COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA **22** (4) (2002) 591–596

ON THE CHROMATIC NUMBER OF TRIANGLE-FREE GRAPHS OF LARGE MINIMUM DEGREE

CARSTEN THOMASSEN

Received September 14, 1999

We prove that, for each fixed real number $c > 1/3$, the triangle-free graphs of minimum degree at least cn (where n is the number of vertices) have bounded chromatic number. This problem was raised by Erdős and Simonovits in 1973 who pointed out that there is no such result for $c < 1/3$.

1. Introduction

It is well-known that there exist triangle-free graphs of arbitrarily large chromatic number, see e.g. Bondy and Murty [3] or Jensen and Toft [9]. Hajnal (see [5]) used the Kneser graphs to show that such graphs may have minimum degree close to $n/3$. Erdős and Simonovits [5] conjectured that this is best possible. For each natural number t, let c_t be the smallest number such that every triangle-free graph with n vertices and minimum degree $> c_t n$ has chromatic number $< t$. We prove that $c_t \to 1/3$ as $t \to \infty$.

2. Generalized pentagons

Our terminology is that of Bondy and Murty [3]. In particular, $\alpha(G)$ denotes the independence number, that is the largest number of vertices in a vertex set in which no two vertices are joined by an edge. Also, if v is a vertex in G, then $N(v,G)$ denotes the set of neighbors of v. The *degree of v in G* is the number of neighbors of v in G and is denoted $d(v,G)$. Two vertices of G

Mathematics Subject Classification (2000): 05C15, 05C35 are G-*equivalent* if they are nonadjacent and have the same neighbors. We say that G is α-*dominated* if, for every set S of $\alpha(G)$ vertices in G, there is a vertex v in G such that $N(v,G) \subset S$. We define a *generalized pentagon* as follows: The 5-cycle (pentagon) is a generalized pentagon. Suppose that G is a generalized pentagon and that v is a vertex of G. We define a new graph P by adding three new vertices w,x,y to G. We add all edges from w to $N(v,G)$ and we add the edges of the path $vxyw$. The idea of the proof is to find a large generalized pentagon in the dense triangle-free graph under consideration and then show that either this generalized pentagon contains a vertex of not too large degree, or else it has a coloring in few colors such that that coloring can be extended to a coloring of the whole graph in few colors. For that we need some properties of generalized pentagons.

Proposition 2.1. *Let P be a generalized pentagon with $3k-1$ vertices. Then the following hold.*

(a) *P has no triangle.*

(b) $\alpha(P)=k$.

(c) *P is α-dominated.*

(d) *If P is a spanning subgraph of a triangle-free graph H, then H has maximum degree at most k.*

Proof. The proof is by induction on k. For $k=2$ the statement is trivial, so we proceed to the induction step. Let P be obtained from G by adding the three vertices w,x,y where w and v have the same neighbors within G, and P contains the path $vxyw$. By the induction hypothesis, G satisfies (a), (b),(c), (d) with $k-1$ instead of k. It is easy to see that P satisfies (a).

To prove (b), we first note that $\alpha(P) > \alpha(G) = k-1$. Now let S be a largest set of independent vertices in P. We may assume that S contains at most one of the vertices w,x,y since w and v may interchange roles. By the induction hypothesis, $S \cap V(G)$ has at most $k-1$ vertices. This proves (b).

We now prove (c). Recall that the above set S contains at most one of the vertices w,x,y. If S contains x or y but not v, then we apply the induction hypothesis to G and $S \cap V(G)$. If S contains y and v, then S contains $N(x,P)$. Finally, if S contains w, then S also contains v and again, we apply the induction hypothesis to G and $S \cap V(G)$. This proves (c).

