,y\}]$ is not hamiltonian. Then there exist a labeling $C=a_1b_1a_2b_2\ldots a_mb_ma_1$ and an integer $r\in\{1,2,\ldots,m\}$ such that $N(x,C)=\{b_1,b_2,\ldots,b_r\}$ and $N(y,C)=\{a_1,a_{r+1},a_{r+2},\ldots,a_m\}$. Moreover, $a_ib_j\notin E$ for all $i\in\{2,\ldots,r\}$ and $j\in\{r+1,\ldots,m\}$.*

Proof. Let $C=a_1b_1a_2b_2\ldots a_mb_ma_1$. Then $d(x,C)+d(y,C)=\sum_{i=1}^{m}(d(x,a_ib_i)+d(y,a_ib_i))\geq m+1$. This implies that there exists $i\in\{1,2,\ldots,m\}$, say $i=1$, such that $d(x,a_1b_1)+d(y,a_1b_1)=2$. Say $\{xb_1,ya_1\}\subseteq E$. Then we see that $\{xb_i,ya_i\}\not\subseteq E$ for all $i\in\{2,3,\ldots,m\}$ for otherwise $G[V(C)\cup\{x,y\}]$ is hamiltonian. It follows that $d(x,a_ib_i)+d(y,a_ib_i)=1$ for all $i\in\{2,3,\ldots,m\}$.

Let r be the largest number in $\{1,2,\ldots,m\}$ such that $\{b_1,b_2,\ldots,b_r\}\subseteq N(x,C)$. If $r=m$, the lemma holds. We assume $r<m$. Then $ya_{r+1}\in E$ as $xb_{r+1}\notin E$. Let s be the largest number in $\{r+1,r+2,\ldots m\}$ such that $\{a_{r+1},a_{r+2},\ldots,a_s\}\subseteq N(y,C)$. We claim that $s=m$. If it is not true, then $xb_{s+1}\in E$ as $ya_{s+1}\notin E$, and we let t be the largest number in $\{s+1,s+2,\ldots,m\}$ such that $\{b_{s+1},b_{s+2},\ldots,b_t\}\subseteq N(x,C)$. If $t<m$, then $ya_{t+1}\in E$ as $xb_{t+1}\notin E$. If $t=m$, we let $a_{t+1}=a_1$ and so we still have $ya_{t+1}\in E$. Thus

$$xb_ra_rb_{r-1}\ldots a_2b_1a_1b_ma_m\ldots b_{t+1}a_{t+1}yb_{r+1}a_{r+1}\ldots a_tb_tx$$

is a hamiltonian cycle of $G[V(C)\cup\{x,y\}]$, a contradiction. Hence $s=m$. If there exist $i\in\{2,3,\ldots,r\}$ and $j\in\{r+1,r+2,\ldots,m\}$ such that $a_ib_j\in E$, then

$$a_ib_ja_jb_{j-1}a_{j-1}\ldots b_{r+1}a_{r+1}ya_{j+1}b_{j+1}\ldots a_mb_ma_1b_1\ldots$$
$$\ldots a_{i-1}b_{i-1}xb_ra_rb_{r-1}a_{r-1}\ldots b_ia_i$$

is a hamiltonian cycle of $G[V(C)\cup\{x,y\}]$, a contradiction. Therefore $a_ib_j\notin E$ for all $i\in\{2,3,\ldots,r\}$ and $j\in\{r+1,r+2,\ldots,m\}$. This proves the lemma. ∎

Lemma 2.3. *Let C be a cycle of order $2m$ in G and P a path of order $2t$ in $G-V(C)$. Let u and w be the two endvertices of P. Suppose that $G[V(C\cup P)]$ does not contain a cycle longer than C. If $d(u,C)>0$ and $d(w,C)>0$, then $m\geq 2t-1+d(u,C)+d(w,C)$.*

Proof. Let $C=a_1b_1\ldots a_mb_ma_1$ and $P=x_1y_1\ldots x_ty_t$ with $\{a_1,x_1\}\subseteq V_1$. We may assume that $y_ta_1\in E$ and $x_1b_j\in E$ for some $j\in\{1,\ldots,m\}$ such that $d(x_1,\{b_1,\ldots,b_{j-1}\})=0$ and $d(y_t,\{a_2,\ldots,a_j\})=0$. As $G[V(C\cup P)]$ does not contain a cycle longer than C, we see that $j\geq t+1$. Let p be the largest number in $\{1,\ldots,m\}$ such that $x_1b_p\in E$. For the same reason, we see that $m-p\geq t$ and $d(y_t,\{a_{p+1},\ldots,a_{p+t}\})=0$. Thus $m-p\geq t+d(y_t,\{a_{p+1},\ldots,a_m\})$. Clearly, $d(x_1,a_ib_i)+d(y_t,a_ib_i)\leq 1$ for each $i\in\{j+1,\ldots,p\}$. Thus $p-j\geq d(x_1,\{b_{j+1},\ldots,b_p\})+d(y_t,\{a_{j+1},\ldots,a_p\})$. As $j\geq t+1$, it follows that $m\geq 2t-1+d(x_1,C)+d(y_t,C)$. ∎

Lemma 2.4. *Let C be a cycle of order $2m$ in G and P a path of order $2t-1$ $(t\geq 2)$ in $G-V(C)$. Let u and w be the two endvertices of P. Suppose that $G[V(C\cup P)]$ does not contain a cycle longer than C. If $d(u,C)>0$, $d(w,C)>0$ and $|N(u,C)\cup N(w,C)|\geq 2$, then $m\geq 2(t-2)+d(u,C)+d(w,C)$.*

Proof. Let $C=a_1b_1\ldots a_mb_ma_1$ and $P=x_1y_1\ldots x_{t-1}y_{t-1}x_t$. W.l.o.g., say $\{a_1,x_1\}\subseteq V_1$. We may assume that $x_1b_1\in E$ and $x_tb_j\in E$ for some $j\in\{2,\ldots,m\}$ such that $d(x_1,\{b_2,\ldots,b_{j-1}\})=0$ and $d(x_t,\{b_2,\ldots,b_{j-1}\})=0$. As $G[V(C\cup P)]$ does not contain a cycle longer than C, we see that $j\geq t+1$. Let p be the largest number in $\{1,\ldots,m\}$ such that $x_tb_p\in E$. For the same reason, we see that $m-p\geq t-1$ and $d(x_1,\{b_{p+1},\ldots,b_{p+t-1}\})=0$. Thus $m-p\geq t-1+d(x_1,\{b_{p+1},\ldots,b_m\})$. Clearly, for each $i\in\{1,\ldots,m\}$, if $x_tb_i\in E$ then $x_1b_{i-1}\notin E$ and $x_1b_{i+1}\notin E$ since $G[V(C\cup P)]$ does not contain a cycle longer than C, where the subscripts are taken modulo m in $\{1,\ldots,m\}$. This implies that $p-j\geq d(x_1,\{b_{j+1},\ldots,b_p\})+d(x_t,\{b_{j+1},\ldots,b_p\})$. As $j\geq t+1$, it follows that $m\geq 2t+d(x_1,\{b_{j+1},\ldots,b_m\})+d(x_t,\{b_{j+1},\ldots,b_m\})$. Consequently, $m\geq 2(t-2)+d(x_1,C)+d(x_t,C)$. ∎

3. Proof of Theorem B

Let k and s be two integers with $s\geq k\geq 2$. Let $G=(V_1,V_2;E)$ be a bipartite graph with $|V_1|=|V_2|=n>2k$ and $\delta(G)\geq s+1$. Suppose, for a contradiction, that G does not contain k vertex-disjoint cycles covering at least

$\min(2n, 4s)$ vertices. By the result of [11] mentioned in the above introduction, G has k vertex-disjoint cycles. Hence $s > k$. We choose k vertex-disjoint cycles $C_1, \ldots, C_k$ such that

$$(1) \qquad \sum_{i=1}^{k} |V(C_i)| \text{ is maximum.}$$

Subject to (1), we choose $C_1, \ldots, C_k$ such that

(2) The length of a longest path of $G - V(\cup_{i=1}^{k} C_i)$ is maximum.