(d) follows from the facts that $\alpha(H) \le \alpha(P) = k$ and every neighborhood in H is an independent set of vertices. ∎

3. Bounding the chromatic number

Theorem 3.2. *Let c be any fixed real number, $c>1/3$. Then the triangle-free graphs of minimum degree $> cn$ (where n is the number of vertices) have bounded chromatic number.*

Proof. Let M be a triangle-free graph with n vertices and minimum degree $> cn$. We shall prove that the chromatic number of M is less than $2(3c-1)^{-(4c-1)/(3c-1)}$. Without loss of generality we may assume that M is maximal triangle-free. In other words, M has diameter 2. If M has chromatic number 2, there is nothing to prove. So we may assume that M has an odd cycle. By considering a path of length 3 in a shortest (and hence chordless) odd cycle and using the fact that M has diameter 2, we conclude that a shortest odd cycle in M is a pentagon. Let P be a largest generalized pentagon in M, and let H be the subgraph of M spanned by P. (That is, H has the same vertices as P, and contains all those edges in M that join two vertices of P.) Assume that P has $3k-1$ vertices where $k>1$.

We now give an upper bound on $|V(H)|$ independent of n.

$$|V(H)| = 3k-1 < 1/(3c-1) \tag{1}$$

To prove (1), we consider the number $s=\sum d(x,M)$ where the sum is taken over all vertices x in H. By Proposition 2.1 (b), the number of edges from $M-V(H)$ to H is at most $k(n-3k+1)$. By Proposition 2.1 (d), H has maximum degree at most k. Hence

$$s \le k(n-3k+1)+k(3k-1) = kn.$$

By the assumption of Theorem 3.2,

$$s > cn(3k-1).$$

By combining these two inequalities, we obtain (1).

We partition $V(G)\setminus V(H)$ into two sets A, B. A is the set of vertices in $V(G)\setminus V(H)$ having precisely k neighbors in H. By Proposition 2.1 (b), every vertex in B has less than k neighbors in H. We now derive an upper bound for $|B|$.

$$|B| < n(c+k(1-3c)) < cn \tag{2}$$

To prove (2) we repeat the proof of (1):

$$cn(3k-1) < s = \sum d(x,M) \le k(3k-1)+k|A|+(k-1)|B| =$$
$$k(3k-1)+k(n-(3k-1)-|B|)+(k-1)|B| = kn-|B|.$$

Hence

$$|B| < kn - cn(3k-1) = n(c + k(1-3c)) < cn$$

which proves (2).

Let $x_1, x_2, \dots, x_{3k-1}$ denote the vertices of H. We partition A into sets $A_1, A_2, \dots, A_{3k-1}$ as follows. Let x be any vertex of A. By Proposition 2.1 (c), there is a vertex x_i in P such that the set of neighbors of x in P contains $N(x_i, P)$ (but not necessarily $N(x_i, H)$). Here, i need not be unique but we choose one such i and we let x belong to A_i. Since any two vertices of A_i have a common neighbor in P it follows that A_i is an independent set for each $i = 1, 2, \dots, 3k-1$.

For each $i = 1, 2, \dots, 3k–1$, we partition $A_i \cup \{x_i\}$ into sets $A_{i,1}, A_{i,2}, \dots, A_{i,q_i}$ as follows: Two vertices u, v of $A_i \cup \{x_i\}$ belong to the same $A_{i,j}$ if and only if u and v have the same M-neighbors in H. Clearly,

(3) $$q_i < (3k-1)^k \text{ for each } i = 1, 2, \dots, 3k-1.$$

We claim that

(4) For each $i = 1, 2, \dots, 3k-1$ and each $j = 1, 2, \dots, q_i$, any two vertices v, w in $A_{i,j}$ are M-equivalent.

Suppose (reductio ad absurdum) that x is a vertex joined to v but not w. By the definition of $A_{i,j}$, v and w have the same neighbors in H and hence x is not in H. Let xyw be a path of length 2 in M. Again, by the definition of $A_{i,j}$, y is not in H. Since v is joined to each vertex in $N(x_i, P)$ (by the definition of A_i), it follows that v may replace x_i in P. In other words, we may assume that $v = x_i$. Adding the path $vxyw$ and all edges from w to $N(x_i, P)$ to P gives a contradiction to the maximality property of P. This proves (4).