Subject to (1) and (2), we choose $C_1, \ldots, C_k$ such that

(3) The number of quadrilaterals in $\{C_1, \ldots, C_k\}$ is maximum.

Subject to (1), (2) and (3), we finally choose $C_1, \ldots, C_k$ such that

$$(4) \qquad \sum_{i=1}^{k} e(G[V(C_i)]) \text{ is maximum.}$$

Let $H = \cup_{i=1}^{k} C_i$ and $D = G - V(H)$. Let P be a longest path of D. Say $|V(C_i)| = 2m_i$ for each $i \in \{1, \ldots, k\}$ and let $m = \sum_{i=1}^{k} m_i$. Then $m < 2s$. We divide our proof into the following two cases.

Case I. The order of P is one, i.e., $e(D) = 0$.

Let $x \in V(D) \cap V_1$ and $y \in V(D) \cap V_2$. Then $d(x, H) + d(y, H) \geq 2s + 2$. By Lemma 2.2, $d(x, C_i) + d(y, C_i) \leq m_i + 1$ for each $i \in \{1, \ldots, k\}$. As $m < 2s$, we see that there exist C_p and C_q in H with $p \neq q$ such that $d(x, C_p) + d(y, C_p) = m_p + 1$ and $d(x, C_q) + d(y, C_q) = m_q + 1$. For the sake of convenience, say $\{p, q\} = \{1, 2\}$. By Lemma 2.2, there exist a labeling of $C_1 = a_1 b_1 \ldots a_{m_1} b_{m_1} a_1$ and $r \in \{1, \ldots, m_1\}$ such that $N(x, C_1) = \{b_1, \ldots, b_r\}$ and $N(y, C_1) = \{a_1, a_{r+1}, \ldots, a_{m_1}\}$. Furthermore, we have that $e(\{a_2, \ldots, a_r\}, \{b_{r+1}, \ldots, b_{m_1}\}) = 0$. Clearly, a_i is replaceable by x and b_j is replaceable by y for each $i \in \{2, \ldots, r\}$ and $j \in \{r+1, \ldots m_1\}$. Thus by (4), $d(a_i, C_1) \geq d(x, C_1)$ and $d(b_j, C_1) \geq d(y, C_1)$ for each $i \in \{2, \ldots, r\}$ and $j \in \{r+1, \ldots, m_1\}$. It follows that if $G_1 = G[\{a_2, \ldots, a_r\} \cup \{b_1, \ldots, b_{r-1}\}]$ and $G_2 = G[\{a_{r+1}, \ldots, a_{m_1}\} \cup \{b_{r+1}, \ldots, b_{m_1}\}]$, then G_1 and G_2 are two complete bipartite graphs. Clearly, $m_1 \geq 3$ for otherwise $G[V(C_1) \cup \{x, y\}]$ contains a quadrilateral and a path of order 2 such that they are vertex-disjoint, contradicting (2). Since $G[V(C_2) \cup \{x, y\}]$ has a hamiltonian path from x to y, we readily see that $G[V(C_1 \cup C_2) \cup \{x, y\} - V(G_i)]$ is hamiltonian for each $i \in \{1, 2\}$. Hence G_1 and G_2 must be of order 2 for otherwise (1) is violated.

Therefore C_1 is of order 6 and $r=2$. Thus $d(a_2,D)=0$ and $d(b_3,D)=0$ for otherwise $G[V(C_1 \cup D)]$ contains a cycle of order 6 and a path of order 2 such that they are vertex-disjoint. Let $Q=xb_1a_2b_2x$, $C_1'=ya_1b_3a_3y$ and $m_1'=2$. Set $C_i'=C_i$ and $m_i'=m_i$ for each $i\in\{2,\dots,k\}$. Say $m'=\sum_{i=1}^{k} m_i'$. Clearly, $m'=m-1\le 2s-2$. Similarly, we have $m_2=3$. By (1) and Lemma 2.1, we have

$$(5)\quad d(u,C_i')+d(w,C_i')\le m_i' \text{ for each } i\in\{1,\dots,k\} \text{ and } uw\in E(Q).$$

As $m'\le 2s-2$, see that $d(u,Q\cup D)+d(w,Q\cup D)\ge 4$ for each $uw\in E(Q)$. First, let us assume that $N(b_1,D)=N(b_2,D)=\{x\}$. Then equality must hold in (5). By Lemma 2.1, $d(u,C_i)=0$ or $d(w,C_i)=0$ for each $i\in\{1,\dots,k\}$ and $uw\in E(Q)$. Then it is easy to see that $G[V(C_2'\cup Q)]$ contains a cycle C_2'' of order 8. Replacing C_1 and C_2 by C_1' and C_2'' in the set $\{C_1,\dots,C_k\}$, we see that (3) is violated while (1) and (2) are maintained.

Therefore, we must have that either $d(b_1,D)\ge 2$ or $d(b_2,D)\ge 2$. W.l.o.g., say the latter holds. Let $P'=xb_1a_2b_2z$ where $z\in V(D)$ and $z\neq x$. By (1) and Lemma 2.4, we have

$$(6)\qquad d(x,C_i')+d(z,C_i')\le m_i' \text{ for each } i\in\{1,\dots,k\}.$$

Since $m'\le 2s-2$, $d(x,D)=0$ and $d(z,D)=0$, equality must hold in (6). Moreover, $d(z,Q)=2$. We claim that for each $i\in\{1,\dots,k\}$, either $d(x,C_i')=0$ or $d(z,C_i')=0$. If this is false, say $d(x,C_i')>0$ and $d(z,C_i')>0$ for some $i\in\{1,\dots,k\}$. By (1) and Lemma 2.4, $N(x,C_i')=N(z,C_i')=\{v\}$ for some $v\in V(C_i')$. Since equality holds in (6), we obtain that $m_i'=2$. Clearly, $G[V(P'\cup C_i')]$ contains a cycle of order 6 and a path of order 3. This would violate (2) while (1) is maintained. Hence the claim holds. As $P''=xb_1zb_2a_2$ and $P'''=a_2b_1xb_2z$ are two paths of $G[V(P')]$ and $d(a_2,D)=0$, we may repeat this argument with P' replaced by either of P'' and P'''. Then equality in (6) must hold when x or z is replaced by a_2, and similar claims follow, too. Clearly, $d(u,\cup_{i=1}^{k}C_i')\ge s-1$ for each $u\in\{x,a_2,z\}$. It follows that $m'\ge 3(s-1)\ge 2s$, a contradiction.

Case II. The order of P is at least 2.

In this case, if u and w are the two endvertices of P, we define $r(P)=d(u,P)+d(w,P)$. We choose a longest path P of D with $r(P)$ as large as possible. When P is of even order, let $P=x_1y_1\dots x_ty_t$ with $x_1\in V_1$. When P is of odd order, let $P=x_1y_1\dots x_{t-1}y_{t-1}x_t$ with $x_1\in V_1$. Set $r_1=d(x_1,P)$ and $r_2=d(w,P)$ where w is the other endvertex of P. Thus $r(P)=r_1+r_2$.

We may assume $r_1 \geq r_2$. Clearly, $r_1 \leq s-1$ for otherwise D has a cycle of order at least $2s$, and then by (1), $m \geq m_1+m_2 \geq 2s$. Thus

$$(7) \qquad d(x_1,H) \geq s+1-r_1 \geq 2 \text{ and } d(w,H) \geq s+1-r_2 \geq 2.$$

We now break into the following two subcases.