For each $i = 1, 2, \dots, 3k-1$ and each $j = 1, 2, \dots, q_i$, we define $B_{i,j}$ as the set of vertices of B that have a neighbor in $A_{i,j}$ but not in any set with a smaller index (in the lexicographic ordering). By (2) and the assumption that all vertices in G have degree $> cn$, every vertex in B is in some set $B_{i,j}$, and by (4), $B_{i,j}$ is a set of independent vertices for each $i = 1, 2, \dots, 3k-1$ and each $j = 1, 2, \dots, q_i$. Now we color M by giving two vertices the same color if and only if they belong to the same set $A_{i,j}$ or $B_{i,j}$, $i = 1, 2, \dots, 3k–1$, $j = 1, 2, \dots, q_i$. By (3) and (1), this number of colors is less than $2(3k–1)^{k+1} < 2(3c-1)^{-(4c-1)/(3c-1)}$. This completes the proof of Theorem 3.2. ∎

4. Remarks and open problems

We refer the reader to Brandt [1] for the history of dense triangle free graphs and open problems in the area.

Below we emphasize two open problems which are particularly relevant to the result of the present paper.

A graph G is *homomorphic* to a graph H if there exists a map $f:V(G)\to V(H)$ such that any two adjacent vertices are mapped to adjacent vertices. Note that f needs neither be $1-1$ nor onto. Also note that the largest clique size in H and the chromatic number of H must be at least as large as the corresponding numbers for G. Theorem 3.2 would follow from an affirmative answer to the following question:

Question 1. Let $c>1/3$ be any fixed constant. Does there exist a finite family $\mathcal{F}_c$ of triangle-free graphs such that every triangle-free graph with n vertices and minimum degree $>cn$ is homomorphic to some graph in $\mathcal{F}_c$?

An affirmative answer to Question 1 was conjectured by Jin [7], and a stronger quantitative version was conjectured by Brandt [2]. The generalized Möbius ladders (that is, the graphs with vertices $p_1,p_2,\dots,p_{3q-1}$, and all edges $p_ip_{i+j}, i=1,2,\dots,3q-1, j=q,q+1,\dots,2q-1$, where the indices are expressed modulo $3q-1$) show that Question 1 has a negative answer for $c=1/3$.

Question 2. Do the graphs with n vertices and minimum degree $>n/3$ have bounded chromatic number?

Erdős and Simonovits [5] conjectured that these graphs are 3-colorable but that was disproved by Häggkvist [6]. Jin [8] conjectured that there is no upper bound for the chromatic number, whereas Brandt [2] conjectures that 4 is an upper bound.

References

[1] S. Brandt: On the structure of dense triangle-free graphs, *Combinatorics, Probability, and Computing,* **8** (1999), 237–245.

[2] S. Brandt: A 4-colour problem for dense triangle-free graphs, *Discrete Math.,* to appear.

[3] J. A. Bondy and U. S. R. Murty: *Graph Theory with Applications,* Macmillan, London, 1976.

[4] P. Erdős: On some problems in graph theory, combinatorial analysis and combinatorial number theory, *Graph Theory and Combinatorics, Proceedings of the Cambridge Combinatorial Conference in Honour of Paul Erdős* (B. Bollobás ed.), Academic Press, London (1984), 1–17.

[5] P. Erdős and M. Simonovits: On a valence problem in extremal graph theory, *Discrete Math.,* **5** (1973), 323–334.

[6] R. Häggkvist: Odd cycles of specified length in non-bipartite graphs, *Annals of Discrete Mathematics* (B. Bollobás ed.), North-Holland, Amsterdam **13** (1982), 89–99.
[7] G. Jin: Triangle-free graphs with high minimal degrees, *Combinatorics, Probability, and Computing,* **2** (1993), 479–490.
[8] G. Jin: Triangle-free 4-chromatic graphs, *Discrete Math.,* **145** (1995), 151–170.
[9] T. Jensen and B. Toft: *Graph Coloring Problems,* John Wiley, New York, 1995.

Carsten Thomassen

Department of Mathematics
Technical University of Denmark
Lyngby
Denmark
C.Thomassen@mat.dtu.dk