Case 2.1. The order of P is even.

In this subcase, $P = x_1y_1,\dots,x_ty_t$, $w = y_t$ and $t \geq r_1$. By (1) and Lemma 2.1, we see that $d(x_1,C_i)+d(y_t,C_i) \leq m_i$ for all $i \in \{1,\dots,k\}$. It follows that $r_1 \geq 2$ as $m \leq 2s-1$. Thus D has a cycle of order at least $2r_1$. Therefore $kr_1 < 2s$, i.e., $r_1 < 2s/k$. Let us assume that there exists C_i in H, say $C_i = C_1$, such that $d(x_1,C_1) > 0$ and $d(y_t,C_1) > 0$. By Lemma 2.3, $m_1 \geq 2t-1+d(x_1,C_1)+d(y_t,C_1)$. Then

$$m \geq 2t-1+d(x_1,H)+d(y_t,H) \geq 2t-1+2(s+1)-(r_1+r_2) \geq 2s+1,$$

a contradiction. Therefore, for each $i \in \{1,\dots,k\}$, either $d(x_1,C_i) = 0$ or $d(y_t,C_i) = 0$. Let p be the largest number in $\{1,\dots,t\}$ such that $x_1y_p \in E$. Set

$$P' = x_py_{p-1}\dots x_2y_1x_1y_px_{p+1}y_{p+1}\dots x_ty_t \text{ and } P'' = x_1y_1x_2\dots y_{p-1}x_p.$$

Then P' is a longest path of D, too. By the maximality of $r(P)$, we see that $d(x_p,P) \leq r_1 \leq p$. As in (7), we must have that $d(x_p,H) \geq s+1-r_1 \geq 2$. Similarly, it also holds that for each $i \in \{1,\dots,k\}$, either $d(x_p,C_i) = 0$ or $d(y_t,C_i) = 0$. Let us consider the relation between P'' and each C_i. Suppose that there exists C_i in H, say $C_i = C_1$, such that $d(x_1,C_1) > 0$, $d(x_p,C_1) > 0$ and $|N(x_1,C_1) \cup N(x_p,C_1)| \geq 2$. By Lemma 2.4, $m_1 \geq 2(p-2)+d(x_1,P)+d(x_p,C_1)$. Then

$$\begin{aligned} m &\geq 2(p-2)+d(x_1,H)+d(x_p,H)+d(y_t,H) \\ &\geq 3(s+1)-2r_1-r_2+2(p-2) \geq 2s, \end{aligned}$$

a contradiction. Therefore for each $i \in \{1,\dots,k\}$, if $d(x_1,C_i) > 0$ and $d(x_p,C_i) > 0$ then $N(x_1,C_i) = N(x_p,C_i) = \{a_i\}$ for some $a_i \in V(C_i)$. Let

$$A = \{i | d(x_1,C_i) > 0 \text{ or } d(x_p,C_i) > 0; 1 \leq i \leq k\}.$$

Set $B = \{1,\dots,k\} - A$. Since $d(x_1,H) \geq 2$, $d(x_p,H) \geq 2$ and $d(y_t,H) \geq 2$, we conclude that $|A| \geq 2$ and $|B| \geq 1$. Thus

$$(8) \qquad \sum_{i \in A} m_i \geq d(x_1,H)+d(x_p,H) \geq 2(s+1)-2r_1;$$

$$(9) \qquad \sum_{i \in B} m_i \geq d(y_t,H) \geq s+1-r_2.$$

We claim that $r_2 \geq 3$. If this is false, then (8) and (9) imply that $r_1 > (s+1)/2$ since $m \leq 2s-1$. Since D has a cycle of order at least $2r_1$, it follows that $m > |A|(s+1)/2 + (s+1-r_2) \geq 2s$, a contradiction. Therefore $r_2 \geq 3$. Let q be the smallest number in $\{1, \ldots, t-1\}$ such that $y_t x_q \in E$. Set $P^{(3)} = x_1 y_1 \ldots x_q y_t x_t y_{t-1} \ldots x_{q+1} y_q$ and $P^{(4)} = y_q x_{q+1} \ldots x_t y_t$. Repeating the above argument with $P^{(3)}$ and $P^{(4)}$ playing the role of P' and P'' , we can readily show that for each C_i in H, either $d(x_1, C_i) = 0$ or $d(y_q, C_i) = 0$. Furthermore, for each C_i in H, if $d(y_q, C_i) > 0$ and $d(y_t, C_i) > 0$, then $N(y_q, C_i) = N(y_t, C_i) = \{b_i\}$ for some $b_i \in V(C_i)$. Assume that $q > p$. Then $x_p y_{p-1} \ldots y_1 x_1 y_p x_{p+1} \ldots x_q y_t x_{t-1} \ldots x_{q+1} y_q$ is a hamiltonian path of $G[V(P)]$. Again, we can show, as above, that for each C_i in H, either $d(x_p, C_i) = 0$ or $d(y_q, C_i) = 0$. Thus $d(y_q, C_i) = 0$ for all $i \in A$. As $d(y_t, H) \geq 2$, it follows that $|B| \geq 2$. Hence $k \geq 4$, and consequently, $r_1 < s/2$. But then

$$\begin{aligned} m &\geq \sum_{i \in A} m_i + \sum_{i \in B} m_i \\ &\geq \sum_{u \in R} d(u, H) \geq 4(s+1) - 2r_1 - 2r_2 > 2s \end{aligned}$$

where $R = \{x_1, x_p, y_q, y_t\}$, a contradiction. Hence $q \leq p$. Clearly, the above calculation is still valid if for each $i \in \{1, \ldots, k\}$, we still have that either $d(x_p, C_i) = 0$ or $d(y_q, C_i) = 0$. Therefore we may assume that there exists C_i in H, say $C_i = C_1$, such that $d(x_p, C_1) > 0$ and $d(y_q, C_1) > 0$. By Lemma 2.3, $m_1 \geq 2\lceil (p+1)/2 \rceil - 1 + d(x_p, C_1) + d(y_q, C_1)$ since D has a path of order at least $2\lceil (p+1)/2 \rceil$ from x_p to y_q. Then

$$\begin{aligned} m &\geq 2\lceil (p+1)/2 \rceil - 1 + \sum_{u \in R} d(u, H) \\ &\geq 4(s+1) + p - 2r_1 - 2r_2 \geq 4s + 4 - 3r_1. \end{aligned}$$

As $k \geq |A| + |B| \geq 3$, we have that $3r_1 < 2s$. It follows that $m > 2s$, a contradiction.

Case 2.2. The order of P is odd.

In this case, $P = x_1 y_1 \ldots x_{t-1} y_{t-1} x_t$, $w = x_t$ and $t \geq r_1 + 1$. Suppose that there exits C_i in H such that $d(x_1, C_i) > 0$, $d(x_t, C_i) > 0$ and $|N(x_1, C_i) \cup N(x_t, C_i)| \geq 2$. By Lemma 2.4, we get

$$m \geq 2(t-2) + d(x_1, H) + d(x_t, H) \geq 2(t-2) + 2(s+1) - r_1 - r_2 \geq 2s,$$

a contradiction.

Therefore for each $i \in \{1, \ldots, k\}$, either $d(x_1, C_i) = 0$, or $d(x_t, C_i) = 0$, or $N(x_1, C_i) = N(x_t, C_i) = \{a_i\}$ for some $a_i \in V(C_i)$. Furthermore, for $i \in$

$\{1,\ldots,k\}$, if $N(x_1,C_i)=N(x_t,C_i)=\{a_i\}$ then a_iPa_i is a cycle of G and therefore $m_i\geq t$ by (1). Since $2s-1\geq m\geq d(x_1,H)+d(x_t,H)$ by Lemma 2.4 and therefore $2s-1\geq 2(s+1)-r_1-r_2$, we see that $r_1\geq 2$ and therefore $t\geq 3$. Let p be the largest number in $\{1,\ldots,t-1\}$ such that $x_1y_p\in E$. Clearly, $p\geq r_1$ and $G[V(P)]$ has a hamiltonian path from x_p to x_t. Thus for each $i\in\{1,\ldots,k\}$, either $d(x_p,C_i)=0$, or $d(x_t,C_i)=0$, or $N(x_p,C_i)=N(x_t,C_i)=\{b_i\}$ for some $b_i\in V(C_i)$. By Lemma 2.4 and the above argument, it follows that $m\geq d(x_1,H)+d(x_p,H)+d(x_t,H)$. If there exists C_i in H such that $d(x_1,C_i)>0$, $d(x_p,C_i)>0$ and $|N(x_1,C_i)\cup N(x_p,C_i)|\geq 2$, then by (7) and Lemma 2.4, we further obtain

$$\begin{aligned} m &\geq 2(p-2)+d(x_1,H)+d(x_p,H)+d(x_t,H)\\ &\geq 2(p-2)+2(s+1-r_1)+(s+1)-r_2\geq 2s, \end{aligned}$$

a contradiction. Therefore for each $i\in\{1,\ldots,k\}$, either $d(x_1,C_i)=0$, or $d(x_p,C_i)=0$, or $N(x_1,C_i)=N(x_p,C_i)=\{c_i\}$ for some $c_i\in V(C_i)$.

If $k=2$, then from the above, we see that $d(x_1,C_i)=d(x_p,C_i)=d(x_t,C_i)=1$ for each $i\in\{1,2\}$. Furthermore, a_1Pa_1 and a_2Pa_2 are two cycles of order $2t$ in G. As $t-1\geq r_1\geq s+1-2=s-1$, we obtain that $m\geq 2t\geq 2s$ by (1), a contradiction.

Therefore $k\geq 3$. We claim that $r_2\geq 3$. On the contrary, say $r_2\leq 2$. Then $d(x_t,H)\geq s+1-r_2\geq s-1$. Since $2s-1\geq m\geq d(x_1,H)+d(x_p,H)+d(x_t,H)\geq 2(s+1-r_1)+s-1$, we get that $r_1\geq (s+2)/2$. Since $m_i\geq r_1$ for each $i\in\{1,\ldots,k\}$, we see that $k=3$ as $m\leq 2s-1$. As $m_1+m_2\geq s+2$, we see that $d(x_t,C_1\cup C_2)>0$ for otherwise $m\geq s+2+d(x_t,H)\geq 2s+1$. Say w.l.o.g. $d(x_t,C_1)>0$. Similarly, we have that $d(x_t,C_2\cup C_3)>0$, say $d(x_t,C_2)>0$. As $3(s-1)\geq 2s$, we must have that $r_1\leq s-2$ for otherwise $m\geq 3r_1\geq 2s$. Thus $d(u,H)\geq 3$ for each $u\in\{x_1,x_p,x_t\}$. It follows from the above argument that $d(u,C_i)=1$ for all $u\in\{x_1,x_p,x_t\}$ and $i\in\{1,2,3\}$. Therefore $r_1=r_2=s-2$. Since a_iPa_i is a cycle of G for each $i\in\{1,2,3\}$, we see, by (1), that $m_i\geq t\geq s-1$ for each $i\in\{1,2,3\}$, and therefore $m\geq 3(s-1)\geq 2s$, a contradiction. This proves that $r_2\geq 3$.

As $r_1\geq r_2\geq 3$, we see $t\geq 4$. Let q be an integer in $\{1,2,\ldots,t-2\}$ such that $x_ty_q\in E$. We first suppose that $q\geq p$. In this situation, it is clear that $G[V(P)]$ has a hamiltonian path from x to x_{q+1} for each $x\in\{x_1,x_p\}$. As argued in the above, we see that for each $i\in\{1,\ldots,k\}$ and $x\in\{x_1,x_p\}$, either $d(x,C_i)=0$, or $d(x_{q+1},C_i)=0$, or $d(x,C_i)=d(x_{q+1},C_i)=1$ with $N(x,C_i)=N(x_{q+1},C_i)$. As $t\geq 4$ and by Lemma 2.4, we readily see

$$2s-1\geq m\geq d(x_1,H)+d(x_p,H)+d(x_{q+1},H)+d(x_t,H)\geq 4(s+1)-2r_1-2r_2.$$

This implies that $r_1>s/2+1$, and thus $k=3$. By Lemma 2.4, $m_3\geq d(x_{q+1},C_1)+d(x_t,C_1)$. Since $m_1+m_2\geq 2r_1$, we must have that

$d(x_{q+1},C_3)+d(x_t,C_3)<2(s-r_1)$. Thus $d(x_{q+1},C_1\cup C_2)+d(x_t,C_1\cup C_2)>2(s+1-r_2)-2(s-r_1)>0$. W.l.o.g., say $d(x_{q+1},C_1)+d(x_t,C_1)>0$. Similarly, we can show that $d(x_{q+1},C_2\cup C_3)+d(x_t,C_2\cup C_3)>0$. W.l.o.g., say $d(x_{q+1},C_2)+d(x_t,C_2)>0$. As $3r_1<2s$, we have that $r_1\le s-2$. Thus $d(x,H)\ge 3$ for each $x\in\{x_1,x_p,x_{q+1},x_t\}$. In summation of the above argument, we conclude that $d(x,C_i)=1$ for all $i\in\{1,2,3\}$ and $x\in\{x_1,x_p,x_{q+1},x_t\}$. It follows that $r_1=r_2=s-2$. Since a_iPa_i is a cycle of G for each $i\in\{1,2,3\}$, we must have that $m_i\ge t\ge s-1$ for each $i\in\{1,2,3\}$, and consequently, $m\ge 3(s-1)\ge 2s$, a contradiction.

Therefore $x_ty_i\notin E$ for all $i\in\{p,\dots,t-1\}$. We may now choose q to be the smallest integer in $\{1,\dots,t-1\}$ with $x_ty_q\in E$. As $r_2\ge 3$, $q\le p-2$. Then $G[V(P)]$ still has a hamiltonian path from x_1 to x_{q+1}. If it is still the case that for each $i\in\{1,\dots,k\}$, either $d(x_p,C_i)=0$, or $d(x_{q+1},C_i)=0$, or $d(x_p,C_i)=d(x_{q+1},C_i)=1$ with $N(x_p,C_i)=N(x_{q+1},C_i)$, then the above argument still prevails and it follows that $m\ge 3(s-1)\ge 2s$. Therefore, there must exist C_i in H, say C_1, such that $d(x_p,C_1)>0$, $d(x_{q+1},C_1)>0$ and $|N(x_p,C_1)\cup N(x_{q+1},C_1)|\ge 2$. Clearly, $G[V(P)]$ has a path of order at least $2\lceil (p+1)/2\rceil -1$ from x_p to x_{q+1}. By Lemma 2.4, $m_1\ge 2(\lceil (p+1)/2\rceil -2)+d(x_p,C_1)+d(x_{q+1},C_1)$. By Lemma 2.4, it follows

$$\begin{aligned} m &\ge 2(\lceil (p+1)/2\rceil -2)+d(x_1,H)+d(x_p,H)+d_{q+1},H)+d(x_t,H)\\ &\ge p-3+4(s+1)-2r_1-2r_2\\ &\ge 4s+1-3r_1. \end{aligned}$$

Since $3r_1<2s$ by (1), it follows that $m\ge 4s+1-3r_1\ge 2s+2$, a contradiction. This proves the theorem.

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Hong Wang

Department of Mathematics
The University of Idaho
Moscow, Idaho 83844
USA
hwang@uidaho.edu